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On the well-posedness of higher order viscous Burgers' equations ☆

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ABSTRACT

We consider higher order viscous Burgers' equations with generalized nonlinearity and study the associated initial value problems for given data in the L^2-based Sobolev spaces. We introduce appropriate time weighted spaces to derive multilinear estimates and use them in the *contraction mapping principle* argument to prove local well-posedness for data with Sobolev regularity below L^2. We also prove ill-posedness for this type of models and show that the local well-posedness results are sharp in some particular cases viz., when the orders of dissipation p, and nonlinearity $k+1$, satisfy a relation $p = 2k+1$.

1. Introduction

In continuation to our recent works [5,6], here we consider higher order viscous Burgers' equations with generalized nonlinearity which are also known as generalized Korteweg–de Vries (KdV) type equations with dissipative perturbation. These sort of models are well studied in the recent literature, see for example [7,14,19,21] and references therein. The authors in [14,21] considered generalization in the dissipative part, while the authors in [7,19] studied generalization in the nonlinearity. In this work, we are interested in considering generalization in both dissipative as well as nonlinear parts and address the well-posedness issues for the initial value problems (IVPs),

$$\begin{cases} v_t + v_{xxx} + \eta L v + (v^{k+1})_x = 0, \quad x \in \mathbb{R},\ t \geqslant 0,\ k \in \mathbb{N},\ k > 1, \\ v(x,0) = v_0(x), \end{cases} \tag{1.1}$$

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and

$$\begin{cases} u_t + u_{xxx} + \eta L u + (u_x)^{k+1} = 0, & x \in \mathbb{R},\ t \geqslant 0,\ k \in \mathbb{N},\ k > 1, \\ u(x,0) = u_0(x), \end{cases} \tag{1.2}$$

where $\eta > 0$ is a constant; $u = u(x,t)$, $v = v(x,t)$ are real-valued functions and the linear operator L is defined via the Fourier transform by $\widehat{Lf}(\xi) = -\Phi(\xi)\widehat{f}(\xi)$.

The Fourier symbol $\Phi(\xi)$ is of the form

$$\Phi(\xi) = -|\xi|^p + \Phi_1(\xi), \tag{1.3}$$

where $p \in \mathbb{R}^+$ and $|\Phi_1(\xi)| \leqslant C(1 + |\xi|^q)$ with $0 \leqslant q < p$. The symbol $\Phi(\xi)$ is a real-valued function which is bounded above; i.e., there is a constant C such that $\Phi(\xi) < C$ (see Lemma 2.2 below). We note that, a particular case of $\Phi(\xi)$ in the form

$$\tilde{\Phi}(\xi) = \sum_{j=0}^{n} \sum_{i=0}^{2m} c_{i,j} \xi^i |\xi|^j, \quad c_{i,j} \in \mathbb{R},\ c_{2m,n} = -1, \tag{1.4}$$

with $p := 2m + n$, has been considered in our earlier work [4].

We observe that, if u is a solution of (1.2) then $v = u_x$ is a solution of (1.1) with initial data $v_0 = (u_0)_x$. For this reason Eq. (1.1) is called the derivative equation of (1.2).

As mentioned above, we are interested in studying the well-posedness issues to the IVPs (1.1) and (1.2) for given data in the low regularity Sobolev spaces $H^s(\mathbb{R})$. Recall that, for $s \in \mathbb{R}$, the L^2-based Sobolev spaces $H^s(\mathbb{R})$ are defined by

$$H^s(\mathbb{R}) := \bigl\{ f \in \mathcal{S}'(\mathbb{R}):\ \|f\|_{H^s} < \infty \bigr\},$$

where

$$\|f\|_{H^s} := \bigl\| \langle \xi \rangle^s \widehat{f} \bigr\|_{L^2_\xi},$$

with $\langle \cdot \rangle = 1 + |\cdot|$, and $\widehat{f}(\xi)$ is the usual Fourier transform given by

$$\widehat{f}(\xi) \equiv \mathcal{F}(f)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx.$$

The factor $\frac{1}{\sqrt{2\pi}}$ in the definition of the Fourier transform does not alter our analysis, so we will omit it.

We use the standard notion of well-posedness. More precisely, we say that an IVP for given data in a Banach space X is locally well-posed, if there exist a certain time interval $[0,T]$ and a unique solution depending continuously upon the initial data, and the solution satisfies the persistence property; i.e., the solution describes a continuous curve in X in the time interval $[0,T]$. If the above properties are true for any time interval, we say that the IVP is globally well-posed, and if any one of the above properties fails to hold, we say that the IVP is ill-posed.

In our recent work [5], we considered dissipative perturbation of KdV type equations (i.e., (1.1) and (1.2) with $k = 1$) and proved sharp local well-posedness results for given data with Sobolev regularity below L^2. The IVPs (1.1) and (1.2) with general nonlinearity $k > 0$ are considered in [6] to obtain local well-posedness in H^s, $s \geqslant -1$ and $s \geqslant 0$ respectively.

The sharp local well-posedness results in [5] were obtained by using the *contraction mapping principle* in suitably defined time weighted function spaces. The motivation behind the introduction of time weighted function spaces is the work of Dix in [12], where the author proved sharp local well-posedness for Burgers' equation in H^s, $s > -\frac{1}{2}$ by showing that uniqueness fails whenever $s < -\frac{1}{2}$. We could not handle the higher order nonlinearity in the time weighted function spaces introduced in [5]. In this work, we suitably modify the spaces introduced in [5] to deal higher order nonlinearity. More precisely, for $s \in \mathbb{R}$, $p \geqslant 2k+1$ and $t \in [0,T]$ with $0 < T \leqslant 1$, we define $\delta_k := \frac{k-1}{2(k+1)}$, $\alpha_k := \frac{\delta_k+|s|}{p}$, $\beta_k := \frac{\delta_k+|s|+1}{p}$, and introduce two Banach spaces X_T^s and Y_T^s as follows

$$X_T^s := \bigl\{f \in C\bigl([0,T]; H^s(\mathbb{R})\bigr) \colon \ \|f\|_{X_T^s} < \infty\bigr\}, \tag{1.5}$$

$$Y_T^s := \bigl\{f \in C\bigl([0,T]; H^s(\mathbb{R})\bigr) \colon \ \|f\|_{Y_T^s} < \infty\bigr\}, \tag{1.6}$$

where

$$\|f\|_{X_T^s} := \sup_{t\in[0,T]} \bigl\{\bigl\|f(t)\bigr\|_{H^s} + t^{\alpha_k}\bigl\|\widehat{f(t)}\bigr\|_{L_\xi^{\frac{k+1}{k}}}\bigr\}, \tag{1.7}$$

and

$$\|f\|_{Y_T^s} := \sup_{t\in[0,T]} \bigl\{\bigl\|f(t)\bigr\|_{H^s} + t^{\beta_k}\bigl\|\widehat{\partial_x f(t)}\bigr\|_{L_\xi^{\frac{k+1}{k}}}\bigr\}. \tag{1.8}$$

Our plan is to derive some multilinear estimates in the spaces X_T^s and Y_T^s and use them to apply the contraction mapping argument so as to prove the local well-posedness results for the IVPs (1.1) and (1.2) respectively.

In sequel, we state the main results of this work. The first result deals with the local well-posedness for the IVP (1.1).

Theorem 1.1. *Let $\eta > 0$ be fixed, $k > 1$ and $\Phi(\xi)$ be as given by (1.3) with $p \geqslant 2k+1$ as the order of the leading term, then for any data $v_0 \in H^s(\mathbb{R})$ there exist a time $T = T(\|v_0\|_{H^s}, \eta)$ and a unique solution v to the IVP (1.1) in $C([0,T], H^s(\mathbb{R}))$, whenever $s > \frac{1}{2} - \frac{p+1}{k+1}$. Moreover, the map $v_0 \mapsto v(t)$ is smooth from $H^s(\mathbb{R})$ to $C([0,T]; H^s(\mathbb{R})) \cap X_T^s$.*

The second result deals the same for the IVP (1.2).

Theorem 1.2. *Let $\eta > 0$ be fixed, $k > 1$ and $\Phi(\xi)$ be as given by (1.3) with $p \geqslant 2k+1$ as the order of the leading term, then for any data $u_0 \in H^s(\mathbb{R})$, there exist a time $T = T(\|u_0\|_{H^s}, \eta)$ and a unique solution u to the IVP (1.2) in $C([0,T], H^s(\mathbb{R}))$, whenever $s > \frac{3}{2} - \frac{p+1}{k+1}$. Moreover, the map $u_0 \mapsto u(t)$ is smooth from $H^s(\mathbb{R})$ to $C([0,T]; H^s(\mathbb{R})) \cap Y_T^s$.*

The method of proof of these theorems is very simple, and does not depend on the dispersive term. This method can be applied to address the model obtained from (1.1) by replacing v_{xxx} by more general dispersive term $L_1 v$, where $\widehat{L_1 v} = i\sigma(\xi)\widehat{v}(\xi)$ with σ real. However, if the order of dissipation is lower than that of dispersion ($p < 2k+1$), then there is a role of dispersive part and this method does not work. In this situation, Bourgain's approach [2] can be applied to get better well-posedness results, see for example [18] and [21] and references therein. When $p = 2k+1$, there is a balance between dispersive and dissipative effects and local well-posedness result holds up to scaling critical regularity $s_c = -\frac{3}{2}$, see below. We observe that, in the proof of ill-posedness result, presence of dispersive term v_{xxx} is relevant, as can be seen in proofs of Theorems 1.3 and 1.4. The same analysis is true in the case of the IVP (1.2) as well.

