J. Math. Anal. Appl. 422 (2015) 72–83

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Orthogonally additive polynomials on Fourier algebras ☆

J. Alaminos, J. Extremera, A.R. Villena *

Departamento de Análisis Matemático, Facultad de Ciencias, Universidad de Granada, 18071 Granada, Spain

ARTICLE INFO

Article history:
Received 19 May 2014
Available online 19 August 2014
Submitted by D. Blecher

Keywords:
Fourier algebra
Figà–Talamanca–Herz algebra
C^*-algebra
Orthogonally additive polynomial
Orthosymmetric multilinear map

ABSTRACT

We show that an n-homogeneous polynomial P on the Fourier algebra $A(G)$ of a locally compact group G can be represented in the form $P(f) = \langle T, f^n\rangle$ $(f \in A(G))$ for some T in the group von Neumann algebra $VN(G)$ of G if and only if it is orthogonally additive and completely bounded.

1. Introduction

Let A be a Banach algebra. A map Φ from A onto a Banach space Y is said to be *orthogonally additive* if $\Phi(a+b) = \Phi(a)+\Phi(b)$ whenever $a, b \in A$ are such that $ab = ba = 0$. In the case where A is a C^*-algebra, it is known that every continuous orthogonally additive n-homogeneous polynomial P on A can be represented in the form $P(a) = \langle\omega, a^n\rangle$ $(a \in A)$ for some $\omega \in A^*$ (see [5,7,22] for commutative C^*-algebras and [21] for arbitrary C^*-algebras). Our purpose is to investigate whether this representation still holds true for orthogonally additive n-homogeneous polynomials on the Fourier algebra $A(G)$ of a locally compact group G. We refer the reader to [13] for the basic properties of $A(G)$. We recall that $A(G)$ is a regular, Tauberian, semisimple, commutative Banach algebra whose character space is identified with G by point evaluation and the dual of $A(G)$ can be identified with the group von Neumann algebra $VN(G)$ of G. The next example shows that the required representation may fail to hold for some groups.

Example 1.1. Let $\mathbb{F}_2$ be the free group on two generators $\{a, b\}$. It is clear that the set $\{a^n b^n : n \in \mathbb{N}\}$ satisfies the Leinert condition [18, Definition] and [18, (2.1)] then shows that there exists $C > 0$ such that $\sum_{n=1}^{\infty} |f(a^n b^n)|^2 \leq C\|f\|^2_{A(\mathbb{F}_2)}$ for each $f \in A(\mathbb{F}_2)$. This allows to define a continuous orthogonally additive

☆ The authors were supported by MINECO Grant MTM2012-31755 and Consejería Economía, Innovación, Ciencia y Empleo, Junta de Andalucia Grants FQM-185 and P09-FQM-4911.

* Corresponding author.
E-mail addresses: alaminos@ugr.es (J. Alaminos), jlizana@ugr.es (J. Extremera), avillena@ugr.es (A.R. Villena).

http://dx.doi.org/10.1016/j.jmaa.2014.08.035
0022-247X/

2-homogeneous polynomial $P: A(\mathbb{F}_2) \to \mathbb{C}$ by $P(f) = \sum_{n=1}^{\infty}(f(a^n b^n))^2$ for each $f \in A(\mathbb{F}_2)$. Suppose that P can be represented by some $T \in VN(\mathbb{F}_2)$. For all $g, h \in A(\mathbb{F}_2)$ we have

$$\begin{aligned}\sum_{n=1}^{\infty} g(a^n b^n) h(a^n b^n) &= \frac{1}{2}\big(P(g+h) - P(g) - P(h)\big)\\ &= \frac{1}{2}\big(\langle T, (g+h)^2\rangle - \langle T, g^2\rangle - \langle T, h^2\rangle\big)\\ &= \langle T, gh\rangle.\end{aligned}$$

Consequently, $\sum_{n=1}^{\infty} f(a^n b^n) = \langle T, f\rangle$ for each f in the linear span B of the set $\{gh : g, h \in A(\mathbb{F}_2)\}$. We mention in passing that, by [18, (2.3)], $B \neq A(\mathbb{F}_2)$. Let λ be the left regular representation of $\mathbb{F}_2$ on $L^2(\mathbb{F}_2)$. By [1, Theorem IV J], we have $\|\sum_{n=1}^{N} \lambda(a^n b^n)\|_{VN(\mathbb{F}_2)} = 2\sqrt{N-1}$ for each $N \geq 3$. Since $VN(\mathbb{F}_2) = A(\mathbb{F}_2)^*$ and B is dense in $A(\mathbb{F}_2)$, we conclude that there exists $f \in B$ such that $\|f\|_{A(\mathbb{F}_2)} = 1$ and

$$\sqrt{N-1} < \left|\left\langle \sum_{n=1}^{N} \lambda(a^n b^n), f \right\rangle\right| = \left|\sum_{n=1}^{N} f(a^n b^n)\right| = \big|\langle T, f\rangle\big| \leq \|T\|_{VN(\mathbb{F}_2)},$$

a contradiction.

In order to have a right view of the representation problem it seems to be appropriate to take into account the structure of operator space of $A(G)$. We refer the reader to [12] for the necessary background from operator space theory. The duality between $A(G)$ and $VN(G)$ equips $A(G)$ with a natural structure of operator space. Further, with this structure, $A(G)$ becomes a completely contractive Banach algebra (see [12, Sections 16.1 and 16.2]). This implies that the polynomial $f \mapsto \langle T, f^n\rangle$ is not merely continuous, but actually completely bounded for each $T \in VN(G)$.

In Section 2 we show that a (complex-valued) n-homogeneous polynomial P on the Fourier algebra $A(G)$ of a locally compact group G can be represented in the form $P(f) = P_T^n(f) := \langle T, f^n\rangle$ ($f \in A(G)$) for some $T \in VN(G)$ if and only if it is orthogonally additive and completely bounded. In fact, the map $T \mapsto P_T^n$ is shown to be a completely isometric isomorphism from $VN(G)$ onto the space $\mathcal{P}_{cbo}^n(A(G), \mathbb{C})$ of all completely bounded orthogonally additive (complex-valued) n-homogeneous polynomials on $A(G)$. Section 3 reveals that the preceding theory applies to other Banach algebras such as the Figà–Talamanca–Herz algebras and the commutative C^*-algebras.

It should be pointed out that the representation of orthogonally additive polynomials has been widely discussed in the context of Banach lattices (see [16] and the references therein).

1.1. Notation

Let $X, X_1, \ldots, X_n$, and Y be Banach spaces. We write $\mathcal{B}^n(X_1, \ldots, X_n; Y)$ for the Banach space of all continuous n-linear maps from $X_1 \times \cdots \times X_n$ into Y. We write $\mathcal{B}^n(X, Y)$ in the case where $X_1 = \cdots = X_n = X$. As usual, we abbreviate $\mathcal{B}^1(X, Y)$ to $\mathcal{B}(X, Y)$, $\mathcal{B}(X, X)$ to $\mathcal{B}(X)$, and $\mathcal{B}(X, \mathbb{C})$ to X^*. We write $\langle \cdot, \cdot \rangle$ for the dual pairing of X and X^*. A map $P: X \to Y$ is a continuous n-homogeneous polynomial if there exists $\varphi \in \mathcal{B}^n(X, Y)$ (which is unique if it is required to be symmetric) such that $P(x) = \varphi(x, \ldots, x)$ for each $x \in X$. Let $\mathcal{P}^n(X, Y)$ denote the space of all continuous n-homogeneous polynomials from X into Y. This is a Banach space equipped with the norm $\|P\| = \sup_{\|x\|=1} \|P(x)\|$. From the polarization formula, it follows that $\|P\| \leq \|\varphi\| \leq \frac{n^n}{n!}\|P\|$, where φ is the symmetric n-linear map associated with P.

