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Homogenization for dislocation based gradient visco-plasticity ☆

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ABSTRACT

In this work we study the homogenization for infinitesimal dislocation based gradient viscoplasticity with linear kinematic hardening and general non-associative monotone plastic flows. The constitutive equations in the models we study are assumed to be only of monotone type. Based on the generalized version of Korn's inequality for incompatible tensor fields (the non-symmetric plastic distortion) due to Neff/Pauly/Witch, we derive uniform estimates for the solutions of quasistatic initial-boundary value problems under consideration and then using a modified unfolding operator technique and a monotone operator method we obtain the homogenized system of equations. A new unfolding result for the Curl Curl-operator is presented in this work as well. The proof of the last result is based on the Helmholtz–Weyl decomposition for vector fields in general L^q-spaces.

1. Introduction

We study the homogenization of quasistatic initial-boundary value problems arising in gradient viscoplasticity. The models we study use rate-dependent constitutive equations with internal variables to describe the deformation behavior of metals at infinitesimally small strain.

Our focus is on a phenomenological model on the macroscale not including the case of single crystal plasticity. Our model has been first presented in [42]. It is inspired by the early work of Menzel and Steinmann [38]. Contrary to more classical strain gradient approaches, the model features from the outset a non-symmetric plastic distortion field $p \in \mathcal{M}^3$ [10], a dislocation based energy storage based solely on $|\mathrm{Curl}\, p|$ (and not ∇p) and therefore second gradients of the plastic distortion in the form of $\mathrm{Curl}\,\mathrm{Curl}\, p$ acting as dislocation based kinematical backstresses. We only consider energetic length scale effects and not higher gradients in the dissipation.

Uniqueness of classical solutions in the subdifferential case (associated plasticity) for rate-independent and rate-dependent formulations is shown in [41]. The existence question for the rate-independent model in

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terms of a weak reformulation is addressed in [42]. The rate-independent model with isotropic hardening is treated in [21,42]. The well-posedness of a rate-dependent variant without isotropic hardening is presented in [49,50]. First numerical results for a simplified rate-independent irrotational formulation (no plastic spin, symmetric plastic distortion p) are presented in [46]. In [26,55] well-posedness for a rate-independent model of Gurtin and Anand [28] is shown under the decisive assumption that the plastic distortion is symmetric (the irrotational case), in which case one may really speak of a strain gradient plasticity model, since the full gradient acts on the symmetric plastic strain.

Let us shortly revisit the modeling ingredients of the gradient plasticity model under consideration. This part does not contain new results but is added for clarity of exposition. As usual in infinitesimal plasticity theory, the basic variables are the displacement $u : \Omega \to \mathbb{R}^3$ and the plastic distortion $p : \Omega \to \mathbb{R}^{3\times 3}$. We split the total displacement gradient ∇u into non-symmetric elastic and non-symmetric plastic distortions

$$\nabla u = e + p.$$

For invariance reasons, the elastic energy contribution may only depend on the symmetric elastic strains $\operatorname{sym} e = \operatorname{sym}(\nabla u - p)$. For more on the basic invariance questions related to this issue dictating this type of behavior, see [59,40]. We assume as well plastic incompressibility $\operatorname{tr} p = 0$, as is usual. The thermodynamic potential of our model is therefore written as

$$\int_\Omega \Big(\underbrace{\mathbb{C}[x]\big(\operatorname{sym}(\nabla u - p)\big)\big(\operatorname{sym}(\nabla u - p)\big)}_{\text{elastic energy}} + \underbrace{\frac{C_1[x]}{2}|\operatorname{dev}\operatorname{sym} p|^2}_{\text{kinematical hardening}} + \underbrace{\frac{C_2}{2}|\operatorname{Curl} p|^2}_{\text{dislocation storage}} + \underbrace{u \cdot b}_{\text{external volume forces}} \Big) dx \tag{1}$$

The positive definite elasticity tensor $\mathbb{C}$ is able to represent the elastic anisotropy of the material. The plastic flow has the form

$$\partial_t p \in g\big(\sigma - C_1[x]\operatorname{dev}\operatorname{sym} p - C_2 \operatorname{Curl}\operatorname{Curl} p\big), \tag{2}$$

where $\sigma = \mathbb{C}[x]\operatorname{sym}(\nabla u - p)$ is the elastic symmetric Cauchy stress of the material and g is a multivalued monotone flow function which is not necessary the subdifferential of a convex plastic potential (associative plasticity). This ensures the validity of the second law of thermodynamics, see [42].

In this generality, our formulation comprises certain non-associative plastic flows in which the yield condition and the flow direction are independent and governed by distinct functions. Moreover, the flow function g is supposed to induce a rate-dependent response as all materials are, in reality, rate-dependent.

Clearly, in the absence of energetic length scale effects (i.e. $C_2 = 0$), the $\operatorname{Curl}\operatorname{Curl} p$-term is absent. In general we assume that g maps symmetric tensors to symmetric tensors. Thus, for $C_2 = 0$ the plastic distortion remains always symmetric and the model reduces to a classical plasticity model. Therefore, the energetic length scale is solely responsible for the plastic spin (the non-symmetry of p) in the model.