We note that, the above results improve the ones obtained in [6] where the IVPs (1.1) and (1.2) are respectively proved to be locally well-posed in H^s for $s > -1$ and $s > 0$. However, as can be seen in the proofs of Propositions 2.11 and 2.12 (see below), the method we developed here only holds for $s \leqslant 0$. On

Table 1
Well-posedness results for the gKdV.

k	Scaling	Well-posedness result
1	$-3/2$	$s \geqslant -3/4$
2	$-1/2$	$s \geqslant 1/4$
3	$-1/6$	$s \geqslant -1/6$
$\geqslant 4$	$1/2 - 2/k$	$s \geqslant 1/2 - 2/k$

the other hand, the order of dissipation considered in [6] ($p \geqslant \frac{3}{2}k + 1$) is lower than the one considered here. Therefore, in some sense, the main well-posedness results of this article and [6] complement each other.

At this point, it is natural to ask whether the local well-posedness results given by Theorems 1.1 and 1.2 are optimal in the sense that if one lowers the Sobolev regularity of the given data below the one given by these theorems, the IVPs (1.1) and (1.2) are ill-posed. In recent literature [3,9,20,22], these sort of questions are addressed by proving that the application data-solution is not smooth in certain range of Sobolev regularity s. This notion of ill-posedness makes sense because if one proves local well-posedness by using the contraction mapping principle, the application data-solution is always smooth.

To have more insight about the well-posedness and the ill-posedness issues, we make an analysis by using scaling argument. Talking heuristically, semilinear evolution equations like viscous Burgers, Korteweg–de Vries (KdV), nonlinear Schrödinger (NLS) and wave equations are usually expected to be well-posed for given data with Sobolev regularity up to scaling and ill-posed below scaling. However, this is not always true, as can be seen in the generalized KdV (gKdV) case. For $\eta = 0$, the IVP (1.1) turns out to be the gKdV equation

$$\begin{cases} v_t + v_{xxx} + (v^{k+1})_x = 0, \quad x \in \mathbb{R},\ t \geqslant 0,\ k \in \mathbb{R},\ k > 1, \\ v(x,0) = v_0(x), \end{cases} \tag{1.9}$$

which satisfies the scaling property. Talking more precisely, if $v(x,t)$ is a solution of the gKdV with initial data $v_0(x)$ then for $\lambda > 0$, so is $v^\lambda(x,t) = \lambda^{\frac{2}{k}} v(\lambda x, \lambda^3 t)$ with initial data $v^\lambda(x,0) = \lambda^{\frac{2}{k}} v(\lambda x, 0)$. Note that, the homogeneous Sobolev norm of the initial data remains invariant if $s - \frac{1}{2} + \frac{2}{k} = 0$. This suggests that the scaling Sobolev regularity is $\frac{1}{2} - \frac{2}{k}$. For the gKdV equation, Table 1 shows the known well-posedness results and their relation to scaling index.

The best well-posedness results for the IVP (1.9) shown in Table 1 are obtained by Kenig et al. [15,16] (for $k = 1, 2$ and $k \geqslant 4$) and Grünrock [13] (for $k = 3$). These results are sharp since the flow-map $u_0 \to u(t)$ is not locally uniformly continuous from $\dot{H}^s(\mathbb{R})$ to $\dot{H}^s(\mathbb{R})$, for $s < s_k$ with $s_1 = -\frac{3}{4}$, $s_2 = \frac{1}{4}$, $s_3 = -\frac{1}{6}$, and for $k \geqslant 4$, $s_k = \frac{1}{2} - \frac{2}{k}$ (see [1,17]).

Generally, for dissipative problem, the scaling index is better in the sense that one can lower the regularity requirement on the data to get well-posedness. As can be seen in the proofs of Theorems 1.1 and 1.2 (below), our method depends on the leading order of L. If we discard the third order derivative (dispersive part) and consider the dissipative operator L with the Fourier symbol $|\xi|^p$, with $p \geqslant 2k + 1$ in (1.1), i.e.,

$$\begin{cases} v_t + \eta L v + (v^{k+1})_x = 0, \qquad \widehat{Lv}(\xi) = |\xi|^p \widehat{v}(\xi) \\ v(x,0) = v_0(x), \end{cases} \tag{1.10}$$

it is easy to check that, if $v(x,t)$ solves (1.10) with initial data $v(x,0)$, then for $\lambda > 0$ so does $v^\lambda(x,t) = \lambda^{\frac{p-1}{k}} v(\lambda x, \lambda^p t)$ with initial data $v^\lambda(x,0) = \lambda^{\frac{p-1}{k}} v(\lambda x, 0)$. Note that

$$\left\| v^\lambda(0) \right\|_{\dot{H}^s} = \lambda^{p-1-\frac{k}{2}+ks} \left\| v(0) \right\|_{\dot{H}^s}. \tag{1.11}$$

From (1.11) we see that the scaling index for this particular situation is $s_c := \frac{1}{2} - \frac{p-1}{k}$. Observe that, for $p = 2k + 1$ we get $s_c = -\frac{3}{2}$ for any value of k, which is much lower than that for the gKdV equation if

$k > 1$. Also, note that for $p = 2k+1$, one has $\frac{1}{2} - \frac{p+1}{k+1} = -\frac{3}{2}$. With this observation we see that Theorem 1.1 provides the local well-posedness result up to the Sobolev regularity given by scaling for $p = 2k+1$. However, for $p > 2k+1$, the regularity requirement for local well-posedness is higher than s_c. Since the regularity requirement for the IVP (1.2) is higher than 1 to that for the IVP (1.1), we see that the scaling index for this is $s_c + 1$. Therefore, for $p = 2k+1$, the well-posedness result given by Theorem 1.2 also goes up to the scaling index $-\frac{1}{2}$.

Having discussed scaling argument, our next task is to check if the local results in Theorems 1.1 and 1.2 are sharp. Recently, Molinet et al. [19] introduced a technique to prove that the generalized Burgers equation is "ill-posed" by showing that the mapping data-solution fails to be C^k for certain range of Sobolev regularity. This method is further adapted to address such issue for the modified KdV–Burgers equation in [8]. Here, we modify the technique introduced in [19] (see the use of Lemma 4.1 below) to address such issue considering generalized nonlinearity as well as generalized dissipation and obtain the following "ill-posedness" results for the IVPs (1.1) and (1.2) respectively in the sense that the mapping data-solution fails to be C^k and C^{k+1}.

Theorem 1.3. *Let $s < \frac{1}{2} - \frac{p-1}{k}$, then there does not exist any $T > 0$ such that the IVP (1.1) admits a unique local solution defined in the interval $[0, T]$ such that the flow-map*

$$v_0 \mapsto v(t), \quad t \in [0, T], \tag{1.12}$$

is C^{k+1}-differentiable at the origin from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R}))$.

Theorem 1.4. *Let $s < \frac{3}{2} - \frac{p-1}{k}$, then there does not exist any $T > 0$ such that the IVP (1.2) admits a unique local solution defined in the interval $[0, T]$ such that the flow-map*

$$u_0 \mapsto u(t), \quad t \in [0, T], \tag{1.13}$$

is C^{k+1}-differentiable at the origin from $H^s(\mathbb{R})$ to $C([0, T]; H^s(\mathbb{R}))$.

Remark 1.5. We observe that, if $p = 2k+1$, one has $\frac{1}{2} - \frac{p+1}{k+1} = -\frac{3}{2} = \frac{1}{2} - \frac{p-1}{k}$ and $\frac{3}{2} - \frac{p+1}{k+1} = -\frac{1}{2} = \frac{3}{2} - \frac{p-1}{k}$. In view of this observation and the results of Theorems 1.3 and 1.4, the local well-posedness results given by Theorems 1.1 and 1.2 are sharp for $p = 2k+1$. However, for $p > 2k+1$ one has that $\frac{1}{2} - \frac{p+1}{k+1} > s_c$ and $\frac{3}{2} - \frac{p+1}{k+1} > s_c + 1$. Therefore, the well-posedness or ill-posedness issue of the IVPs (1.1) and (1.2) for values of s respectively in $[s_c, \frac{1}{2} - \frac{p+1}{k+1}]$ and $[s_c + 1, \frac{3}{2} - \frac{p+1}{k+1}]$ is an open problem.

A detailed explanation about the particular examples that belong to the classes considered in (1.1) and (1.2) and the known well-posedness results about them are presented in our earlier works [4,6] and references therein.

Now, we comment about the global well-posedness. In [7], the IVP (1.1) is proved to be globally well-posed for given data in $H^s(\mathbb{R})$, $s \geqslant 1$, $k = 1, 2, 3$. As the models under consideration do not have conserved quantities, the global well-posedness results have been proved by constructing appropriate *a priori* estimates. However, for given data in $H^s(\mathbb{R})$, $s < 1$ no *a priori* estimates are available. Also, the lack of conserved quantities prevent us to use the recently introduced *I-method* in [10,11], to obtain global solution for the low regularity data. It would be interesting to develop some new method to address global well-posedness issues.

This paper is organized as follows: In Section 2, we prove some preliminary estimates. Section 3 is dedicated to prove the local well-posedness results. Finally, we prove ill-posedness results in Section 4.

2. Linear and nonlinear estimates

This section is devoted to obtain linear and nonlinear estimates that are essential in the proof of the main results. We start with following estimate that the Fourier symbol defined in (1.3) satisfies.

Lemma 2.1. *There exists $M > 0$ large such that for all $|\xi| \geqslant M$, one has that*

$$\Phi(\xi) = -|\xi|^p + \Phi_1(\xi) < -1, \tag{2.1}$$

$$\frac{|\Phi_1(\xi)|}{|\xi|^p} \leqslant \frac{1}{2}, \tag{2.2}$$

and

$$\left|\Phi(\xi)\right| \geqslant \frac{|\xi|^p}{2}. \tag{2.3}$$

Proof. The inequalities (2.1) and (2.2) are direct consequences of

$$\lim_{\xi\to\infty} \frac{\Phi_1(\xi)+1}{|\xi|^p} = 0 \quad \text{and} \quad \lim_{\xi\to\infty} \frac{|\Phi_1(\xi)|}{|\xi|^p} = 0,$$

respectively.