Throughout this paper we confine ourselves to complex-valued polynomials on a Banach algebra A. Of course, for any $\omega \in A^*$, the map $P_\omega^n: A \to \mathbb{C}$ defined by

$$P_\omega^n(a) = \langle \omega, a^n \rangle \quad (a \in A)$$

is a continuous orthogonally additive n-homogeneous polynomial. The symmetric n-linear form $\varphi_\omega^n: A^n \to \mathbb{C}$ associated with P_ω^n is given by

$$\varphi_\omega^n(a_1, \ldots, a_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \langle \omega, a_{\sigma(1)} \cdots a_{\sigma(n)} \rangle \quad (a_1, \ldots, a_n \in A),$$

which becomes simply

$$\varphi_\omega^n(a_1, \ldots, a_n) = \langle \omega, a_1 \cdots a_n \rangle \quad (a_1, \ldots, a_n \in A),$$

in the case where A is commutative.

We now suppose that $X, X_1, \ldots, X_n$, and Y are operator spaces. As usual, $\mathbb{M}_k(X)$ denotes the space of $k \times k$ matrices with entries in X. This may also be thought of as the algebraic tensor product $\mathbb{M}_k \otimes X$, where $\mathbb{M}_k = \mathbb{M}_k(\mathbb{C})$. We identify matrices of matrices with simple matrices in the usual way. Let $\varphi: X_1 \times \cdots \times X_n \to Y$ be an n-linear map and $k_1, \ldots, k_n \in \mathbb{N}$. Then the $(k_1, \ldots, k_n)$-amplification

$$\varphi^{(k_1,\ldots,k_n)}: \mathbb{M}_{k_1}(X_1) \times \cdots \times \mathbb{M}_{k_n}(X_n) \to \mathbb{M}_{k_1 \cdots k_n}(Y)$$

of φ is defined through

$$\varphi^{(k_1,\ldots,k_n)}(\alpha_1 \otimes x_1, \ldots, \alpha_n \otimes x_n) = \alpha_1 \otimes \cdots \otimes \alpha_n \otimes \varphi(x_1, \ldots, x_n)$$

for all $(\alpha_1, \ldots \alpha_n) \in \mathbb{M}_{k_1}(\mathbb{C}) \times \cdots \times \mathbb{M}_{k_n}(\mathbb{C})$ and $(x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n$. The map φ is said to be completely bounded if

$$\|\varphi\|_{cb} = \sup\bigl\{\bigl\|\varphi^{(k_1,\ldots,k_n)}\bigr\|_{\mathcal{B}^n(\mathbb{M}_{k_1}(X_1),\ldots,\mathbb{M}_{k_n}(X_n);\mathbb{M}_{k_1\cdots k_n}(Y))} : k_1, \ldots, k_n \in \mathbb{N}\bigr\} < \infty.$$

This is the same as asserting that the linearization of φ from $X_1 \otimes \cdots \otimes X_n$ into Y determines a completely bounded linear map $\widehat{\varphi}$ from the operator space projective tensor product $X_1 \widehat{\otimes} \cdots \widehat{\otimes} X_n$ into Y. We write $\mathcal{CB}^n(X_1, \ldots, X_n; Y)$ for the linear space of all completely bounded n-linear maps from $X_1 \times \cdots \times X_n$ into Y. This is an operator space with matrix norms coming from the identification $\mathbb{M}_k(\mathcal{CB}^n(X_1, \ldots, X_n; Y)) = \mathcal{CB}^n(X_1, \ldots, X_n; \mathbb{M}_k(Y))$. We write $\mathcal{CB}^n(X, Y)$ in the case where $X_1 = \cdots = X_n = X$ and we abbreviate $\mathcal{CB}^1(X, Y)$ to $\mathcal{CB}(X, Y)$. The map $\varphi \mapsto \widehat{\varphi}$ is a completely isometric isomorphism from $\mathcal{CB}^n(X_1, \ldots, X_n; Y)$ onto $\mathcal{CB}(X_1 \widehat{\otimes} \cdots \widehat{\otimes} X_n, Y)$ and there are natural completely isometric isomorphisms from $\mathcal{CB}^n(X_1, \ldots, X_n; Y)$ onto $\mathcal{CB}(X_i, \mathcal{CB}^{n-1}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n; Y))$. An n-homogeneous polynomial $P: X \to Y$ is completely bounded if the symmetric n-linear form associated with P is completely bounded. We write $\mathcal{P}_{cb}^n(X, Y)$ for the linear space of all completely bounded n-homogeneous polynomials from X into Y. This is an operator space with the structure inherited from $\mathcal{CB}^n(X, Y)$ through the identification of $\mathcal{P}_{cb}^n(X, Y)$ with the symmetric completely bounded n-linear maps. We refer the reader to [10] for further details about completely bounded polynomials.

2. Orthogonally additive polynomials on Fourier algebras

Let A be a Banach algebra, and let n be an integer with $n \geq 2$. Motivated by [6] we call an n-linear form $\varphi: A^n \to \mathbb{C}$ *orthosymmetric* if $\varphi(a_1, \ldots, a_n) = 0$, whenever $(a_1, \ldots, a_n) \in A^n$ is such that $a_i a_j = a_j a_i = 0$ for some $i, j \in \{1, \ldots, n\}$. Guided by [16] we also consider the following variant of the newly quoted orthosymmetry. An n-tuple $(a_1, \ldots, a_n) \in A^n$ is said to be *partitionally orthogonal* if there exists a partition $\{\Lambda_1, \ldots, \Lambda_m\}$ of the set $\{1, \ldots, n\}$ with $2 \leq m \leq n$ such that $a_i a_j = a_j a_i = 0$ whenever $i \in \Lambda_k$ and $j \in \Lambda_l$ with $k, l \in \{1, \ldots, m\}$ and $k \neq l$. An n-linear form $\varphi: A^n \to \mathbb{C}$ is said to be *partitionally orthosymmetric* if $\varphi(a_1, \dots, a_n) = 0$ whenever $(a_1, \dots, a_n) \in A^n$ is partitionally orthogonal. It is easy to check that the orthosymmetry of φ implies the partitional orthosymmetry of φ.

It seems appropriate to mention in passing that the orthosymmetric bilinear maps have been intensively used in [2–4] to study the disjointness preserving linear maps on a variety of Banach algebras.