Regarding the boundary conditions necessary for the formulation of the higher order theory we assume that the so-called micro-hard boundary condition (see [29]) is specified, namely

$$p \times n|_{\partial\Omega} = 0.$$

This is the correct boundary condition for tensor fields in $L^2_{\operatorname{Curl}}$-spaces which admits tangential traces. We combine this with a new inequality extending Korn's inequality to incompatible tensor fields, namely

$$\exists C = C(\Omega) > 0\ \forall p \in L^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3): \quad p \times n|_{\partial\Omega} = 0:$$
$$\underbrace{\|p\|_{L^2(\Omega)}}_{\text{plastic distortion}} \leq C(\Omega)\Big(\underbrace{\|\mathrm{sym}\, p\|_{L^2(\Omega)}}_{\text{plastic strain}} + \underbrace{\|\mathrm{Curl}\, p\|_{L^2(\Omega)}}_{\text{dislocation density}}\Big). \tag{3}$$

Here, the domain Ω needs to be **sliceable**, i.e. cuttable into finitely many simply connected subdomains with Lipschitz boundaries. This inequality has been derived in [43–45] and is precisely motivated by the well-posedness question for our model [42]. The inequality (3) expresses the fact that controlling the plastic strain $\mathrm{sym}\, p$ and the dislocation density $\mathrm{Curl}\, p$ in $L^2(\Omega)$ gives a control of the plastic distortion p in $L^2(\Omega)$ provided the correct boundary conditions are specified: namely the micro-hard boundary condition. Since we assume that $\mathrm{tr}(p) = 0$ (plastic incompressibility) the quadratic terms in the thermodynamic potential provide a control of the right hand side in (3).

It is worthy to note that with g only monotone and not necessarily a subdifferential the powerful energetic solution concept [37,26,35] cannot be applied. In our model we face the combined challenge of a gradient plasticity model based on the dislocation density tensor $\mathrm{Curl}\, p$ involving the plastic spin, a general non-associative monotone flow-rule and a rate-dependent response.

Setting of the homogenization problem Let $\Omega \subset \mathbb{R}^3$ be an open bounded set, the set of material points of the solid body, with a C^2-boundary and $Y \subset \mathbb{R}^3$ be a set having the paving property with respect to a basis (b_1, b_2, b_3) defining the periods, a reference cell. By T_e we denote a positive number (time of existence), which can be chosen arbitrarily large, and for $0 < t \leq T_e$

$$\Omega_t = \Omega \times (0, t).$$

The sets, $\mathcal{M}^3$ and $\mathcal{S}^3$ denote the sets of all 3×3–matrices and of all symmetric 3×3–matrices, respectively. Let $\mathfrak{sl}(3)$ be the set of all traceless 3×3–matrices, i.e.

$$\mathfrak{sl}(3) = \{v \in \mathcal{M}^3 \mid \mathrm{tr}\, v = 0\}.$$

Unknown in our small strain formulation are the displacement $u_\eta(x,t) \in \mathbb{R}^3$ of the material point x at time t and the non-symmetric infinitesimal plastic distortion $p_\eta(x,t) \in \mathfrak{sl}(3)$.

The model equations of the problem are

$$-\mathrm{div}_x\, \sigma_\eta(x,t) = b(x,t), \tag{4}$$
$$\sigma_\eta(x,t) = \mathbb{C}[x/\eta]\big(\mathrm{sym}\big(\nabla_x u_\eta(x,t) - p_\eta(x,t)\big)\big), \tag{5}$$
$$\partial_t p_\eta(x,t) \in g\big(x/\eta, \Sigma^{\mathrm{lin}}_\eta(x,t)\big), \quad \Sigma^{\mathrm{lin}}_\eta = \Sigma^{\mathrm{lin}}_{\mathrm{e},\eta} + \Sigma^{\mathrm{lin}}_{\mathrm{sh},\eta} + \Sigma^{\mathrm{lin}}_{\mathrm{curl},\eta}, \tag{6}$$
$$\Sigma^{\mathrm{lin}}_{\mathrm{e},\eta} = \sigma_\eta, \quad \Sigma^{\mathrm{lin}}_{\mathrm{sh},\eta} = -C_1[x/\eta]\,\mathrm{dev}\,\mathrm{sym}\, p_\eta, \quad \Sigma^{\mathrm{lin}}_{\mathrm{curl},\eta} = -C_2\,\mathrm{Curl}\,\mathrm{Curl}\, p_\eta,$$

which must be satisfied in $\Omega \times [0, T_e)$. Here, $C_2 \geq 0$ is a given material constant independent of η and $\Sigma^{\mathrm{lin}}_\eta$ is the infinitesimal Eshelby stress tensor driving the evolution of the plastic distortion p_η and η is a scaling parameter of the microstructure. The homogeneous initial condition and Dirichlet boundary condition are

$$p_\eta(x,0) = 0, \quad x \in \Omega, \tag{7}$$
$$p_\eta(x,t) \times n(x) = 0, \quad (x,t) \in \partial\Omega \times [0, T_e), \tag{8}$$
$$u_\eta(x,t) = 0, \quad (x,t) \in \partial\Omega \times [0, T_e), \tag{9}$$

where n is a normal vector on the boundary $\partial\Omega$.[1] For simplicity we consider only homogeneous boundary condition and we assume that the cell of periodicity is given by $Y = [0,1)^3$. Then, we assume that $C_1 : Y \to \mathbb{R}$, a given material function, is measurable, periodic with the periodicity cell Y and satisfies the inequality

$$C_1[y] \geq \alpha_1 > 0 \tag{10}$$

for all $y \in Y$ and some positive constant α_1. For every $y \in Y$ the elasticity tensor $\mathbb{C}[y] : \mathcal{S}^3 \to \mathcal{S}^3$ is linear symmetric and such that there exist two positive constants $0 < \alpha \leq \beta$ satisfying

$$\alpha|\xi|^2 \leq \mathbb{C}_{ijkl}[y]\xi_{kl}\xi_{ij} \leq \beta|\xi|^2 \quad \text{for any } \xi \in \mathcal{S}^3. \tag{11}$$

We assume that the mapping $y \mapsto \mathbb{C}[y] : \mathbb{R}^3 \to \mathcal{S}^3$ is measurable and periodic with the same periodicity cell Y. Due to the above assumption ($C_1 > 0$), the classical linear kinematic hardening is included in the model. Here, the nonlocal backstress contribution is given by the dislocation density motivated term $\Sigma^{\text{lin}}_{\text{curl},\eta} = -C_2 \operatorname{Curl}\operatorname{Curl} p_\eta$ together with corresponding Neumann conditions.