The estimate (2.3) follows from (2.1) and (2.2). In fact, for $|\xi| > M$

$$\left|\Phi(\xi)\right| = |\xi|^p - \Phi_1(\xi) \geqslant \frac{|\xi|^p}{2}, \tag{2.4}$$

and this concludes the proof of the (2.3). □

Lemma 2.2. *The Fourier symbol $\Phi(\xi)$ given by (1.3) is bounded from above and the following estimate holds true*

$$\left\|e^{t\Phi(\xi)}\right\|_{L^\infty} \leqslant e^{TC_M}. \tag{2.5}$$

Proof. From Lemma 2.1, there is $M > 1$ large enough such that for $|\xi| \geqslant M$ one has $\Phi(\xi) < -1$. Consequently, $e^{t\Phi(\xi)} \leqslant e^{-t} \leqslant 1$. Now for $|\xi| < M$, it is easy to get $\Phi(\xi) < C_M$, so that $e^{t\Phi(\xi)} \leqslant e^{TC_M}$. Therefore, in any case

$$\left\|e^{t\Phi(\xi)}\right\|_{L^\infty} \leqslant e^{TC_M}. \qquad \square$$

The following result is an elementary fact from calculus.

Lemma 2.3. *Let $f(t) = t^a e^{tb}$ with $a > 0$ and $b < 0$, then for all $t \geqslant 0$ one has*

$$f(t) \leqslant \left(\frac{a}{|b|}\right)^a e^{-a}. \tag{2.6}$$

Lemma 2.4 *(Generalized Young's Inequality). Let $n \geqslant 1$, $1 < r \leqslant \infty$, and $r_i > 1$ such that*

$$\frac{1}{r} + n = \sum_{i=1}^{n+1} \frac{1}{r_i}.$$

Then there exists a constant c_n such that for any $u_1, u_2, \dots, u_{n+1} \in \mathbb{S}(\mathbb{R})$, we have

$$\|u_1 * u_2 * \cdots * u_{n+1}\|_{L^r} \leqslant c_n \|u_1\|_{L^{r_1}} \|u_2\|_{L^{r_2}} \cdots \|u_{n+1}\|_{L^{r_{n+1}}}. \tag{2.7}$$

Proof. This inequality is obtained immediately, using the classical Young's inequality and induction argument. □

Now we consider the IVP associated to the linear parts of (1.1) and (1.2),

$$\begin{cases} w_t + w_{xxx} + \eta L w = 0, & x, t \geqslant 0, \\ w(0) = w_0. \end{cases} \tag{2.8}$$

The solution to (2.8) is given by $w(x,t) = V(t)w_0(x)$ where the semigroup $V(t)$ is defined as

$$\widehat{V(t)w_0}(\xi) = e^{it\xi^3 + \eta t \Phi(\xi)} \widehat{w_0}(\xi). \tag{2.9}$$

In what follows we prove some estimates satisfied by the group defined in (2.9).

Lemma 2.5. *Let $0 < T \leqslant 1$, $k > 1$ and $t \in [0, T]$. Then for all $s \in \mathbb{R}$, we have*

$$\|V(t)w_0\|_{X_T^s} \lesssim \|w_0\|_{H^s}, \tag{2.10}$$

where the implicit constant depends on M with M as in Lemma 2.1, k and p.

Proof. For $s > 0$ the proof is easy, so we give details only for $s \leqslant 0$. We start by estimating the first component of the X_T^s-norm. We have that

$$\|V(t)w_0\|_{H^s} = \|\langle\xi\rangle^s e^{t\Phi(\xi)} \widehat{w_0}(\xi)\|_{L^2} \leqslant \|e^{t\Phi(\xi)}\|_{L^\infty} \|w_0\|_{H^s}. \tag{2.11}$$

Using (2.5) in (2.11), we get

$$\|V(t)w_0\|_{H^s} \leqslant e^{TC_M} \|w_0\|_{H^s}. \tag{2.12}$$

Now, we move to estimate the second component of the X_T^s-norm. We have

$$\begin{aligned} t^{\alpha_k} \|\widehat{V(t)w_0}\|_{L^{\frac{k+1}{k}}} &= t^{\alpha_k} \|e^{t\Phi(\xi)} \widehat{w_0}\|_{L^{\frac{k+1}{k}}} \\ &= t^{\alpha_k} \|\langle\xi\rangle^{-s} e^{t\Phi(\xi)} \langle\xi\rangle^s \widehat{w_0}\|_{L^{\frac{k+1}{k}}} \\ &\leqslant t^{\alpha_k} \|\langle\xi\rangle^{-s} e^{t\Phi(\xi)}\|_{L^{\frac{2(k+1)}{k-1}}} \|\langle\xi\rangle^s \widehat{w_0}\|_{L^2} \\ &\leqslant t^{\alpha_k} \|\langle\xi\rangle^{|s|} e^{t\Phi(\xi)}\|_{L^{\frac{2(k+1)}{k-1}}} \|w_0\|_{H^s}. \end{aligned} \tag{2.13}$$

To obtain an estimate for $\|\langle\xi\rangle^{|s|} e^{t\Phi(\xi)}\|_{L^{\frac{2(k+1)}{k-1}}}$ we consider M as in Lemma 2.1 and write it as

$$\|\langle\xi\rangle^{|s|} e^{t\Phi(\xi)}\|_{L^{\frac{2(k+1)}{k-1}}} \leqslant \|\langle\xi\rangle^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| \leqslant M\}}\|_{L^{\frac{2(k+1)}{k-1}}} + \|\langle\xi\rangle^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi| > M\}}\|_{L^{\frac{2(k+1)}{k-1}}}. \tag{2.14}$$

The low frequency part in (2.14) is bounded by some constant C_M. For the high frequency part we use the estimate (2.3) from Lemma 2.1 to get $\Phi(\xi) \leqslant -\frac{|\xi|^p}{2}$ and the estimate $\langle\xi\rangle^{|s|} \lesssim |\xi|^{|s|}$, to write it as

$$\begin{aligned}
\left\|\langle\xi\rangle^{|s|}e^{t\Phi(\xi)}\chi_{\{|\xi|>M\}}\right\|_{L^{\frac{2(k+1)}{k-1}}} &= \left(\int\limits_{\chi_{\{|\xi|>M\}}} \langle\xi\rangle^{-s\frac{2(k+1)}{k-1}} e^{t\Phi(\xi)\frac{2(k+1)}{k-1}}\,d\xi\right)^{\frac{k-1}{2(k+1)}} \\
&\leqslant \left(\int\limits_{\chi_{\{|\xi|>M\}}} |\xi|^{-s\frac{2(k+1)}{k-1}} e^{-t|\xi|^p\frac{(k+1)}{k-1}}\,d\xi\right)^{\frac{k-1}{2(k+1)}} \\
&\leqslant \left(\int\limits_{\mathbb{R}} |\xi|^{-s\frac{2(k+1)}{k-1}} e^{-t|\xi|^p\frac{(k+1)}{k-1}}\,d\xi\right)^{\frac{k-1}{2(k+1)}}. \qquad (2.15)
\end{aligned}$$

Now, making change of variable $x = \xi t^{\frac{1}{p}}$, we get from (2.15)

$$\begin{aligned}
\left\|\langle\xi\rangle^{|s|}e^{t\Phi(\xi)}\chi_{\{|\xi|>M\}}\right\|_{L^{\frac{2(k+1)}{k-1}}} &\leqslant \left(\int\limits_{\mathbb{R}} |x|^{-s\frac{2(k+1)}{k-1}} t^{s\frac{2(k+1)}{p(k-1)}} e^{-|x|^p\frac{(k+1)}{k-1}} t^{-\frac{1}{p}}\,dx\right)^{\frac{k-1}{2(k+1)}} \\
&= t^{\frac{s}{p}-\frac{k-1}{2p(k+1)}}\left(\int\limits_{\mathbb{R}} |x|^{-s\frac{2(k+1)}{k-1}} e^{-|x|^p\frac{(k+1)}{k-1}}\,dx\right)^{\frac{k-1}{2(k+1)}} \\
&< Ct^{\frac{s}{p}-\frac{k-1}{2p(k+1)}}. \qquad (2.16)
\end{aligned}$$

Since $t \in [0, T]$ with $T \leqslant 1$, $s \leqslant 0$ and

$$\alpha_k + \frac{s}{p} - \frac{k-1}{2p(k+1)} = 0, \qquad (2.17)$$

we get from (2.14) and (2.16) that

$$t^{\alpha_k}\left\|\langle\xi\rangle^{|s|}e^{t\Phi(\xi)}\right\|_{L^{\frac{2(k+1)}{k-1}}} \leqslant C_{M,k,p}. \qquad (2.18)$$

Inserting (2.18) in (2.12) we obtain the required estimate (2.10). □

Lemma 2.6. *Let $0 < T \leqslant 1$ and $t \in [0, T]$. Then for all $s \in \mathbb{R}$, we have*

$$\left\|V(t)w_0\right\|_{Y_T^s} \lesssim \|w_0\|_{H^s}, \qquad (2.19)$$

where the implicit constant depends on k, p and M with M as in Lemma 2.1.