Lemma 2.1. *Let A be a Banach algebra, and let n be an integer with $n \geq 2$. Then for any $\omega \in A^*$ the n-linear form φ_ω^n is orthosymmetric and $\|\varphi_\omega^n\|_{\mathcal{B}^n(A,\mathbb{C})} \leq \|\omega\|_{A^*}$. If A has an approximate identity of bound C, then $\|\omega\|_{A^*} \leq C^{n-1}\|\varphi_\omega^n\|_{\mathcal{B}^n(A,\mathbb{C})}$.*

Proof. It is easily seen that φ_ω^n is orthosymmetric and that $\|\varphi_\omega^n\|_{\mathcal{B}^n(A,\mathbb{C})} \leq \|\omega\|_{A^*}$.

Suppose that $(\rho_\lambda)_{\lambda\in\Lambda}$ is an approximate identity of bound C. It is clear that $\langle\omega, a\rangle = \lim_{\lambda\in\Lambda} \varphi_\omega^n(a, \rho_\lambda, \dots, \rho_\lambda)$ for each $a \in A$. Moreover, $|\varphi_\omega^n(a, \rho_\lambda, \dots, \rho_\lambda)| \leq \|\varphi_\omega^n\|_{\mathcal{B}^n(A,\mathbb{C})} C^{n-1}\|a\|_A$ for all $a \in A$ and $\lambda \in \Lambda$. Hence, $|\langle\omega, a\rangle| \leq \|\varphi_\omega^n\|_{\mathcal{B}^n(A,\mathbb{C})} C^{n-1}\|a\|_A$ and therefore $\|\omega\|_{A^*} \leq C^{n-1}\|\varphi_\omega^n\|_{\mathcal{B}^n(A,\mathbb{C})}$. $\square$

Remark 2.2.

(1) If A has an approximate identity of bound 1, then $\|\varphi_\omega^n\|_{\mathcal{B}^n(A,\mathbb{C})} = \|\omega\|_{A^*}$ for each $\omega \in A^*$. Of course, $\|P_\omega^n\|_{\mathcal{P}^n(A,\mathbb{C})} \leq \|\omega\|_{A^*}$. Nevertheless, this latter inequality may fail to be an equality (see Example 2.7).

(2) It is worth noting that the Fourier algebra $A(G)$ of any amenable group G has an approximate identity of bound 1 (further, $A(G)$ has a bounded approximate identity precisely when G is amenable) [19].

The next result reveals the reason of considering the partitional orthosymmetry.

Lemma 2.3. *Let A be a Banach algebra. Let $P: A \to \mathbb{C}$ be a continuous n-homogeneous polynomial for some integer n with $n \geq 2$, and let $\varphi: A^n \to \mathbb{C}$ be the symmetric continuous n-linear form associated with P. Then the following assertions are equivalent:*

(1) *the polynomial P is orthogonally additive;*

(2) *the form φ is partitionally orthosymmetric.*

Proof. Let $(a_1, \dots, a_n) \in A^n$ partitionally orthogonal. Let $\{\Lambda_1, \dots, \Lambda_m\}$ be the corresponding partition from the definition. Let $z_1, \dots, z_n \in \mathbb{C}$. Then the elements $\sum_{i\in\Lambda_j} z_i a_i$ with $j \in \{1, \dots, m\}$ are mutually orthogonal and therefore

$$P\left(\sum_{i=1}^n z_i a_i\right) = \sum_{j=1}^m P\left(\sum_{i\in\Lambda_j} z_i a_i\right).$$

The coefficient of the monomial $z_1 \cdots z_n$ on the left side of the identity is $n!\varphi(a_1, \dots, a_n)$ while the coefficient of $z_1 \cdots z_n$ of each and every summand on the right side is zero because $m \geq 2$. We thus get $\varphi(a_1, \dots, a_n) = 0$. $\square$

Theorem 2.4. *Let G be a locally compact group, and let n be an integer with $n \geq 2$. Then the following statements hold:*

(1) *a completely bounded n-linear form on $A(G)$ is orthosymmetric if and only if it is partitionally orthosymmetric;*

(2) *the map $T \mapsto \varphi_T^n$ is a completely isometric isomorphism from $VN(G)$ onto the space $\mathcal{CB}_o^n(A(G), \mathbb{C})$ of all completely bounded orthosymmetric n-linear forms on $A(G)$;*

(3) *the map* $T \mapsto P_T^n$ *is a completely isometric isomorphism from* $VN(G)$ *onto the space* $\mathcal{P}^n_{cbo}(A(G), \mathbb{C})$ *of all completely bounded orthogonally additive* n*-homogeneous polynomials on* $A(G)$*.*

Proof. Let $M_n: A(G)^n \to A(G)$ be the n-linear map defined by

$$M_n(f_1, \dots, f_n) = f_1 \cdots f_n$$

for all $f_1, \dots, f_n \in A(G)$. It is well known that M_2 is completely bounded with $\|M_2\|_{\mathcal{CB}^2(A(G),A(G))} \leq 1$. Furthermore, it is clear that

$$M_n^{(k_1,\dots,k_n)}(F_1, \dots F_n) = M_2^{(k_1,k_2\cdots k_n)}\big(F_1, M_{n-1}^{(k_2,\dots,k_n)}(F_2, \dots, F_n)\big)$$

for all $k_1, \dots, k_n \in \mathbb{N}$ and $F_1 \in \mathbb{M}_{k_1}(A(G)), \dots, F_{k_n} \in \mathbb{M}_{k_n}(A(G))$. By using an inductive procedure we deduce that $\|M_n\|_{\mathcal{CB}^n(A(G),A(G))} \leq 1$.

Let H be a locally compact group. If $f \in A(G)$ and $g \in A(H)$, then the function $(s,t) \mapsto f(s)g(t)$ on $G \times H$ lies in $A(G \times H)$. By [11], this induces a canonical completely isometric isomorphism from $A(G)\widehat{\otimes}A(H)$ onto $A(G \times H)$. Consequently, the map $\Psi: A(G)\widehat{\otimes} \cdots \widehat{\otimes}A(G) \to A(G^n)$ defined through

$$\Psi(f_1 \otimes \cdots \otimes f_n)(t_1, \dots, t_n) = f_1(t_1) \dots f_n(t_n)$$

for all $f_1, \dots, f_n \in A(G)$ and $t_1, \dots, t_n \in G$ is a completely isometric isomorphism.

Let $T \in VN(G)$. It is clear that φ_T^n is partitionally orthogonal. Further, we have $\varphi_T^n = T \circ M_n$, and therefore $\varphi_T^n \in \mathcal{CB}^n(A(G), \mathbb{C})$.

Let $[T_{ij}] \in \mathbb{M}_k(VN(G))$. Then

$$\begin{aligned}\big\|\big[\varphi^n_{T_{ij}}\big]\big\|_{\mathbb{M}_k(\mathcal{CB}^n(A(G),\mathbb{C}))} &= \big\|[T_{ij}] \circ M_n\big\|_{\mathcal{CB}^n(A(G),\mathbb{M}_k)} \\ &\leq \big\|[T_{ij}]\big\|_{\mathcal{CB}(A(G),\mathbb{M}_k)}\|M_n\|_{\mathcal{CB}^n(A(G),A(G))} \\ &\leq \big\|[T_{ij}]\big\|_{\mathbb{M}_k(VN(G))}.\end{aligned}$$

Let φ be a completely bounded partitionally orthosymmetric n-linear form on $A(G)$. Our next objective is to show that $\varphi = \varphi_T^n$ for some $T \in VN(G)$, which establishes the orthosymmetry of φ. Let us also observe that such a T is necessarily unique because the linear span of the set $\{f_1 \cdots f_n: f_1, \dots, f_n \in A(G)\}$ is dense in $A(G)$.