For the model we require that the nonlinear constitutive mapping $v \mapsto g(y,v) : \mathcal{M}^3 \to 2^{\mathfrak{sl}(3)}$ is monotone for all $y \in Y$, i.e. it satisfies

$$0 \leq (v_1 - v_2) \cdot \left(v_1^* - v_2^*\right), \tag{12}$$

for all $v_i \in \mathcal{M}^3$, $v_i^* \in g(y, v_i)$, $i = 1, 2$ and all $y \in Y$. We also require that

$$0 \in g(y, 0), \quad \text{a.e. } y \in Y. \tag{13}$$

The mapping $y \mapsto g(y, \cdot) : \mathbb{R}^3 \to 2^{\mathfrak{sl}(3)}$ is periodic with the same periodicity cell Y. Given are the volume force $b(x,t) \in \mathbb{R}^3$ and the initial datum $p^{(0)}(x) \in \mathfrak{sl}(3)$.

Remark 1.1. It is well known that classical viscoplasticity (without gradient effects) gives rise to a well-posed problem. We extend this result to our formulation of rate-dependent gradient plasticity. The presence of the classical linear kinematic hardening in our model is related to $C_1 > 0$ whereas the presence of the nonlocal gradient term is always related to $C_2 > 0$.

The development of the homogenization theory for the quasi-static initial boundary value problem of monotone type in the classical elasto/visco-plasticity introduced by Alber in [2] has started with the work [3], where the homogenized system of equations has been derived using the formal asymptotic ansatz. In the following work [4] Alber justified the formal asymptotic ansatz for the case of positive definite free energy,[2] employing the energy method of Murat–Tartar, yet only for local smooth solutions of the homogenized problem. It is shown there that the solutions of elasto/visco-plasticity problems can be approximated in the $L^2(\Omega)$-norm by the smooth functions constructed from the solutions of the homogenized problem. Later in [47], under the assumption that the free energy is positive definite, it is proved that the difference of the solutions of the microscopic problem and the solutions constructed from the homogenized problem, which both need not be smooth, tends to zero in the $L^2(\Omega\times Y)$-norm, where Y is the periodicity cell. Based on the results obtained in [47], in [5] the convergence in $L^2(\Omega\times Y)$ is replaced by convergence in $L^2(\Omega)$. In the meantime, for the rate-independent problems in plasticity similar results are obtained in [39] using the unfolding operator method (see Section 3) and methods of energetic solutions due to Mielke. For special rate-dependent models of monotone type, namely for rate-dependent generalized standard materials, the two-scale conver-

[1] Here, $v \times n$ with $v \in \mathcal{M}^3$ and $n \in \mathbb{R}^3$ denotes a row by column operation.

[2] Positive definite energy corresponds to linear kinematic hardening behavior of materials.

gence of the solutions of the microscopic problem to the solutions of the homogenized problem has been shown in [61,62]. The homogenization of the Prandtl–Reuss model is performed in [57,62]. In [48] the author considered the rate-dependent problems of monotone type with constitutive functions g, which need not be subdifferentials, but which belong to the class of functions $\mathbb{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$ introduced in Section 5. Using the unfolding operator method and in particular the homogenization methods developed in [18], for this class of functions the homogenized equations for the viscoplactic problems of monotone type are obtained in [48].

In the present work the construction of the homogenization theory for the initial boundary value problem (4)–(9) is based on the existence result derived in [50] (see Theorem 5.6) and on the homogenization techniques developed in [48] for classical viscoplasticity of monotone type. The existence result in [50] extends the well-posedness for infinitesimal dislocation based gradient viscoplasticity with linear kinematic hardening from the subdifferential case (see [49]) to general non-associative monotone plastic flows for sliceable domains. In this work we also assume that the domain Ω is sliceable and that the monotone function $g : \mathbb{R}^3 \times \mathcal{M}^3 \to 2^{\mathfrak{sl}(3)}$ belongs to the class $\mathbb{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$. For sliceable domains Ω, based on the inequality (3), we are able to derive then uniform estimates for the solutions of (4)–(9) in Lemma 5.8. Using the uniform estimates for the solutions of (4)–(9), the unfolding operator method and the homogenization techniques developed in [18,48], for the class of functions $\mathbb{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$ we obtain easily the homogenized equations for the original problem under consideration (see Theorem 5.7). The distinguish feature of this work is that we use a variant of the unfolding operator due to Francu (see [24,25]) and not the one defined in [17]. The modified unfolding operator helps to resolve the problems connecting with the need of the careful treatment of the boundary layer in the definition of the unfolding operator in [17]. To the best our knowledge this is the first homogenization result obtained for the problem (4)–(9). We note that similar homogenization results for the strain-gradient model of Fleck and Willis [22] are derived in [23,27,31] using the unfolding method together with the Γ-convergence method in the rate-independent setting. In [23] the authors, based on the assumption that the model under consideration is of rate-independent type, are able to treat the case when C_2 is a Y-periodic function as well. In the rate-independent setting this is possible due to the fact that the whole system (4)–(9) can be rewritten as a standard variational inequality (see [30]) and then the subsequent usage of the techniques of the convex analysis enable the passage to the limit in the model equations. Contrary to this, in the rate-independent case this reduction to a single variational inequality is not possible and one is forced to use the monotonicity argument to study the asymptotic behavior of the third term $\Sigma^{\text{lin}}_{\text{curl},\eta}$ in (6).