Proof. The estimate for the first component of the Y_T^s-norm has already been obtained in (2.12). In what follows, we estimate the second component of the Y_T^s-norm. We only consider the case when $s < 0$. In the case when $s \geqslant 0$ the estimates follow easily. We have

$$\begin{aligned}
t^{\beta_k}\left\|\xi\widehat{V(t)w_0}(\xi)\right\|_{L^{\frac{k+1}{k}}} &= t^{\beta_k}\left\|\xi e^{t\Phi(\xi)}\widehat{w_0}\right\|_{L^{\frac{k+1}{k}}} \\
&= t^{\beta_k}\left\|\xi\langle\xi\rangle^{-s}e^{t\Phi(\xi)}\langle\xi\rangle^{s}\widehat{w_0}\right\|_{L^{\frac{k+1}{k}}} \\
&\leqslant t^{\beta_k}\left\|\xi\langle\xi\rangle^{|s|}e^{t\Phi(\xi)}\right\|_{L^{r_k}}\|w_0\|_{H^s}, \qquad (2.20)
\end{aligned}$$

where $r_k = 1/\delta_k$. Now, let M large as in Lemma 2.1, we obtain

$$\begin{aligned}
t^{\beta_k}\left\|\xi\langle\xi\rangle^{|s|}e^{t\Phi(\xi)}\right\|_{L^{r_k}} &\leqslant t^{\beta_k}\left\|\xi\langle\xi\rangle^{|s|}e^{t\Phi(\xi)}\chi_{\{|\xi|\leqslant M\}}\right\|_{L^{r_k}} + t^{\beta_k}\left\|\xi\langle\xi\rangle^{|s|}e^{t\Phi(\xi)}\chi_{\{|\xi|>M\}}\right\|_{L^{r_k}} \\
&=: J_1 + J_2. \qquad (2.21)
\end{aligned}$$

Since $0 \leqslant t \leqslant T \leqslant 1$, we have

$$J_1 \leqslant C_{M,p} t^{\beta_k} \leqslant C_{M,p}. \tag{2.22}$$

Now, we move to estimate the high-frequency part J_2. For this, we use Lemma 2.3 thus $\Phi(\xi) < -|\xi|^p/2$ if $|\xi| > M$, we get

$$J_2 = t^{\beta_k} \big\| \xi \langle \xi \rangle^{|s|} e^{t\Phi(\xi)} \chi_{\{|\xi|>M\}} \big\|_{L^{r_k}} \leqslant t^{\beta_k} \big\| \xi \langle \xi \rangle^{|s|} e^{-t|\xi|^p/2} \chi_{\{|\xi|>M\}} \big\|_{L^{r_k}}. \tag{2.23}$$

Since $M > 1$ is large, $\langle \xi \rangle^{|s|} \lesssim |\xi|^{|s|}$, a change of the variables $x = t^{1/p}\xi$, yields

$$J_2 \leqslant t^{\beta_k} \big\| |\xi|^{1+|s|} e^{-t|\xi|^p/2} \big\|_{L^{r_k}} \leqslant t^{\beta_k} t^{-\beta_k} \big\| |x|^{1+|s|} e^{-|x|^p/2} \big\|_{L^{r_k}}, \tag{2.24}$$

and consequently

$$J_2 \leqslant C. \tag{2.25}$$

The conclusion of the lemma follows from (2.20), (2.21), (2.22) and (2.25). □

In what follows we present two technical lemmas.

Lemma 2.7. *Let $k > 1$, $p \geqslant 2k+1$ and $\frac{3}{2} - \frac{p+1}{k+1} < s \leqslant 0$, then there exists a number $a_0 > 0$ such that*

$$\frac{1+2s}{p} < a_0 < 2\big(1 - (k+1)\beta_k\big). \tag{2.26}$$

Proof. In order to prove (2.26), it suffices to prove that

$$\frac{1+2s}{p} < 2 - \frac{2(k+1)(1+\delta_k - s)}{p}, \tag{2.27}$$

this inequality is equivalent with

$$s > \frac{3}{2} - \frac{p-1}{k},$$

which is true because

$$s > \frac{3}{2} - \frac{p+1}{k+1} \geqslant \frac{3}{2} - \frac{p-1}{k},$$

where the last inequality is equivalent with $p \geqslant 2k+1$.

Now, to show that a_0 can be chosen positive, we observe that

$$2 - \frac{2(k+1)(1+\delta_k - s)}{p} > 0 \quad \Longleftrightarrow \quad s > \frac{3}{2} - \frac{p+1}{k+1},$$

which is true by hypothesis of the lemma. □

Now we state another technical result whose proof is very similar to that of Lemma 2.7.

Lemma 2.8. *Let $k > 1$, $p \geqslant 2k+1$ and $\frac{1}{2} - \frac{p+1}{k+1} < s \leqslant 0$, then there exists a number $a_1 > 0$ such that*

$$\frac{3+2s}{p} < a_1 < 2\big(1-(k+1)\alpha_k\big). \tag{2.28}$$

Lemma 2.9. *Let $k > 1$, $p \geqslant 2k+1$ and $\frac{1}{2} - \frac{p+1}{k+1} < s \leqslant 0$ and a_1 as in Lemma 2.8, then for $0 < T \leqslant 1$ and $\tau \in (0, T]$, we have*

$$\big\|\xi\langle\xi\rangle^s e^{\tau\Phi(\xi)}\big\|_{L^2_\xi} \lesssim \frac{1}{\tau^{\frac{a_1}{2}}}. \tag{2.29}$$

Proof. Let M be as in Lemma 2.1. We decompose the integral as

$$\big\|\xi\langle\xi\rangle^s e^{\tau\Phi(\xi)}\big\|^2_{L^2_\xi} = \int\limits_{|\xi|\leqslant M} \xi^2\langle\xi\rangle^{2s} e^{2\tau\Phi(\xi)}\,d\xi + \int\limits_{|\xi|\geqslant M} \xi^2\langle\xi\rangle^{2s} e^{2\tau\Phi(\xi)}\,d\xi =: I_1 + I_2. \tag{2.30}$$

In the first integral, since $a_1 > 0$ and $\tau \in [0,1]$ we have

$$I_1 \leqslant \int\limits_{|\xi|\leqslant M} M^2\langle M\rangle^{2|s|} e^{2\tau C_{M,p}}\,d\xi \leqslant 2M^3\langle M\rangle^{2|s|} e^{2C_{M,p}} \leqslant C_{M,p,s}\frac{1}{\tau^{a_1}}. \tag{2.31}$$

Now, we consider the second integral in (2.30). We have $\langle\xi\rangle^s \lesssim |\xi|^s$. For sufficiently large M, considering $b = 2\Phi(\xi) < 0$ (see Lemma 2.1), and $a = a_1 > 0$, one can get, using the estimate (2.6) that

$$I_2 \leqslant \int\limits_{|\xi|>M} |\xi|^{2s+2} e^{2\tau\Phi(\xi)}\,d\xi \lesssim \int\limits_{|\xi|>M} \frac{1}{|\xi|^{-2s-2}}\frac{1}{\tau^{a_1}|\Phi(\xi)|^{a_1}}\,d\xi. \tag{2.32}$$

Using (2.3) in (2.32), one obtains

$$I_2 \lesssim \frac{1}{\tau^{a_1}}\int\limits_{|\xi|>M} \frac{1}{|\xi|^{-2s-2}}\frac{1}{|\xi|^{pa_1}}\,d\xi \lesssim \frac{1}{\tau^{a_1}}, \tag{2.33}$$

where in the last inequality the fact that $-2s-2+pa_1 > 1$ has been used.

Inserting (2.31) and (2.33) in (2.30) we obtain the required estimate (2.29). □

Lemma 2.10. *Let $k > 1$, $p \geqslant 2k+1$ and $\frac{3}{2} - \frac{p+1}{k+1} < s \leqslant 0$ and a_0 as in Lemma 2.7, then*

$$\big\|\langle\xi\rangle^s e^{\tau\Phi(\xi)}\big\|_{L^2} \lesssim \frac{1}{\tau^{\frac{a_0}{2}}}, \tag{2.34}$$

and

$$\big\|\xi e^{\tau\Phi(\xi)}\big\|_{L^{\frac{k+1}{k}}} \lesssim_k \frac{1}{\tau^{\frac{2k+1}{(k+1)p}}}, \tag{2.35}$$

where $0 < \tau \leqslant T \leqslant 1$.

Proof. For M large as in Lemma 2.1, we have

$$\big\|\langle\xi\rangle^s e^{\tau\Phi(\xi)}\big\|^2_{L^2} = \int\limits_{|\xi|\leqslant M} \langle\xi\rangle^{2s} e^{2\tau\Phi(\xi)}\,d\xi + \int\limits_{|\xi|>M} \langle\xi\rangle^{2s} e^{2\tau\Phi(\xi)}\,d\xi =: A + B.$$

Now, for $\tau \in (0,T]$ and a_0 as in Lemma 2.7, one has

$$A \leqslant C_M e^{TC_{M,p}} \leqslant \frac{C_M e^{TC_{M,p}}}{\tau^{a_0}}. \tag{2.36}$$

To obtain an estimate for the high frequency part B, we use estimate (2.6) with $a = a_0 > 0$ and $b = 2\Phi(\xi) < 0$, to obtain

$$B \lesssim \int\limits_{|\xi|>M} \frac{|\xi|^{2s}}{\tau^{a_0}} \frac{(a_0 e^{-1})^{a_0}}{|\Phi(\xi)|^{a_0}}\, d\xi \lesssim \int\limits_{|\xi|>M} \frac{1}{|\xi|^{pa_0-2s}} \frac{1}{\tau^{a_0}}\, d\xi \lesssim \frac{1}{\tau^{a_0}}, \tag{2.37}$$

where Lemma 2.1 and $pa_0 - 2s > 1$ have been used. This concludes the proof of (2.34).