Let I be the closed linear subspace of $A(G^n)$ generated by the set K of all functions of the form $\Psi(f_1 \otimes \cdots \otimes f_n)$, where $(f_1, \dots, f_n) \in A(G)^n$ is partitionally orthogonal. It is clear that $\Psi(f_1 \otimes \cdots \otimes f_n)K \subset K$ for each $(f_1, \dots, f_n) \in A(G)^n$ and, since $\Psi(A(G) \otimes \cdots \otimes A(G))$ is dense in $A(G^n)$, it follows that I is an ideal of $A(G^n)$. Since φ is partitionally orthosymmetric, it follows that $I \subset \ker(\widehat{\varphi} \circ \Psi^{-1})$. Consequently, there exists a continuous linear functional $\widetilde{\varphi}: A(G^n)/I \to \mathbb{C}$ such that $\widehat{\varphi} \circ \Psi^{-1} = \widetilde{\varphi} \circ Q$, where $Q: A(G^n) \to A(G^n)/I$ is the quotient homomorphism.

The task is now to prove that

$$I = \big\{F \in A\big(G^n\big) : F(t, \dots, t) = 0 \text{ for each } t \in G\big\}.$$

To this end, we first show that the hull

$$h(I) = \big\{(t_1, \dots, t_n) \in G^n : F(t_1, \dots, t_n) = 0 \text{ for each } F \in I\big\}$$

of I is the set $\Delta = \{(t, \dots, t) : t \in G\}$. It is immediate to check that $\Delta \subset h(I)$. Conversely, assume that $(t_1, \dots, t_n) \in G^n \setminus \Delta$. Then there exists a partition $\{\Lambda_1, \dots, \Lambda_m\}$ of the set $\{1, \dots, n\}$ and pairwise different elements $s_1, \dots, s_m \in G$ with $2 \le m \le n$ such that $t_i = s_k$ whenever $i \in \Lambda_k$ for some $k \in \{1, \dots, m\}$. Let $U_1, \dots, U_m$ pairwise disjoint open subsets of G with $s_k \in U_k$ ($k \in \{1, \dots, m\}$) and let $g_1, \dots, g_m \in A(G)$ with $\operatorname{supp}(g_k) \subset U_k$ and $g_k(s_k) = 1$ ($k \in \{1, \dots, m\}$). We define $f_1, \dots, f_n \in A(G)$ by $f_i = g_k$ whenever $i \in \Lambda_k$ for some $k \in \{1, \dots, m\}$. Then $(f_1, \dots, f_n)$ is partitionally orthogonal so that $\Psi(f_1 \otimes \dots \otimes f_n) \in K$. Since $\Psi(f_1 \otimes \dots \otimes f_n)(t_1, \dots, t_n) = 1$, it follows that $(t_1, \dots, t_n) \in G^n \setminus h(I)$, which completes the proof of the property $h(I) = \Delta$. Since Δ is a closed subgroup of G^n, we conclude that Δ is a set of synthesis for $A(G^n)$ [24, Theorem 3]. This means that $\{F \in A(G^n) : F(t, \dots, t) = 0 \text{ for each } t \in G\}$ is the only closed ideal whose hull equal to Δ and therefore it is equal to I, as claimed.

On account of [14, Proposition 4.2], the restriction map $\Gamma: A(G^n)/I \to A(G)$ defined by $\Gamma(Q(F))(t) = F(t, \dots, t)$ for all $F \in A(G^n)$ and $t \in G$ is a completely isometric isomorphism. Further, it should be observed that clearly $(\Gamma \circ Q \circ \Psi)(f_1 \otimes \dots \otimes f_n) = f_1 \cdots f_n$ for each $(f_1, \dots, f_n) \in A(G)^n$. Define $T = \widetilde{\varphi} \circ \Gamma^{-1} \in A(G)^*$. Then

$$\begin{aligned}
\varphi_T^n(f_1, \dots, f_n) &= \left\langle \widetilde{\varphi} \circ \Gamma^{-1}, (\Gamma \circ Q \circ \Psi)(f_1 \otimes \dots \otimes f_n) \right\rangle \\
&= \langle \widetilde{\varphi} \circ Q \circ \Psi, f_1 \otimes \dots \otimes f_n \rangle \\
&= \left\langle \widehat{\varphi} \circ \Psi^{-1} \circ \Psi, f_1 \otimes \dots \otimes f_n \right\rangle \\
&= \langle \widehat{\varphi}, f_1 \otimes \dots \otimes f_n \rangle \\
&= \varphi(f_1, \dots, f_n)
\end{aligned}$$

for each $(f_1, \dots, f_n) \in A(G)^n$.

Let $[\varphi_{ij}] \in \mathbb{M}_k(\mathcal{CB}_o^n(A(G), \mathbb{C}))$ and take $T_{ij} = \widetilde{\varphi_{ij}} \circ \Gamma^{-1}$ for all $i, j \in \{1, \dots, k\}$. Using that Q is a complete quotient map and that Γ, Ψ, and $\widehat{\ }$ are completely isometric isomorphisms, we obtain

$$\begin{aligned}
\left\|[T_{ij}]\right\|_{\mathbb{M}_k(VN(G))} &= \left\|[T_{ij}]\right\|_{\mathcal{CB}(A(G), \mathbb{M}_k)} = \left\|[T_{ij}]^{(k)}\right\|_{\mathcal{B}(\mathbb{M}_k(A(G)), \mathbb{M}_k(\mathbb{M}_k))} \\
&= \left\|[\widetilde{\varphi_{ij}}]^{(k)} \circ \left(\Gamma^{-1}\right)^{(k)}\right\|_{\mathcal{B}(\mathbb{M}_k(A(G)), \mathbb{M}_k(\mathbb{M}_k))} \\
&\le \left\|[\widetilde{\varphi_{ij}}]^{(k)}\right\|_{\mathcal{B}(\mathbb{M}_k(A(G^n)/I), \mathbb{M}_k(\mathbb{M}_k))} \left\|\left(\Gamma^{-1}\right)^{(k)}\right\|_{\mathcal{B}(\mathbb{M}_k(A(G)), \mathbb{M}_k(A(G^n)/I))} \\
&= \left\|[\widetilde{\varphi_{ij}}]^{(k)}\right\|_{\mathcal{B}(\mathbb{M}_k(A(G^n)/I), \mathbb{M}_k(\mathbb{M}_k))} = \left\|[\widetilde{\varphi_{ij}}]^{(k)} \circ Q^{(k)}\right\|_{\mathcal{B}(\mathbb{M}_k(A(G^n)), \mathbb{M}_k(\mathbb{M}_k))} \\
&= \left\|[\widehat{\varphi_{ij}}]^{(k)} \circ \left(\Psi^{-1}\right)^{(k)}\right\|_{\mathcal{B}(\mathbb{M}_k(A(G^n)), \mathbb{M}_k(\mathbb{M}_k))} \\
&\le \left\|[\widehat{\varphi_{ij}}]^{(k)}\right\|_{\mathcal{B}(\mathbb{M}_k(A(G)\widehat{\otimes}\cdots\widehat{\otimes}A(G)), \mathbb{M}_k(\mathbb{M}_k))} \left\|\left(\Psi^{-1}\right)^{(k)}\right\|_{\mathcal{B}(\mathbb{M}_k(A(G^n)), \mathbb{M}_k(A(G)\widehat{\otimes}\cdots\widehat{\otimes}A(G)))} \\
&= \left\|[\widehat{\varphi_{ij}}]^{(k)}\right\|_{\mathcal{B}(\mathbb{M}_k(A(G)\widehat{\otimes}\cdots\widehat{\otimes}A(G)), \mathbb{M}_k(\mathbb{M}_k))} \le \left\|[\widehat{\varphi_{ij}}]\right\|_{\mathcal{CB}(A(G)\widehat{\otimes}\cdots\widehat{\otimes}A(G), \mathbb{M}_k)} \\
&= \left\|[\widehat{\varphi_{ij}}]\right\|_{\mathbb{M}_k(\mathcal{CB}(A(G)\widehat{\otimes}\cdots\widehat{\otimes}A(G), \mathbb{C}))} = \left\|[\varphi_{ij}]\right\|_{\mathbb{M}_k(\mathcal{CB}^n(A(G), \mathbb{C}))}.
\end{aligned}$$