Notation Suppose that Ω is a bounded domain with a C^2-boundary $\partial\Omega$. Throughout the whole work we choose the numbers q, q^* satisfying the following conditions

$$1 < q, q^* < \infty \quad \text{and} \quad 1/q + 1/q^* = 1,$$

and $|\cdot|$ denotes a norm in $\mathbb{R}^k$. Moreover, the following notations are used in this work. The space $W^{m,q}(\Omega, \mathbb{R}^k)$ with $q \in [1, \infty]$ consists of all functions in $L^q(\Omega, \mathbb{R}^k)$ with weak derivatives in $L^q(\Omega, \mathbb{R}^k)$ up to order m. If m is not integer, then $W^{m,q}(\Omega, \mathbb{R}^k)$ denotes the corresponding Sobolev–Slobodecki space. We set $H^m(\Omega, \mathbb{R}^k) = W^{m,2}(\Omega, \mathbb{R}^k)$. The norm in $W^{m,q}(\Omega, \mathbb{R}^k)$ is denoted by $\| \cdot \|_{m,q,\Omega}$ ($\| \cdot \|_q := \| \cdot \|_{0,q,\Omega}$). The operator Γ_0 defined by

$$\Gamma_0 : v \in W^{1,q}\big(\Omega, \mathbb{R}^k\big) \mapsto W^{1-1/q,q}\big(\partial\Omega, \mathbb{R}^k\big)$$

denotes the usual trace operator. The space $W^{m,q}_0(\Omega, \mathbb{R}^k)$ with $q \in [1, \infty]$ consists of all functions v in $W^{m,q}(\Omega, \mathbb{R}^k)$ with $\Gamma_0 v = 0$. One can define the bilinear form on the product space $L^q(\Omega, \mathcal{M}^3) \times L^{q^*}(\Omega, \mathcal{M}^3)$ by

$$(\xi, \zeta)_\Omega = \int_\Omega \xi(x) \cdot \zeta(x) dx.$$

The space

$$L^q_{\mathrm{Curl}}(\Omega,\mathcal{M}^3)=\{v\in L^q(\Omega,\mathcal{M}^3)\mid \mathrm{Curl}\, v\in L^q(\Omega,\mathcal{M}^3)\}$$

is a Banach space with respect to the norm

$$\|v\|_{q,\mathrm{Curl}}=\|v\|_q+\|\mathrm{Curl}\, v\|_q.$$

The well known result on the generalized trace operator (see [58, Section II.1.2]) can be easily adopted to the functions with values in $\mathcal{M}^3$. Then, according to this result, there is a bounded operator Γ_n on $L^q_{\mathrm{Curl}}(\Omega,\mathcal{M}^3)$

$$\Gamma_n : v\in L^q_{\mathrm{Curl}}(\Omega,\mathcal{M}^3)\mapsto \left(W^{1-1/q^*,q^*}(\partial\Omega,\mathcal{M}^3)\right)^*$$

with

$$\Gamma_n v=v\times n|_{\partial\Omega}\quad \text{if } v\in C^1(\bar{\Omega},\mathcal{M}^3),$$

where X^* denotes the dual of a Banach space X. Next,

$$L^q_{\mathrm{Curl},0}(\Omega,\mathcal{M}^3)=\{w\in L^q_{\mathrm{Curl}}(\Omega,\mathcal{M}^3)\mid \Gamma_n(w)=0\}.$$

Let us define spaces $V^q(\Omega,\mathcal{M}^3)$ and $X^q(\Omega,\mathcal{M}^3)$ by

$$\begin{aligned}
V^q(\Omega,\mathcal{M}^3)&=\{v\in L^q(\Omega,\mathcal{M}^3)\mid \mathrm{div}\, v,\mathrm{Curl}\, v\in L^q(\Omega,\mathcal{M}^3),\Gamma_n v=0\},\\
X^q(\Omega,\mathcal{M}^3)&=\{v\in L^q(\Omega,\mathcal{M}^3)\mid \mathrm{div}\, v,\mathrm{Curl}\, v\in L^q(\Omega,\mathcal{M}^3),\Gamma_0 v=0\},
\end{aligned}$$

which are Banach spaces with respect to the norm

$$\|v\|_{V^q}\left(\|v\|_{X^q}\right)=\|v\|_q+\|\mathrm{Curl}\, v\|_q+\|\mathrm{div}\, v\|_q.$$

According to [34, Theorem 2][3] the spaces $V^q(\Omega,\mathcal{M}^3)$ and $X^q(\Omega,\mathcal{M}^3)$ are continuously imbedded into $W^{1,q}(\Omega,\mathcal{M}^3)$. We define $V^q_\sigma(\Omega,\mathcal{M}^3)$ and $X^q_\sigma(\Omega,\mathcal{M}^3)$ by

$$\begin{aligned}
V^q_\sigma(\Omega,\mathcal{M}^3)&:=\{v\in V^q(\Omega,\mathcal{M}^3)\mid \mathrm{div}\, v=0\},\\
X^q_\sigma(\Omega,\mathcal{M}^3)&:=\{v\in X^q(\Omega,\mathcal{M}^3)\mid \mathrm{div}\, v=0\},
\end{aligned}$$

and denote by $V^q_{har}(\Omega,\mathcal{M}^3)$ and $X^q_{har}(\Omega,\mathcal{M}^3)$ the L^q-spaces of harmonic functions on Ω as

$$\begin{aligned}
V^q_{har}(\Omega,\mathcal{M}^3)&:=\{v\in V^q_\sigma(\Omega,\mathcal{M}^3)\mid \mathrm{Curl}\, v=0\},\\
X^q_{har}(\Omega,\mathcal{M}^3)&:=\{v\in X^q_\sigma(\Omega,\mathcal{M}^3)\mid \mathrm{Curl}\, v=0\}.
\end{aligned}$$