In order to prove (2.35) we proceed as above. For M large as in Lemma 2.1, we obtain

$$\left\|\xi e^{\tau\Phi(\xi)}\right\|_{L^{\frac{k+1}{k}}} \leqslant \left(\int\limits_{|\xi|\leqslant M} |\xi|^{\frac{k+1}{k}} e^{\frac{k+1}{k}\tau\Phi(\xi)}\, d\xi \right)^{\frac{k}{k+1}} + \left(\int\limits_{|\xi|>M} |\xi|^{\frac{k+1}{k}} e^{\frac{k+1}{k}\tau\Phi(\xi)}\, d\xi \right)^{\frac{k}{k+1}} := D + E. \tag{2.38}$$

Considering $0 < \tau \leqslant T \leqslant 1$ in the low frequency part D we have

$$D \leqslant C_{k,M} \lesssim \frac{1}{\tau^{\frac{2k+1}{(k+1)p}}}. \tag{2.39}$$

Using Lemma 2.1 in the high frequency part D ($\Phi(\xi) \leqslant -|\xi|^p/2$ if $|\xi| > M$) we arrive to

$$E \leqslant \left(\int\limits_{|\xi|>M} |\xi|^{\frac{k+1}{k}} e^{-\frac{\tau(k+1)|\xi|^p}{2k}}\, d\xi \right)^{\frac{k}{k+1}} \leqslant \left(\int\limits_{\mathbb{R}} |\xi|^{\frac{k+1}{k}} e^{-\frac{\tau(k+1)|\xi|^p}{2k}}\, d\xi \right)^{\frac{k}{k+1}}. \tag{2.40}$$

A change of variables $x = \tau^{1/p}\xi$ in the RHS of (2.40), gives

$$E \lesssim \frac{1}{\tau^{\frac{2k+1}{(k+1)p}}}. \tag{2.41}$$

Now, plugging (2.39) and (2.41) in (2.38) we get the desired estimate (2.35). □

Proposition 2.11. *Let $k \in \mathbb{R}$, $k > 1$, $\frac{1}{2} - \frac{p+1}{k+1} < s \leqslant 0$, $p \geqslant 2k+1$, $0 < T \leqslant 1$ and $t \in [0,T]$. Then we have*

$$\left\| \int_0^t V(t-t')\partial_x\left(u^{k+1}\right)(t')\, dt' \right\|_{X_T^s} \lesssim T^\alpha \|u\|_{X_T^s}^{k+1}, \tag{2.42}$$

where $\alpha > 0$.

Proof. Using the definition of $V(t)$ and Minkowski's inequality, we have

$$\begin{aligned} \left\| \int_0^t V(t-t')\partial_x\left(u^{k+1}\right)(t')\, dt' \right\|_{H^s} &\leqslant \int_0^t \left\| \xi\langle\xi\rangle^s e^{(t-t')\Phi(\xi)} \widehat{u^{k+1}(t')}\, dt' \right\|_{L^2_\xi} \\ &\leqslant \int_0^t \left\| \xi\langle\xi\rangle^s e^{(t-t')\Phi(\xi)} \right\|_{L^2_\xi} \left\| \widehat{u^{k+1}(t')} \right\|_{L^\infty_\xi} dt'. \end{aligned} \tag{2.43}$$

The generalized Young's inequality from Lemma 2.4 and the definition of X_T^s norm yield

$$\big\|\widehat{u^{k+1}(t')}\big\|_{L_\xi^\infty} \leqslant C_k\big\|\widehat{u(t')}\big\|_{L_\xi^{\frac{k+1}{k}}}^{k+1} \leqslant C_k t'^{-(k+1)\alpha_k}\|u\|_{X_T^s}^{k+1}. \tag{2.44}$$

Now, using estimate (2.29) we obtain, that

$$\big\|\xi\langle\xi\rangle^s e^{(t-t')\Phi(\xi)}\big\|_{L_\xi^2} \lesssim \frac{1}{(t-t')^{\frac{a_1}{2}}}. \tag{2.45}$$

Inserting (2.45) and (2.44) in (2.43), we get

$$\left\|\int_0^t V(t-t')\partial_x\big(u^{k+1}\big)(t')\,dt'\right\|_{H^s} \lesssim \|u\|_{X_T^s}^{k+1}\int_0^t \frac{1}{(t-t')^{\frac{a_1}{2}}}\frac{1}{t'^{(k+1)\alpha_k}}\,dt'. \tag{2.46}$$

Making a change of variables $t' = t\tau$, we get

$$\begin{aligned}\left\|\int_0^t V(t-t')\partial_x\big(u^{k+1}\big)(t')\,dt'\right\|_{H^s} &\lesssim t^{1-\frac{a_1}{2}-\alpha_k(k+1)}\|u\|_{X_T^s}^{k+1}\int_0^1 \frac{1}{(1-\tau)^{\frac{a_1}{2}}}\frac{1}{\tau^{(k+1)\alpha_k}}\,d\tau\\ &\lesssim t^{1-\frac{a_1}{2}-\alpha_k(k+1)}\|u\|_{X_T^s}^{k+1},\end{aligned} \tag{2.47}$$

where in the last inequality $a_1 < 2$ and $\alpha_k(k+1) < 1$ have been used.

Similarly

$$\begin{aligned}t^{\alpha_k}\left\|\mathcal{F}_x\left(\int_0^t V(t-t')\partial_x\big(u^{k+1}\big)(t')\,dt'\right)\right\|_{L^{\frac{k+1}{k}}} &\lesssim t^{\alpha_k}\int_0^t \big\|e^{(t-t')\Phi(\xi)}\xi\widehat{u^{k+1}(t')}\big\|_{L^{\frac{k+1}{k}}}\,dt'\\ &\lesssim \|u\|_{X_T^s}^{k+1} t^{\alpha_k}\int_0^t \frac{1}{t'^{\alpha_k(k+1)}}\big\|\xi e^{(t-t')\Phi(\xi)}\big\|_{L^{\frac{k+1}{k}}}\,dt'.\end{aligned} \tag{2.48}$$

From (2.35), we have

$$\big\|\xi e^{(t-t')\Phi(\xi)}\big\|_{L^{\frac{k+1}{k}}} \lesssim (t-t')^{-\frac{2k+1}{p(k+1)}}. \tag{2.49}$$

Inserting (2.49) in (2.48), we obtain

$$t^{\alpha_k}\left\|\mathcal{F}_x\left(\int_0^t V(t-t')\partial_x\big(u^{k+1}\big)(t')\,dt'\right)\right\|_{L^{\frac{k+1}{k}}} \lesssim t^{\alpha_k}\int_0^t \frac{1}{(t-t')^{\frac{2k+1}{p(k+1)}}}\frac{1}{t'^{\alpha_k(k+1)}}\,dt'. \tag{2.50}$$

Again, making a change of variables $t' = t\tau$, one has

$$\begin{aligned}t^{\alpha_k}\left\|\mathcal{F}_x\int_0^t V(t-t')\partial_x\big(u^{k+1}\big)(t')\,dt'\right\|_{L^{\frac{k+1}{k}}} &\lesssim t^{1+\alpha_k-\frac{2k+1}{p(k+1)}-\alpha_k(k+1)}\int_0^1 \frac{1}{(1-\tau)^{\frac{2k+1}{p(k+1)}}}\frac{1}{\tau^{\alpha_k(k+1)}}\,d\tau\\ &\lesssim t^{1-k\alpha_k-\frac{2k+1}{p(k+1)}}.\end{aligned} \tag{2.51}$$

We notice that

$$1 - k\alpha_k - \frac{2k+1}{p(k+1)} > 0 \quad \Longleftrightarrow \quad s > \frac{1}{2} - \frac{p-1}{k}. \tag{2.52}$$

The second inequality in (2.52) holds true, since $p \geqslant 2k+1$, implies that

$$s > \frac{1}{2} - \frac{p+1}{k+1} \geqslant \frac{1}{2} - \frac{p-1}{k},$$

and this completes the proof. □

Proposition 2.12. *Let $k > 1$, $p \geqslant 2k+1$, $\frac{3}{2} - \frac{p+1}{k+1} < s \leqslant 0$, $0 < T \leqslant 1$ and $t \in [0,T]$. Then we have*

$$\left\| \int_0^t V(t-t')(u_x)^{k+1}(t')\,dt' \right\|_{Y_T^s} \lesssim T^\theta \|u\|_{Y_T^s}^{k+1}, \tag{2.53}$$

where $\theta > 0$.

Proof. The proof of this proposition is analogous to that of Proposition 2.11. So we only give a sketch. As in the proof of Proposition 2.11, using (2.34) one gets

$$\left\| \int_0^t V(t-t')(u_x)^{k+1}(t')\,dt' \right\|_{H^s} \lesssim t^{1-\frac{a_0}{2}-(k+1)\beta_k} \|u\|_{Y_T^s}^{k+1} \int_0^1 \frac{1}{|1-\tau|^{\frac{a_0}{2}} |\tau|^{(k+1)\beta_k}}\,d\tau. \tag{2.54}$$

For our choice of a_0 as in Lemma 2.7, $p \geqslant 2k+1$ and $s > \frac{3}{2} - \frac{p+1}{k+1}$ one has $a_0 < 2$ and $(k+1)\beta_k < 1$. Therefore the integral in the RHS of (2.54) is finite. Also we have

$$1 - \frac{a_0}{2} - (k+1)\beta_k > 0, \tag{2.55}$$

so it is easy to deduce that

$$\left\| \int_0^t V(t-t')(u_x)^{k+1}(t')\,dt' \right\|_{H^s} \lesssim t^{1-\frac{a_0}{2}-(k+1)\beta_k} \|u\|_{Y_T^s}^{k+1}. \tag{2.56}$$

To estimate the second part of the Y_T^s-norm, we use Minkowski's inequality, Young's inequality and (2.35), to get

$$t^{\beta_k} \left\| \xi \mathcal{F}\left(\int_0^t \partial_x V(t-t')(u_x)^{k+1}(t')\,dt' \right)(\xi) \right\|_{L^{\frac{k+1}{k}}} \lesssim t^{1-k\beta_k-\gamma} \|u\|_{Y_T^s}^{k+1} \int_0^1 \frac{1}{|1-\tau|^{\gamma} |\tau|^{(k+1)\beta_k}}\,d\tau, \tag{2.57}$$

where $\gamma = \frac{2k+1}{(k+1)p}$.