This completes the proof of the statement (2).

Finally, Lemma 2.3 and the statement (2) in the theorem now yield (3). □

We will now show that if G is almost abelian, then we are allowed to ignore the operator space structure of $A(G)$.

Lemma 2.5. *Let G be a locally compact group. Let X be an operator space, and let $\varphi: A(G)^n \to X$ be a bounded n-linear map for some positive integer n. Suppose that G has an abelian subgroup of finite index. Then φ is completely bounded.*

Proof. The case $n = 1$ follows from [14, Theorem 4.5].

We now turn to the case $n \geq 2$. Throughout the proof, $\otimes_\pi$ denotes the Banach space projective tensor product. The linearization of φ on $A(G) \otimes \cdots \otimes A(G)$ gives a bounded linear map $\varphi_\pi \colon A(G) \otimes_\pi \cdots \otimes_\pi A(G) \to X$. By [20, Theorem 1], the canonical map $\Phi \colon A(G) \otimes_\pi \cdots \otimes_\pi A(G) \to A(G^n)$ is a (Banach space) isomorphism (though not necessarily isometric). Since G^n has an abelian subgroup of finite index, [14, Theorem 4.5] shows that every bounded linear map from $A(G^n)$ into any operator space is completely bounded. Consequently, the map $\varphi_\pi \circ \Phi^{-1}$ is completely bounded. We finally observe that $\varphi = \varphi_\pi \circ \Phi^{-1} \circ \Psi \circ \Theta$, where Ψ is the completely isometric isomorphism from $A(G)\widehat{\otimes} \cdots \widehat{\otimes} A(G)$ onto $A(G^n)$ used in the proof of Theorem 2.4 and $\Theta \colon A(G)^n \to A(G)\widehat{\otimes} \cdots \widehat{\otimes} A(G)$ is the completely bounded n-linear map defined by $\Theta(f_1, \dots, f_n) = f_1 \otimes \cdots \otimes f_n$ for each $(f_1, \dots, f_n) \in A(G)^n$. $\square$

Theorem 2.4 together with Lemmas 2.5 and 2.1 and Remark 2.2 now yield the next corollary. It may be worth noting that if the group G has an abelian subgroup of finite index, then G is amenable.

Corollary 2.6. *Let G be a locally compact group, and let n be an integer with $n \geq 2$. Suppose that G has an abelian subgroup of finite index. Then the following statements hold:*

(1) *a continuous n-linear form on $A(G)$ is orthosymmetric if and only if it is partitionally orthosymmetric;*
(2) *the map $T \mapsto \varphi_T^n$ is an isometry from $VN(G)$ onto the space $\mathcal{B}_o^n(A(G), \mathbb{C})$ of all continuous orthosymmetric n-linear forms on $A(G)$;*
(3) *the map $T \mapsto P_T^n$ is an isomorphism from $VN(G)$ onto the space $\mathcal{P}_o^n(A(G), \mathbb{C})$ of all continuous orthogonally additive n-homogeneous polynomials on $A(G)$.*

Example 2.7. Let $\mathbb{T}$ be the circle group. The Fourier transform gives an isometric isomorphism from $A(\mathbb{T})$ onto the group algebra $\ell^1(\mathbb{Z})$ so that $VN(\mathbb{T})$ is identified with $\ell^\infty(\mathbb{Z})$. Let $T \in VN(\mathbb{T})$ be defined by $\langle T, f \rangle = \widehat{f}(1)$ for each $f \in A(\mathbb{T})$. It is a simple matter to check that $\|T\| = 1$. Our next goal is tho show that $\|P_T^2\| = 1/2$. To this end, we first observe that if $0 \leq \alpha, \beta$ and $\alpha + \beta \leq 1$, then $4\alpha\beta \leq (\alpha + \beta)^2 \leq \alpha + \beta$. Let $f \in A(\mathbb{T})$ be such that $\|f\|_{A(\mathbb{T})} = 1$. Then $|\widehat{f}(1 - k)| + |\widehat{f}(k)| \leq \|f\|_{A(\mathbb{T})} = 1$ for each $k \in \mathbb{Z}$ and therefore

$$\begin{aligned}
\left|P_T^2(f)\right| &= \left|\widehat{f^2}(1)\right| = \left|(\widehat{f} * \widehat{f})(1)\right| = \left|\sum_{k=-\infty}^{+\infty} \widehat{f}(1-k)\widehat{f}(k)\right| \\
&\leq \sum_{k=-\infty}^{+\infty} \left|\widehat{f}(1-k)\right|\left|\widehat{f}(k)\right| \leq \sum_{k=-\infty}^{+\infty} \frac{1}{4}\left(\left|\widehat{f}(1-k)\right| + \left|\widehat{f}(k)\right|\right) \\
&= \frac{1}{2} \sum_{k=-\infty}^{+\infty} \left|\widehat{f}(k)\right| = \frac{1}{2}.
\end{aligned}$$

3. Application to other Banach algebras

We will now illustrate how the preceding theory applies to other Banach algebras such as the Figà–Talamanca–Herz algebras and the commutative C^*-algebras.

3.1. Figà–Talamanca–Herz algebras

We now pay attention to a significant class of Banach function algebras associated with a locally compact group G. Let $p \in]1, \infty[$. Then $A_p(G)$ is the *Figà–Talamanca–Herz algebra* of G. $A_p(G)$ is a regular, Tauberian, semisimple, commutative Banach algebra whose character space is identified with G by point evaluation (see [8, pp. 493–494] and [15]). It should be pointed out that $A_2(G)$ agrees with $A(G)$. The dual space of $A_p(G)$ is the Banach algebra $PM_q(G)$ of q-pseudomeasures on G, where $\frac{1}{p}+\frac{1}{q}=1$. It is worth noting that $A_p(G)$ has an approximate identity of bound 1 for any amenable group G (see [15, Theorem 6]).