Then the spaces $V^q_{har}(\Omega,\mathcal{M}^3)$ and $X^q_{har}(\Omega,\mathcal{M}^3)$ for every fixed q, $1<q<\infty$, coincides with the spaces $V_{har}(\Omega,\mathcal{M}^3)$ and $X_{har}(\Omega,\mathcal{M}^3)$ given by

$$\begin{aligned}
V_{har}(\Omega,\mathcal{M}^3)&=\{v\in C^\infty(\bar{\Omega},\mathcal{M}^3)\mid \mathrm{div}\, v=0,\mathrm{Curl}\, v=0 \text{ with } v\cdot n=0 \text{ on } \partial\Omega\},\\
X_{har}(\Omega,\mathcal{M}^3)&=\{v\in C^\infty(\bar{\Omega},\mathcal{M}^3)\mid \mathrm{div}\, v=0,\mathrm{Curl}\, v=0 \text{ with } v\times n=0 \text{ on } \partial\Omega\},
\end{aligned}$$

[3] This theorem has to be applied to each row of a function with values in $\mathcal{M}^3$ to obtain the desired result.

respectively (see [34, Theorem 2.1(1)]). The spaces $V_{har}(\Omega, \mathcal{M}^3)$ and $X_{har}(\Omega, \mathcal{M}^3)$ are finite dimensional vector spaces ([34, Theorem 1]).

We also define the space $Z^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$ by

$$Z^q_{\mathrm{Curl}}(\Omega, \mathcal{M}^3) = \{ v \in L^q_{\mathrm{Curl},0}(\Omega, \mathcal{M}^3) \mid \mathrm{Curl}\,\mathrm{Curl}\, v \in L^q(\Omega, \mathcal{M}^3) \},$$

which is a Banach space with respect to the norm

$$\|v\|_{Z^q_{\mathrm{Curl}}} = \|v\|_{q,\mathrm{Curl}} + \|\mathrm{Curl}\,\mathrm{Curl}\, v\|_q.$$

The space $W^{m,q}_{per}(Y, \mathbb{R}^k)$ denotes the Banach space of Y-periodic functions in $W^{m,q}_{loc}(\mathbb{R}^k, \mathbb{R}^k)$ equipped with the $W^{m,q}(Y, \mathbb{R}^k)$-norm.

For functions v defined on $\Omega \times [0, \infty)$ we denote by $v(t)$ the mapping $x \mapsto v(x, t)$, which is defined on Ω. The space $L^q(0, T_e; X)$ denotes the Banach space of all Bochner-measurable functions $u : [0, T_e) \to X$ such that $t \mapsto \|u(t)\|^q_X$ is integrable on $[0, T_e)$. Finally, we frequently use the spaces $W^{m,q}(0, T_e; X)$, which consist of Bochner measurable functions having q-integrable weak derivatives up to order m.

2. Maximal monotone operators

In this section we recall some basics about monotone and maximal monotone operators. For more details see [9,32,53], for example.

Let V be a reflexive Banach space with the norm $\|\cdot\|$, V^* be its dual space with the norm $\|\cdot\|_*$. The brackets $\langle \cdot, \cdot \rangle$ denotes the dual pairing between V and V^*. Under V we shall always mean a reflexive Banach space throughout this section. For a multivalued mapping $A : V \to 2^{V^*}$ the sets

$$D(A) = \{ v \in V \mid Av \neq \emptyset \}$$

and

$$Gr\, A = \{ [v, v^*] \in V \times V^* \mid v \in D(A),\ v^* \in Av \}$$

are called the *effective domain* and the *graph* of A, respectively.

Definition 2.1. A mapping $A : V \to 2^{V^*}$ is called *monotone* if and only if the inequality holds

$$\langle v^* - u^*, v - u \rangle \geq 0 \quad \forall [v, v^*], [u, u^*] \in Gr\, A.$$

A monotone mapping $A : V \to 2^{V^*}$ is called *maximal monotone* iff the inequality

$$\langle v^* - u^*, v - u \rangle \geq 0 \quad \forall [u, u^*] \in Gr\, A$$

implies $[v, v^*] \in Gr\, A$.

A mapping $A : V \to 2^{V^*}$ is called *generalized pseudomonotone* iff the set Av is closed, convex and bounded for all $v \in D(A)$ and for every pair of sequences $\{v_n\}$ and $\{v^*_n\}$ such that $v^*_n \in Av_n$, $v_n \rightharpoonup v_0$, $v^*_n \rightharpoonup v^*_0 \in V^*$ and

$$\limsup_{n \to \infty} \langle v^*_n, v_n - v_0 \rangle \leq 0,$$

we have that $[v_0, v^*_0] \in Gr\, A$ and $\langle v^*_n, v_n \rangle \to \langle v^*_0, v_0 \rangle$.

A mapping $A : V \to 2^{V^*}$ is called *strongly coercive* iff either $D(A)$ is bounded or $D(A)$ is unbounded and the condition

$$\frac{\langle v^*, v - w\rangle}{\|v\|} \to +\infty \quad \text{as } \|v\| \to \infty,\ \left[v, v^*\right] \in Gr\, A,$$

is satisfied for each $w \in D(A)$.