For our choice of $k \geqslant 1$, $p \geqslant 2k+1$ and $s > \frac{3}{2} - \frac{p+1}{k+1}$ the integral in the RHS of (2.57) is finite. Therefore, from (2.57), we obtain

$$t^{\beta_k} \left\| \xi \mathcal{F}\left(\int_0^t \partial_x V(t-t')(u_x)^{k+1}(t')\,dt' \right)(\xi) \right\|_{L^{\frac{k+1}{k}}} \lesssim t^{1-k\beta_k-\gamma} \|u\|_{Y_T^s}^{k+1}. \tag{2.58}$$

Combining (2.56) and (2.58) we get the required estimate (2.53). Observe that

$$1 - k\beta_k - \gamma > 0 \quad \Longleftrightarrow \quad s > \frac{3}{2} - \frac{p-1}{k},$$

and the second inequality holds true, since $p \geqslant 2k+1$, implies that

$$s > \frac{3}{2} - \frac{p+1}{k+1} \geqslant \frac{3}{2} - \frac{p-1}{k}. \qquad \square$$

3. Proof of the well-posedness results

This section we will use the linear and nonlinear estimates to provide proofs of the local well-posedness results stated in Theorems 1.1 and 1.2.

Proof of Theorem 1.1. The case $s \geqslant 0$, was considered in our earlier work [6]. Thus, from here onwards, we only consider the case when $\frac{1}{2} - \frac{p+1}{k+1} < s \leqslant 0$.

Now consider the IVP (1.1) in its equivalent integral form

$$v(t) = V(t)v_0 - \int_0^t V(t-t')(v^{k+1})_x(t')\,dt', \tag{3.1}$$

where $V(t)$ is the semigroup associated with the linear part given by (2.9).

We define an application

$$\Psi(v)(t) = V(t)v_0 - \int_0^t V(t-t')(v^{k+1})_x(t')\,dt'. \tag{3.2}$$

For $s > \frac{1}{2} - \frac{p+1}{k+1}$, $r > 0$ and $0 < T \leqslant 1$, let us define a ball

$$B_r^T = \{f \in X_T^s;\ \|f\|_{X_T^s} \leqslant r\}.$$

We will prove that there exist $r > 0$ and $0 < T \leqslant 1$ such that the application Ψ maps B_r^T into B_r^T and is a contraction. Let $v \in B_r^T$. By using Lemma 2.5 and Proposition 2.11, we get

$$\|\Psi(v)\|_{X_T^s} \leqslant c\|v_0\|_{H^s} + cT^\alpha \|v\|_{X_T^s}^{k+1}, \tag{3.3}$$

where $\alpha > 0$.

Now, using the definition of B_r^T, one obtains

$$\|\Psi(v)\|_{X_T^s} \leqslant \frac{r}{4} + cT^\alpha r^{k+1} \leqslant \frac{r}{2}, \tag{3.4}$$

where we have chosen $r = 4c\|v_0\|_{H^s}$ and $cT^\alpha r^k = 1/4$. Therefore, from (3.4) we see that the application Ψ maps B_r^T into itself. A similar argument proves that Ψ is a contraction. Hence Ψ has a fixed point v which is a solution of the IVP (1.1) such that $v \in C([0,T], H^s(\mathbb{R}))$. The rest of the proof follows standard argument, see for example [15]. $\square$

Proof of Theorem 1.2. The proof of this theorem is similar to the one presented for Theorem 1.1. Here, we will use the estimates from Lemma 2.6 and Proposition 2.12. So, we omit the details. $\square$

4. Ill-posedness result

In this section we will use the ideas presented in [18] and [19] to prove the ill-posedness results stated in Theorems 1.3 and 1.4. The idea is to prove that there are no spaces X_T^s and Y_T^s that are continuously embedded in $C([0,T];H^s(\mathbb{R}))$ on which a contraction mapping argument can be applied.

We start with a lemma of elementary calculus which serve as important tools in the proof of Propositions 4.3 and 4.4.

Lemma 4.1. *Let $g:\mathbb{R}^n\to\mathbb{R}$ be a continuous function and $f:\mathbb{R}^n\to\mathbb{R}$ be a positive function. If for any $x\in\mathbb{R}^n$, $|g(x)|\geqslant c_0>0$, then*

$$\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right|\geqslant c_0\int_{\mathbb{R}^n} f(x)\,dx. \tag{4.1}$$

Remark 4.2. Observe that the estimate (4.1) in Lemma 4.1 is false if g is a complex valued function. In fact, if we consider $n=1$, $g(x)=e^{ix}$ and $f(x)=\chi_{[-\pi,\pi]}(x)$, the hypotheses of Lemma 4.1 are satisfied but the estimate (4.1) does not hold.

The following proposition plays a central role in the proof of the ill-posedness result stated in Theorem 1.3.

Proposition 4.3. *Let $k\in\mathbb{Z}^+$, $k>1$, $s<\frac{1}{2}-\frac{p-1}{k}$ and $T>0$. Then there does not exist a space X_T^s continuously embedded in $C([0,T];H^s(\mathbb{R}))$ such that*

$$\big\|V(t)v_0\big\|_{X_T^s}\lesssim\|v_0\|_{H^s}, \tag{4.2}$$

$$\left\|\int_0^t V\big(t-t'\big)\partial_x\big(v\big(t'\big)\big)^{k+1}\,dt'\right\|_{X_T^s}\lesssim\|v\|_{X_T^s}^{k+1}. \tag{4.3}$$

Proof. The proof follows a contradiction argument. If possible, suppose that there exists a space X_T^s that is continuously embedded in $C([0,T];H^s(\mathbb{R}))$ such that the estimates (4.2) and (4.3) hold true. If we consider $v=V(t)v_0$ then from (4.2) and (4.3), we get

$$\left\|\int_0^t V\big(t-t'\big)\partial_x\big[V\big(t'\big)v_0\big]^{k+1}\,dt'\right\|_{H^s}\lesssim\|v_0\|_{H^s}^{k+1}. \tag{4.4}$$

The main idea to complete the proof is to find appropriate initial data v_0 for which the estimate (4.4) fails to hold whenever $s<\frac{1}{2}-\frac{p-1}{k}$.

Let $N\gg 1$, and A and B, $A<B$ be two positive numbers to be chosen later (see (4.16)). Define an initial data via Fourier transform

$$\widehat{v_0}(\xi):=N^{-\frac{2s+1}{2}}\chi_{[A,B]}(\xi/N), \tag{4.5}$$

a simple calculation shows that, for all $s\in\mathbb{R}$, one has $\|v_0\|_{H^s}\sim 1$.

Now, we move to calculate the H^s norm of $f(x,t)$, where

$$f(x,t):=\int_0^t V\big(t-t'\big)\partial_x\big[V\big(t'\big)v_0\big]^{k+1}\,dt'. \tag{4.6}$$

Taking the Fourier transform in the space variable x, we get

$$\begin{aligned}\widehat{f(t)}(\xi) &= \int_0^t e^{i(t-t')\xi^3+(t-t')\Phi(\xi)} i\xi \mathcal{F}_x\big[V(t')v_0\big]^{k+1}(\xi)\,dt' \\ &= e^{it\xi^3+t\Phi(\xi)} i\xi \int_0^t e^{-it'\xi^3-t'\Phi(\xi)} \mathcal{F}_x\big[V(t')v_0\big]^{k+1}(\xi)\,dt'. \end{aligned} \tag{4.7}$$

Defining

$$\varphi_1 := \varphi_1(\xi,\xi_1,\dots,\xi_k) = (\xi-\xi_1-\dots-\xi_k)^3 + \sum_{j=1}^k \xi_j^3 \tag{4.8}$$

and

$$\varphi_2 := \varphi_2(\xi,\xi_1,\dots,\xi_k) = \Phi(\xi-\xi_1-\dots-\xi_k) + \sum_{j=1}^k \Phi(\xi_j), \tag{4.9}$$

one can obtain

$$\begin{aligned}\mathcal{F}_x\big[V(t')v_0\big]^{k+1}(\xi) = N^{-(k+1)\frac{2s+1}{2}} &\int_{\mathbb{R}^k} e^{t'[i\varphi_1+\varphi_2]}\chi_{[A,B]}\left(\frac{\xi-\xi_1-\dots-\xi_k}{N}\right) \\ &\times \prod_{j=1}^k \chi_{[A,B]}\left(\frac{\xi_j}{N}\right) d\xi_1\cdots d\xi_k. \end{aligned} \tag{4.10}$$

Let $\mathcal{M}_{N,\xi}(t) = N^{-(k+1)\frac{2s+1}{2}} e^{it\xi^3+t\Phi(\xi)} i\xi$. Inserting (4.10) in (4.7), and using Fubini's Theorem, we have

$$\begin{aligned}\widehat{f(t)}(\xi) &= \mathcal{M}_{N,\xi}(t)\int_{\mathbb{R}^k} \chi_{[A,B]}\left(\frac{\xi-\xi_1-\dots-\xi_k}{N}\right)\prod_{j=1}^k \chi_{[A,B]}\left(\frac{\xi_j}{N}\right)\int_0^t e^{-it'\xi^3-t'\Phi(\xi)}e^{t'[i\varphi_1+\varphi_2]}\,dt'\,d\xi_1\cdots d\xi_k \\ &= \mathcal{M}_{N,\xi}(t)\int_{\mathbb{R}^k} \chi_{[A,B]}\left(\frac{\xi-\xi_1-\dots-\xi_k}{N}\right)\prod_{j=1}^k \chi_{[A,B]}\left(\frac{\xi_j}{N}\right)\int_0^t e^{t'[i(\varphi_1-\xi^3)+\varphi_2-\Phi(\xi)]}\,dt'\,d\xi_1\cdots d\xi_k \\ &= \mathcal{M}_{N,\xi}(t)\int_{\mathbb{R}^k} \chi_{[A,B]}\left(\frac{\xi-\xi_1-\dots-\xi_k}{N}\right)\prod_{j=1}^k \chi_{[A,B]}\left(\frac{\xi_j}{N}\right)\frac{e^{t[i(\varphi_1-\xi^3)+\varphi_2-\Phi(\xi)]}-1}{i(\varphi_1-\xi^3)+\varphi_2-\Phi(\xi)}\,d\xi_1\cdots d\xi_k. \end{aligned} \tag{4.11}$$