There have been several attempts to equip $A_p(G)$ with an operator space structure [9,17,23]. Here we consider the structure defined in [17] which turns $A_p(G)$ into a quantized Banach algebra (though the multiplication is not known to be completely contractive) and a quantized Banach $A(G)$-module.

Lemma 3.1. *Let G be a locally compact group, and let $p \in]1,+\infty[$. Let $\varphi: A_p(G)^n \to \mathbb{C}$ be a completely bounded partitionally orthogonal n-linear form for some integer n with $n \geq 2$. Then the following statements hold:*

(1) *φ is symmetric;*
(2) *$\varphi(gf_1,\dots,f_n)=\varphi(f_1,\dots,gf_n)$ for all $f_1,\dots,f_n,g \in A_p(G)$.*

Furthermore, in the case where G has an abelian subgroup of finite index, the complete boundedness of φ can be replaced by the continuity of φ.

Proof. Let σ be a permutation of the set $\{1,\dots,n\}$. We claim that

$$\varphi(f_{\sigma(1)}g_1,\dots,f_{\sigma(n)}g_n)=\varphi(f_1g_1,\dots,f_ng_n) \tag{3.1}$$

and

$$\varphi(hf_1g_1,\dots,f_ng_n)=\varphi(f_1g_1,\dots,hf_ng_n) \tag{3.2}$$

for all $f_1,\dots,f_n,h \in A(G)$, $g_1,\dots,g_n \in A_p(G)$. Fix $(g_1,\dots,g_n) \in A_p(G)^n$ and define $\psi: A(G)^n \to \mathbb{C}$ by

$$\psi(f_1,\dots,f_n)=\varphi(f_1g_1,\dots,f_ng_n)$$

for each $(f_1,\dots,f_n) \in A(G)^n$. Then ψ is a completely bounded partitionally orthosymmetric n-linear form. On account of Theorem 2.4, $\psi=\varphi_T^n$ for some $T \in VN(G)$. In the case where G an abelian subgroup of finite index and φ is merely continuous, ψ is merely continuous and we apply Corollary 2.6 instead of Theorem 2.4. Consequently,

$$\psi(f_{\sigma(1)},\dots,f_{\sigma(n)})=\langle T,f_{\sigma(1)}\cdots f_{\sigma(n)}\rangle=\langle T,f_1\cdots f_n\rangle=\psi(f_1,\dots,f_n),$$

which gives (3.1). Further,

$$\psi(hf_1,\dots,f_n)=\big\langle T,(hf_1)\cdots f_n\big\rangle=\big\langle T,f_1\cdots(hf_n)\big\rangle=\psi(f_1,\dots,hf_n),$$

which yields (3.2). According to (3.1), we have

$$\varphi(f_{\sigma(1)}g_{\sigma(1)},\dots,f_{\sigma(n)}g_{\sigma(n)})=\varphi(f_1g_{\sigma(1)},\dots,f_ng_{\sigma(n)})=\varphi(f_1g_1,\dots,f_ng_n)$$

for all $f_1,\dots,f_n,g_1,\dots,g_n \in A(G)\cap A_p(G)$. This shows that φ is symmetric when restricted to the linear span J of $(A(G)\cap A_p(G))(A(G)\cap A_p(G))$. Similarly, (3.2) shows that (2) holds whenever $f_1,\dots,f_n,g \in J$. Since φ is continuous and J is dense in $A_p(G)$, the statements (1) and (2) follow. □

Theorem 3.2. *Let G be an amenable locally compact group, and let $p,q \in]1,+\infty[$ be such that $\frac{1}{p}+\frac{1}{q}=1$. Let n be an integer with $n \geq 2$. Then the following statements hold:*

(1) *a completely bounded n-linear form on $A_p(G)$ is orthosymmetric if and only if it is partitionally orthosymmetric;*
(2) *the map $T \mapsto \varphi_T^n$ is a complete isomorphism from $PM_q(G)$ onto the space $\mathcal{CB}_o^n(A_p(G), \mathbb{C})$ of all completely bounded orthosymmetric n-linear forms on $A_p(G)$;*
(3) *the map $T \mapsto P_T^n$ is a complete isomorphism from $PM_q(G)$ onto the space $\mathcal{P}_{cbo}^n(A_p(G), \mathbb{C})$ of all completely bounded orthogonally additive n-homogeneous polynomials on $A_p(G)$.*

Proof. Let φ be a completely bounded partitionally orthosymmetric n-linear form on $A_p(G)$. We claim that φ is orthosymmetric. Certainly, we can assume that $n \geq 3$. Moreover, since φ is known to be symmetric (Lemma 3.1), we are reduced to proving that $\varphi(f_1, f_2, \dots, f_n) = 0$, whenever $(f_1, f_2, \dots, f_n) \in A_p(G)^n$ is such that $f_1 f_2 = 0$. Let $(f_1, f_2, \dots, f_n) \in A_p(G)^n$ as above. On account of Lemma 3.1, we have

$$\begin{aligned}\varphi(f_1, f_2, \dots, f_{n-1}, gh) &= \varphi(gf_1, f_2, \dots, f_{n-1}, h) \\ &= \varphi(g, f_1 f_2, \dots, f_{n-1}, h) = 0\end{aligned}$$

for all $g, h \in A_p(G)$. Consequently, $\varphi(f_1, f_2, \dots, f_{n-1}, g) = 0$ whenever g lies in the linear span J of the set $A_p(G)A_p(G)$. Since φ is continuous and J is dense in $A_p(G)$, it follows that $\varphi(f_1, f_2, \dots, f_{n-1}, f_n) = 0$, as required. This yields the statement (1).

Let $M_n : A_p(G)^n \to A_p(G)$ be the n-linear map defined by

$$M_n(f_1, \dots, f_n) = f_1 \cdots f_n$$

for all $f_1, \dots, f_n \in A_p(G)$. Write $\mu = \|M_2\|_{\mathcal{CB}^2(A_p(G), A_p(G))}$. Then the same argument as in the beginning of the proof of Theorem 2.4 applies to show that $\|M_n\|_{\mathcal{CB}^n(A_p(G), A_p(G))} \leq \mu^{n-1}$.