It is well known ([53, p. 105]) that if A is a maximal monotone operator, then for any $v \in D(A)$ the image Av is a closed convex subset of V^* and the graph $Gr\, A$ is demi-closed.[4] A maximal monotone operator is also generalized pseudomonotone (see [9,32,53]).

Remark 2.2. We recall that the subdifferential of a lower semi-continuous and convex function is maximal monotone (see [54, Theorem 2.25]).

Definition 2.3. The *duality mapping* $J : V \to 2^{V^*}$ is defined by

$$J(v) = \left\{v^* \in V^* \mid \left\langle v^*, v\right\rangle = \|v\|^2 = \left\|v^*\right\|_*^2\right\}$$

for all $v \in V$.

Without loss of generality (due to Asplund's theorem) we can assume that both V and V^* are strictly convex, i.e. that the unit ball in the corresponding space is strictly convex. In virtue of [9, Theorem II.1.2], the equation

$$J(v_\lambda - v) + \lambda A v_\lambda \ni 0$$

has a solution $v_\lambda \in D(A)$ for every $v \in V$ and $\lambda > 0$ if A is maximal monotone. The solution is unique (see [9, p. 41]).

Definition 2.4. Setting

$$v_\lambda = j_\lambda^A v \quad \text{and} \quad A_\lambda v = -\lambda^{-1} J(v_\lambda - v)$$

we define two single valued operators: the *Yosida approximation* $A_\lambda : V \to V^*$ and the *resolvent* $j_\lambda^A : V \to D(A)$ with $D(A_\lambda) = D(j_\lambda^A) = V$.

By the definition, one immediately sees that $A_\lambda v \in A(j_\lambda^A v)$. For the main properties of the Yosida approximation we refer to [9,32,53] and mention only that both are continuous operators and that A_λ is bounded and maximal monotone.

Convergence of maximal monotone graphs In the presentation of the next subsections we follow the work [18], where the reader can also find the proofs of the results mentioned here.

The derivation of the homogenized equations for the initial boundary value problem (4)–(9) is based on the notion of the convergence of the graphs of maximal monotone operators. According to Brezis [11] and Attouch [8], the convergence of the graphs of maximal monotone operators is defined as follows.

[4] A set $A \in V \times V^*$ is demi-closed if v_n converges strongly to v_0 in V and v_n^* converges weakly to v_0^* in V^* (or v_n converges weakly to v_0 in V and v_n^* converges strongly to v_0^* in V^*) and $[v_n, v_n^*] \in Gr\, A$, then $[v, v^*] \in Gr\, A$.

Definition 2.5. Let A^n, $A : V \to 2^{V^*}$ be maximal monotone operators. The sequence A^n converges to A as $n \to \infty$, ($A^n \rightarrowtail A$), if for every $[v, v^*] \in Gr\, A$ there exists a sequence $[v_n, v_n^*] \in Gr\, A^n$ such that $[v_n, v_n^*] \to [v, v^*]$ strongly in $V \times V^*$ as $n \to \infty$.

Obviously, if A^n and A are everywhere defined, continuous and monotone, then the pointwise convergence, i.e. if for every $v \in V$, $A^n(v) \to A(v)$, implies the convergence of the graphs. The converse is true in finite-dimensional spaces.

The next theorem is the main mathematical tool in the derivation of the homogenized equations for the problem (4)–(9).

Theorem 2.6. *Let A^n, $A : V \to 2^{V^*}$ be maximal monotone operators, and let $[v_n, v_n^*] \in Gr\, A^n$ and $[v, v^*] \in V \times V^*$. If, as $n \to \infty$, $A^n \rightarrowtail A$, $v_n \rightharpoonup v_0$, $v_n^* \rightharpoonup v_0^* \in V^*$ and*

$$\limsup_{n\to\infty} \langle v_n^*, v_n \rangle \le \langle v_0^*, v_0 \rangle, \tag{14}$$

then $[v_0, v_0^*] \in Gr\, A$ and

$$\liminf_{n\to\infty} \langle v_n^*, v_n \rangle = \langle v_0^*, v_0 \rangle.$$

Proof. See [18, Theorem 2.8]. □

Remark 2.7. We note that if a sequence $[v_n, v_n^*] \in Gr\, A^n$ in the definition of the graph convergence of maximal monotone operators converges strongly to some $[v, v^*]$ in $V \times V^*$ as $n \to \infty$, then the condition (14) is satisfied and due to Theorem 2.6 the limit $[v, v^*]$ belongs to the graph of the operator A.

The convergence of the graphs of multi-valued maximal monotone operators can be equivalently stated in term of the pointwise convergence of the corresponding single-valued Yosida approximations and resolvents.

Theorem 2.8. *Let A^n, $A : V \to 2^{V^*}$ be maximal monotone operators and $\lambda > 0$. The following statements are equivalent:*

(a) $A^n \rightarrowtail A$ *as* $n \to \infty$*;*
(b) for every $v \in V$, $j_\lambda^{A^n} v \to j_\lambda^A v$ *as* $n \to \infty$*;*
(c) for every $v \in V$, ${A^n}_\lambda v \to A_\lambda v$ *as* $n \to \infty$*;*
(d) $A_\lambda^n \rightarrowtail A_\lambda$ *as* $n \to \infty$*.*

Moreover, the convergences $j_\lambda^{A^n} v \to j_\lambda^A v$ and ${A^n}_\lambda v \to A_\lambda v$ are uniform on strongly compact subsets of V.