Observe that

$$\begin{aligned}&\left|\int_{\mathbb{R}^k} \chi_{[A,B]}\left(\frac{\xi-\xi_1-\dots-\xi_k}{N}\right)\prod_{j=1}^k \chi_{[A,B]}\left(\frac{\xi_j}{N}\right)\frac{e^{t[i(\varphi_1-\xi^3)+\varphi_2-\Phi(\xi)]}-1}{i(\varphi_1-\xi^3)+\varphi_2-\Phi(\xi)}\,d\xi_1\cdots d\xi_k\right| \\ &\quad\geqslant \left|\int_{\mathbb{R}^k} \chi_{[A,B]}\left(\frac{\xi-\xi_1-\dots-\xi_k}{N}\right)\prod_{j=1}^k \chi_{[A,B]}\left(\frac{\xi_j}{N}\right)\mathrm{Re}\left\{\frac{e^{t[i(\varphi_1-\xi^3)+\varphi_2-\Phi(\xi)]}-1}{i(\varphi_1-\xi^3)+\varphi_2-\Phi(\xi)}\right\}d\xi_1\cdots d\xi_k\right|. \end{aligned} \tag{4.12}$$

Now, in order to apply Lemma 4.1 (see Remark 4.2), we need to estimate the term

$$\mathrm{Re}\left\{\frac{e^{t[i(\varphi_1-\xi^3)+\varphi_2-\Phi(\xi)]}-1}{i(\varphi_1-\xi^3)+\varphi_2-\Phi(\xi)}\right\} = \frac{\omega_2(\cos(t\omega_1)e^{t\omega_2}-1)+\omega_1\sin(t\omega_1)e^{t\omega_2}}{\omega_1^2+\omega_2^2}, \tag{4.13}$$

where $\omega_1 := \varphi_1 - \xi^3$ and $\omega_2 := \varphi_2 - \Phi(\xi)$.

We consider $N > \frac{M}{A(k+1)}$, where M is as in Lemma 2.1 and use (2.1) and (2.3), to obtain

$$\begin{aligned}\omega_2 = \varphi_2 - \Phi(\xi) &= \Phi(\xi-\xi_1-\cdots-\xi_k) + \sum_{j=1}^k \Phi(\xi_j) - \Phi(\xi)\\ &\geqslant -|\xi-\xi_1-\cdots-\xi_k|^p + \Phi_1(\xi-\xi_1-\cdots-\xi_k) - \sum_{j=1}^k\big(|\xi_j|^p - \Phi_1(\xi_j)\big) + \frac{|\xi|^p}{2}\\ &\geqslant -(k+1)(BN)^p + \Phi_1(\xi-\xi_1-\cdots-\xi_k) + \sum_{j=1}^k \Phi_1(\xi_j) + \frac{(k+1)^p(AN)^p}{2}. \end{aligned} \tag{4.14}$$

Since Φ_1 is a polynomial of degree q, with $q < p$, we obtain from (4.14) that

$$\omega_2 = \varphi_2 - \Phi(\xi) \gtrsim N^p, \tag{4.15}$$

provided

$$\frac{(k+1)^p(AN)^p}{2} - (k+1)(BN)^p > \frac{(k+1)^p(AN)^p}{4} \quad \Longleftrightarrow \quad A > \frac{4^{1/p}}{(k+1)^{1-1/p}}B. \tag{4.16}$$

We also have $|\omega_1| \lesssim N^3$. So, considering

$$t := t_0 = CN^{-p}, \quad C > 1, \tag{4.17}$$

where $p \geqslant 2k+1$, $k > 1$ and N large, one obtains

$$\cos(t_0\omega_1)e^{t_0\omega_2} - 1 \geqslant \frac{e^{t_0\omega_2}}{4}. \tag{4.18}$$

In this way, using triangular inequality and (4.15), (4.17) and (4.18) we have for N large

$$\begin{aligned}\left|\mathrm{Re}\left\{\frac{e^{t_0[i(\varphi_1-\xi^3)+\varphi_2-\Phi(\xi)]}-1}{i(\varphi_1-\xi^3)+\varphi_2-\Phi(\xi)}\right\}\right| &= \frac{|\omega_2(\cos(t_0\omega_1)e^{t_0\omega_2}-1)+\omega_1\sin(t_0\omega_1)e^{t_0\omega_2}|}{\omega_1^2+\omega_2^2}\\ &\geqslant \frac{|\omega_2(\cos(t_0\omega_1)e^{t_0\omega_2}-1)| - |\omega_1\sin(t_0\omega_1)e^{t_0\omega_2}|}{\omega_1^2+\omega_2^2}\\ &\geqslant \frac{|\omega_2(\cos(t_0\omega_1)e^{t_0\omega_2}-1)|}{2(\omega_1^2+\omega_2^2)}\\ &\gtrsim \frac{\omega_2 e^{t_0\omega_2}}{\omega_1^2+\omega_2^2}\\ &\gtrsim \frac{t_0\omega_2^2}{\omega_1^2+\omega_2^2}\\ &\gtrsim t_0, \end{aligned} \tag{4.19}$$

where in the last inequality, we used the fact $\omega_1 \leqslant \omega_2$, for $p \geqslant 2k+1 > 3$.

Now using Lemma 4.1 and combining (4.11), (4.12) and (4.19), one gets

$$\begin{aligned}
\big|\widehat{f(t_0)}(\xi)\big| &\gtrsim t_0\big|\mathcal{M}_{N,\xi}(t_0)\big| \int\limits_{\mathbb{R}^k} \chi_{[A,B]}\left(\frac{\xi-\xi_1-\cdots-\xi_k}{N}\right)\prod_{j=1}^{k}\chi_{[A,B]}\left(\frac{\xi_j}{N}\right) d\xi_1\cdots d\xi_k \\
&= t_0\big|\mathcal{M}_{N,\xi}(t_0)\big|N^k \int\limits_{\mathbb{R}^k} \chi_{[A,B]}\left(\frac{\xi}{N}-x_1-\cdots-x_k\right)\prod_{j=1}^{k}\chi_{[A,B]}(x_j)\, dx_1\cdots dx_k \\
&\gtrsim t_0N^k\big|\mathcal{M}_{N,\xi}(t_0)\big|\chi_{[A,B]}*\overset{k+1}{\cdots}*\chi_{[A,B]}\left(\frac{\xi}{N}\right).
\end{aligned} \tag{4.20}$$

Let

$$h(\xi) := \chi_{[A,B]}*\overset{k+1}{\cdots}*\chi_{[A,B]}(\xi), \tag{4.21}$$

and consider N very large (in order to apply Lemma 2.1) to get

$$\begin{aligned}
\big\|f(t_0)\big\|_{H^s}^2 &\geqslant CN^{-(k+1)(2s+1)}N^{2k}\int\limits_{\mathbb{R}} \langle\xi\rangle^{2s}t_0^2e^{2t_0\Phi(\xi)}\xi^2\left|h\left(\frac{\xi}{N}\right)\right|^2 d\xi \\
&= CN^{-(k+1)(2s+1)}N^{2k}\int\limits_{A(k+1)N}^{B(k+1)N} \langle\xi\rangle^{2s}t_0^2e^{-4t_0|\xi|^p}\xi^2\left|h\left(\frac{\xi}{N}\right)\right|^2 d\xi \\
&\geqslant CN^{-(k+1)(2s+1)}N^{2k}\int\limits_{A(k+1)N}^{B(k+1)N} \big(t_0^2|\xi|^{2p}\big)e^{-4t_0|\xi|^p}|\xi|^{2s+2-2p}\left|h\left(\frac{\xi}{N}\right)\right|^2 d\xi \\
&= CN^{-(k+1)(2s+1)}N^{2k}N^{2s+3}\int\limits_{A(k+1)}^{B(k+1)} \big(t_0^2|\xi|^{2p}\big)e^{-4t_0|\xi|^p}|\xi|^{2s+2-2p}\big|h(\xi)\big|^2 d\xi.
\end{aligned} \tag{4.22}$$

Now, recalling that $t_0 \sim N^{-p}$, we obtain,

$$\sup_{t\in[0,T]}\big\|f(t)\big\|_{H^s}^2 \geqslant \big\|f(t_0)\big\|_{H^s}^2 \geqslant CN^{-(k+1)(2s+1)+2k-2p+2s+3}. \tag{4.23}$$

Since $\|v_0\|_{H^s} \sim 1$, in view of (4.4), the estimate (4.23) provides a contradiction if $-(k+1)(2s+1)+2k+2s-2p+3>0$, i.e., if $s<\frac{1}{2}-\frac{p-1}{k}$, and this concludes the proof. □

Proof of Theorem 1.3. For $v_0 \in H^s(\mathbb{R})$, consider the Cauchy problem

$$\begin{cases} v_t + v_{xxx} + \eta Lv + (v^{k+1})_x = 0, & x\in\mathbb{R},\ t\geqslant 0, \\ v(x,0) = \epsilon v_0(x), \end{cases} \tag{4.24}$$

where $\epsilon > 0$ is a parameter. The solution $v^\epsilon(x,t)$ of (4.24) depends on the parameter ϵ. We can write (4.24) in the equivalent integral equation form as

$$v^\epsilon(t) = \epsilon V(t)v_0 - \int\limits_0^t V\big(t-t'\big)\big(v^{k+1}\big)_x\big(t'\big)\, dt', \tag{4.25}$$

where, $V(t)$ is the unitary group describing the solution of the linear part of the IVP (4.24).