Let $[T_{ij}] \in \mathbb{M}_k(PM_q(G))$. Then

$$\begin{aligned}\big\| [\varphi_{T_{ij}}^n] \big\|_{\mathbb{M}_k(\mathcal{CB}^n(A_p(G), \mathbb{C}))} &= \big\| [T_{ij}] \circ M_n \big\|_{\mathcal{CB}^n(A_p(G), \mathbb{M}_k)} \\ &\leq \big\| [T_{ij}] \big\|_{\mathcal{CB}(A_p(G), \mathbb{M}_k)} \|M_n\|_{\mathcal{CB}^n(A_p(G), A_p(G))}.\end{aligned}$$

Let $\varphi \in \mathcal{CB}_o^n(A_p(G), \mathbb{C})$. According to Lemma 3.1, we have

$$\varphi(g_1 f_1, \dots, g_{n-1} f_{n-1}, f_n) = \varphi(g_1, \dots, g_{n-1}, f_1 \cdots f_n)$$

for all $f_1, \dots, f_n, g_1, \dots, g_{n-1} \in A_p(G)$. Let $(\rho_\lambda)_{\lambda \in \Lambda}$ be an approximate identity of $A_p(G)$ of bound 1. Then

$$\varphi(\rho_\lambda f_1, \dots, \rho_\lambda f_{n-1}, f_n) = \varphi(\rho_\lambda, \dots, \rho_\lambda, f_1 \cdots f_n)$$

for all $(f_1, \dots, f_n) \in A_p(G)^n$ and $\lambda \in \Lambda$. We thus get

$$\varphi(f_1, \dots, f_n) = \lim_{\lambda \in \Lambda} \varphi(\rho_\lambda, \dots, \rho_\lambda, f_1 \cdots f_n)$$

for each $(f_1, \dots, f_n) \in A_p(G)^n$. Let $f \in A_p(G)$. By Cohen's factorization theorem f can be written as a product $f_1 \cdots f_n$ with $f_1, \dots, f_n \in A_p(G)$. Therefore the net $(\varphi(\rho_\lambda, \dots, \rho_\lambda, f))_{\lambda \in \Lambda}$ is convergent. Hence we may define a linear functional T on $A_p(G)$ by $\langle T, f \rangle = \lim_{\lambda \in \Lambda} \varphi(\rho_\lambda, \dots, \rho_\lambda, f)$ for each $f \in A_p(G)$. We now show that T is bounded. Indeed,

$$\big|\varphi(\rho_\lambda, \dots, \rho_\lambda, f)\big| \leq \|\varphi\|_{\mathcal{B}^n(A_p(G), \mathbb{C})} \|f\|_{A_p(G)}$$

for all $f \in A_p(G)$ and $\lambda \in \Lambda$, which implies $|\langle T, f \rangle| \leq \|\varphi\|_{\mathcal{B}^n(A_p(G), \mathbb{C})} \|f\|_{A_p(G)}$. It is clear that $\varphi_T^n = \varphi$.

Let $[\varphi_{ij}] \in \mathbb{M}_k(\mathcal{CB}^n_o(A_p(G), \mathbb{C}))$ and let $T_{ij} \in PM_q(G)$ be defined by $\langle T_{ij}, f\rangle = \lim_{\lambda\in\Lambda} \varphi_{ij}(\rho_\lambda, \dots, \rho_\lambda, f)$ for all $f \in A_p(G)$ and $i, j \in \{1, \dots, k\}$.

$$\|[T_{ij}]\|_{\mathbb{M}_k(PM_q(G))} = \|[T_{ij}]\|_{\mathcal{CB}(A_p(G),\mathbb{M}_k)} = \|[T_{ij}]^{(k)}\|_{\mathcal{B}(\mathbb{M}_k(A_p(G)),\mathbb{M}_k(\mathbb{M}_k))}.$$

Let $[f_{rs}] \in \mathbb{M}_k(A_p(G))$. Then

$$\begin{aligned}\big\|\big[\varphi_{ij}(\rho_\lambda, \dots, \rho_\lambda, f_{rs})\big]\big\|_{\mathbb{M}_k(\mathbb{M}_k)} &= \big\|[\varphi_{ij}]^{(1,\dots,1,k)}\big([\rho_\lambda], \dots, [\rho_\lambda], [f_{rs}]\big)\big\|_{\mathbb{M}_k(\mathbb{M}_k)} \\ &\le \big\|[\varphi_{ij}]^{(1,\dots,1,k)}\big\|_{\mathcal{B}^n(\mathbb{M}_1(A_p(G)),\dots,\mathbb{M}_1(A_p(G)),\mathbb{M}_k(A_p(G));\mathbb{M}_k(\mathbb{M}_k))}\big\|[f_{rs}]\big\|_{\mathbb{M}_k(A_p(G))} \\ &\le \big\|[\varphi_{ij}]\big\|_{\mathcal{CB}^n(A_p(G),\mathbb{M}_k)}\big\|[f_{rs}]\big\|_{\mathbb{M}_k(A_p(G))}.\end{aligned}$$

Hence

$$\big\|[T_{ij}]^{(k)}\big([f_{rs}]\big)\big\|_{\mathbb{M}_k(\mathbb{M}_k)} \le \big\|[\varphi_{ij}]\big\|_{\mathcal{CB}^n(A_p(G),\mathbb{M}_k)}\big\|[f_{rs}]\big\|_{\mathbb{M}_k(A_p(G))}$$

and therefore

$$\big\|[T_{ij}]^{(k)}\big\|_{\mathcal{B}(\mathbb{M}_k(A_p(G)),\mathbb{M}_k(\mathbb{M}_k))} \le \big\|[\varphi_{ij}]\big\|_{\mathcal{CB}^n(A_p(G),\mathbb{M}_k)}.$$

This completes our argument for (2). And lastly, Lemma 2.3 and (2) establish (3). $\square$

Corollary 3.3. *Let G be a locally compact group, and let $p, q \in]1, +\infty[$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let n be an integer with $n \ge 2$. Suppose that G has an abelian subgroup of finite index. Then the following statements hold:*

(1) *a continuous n-linear form on $A_p(G)$ is orthosymmetric if and only if it is partitionally orthosymmetric;*
(2) *the map $T \mapsto \varphi^n_T$ is an isometry from $PM_q(G)$ onto the space $\mathcal{B}^n_o(A_p(G), \mathbb{C})$ of all continuous orthosymmetric n-linear forms on $A_p(G)$;*
(3) *the map $T \mapsto P^n_T$ is an isomorphism from $PM_q(G)$ onto the space $\mathcal{P}^n_o(A_p(G), \mathbb{C})$ of all continuous orthogonally additive n-homogeneous polynomials on $A_p(G)$.*

Proof. Taking into account the extra statement in Lemma 3.1, the same argument as in Theorem 3.2 applies to prove (1) and that every continuous orthosymmetric n-linear form (continuous orthogonally additive n-homogeneous polynomial) on $A_p(G)$ is of the form φ^n_T (respectively, P^n_T) for some $T \in PM_q(G)$. Finally, Lemma 2.1 together with Remark 2.2 show that $\|T\|_{PM_q(G)} = \|\varphi^n_T\|_{\mathcal{B}^n(A_p(G),\mathbb{C})}$ for each $T \in PM_q(G)$. Finally, Lemma 2.3 and (2) give (3). $\square$

3.2. Commutative C^*-algebras

Let A be a unital C^*-algebra, and let u be a unitary element of A. Then the map $f \mapsto f(u)$ gives a homomorphism from $A(\mathbb{T})$ into A. Furthermore, if $f \in A(\mathbb{T})$, then

$$\|f(u)\|_A = \left\|\sum_{k=-\infty}^{+\infty} \widehat{f}(k)u^k\right\|_A \le \sum_{k=-\infty}^{+\infty} \big|\widehat{f}(k)\big|\|u\|_A^k = \sum_{k=-\infty}^{+\infty} \big|\widehat{f}(k)\big| = \|f\|_{A(\mathbb{T})}.$$

Theorem 3.4. *Let A be a commutative C^*-algebra, and let n be an integer with $n \ge 2$. Then the following statements hold:*