Proof. See [18, Theorem 2.9]. □

Canonical extensions of maximal monotone operators In this subsection we present briefly some facts about measurable multi-valued mappings. We assume that V, and hence V^*, is separable and denote the set of maximal monotone operators from V to V^* by $\mathfrak{M}(V \times V^*)$. Further, let $(S, \Sigma(S), \mu)$ be a σ-finite μ-complete measurable space. The notion of measurability for maximal monotone mappings can be defined in terms of the measurability for appropriate single-valued mappings.

Definition 2.9. A function $A : S \to \mathfrak{M}(V \times V^*)$ is measurable iff for every $v \in E$, $x \mapsto j_\lambda^{A(x)} v$ is measurable.

For further reading on measurable multi-valued mappings we refer the reader to [14,18,32,52].

Given a mapping $A : S \to \mathfrak{M}(V \times V^*)$, one can define a monotone graph from $L^p(S,V)$ to $L^q(S,V^*)$, where $1/p + 1/q = 1$, as follows:

Definition 2.10. Let $A : S \to \mathfrak{M}(V \times V^*)$, the canonical extension of A from $L^p(S,V)$ to $L^q(S,V^*)$, where $1/p + 1/q = 1$, is defined by:

$$Gr\, \mathcal{A} = \left\{ \left[v, v^*\right] \in L^p(S,V) \times L^q\left(S, V^*\right) \;\middle|\; \left[v(x), v^*(x)\right] \in Gr\, A(x) \text{ for a.e. } x \in S \right\}.$$

Monotonicity of $\mathcal{A}$ defined in Definition 2.10 is obvious, while its maximality follows from the next proposition.

Proposition 2.11. *Let $A : S \to \mathfrak{M}(V \times V^*)$ be measurable. If $Gr\, \mathcal{A} \neq \emptyset$, then $\mathcal{A}$ is maximal monotone.*

Proof. See [18, Proposition 2.13]. □

We have to point out here that the maximality of $A(x)$ for almost every $x \in S$ does not imply the maximality of $\mathcal{A}$ as the latter can be empty [18]: $S = (0,1)$, and $Gr\, A(x) = \{[v, v^*] \in \mathbb{R} \times \mathbb{R} \mid v^* = x^{-1/q}\}$.

For given mappings A, $A^n : S \to \mathfrak{M}(V \times V^*)$ and their canonical extensions $\mathcal{A}$, $\mathcal{A}^n$, one can ask whether the pointwise convergence $A^n(x) \rightarrowtail A(x)$ implies the convergence of the graphs of the corresponding canonical extensions $\mathcal{A}^n \rightarrowtail \mathcal{A}$. The answer is given by the next theorem.

Theorem 2.12. *Let A, $A^n : S \to \mathfrak{M}(V \times V^*)$ be measurable. Assume*

(a) for almost every $x \in S$, $A^n(x) \rightarrowtail A(x)$ as $n \to \infty$,
(b) $\mathcal{A}$ and $\mathcal{A}^n$ are maximal monotone,
(c) there exist $[\alpha_n, \beta_n] \in Gr\, \mathcal{A}^n$ and $[\alpha, \beta] \in L^p(S,V) \times L^q(S,V^*)$ such that $[\alpha_n, \beta_n] \to [\alpha, \beta]$ strongly in $L^p(S,V) \times L^q(S,V^*)$ as $n \to \infty$,

then $\mathcal{A}^n \rightarrowtail \mathcal{A}$.

Proof. See [18, Proposition 2.16]. □

We note that assumption (c) in Theorem 2.12 cannot be dropped in virtue of Remark 2.16 in [18].

3. The periodic unfolding

The derivation of the homogenized problem for (4)–(9) is based on the periodic unfolding operator method. In 1990, Arbogast, Douglas and Hornung used a so-called dilation operator to study the homogenization of double-porosity periodic medium in [7] (see [12,13] for further applications of the method). This idea has been extended and further developed in [16] for two-scale and multi-scale homogenization under the name of "unfolding method". Nowadays there exists an extensive literature concerning the applications and extensions of the unfolding operator method. We recommend an interested reader to have a look into the following survey papers [15,17] and in the literature cited there. We recall briefly the definition of the unfolding operator due to Cioranescu, Damlamian and Griso [16,17]:

Let $\Omega \subset \mathbb{R}^3$ be an open set and $Y = [0,1)^3$. Let (e_1, e_2, e_3) denote the standard basis in $\mathbb{R}^3$. For $z \in \mathbb{R}^3$, $[z]_Y$ denotes a linear combination $\sum_{j=1}^3 d_j e_j$ with $\{d_1, d_2, d_3\} \in \mathbb{Z}$ such that $z - [z]_Y$ belongs to Y, and set

$$\{z\}_Y := z - [z]_Y \in Y \quad v \in \mathbb{R}^3.$$

Then, for each $x \in \mathbb{R}^3$, one has

$$x = \eta\left(\left[\frac{x}{\eta}\right]_Y + y\right).$$

We use the following notations:

$$\Xi_\eta = \{\xi \in \mathbb{Z}^k \mid \eta(\xi + Y) \subset \Omega\}, \qquad \hat{\Omega}_\eta = \operatorname{int}\left\{\bigcup_{\xi \in \Xi_\eta} (\eta\xi + \eta\overline{Y})\right\}, \qquad \Lambda_\eta = \Omega \setminus \hat{\Omega}_\eta.$$

The set $\hat{\Omega}_\eta$ is the largest union of $\eta(\xi + \overline{Y})$ cells ($\xi \in \mathbb{Z}^3$) included in Ω, while Λ_η is the subset of Ω containing the parts from $\eta(\xi + \overline{Y})$ cells intersecting the boundary $\partial\Omega$.