Differentiating $v^\epsilon(x,t)$ in (4.25) with respect ϵ and evaluating at $\epsilon = 0$ we get

$$\frac{\partial v^\epsilon(x,t)}{\partial \epsilon}\bigg|_{\epsilon=0} = V(t)v_0(x) =: v_1(x) \tag{4.26}$$

and

$$\frac{\partial^{k+1} v^\epsilon(x,t)}{\partial \epsilon^{k+1}}\bigg|_{\epsilon=0} = C_k \int_0^t V\big(t-t'\big)\partial_x\big(v_1^{k+1}\big(x,t'\big)\big)\,dt' =: v_2(x). \tag{4.27}$$

If the flow-map is C^{k+1} at the origin from $H^s(\mathbb{R})$ to $C([-T,T];H^s(\mathbb{R}))$, we must have

$$\|v_2\|_{L_T^\infty H^s(\mathbb{R})} \lesssim \|v_0\|_{H^s(\mathbb{R})}^{k+1}. \tag{4.28}$$

But from Proposition 4.3 we have seen that the estimate (4.28) fails to hold for $s < \frac{1}{2} - \frac{p-1}{k}$ if we consider v_0 given by (4.5) and this completes the proof of the theorem. □

The following result will be fundamental in the proof of Theorem 1.4.

Proposition 4.4. *Let $k \in \mathbb{Z}^+$, $k > 1$, $s < \frac{3}{2} - \frac{p-1}{k}$ and $T > 0$. Then there does not exists a space X_T^s continuously embedded in $C([0,T];H^s(\mathbb{R}))$ such that*

$$\big\|V(t)u_0\big\|_{X_T^s} \lesssim \|u_0\|_{H^s}, \tag{4.29}$$

$$\left\|\int_0^t V\big(t-t'\big)\big(\partial_x u\big(t'\big)\big)^{k+1}\,dt'\right\|_{X_T^s} \lesssim \|u\|_{X_T^s}^{k+1}. \tag{4.30}$$

Proof. The proof follows a contradiction argument. If possible, suppose that there exists a space X_T^s that is continuously embedded in $C([0,T];H^s(\mathbb{R}))$ such that the estimates (4.29) and (4.30) hold true. If we consider $u = V(t)u_0$ then from (4.29) and (4.30), we get

$$\left\|\int_0^t V\big(t-t'\big)\big[\partial_x V\big(t'\big)u_0\big]^{k+1}\,dt'\right\|_{H^s} \lesssim \|u_0\|_{H^s}^{k+1}. \tag{4.31}$$

The main idea to complete the proof is to find an appropriate initial data u_0 for which the estimate (4.31) fails to hold whenever $s < \frac{3}{2} - \frac{p-1}{k}$. We will consider $u_0 := v_0$ with v_0 defined in (4.5).

We define

$$g(x,t) := \int_0^t V\big(t-t'\big)\big[\partial_x V\big(t'\big)u_0\big]^{k+1}\,dt', \tag{4.32}$$

and calculate its H^s norm.

Taking the Fourier transform in the space variable x, we get

$$\begin{aligned}\widehat{g(t)}(\xi) &= \int_0^t e^{i(t-t')\xi^3+(t-t')\Phi(\xi)}\mathcal{F}_x\big[\partial_x V\big(t'\big)u_0\big]^{k+1}(\xi)\,dt' \\ &= e^{it\xi^3+t\Phi(\xi)}\int_0^t e^{-it'\xi^3-t'\Phi(\xi)}\mathcal{F}_x\big[\partial_x V\big(t'\big)u_0\big]^{k+1}(\xi)\,dt'.\end{aligned} \tag{4.33}$$

Considering the same functions φ_1 and φ_2 defined in (4.8) and (4.9), one can obtain

$$\mathcal{F}_x\big[\partial_x V(t')u_0\big]^{k+1}(\xi) = N^{-(k+1)\frac{2s+1}{2}} \int\limits_{\mathbb{R}^k} e^{t'[i\varphi_1+\varphi_2]} i(\xi-\xi_1-\cdots-\xi_k)\chi_{[A,B]} \times \left(\frac{\xi-\xi_1-\cdots-\xi_k}{N}\right) \prod_{j=1}^{k} i\xi_j \chi_{[A,B]}\left(\frac{\xi_j}{N}\right) d\xi_1\cdots d\xi_k. \tag{4.34}$$

Let $\widetilde{\mathcal{M}}_{N,\xi}(t) = N^{-(k+1)\frac{2s+1}{2}} e^{it\xi^3+t\Phi(\xi)}$. Inserting (4.34) in (4.33), and using Fubini's Theorem, we have

$$\begin{aligned}\widehat{g(t)}(\xi) &= \widetilde{\mathcal{M}}_{N,\xi}(t) \int\limits_{\mathbb{R}^k} i(\xi-\xi_1-\cdots-\xi_k)\chi_{[A,B]}\left(\frac{\xi-\xi_1-\cdots-\xi_k}{N}\right) \prod_{j=1}^{k} i\xi_j \chi_{[A,B]}\left(\frac{\xi_j}{N}\right) \\ &\quad \times \int\limits_0^t e^{-it'\xi^3-t'\Phi(\xi)} e^{t'[i\varphi_1+\varphi_2]}\,dt'\,d\xi_1\cdots d\xi_k \\ &= \widetilde{\mathcal{M}}_{N,\xi}(t) i^{k+1} \int\limits_{\mathbb{R}^k} (\xi-\xi_1-\cdots-\xi_k)\chi_{[A,B]}\left(\frac{\xi-\xi_1-\cdots-\xi_k}{N}\right) \prod_{j=1}^{k} \xi_j \chi_{[A,B]}\left(\frac{\xi_j}{N}\right) \\ &\quad \times \frac{e^{t[i(\varphi_1-\xi^3)+\varphi_2-\Phi(\xi)]}-1}{i(\varphi_1-\xi^3)+\varphi_2-\Phi(\xi)}\,d\xi_1\cdots d\xi_k. \end{aligned} \tag{4.35}$$

We consider N, A, B and $t := t_0 \sim N^{-p}$ as in proof of Proposition 4.3 and with a similar argument as in (4.20), one can easily obtain

$$\begin{aligned}\big|\widehat{g(t_0)}(\xi)\big| &\gtrsim t_0 N^{k+1}\big|\widetilde{\mathcal{M}}_{N,\xi}(t_0)\big| \int\limits_{\mathbb{R}^k} \chi_{[A,B]}\left(\frac{\xi-\xi_1-\cdots-\xi_k}{N}\right) \prod_{j=1}^{k} \chi_{[A,B]}\left(\frac{\xi_j}{N}\right) d\xi_1\cdots d\xi_k \\ &= t_0 N^{k+1}\big|\widetilde{\mathcal{M}}_{N,\xi}(t_0)\big| N^k \int\limits_{\mathbb{R}^k} \chi_{[A,B]}\left(\frac{\xi}{N}-x_1-\cdots-x_k\right) \prod_{j=1}^{k} \chi_{[A,B]}(x_j)\,dx_1\cdots dx_k \\ &\gtrsim t_0 N^{k+1} N^k \big|\widetilde{\mathcal{M}}_{N,\xi}(t_0)\big| \chi_{[A,B]} * \overset{k+1}{\cdots} * \chi_{[A,B]}\left(\frac{\xi}{N}\right). \end{aligned} \tag{4.36}$$

In this way, we get

$$\big|\widehat{g(t_0)}(\xi)\big| \geqslant C\big|\widetilde{\mathcal{M}}_{N,\xi}(t_0)\big| t_0 N^{2k+1} h\left(\frac{\xi}{N}\right), \tag{4.37}$$

where h is as in (4.21). Thus, for N very large, we have

$$\big|\widehat{g(t_0)}(\xi)\big| \geqslant C t_0 N^{-(k+1)\frac{2s+1}{2}} e^{-4t_0|\xi|^p} N^{2k+1} h\left(\frac{\xi}{N}\right), \tag{4.38}$$

and

$$\|g(t_0)\|_{H^s}^2 \geqslant CN^{-(k+1)(2s+1)}N^{2(2k+1)} \int_{\mathbb{R}} \langle\xi\rangle^{2s} t_0^2 e^{-4t_0|\xi|^p} \left|h\left(\frac{\xi}{N}\right)\right|^2 d\xi$$

$$\geqslant CN^{-(k+1)(2s+1)}N^{2(2k+1)} \int_{A(k+1)N}^{B(k+1)N} \left(t_0^2|\xi|^{2p}\right) e^{-4t_0|\xi|^p} |\xi|^{2s-2p} \left|h\left(\frac{\xi}{N}\right)\right|^2 d\xi. \tag{4.39}$$

As in (4.22), taking $t_0 \sim N^{-p}$, we obtain that

$$\sup_{t\in[0,T]} \|g(t)\|_{H^s}^2 \geqslant \|g(t_0)\|_{H^s}^2 \geqslant CN^{-(k+1)(2s+1)+2(2k+1)+2s-2p}. \tag{4.40}$$

In view of (4.31), the estimate (4.40) is a contradiction if $-(k+1)(2s+1)+2(2k+1)+2s-2p > 0$, i.e., if $s < \frac{3}{2} - \frac{p-1}{k}$, and this completes the proof of the proposition. □

Proof of Theorem 1.4. Proof of this theorem is very similar to that of Theorem 1.3 and follows by using Proposition 4.4. So, we omit the details. □

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