(1) *a continuous n-linear form on A is orthosymmetric if and only if it is partitionally orthosymmetric;*
(2) *the map $T \mapsto \varphi_T^n$ is an isometry from A^* onto the space $\mathcal{B}_o^n(A,\mathbb{C})$ of all continuous orthosymmetric n-linear forms on A;*
(3) *the map $T \mapsto P_T^n$ is an isomorphism from A^* onto the space $\mathcal{P}_o^n(A,\mathbb{C})$ of all continuous orthogonally additive n-homogeneous polynomials on A.*

Proof. Let φ be a continuous partitionally orthosymmetric n-linear form on A. Fix $(a_1,\dots,a_n) \in A^n$ and a unitary element u of the multiplier algebra $\mathcal{M}(A)$ of A. We define $\psi\colon A(\mathbb{T})^n \to \mathbb{C}$ by

$$\psi(f_1,\dots,f_n) = \varphi\big(a_1 f_1(u),\dots,a_n f_n(u)\big)$$

for each $(f_1,\dots,f_n) \in A(\mathbb{T})^n$. It is a simple matter to check that ψ is a continuous partitionally orthosymmetric n-linear form. By Corollary 2.6, $\psi = \varphi_T^n$ for some $T \in VN(\mathbb{T})$. Let $\mathbf{1}$ and $\mathbf{z}$ stand for the functions on $\mathbb{T}$ defined by $\mathbf{1}(z) = 1$ and $\mathbf{z}(z) = z$ for each $z \in \mathbb{T}$. We thus get

$$\begin{aligned}\varphi(\dots,a_i u,\dots,a_n) &= \psi(\mathbf{1},\dots,\overset{i}{\mathbf{z}},\dots,\mathbf{1}) = \langle T, \mathbf{1}\cdots\overset{i}{\mathbf{z}}\cdots\mathbf{1}\rangle \\ &= \langle T, \mathbf{1}\cdots\mathbf{z}\rangle = \psi(\mathbf{1},\dots,\mathbf{z}) = \varphi(\dots,a_i,\dots,a_n u)\end{aligned}$$

whenever $1 \le i \le n$. Since every element in A is a linear combination of four unitary elements of $\mathcal{M}(A)$, it follows that $\varphi(\dots,a_i b,\dots,a_n) = \varphi(\dots,a_i,\dots,a_n b)$ for all $b \in A$ and $1 \le i \le n$. Hence

$$\varphi(a_1 b_1,\dots,a_{n-1}b_{n-1},a_n) = \varphi(b_1,\dots,b_{n-1},a_1\cdots a_n)$$

for all $a_1,\dots,a_n,b_1,\dots,b_{n-1} \in A$.

We now take an approximate identity $(\rho_\lambda)_{\lambda\in\Lambda}$ of A of bound 1 and the rest of the proof goes through as for Theorem 3.2. We can define $\omega \in A^*$ by $\langle \omega, a\rangle = \lim_{\lambda\in\Lambda}\varphi(\rho_\lambda,\dots,\rho_\lambda,a)$ for each $a \in A$, which satisfies $\varphi = \varphi_\omega^n$. In particular φ is orthosymmetric. Lemma 2.1 together with Remark 2.2 show that $\|\omega\|_{A^*} = \|\varphi_T^n\|_{\mathcal{B}^n(A,\mathbb{C})}$. Finally, Lemma 2.3 and (2) give (3). $\square$

References

[1] C.A. Akemann, P.A. Ostrand, Computing norms in group C^*-algebras, Amer. J. Math. 98 (1976) 1015–1047.
[2] J. Alaminos, M. Brešar, J. Extremera, A.R. Villena, Maps preserving zero products, Studia Math. 193 (2009) 131–159.
[3] J. Alaminos, M. Brešar, J. Extremera, A.R. Villena, Characterizing Jordan maps on C^*-algebras through zero products, Proc. Edinb. Math. Soc. (2) 53 (2010) 543–555.
[4] J. Alaminos, J. Extremera, A.R. Villena, Zero product preserving maps on Banach algebras of Lipschitz functions, J. Math. Anal. Appl. 369 (2010) 94–100.
[5] Y. Benyamini, S. Lassalle, J.G. Llavona, Homogeneous orthogonally additive polynomials on Banach lattices, Bull. Lond. Math. Soc. 38 (2006) 459–469.
[6] K. Boulabiar, G. Buskes, Vector lattice powers: f-algebras and functional calculus, Comm. Algebra 34 (2006) 1435–1442.
[7] D. Carando, S. Lassalle, I. Zalduendo, Orthogonally additive polynomials over $C(K)$ are measures — a short proof, Integral Equations Operator Theory 56 (2006) 597–602.
[8] H.G. Dales, Banach Algebras and Automatic Continuity, London Mathematical Society Monographs. New Series, vol. 24, The Clarendon Press, Oxford University Press, New York, 2000. Oxford Science Publications.
[9] M. Daws, p-Operator spaces and Figà–Talamanca–Herz algebras, J. Operator Theory 63 (2010) 47–83.
[10] S. Dineen, C. Radu, Completely bounded polynomials between operator spaces, Math. Scand. 107 (2010) 249–266.
[11] E.G. Effros, Z.-J. Ruan, On approximation properties for operator spaces, Internat. J. Math. 1 (1990) 163–187.
[12] E.G. Effros, Z.-J. Ruan, Operator Spaces, London Mathematical Society Monographs. New Series, vol. 23, The Clarendon Press, Oxford University Press, New York, 2000.
[13] P. Eymard, L'algèbre de Fourier d'un groupe localement compact, Bull. Soc. Math. France 92 (1964) 181–236.
[14] B. Forrest, P. Wood, Cohomology of the operator space structure of the Fourier algebra and its second dual, Indiana Univ. Math. J. 50 (2001) 1217–1240.
[15] C. Herz, Harmonic synthesis for subgroups, Ann. Inst. Fourier (Grenoble) 23 (1973) 91–123.
[16] A. Ibort, P. Linares, J.G. Llavona, A representation theorem for orthogonally additive polynomials on Riesz spaces, Rev. Mat. Complut. 25 (2012) 21–30.

[17] A. Lambert, M. Neufang, V. Runde, Operator space structure and amenability for Figà–Talamanca–Herz algebras, J. Funct. Anal. 211 (2004) 245–269.
[18] M. Leinert, Faltungsoperatoren auf gewissen diskreten Gruppen, Studia Math. 52 (1974) 149–158.
[19] H. Leptin, Sur l'algèbre de Fourier d'un groupe localement compact, C. R. Math. Acad. Sci. Paris 266 (1968) 1180–1182.
[20] V. Losert, On tensor products of Fourier algebras, Arch. Math. (Basel) 43 (1984) 370–372.
[21] C. Palazuelos, A.M. Peralta, I. Villanueva, Orthogonally additive polynomials on C^*-algebras, Q. J. Math. 59 (2008) 363–374.
[22] D. Pérez-García, I. Villanueva, Orthogonally additive polynomials on spaces of continuous functions, J. Math. Anal. Appl. 306 (2005) 97–105.
[23] V. Runde, Operator Figà–Talamanca–Herz algebras, Studia Math. 155 (2003) 153–170.
[24] M. Takesaki, N. Tatsuuma, Duality and subgroups, II, J. Funct. Anal. 11 (1972) 184–190.