Definition 3.1. Let Y be a reference cell, η be a positive number and a map $v : \Omega \to \mathbb{R}^k$. The unfolding operator $\mathcal{T}_\eta(v) : \Omega \times Y \to \mathbb{R}^k$ is defined by

$$(\mathcal{T}_\eta v)(x, y) := \begin{cases} v\left(\eta\left[\frac{x}{\eta}\right]_Y + \eta y\right), & \text{a.e. } (x, y) \in \hat{\Omega}_\eta \times Y, \\ 0, & \text{a.e. } (x, y) \in \Lambda_\eta \times Y. \end{cases} \tag{15}$$

From Definition 3.1 it easily follows that, for $q \in [1, \infty[$, the operator $\mathcal{T}_\eta$ is linear and continuous from $L^q(\Omega, \mathbb{R}^k)$ to $L^q(\Omega \times Y, \mathbb{R}^k)$ and that or every ϕ in $L^1(\Omega, \mathbb{R}^k)$ one has

$$\frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\eta(\phi)(x, y)dxdy = \int_{\hat{\Omega}_\eta} \phi(x)dx \tag{16}$$

and

$$\left| \int_{\hat{\Omega}_\eta} \phi(x)dx - \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\eta(\phi)(x, y)dxdy \right| \le \int_{\Lambda_\eta} |\phi(x)| dx.$$

Obviously, if $\phi_\eta \in L^1(\Omega, \mathbb{R}^k)$ satisfies

$$\int_{\Lambda_\eta} |\phi_\eta(x)| dx \to 0, \tag{17}$$

then

$$\int_\Omega \phi_\eta(x)dx - \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\eta(\phi_\eta)(x, y)dxdy \to 0.$$

In [17], each sequence ϕ_η fulfilling (17) has been called the sequence satisfying unfolding criterion for integrals and this has been denoted as follows

$$\int_\Omega \phi_\eta(x)dx \overset{\mathcal{T}_\eta}{\simeq} \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\eta(\phi_\eta)(x, y)dxdy.$$

The fact, that we cannot consider the integration on the right hand side in (16) over the whole domain Ω and have to establish the validity of the unfolding criterion for integrals for a sequence of functions, can cause some difficulty due to the necessity of the careful treatment of the boundary layer in (17). In [24,25] this problem has been resolved by extending the unfolding operator by the identity:

$$(\mathcal{T}_\eta v)(x,y) := \begin{cases} v\big(\eta\big[\frac{x}{\eta}\big]_Y + \eta y\big), & \text{a.e. } (x,y) \in \hat{\Omega}_\eta \times Y, \\ v(x), & \text{a.e. } (x,y) \in \Lambda_\eta \times Y. \end{cases} \tag{18}$$

The unfolding operator in (18) conserves the integral, i.e. every ϕ in $L^1(\Omega, \mathbb{R}^k)$ one has

$$\frac{1}{|Y|} \int\limits_{\Omega\times Y} \mathcal{T}_\eta(\phi)(x,y)dxdy = \int\limits_{\Omega} \phi(x)dx,$$

which implies that it is an isometry between $L^q(\Omega, \mathbb{R}^k)$ and $L^q(\Omega \times Y, \mathbb{R}^k)$. In case of a general bounded domain Ω, i.e. when $|\Lambda_\eta| > 0$ and $|\Lambda_\eta| \to 0$, both definitions of the unfolding operator (15) and (18) are equivalent for the sequences, which are bounded in $L^q(\Omega, \mathbb{R}^k)$. For the sequences, which are unbounded in $L^q(\Omega, \mathbb{R}^k)$, the definitions differ (see [25, Section 4]). Since in this work we are dealing only with bounded sequence, we shall not introduce a new notation for the unfolding operator (18) and use the results in [17], which are proved for bounded sequences in $L^q(\Omega, \mathbb{R}^k)$ and the unfolding operator defined by (15).

Proposition 3.2. *Let q belong to $[1, \infty[$.*

(a) For any $v \in L^q(\Omega, \mathbb{R}^k)$, $\mathcal{T}_\eta(v) \to v$ strongly in $L^q(\Omega \times Y, \mathbb{R}^k)$,
(b) Let v_η be a bounded sequence in $L^q(\Omega, \mathbb{R}^k)$ such that $v_\eta \to v$ strongly in $L^q(\Omega, \mathbb{R}^k)$, then

$$\mathcal{T}_\eta(v_\eta) \to v, \quad \textit{strongly in } L^q\big(\Omega \times Y, \mathbb{R}^k\big).$$

(c) For every relatively weakly compact sequence v_η in $L^q(\Omega, \mathbb{R}^k)$, the corresponding $\mathcal{T}_\eta(v_\eta)$ is relatively weakly compact in $L^q(\Omega \times Y, \mathbb{R}^k)$. Furthermore, if

$$\mathcal{T}_\eta(v_\eta) \rightharpoonup \hat{v} \quad \textit{in } L^q\big(\Omega \times Y, \mathbb{R}^k\big),$$

then

$$v_\eta \rightharpoonup \frac{1}{|Y|} \int\limits_Y \hat{v}dy \quad \textit{in } L^q\big(\Omega, \mathbb{R}^k\big).$$

Proof. See [17, Proposition 2.9]. □

Next results present some properties of the restriction of the unfolding operator to the space $W^{1,q}(\Omega, \mathbb{R}^k)$.

Proposition 3.3. *Let q belong to $]1, \infty[$. Let v_η converge weakly in $W^{1,q}(\Omega, \mathbb{R}^k)$ to v. Then*

$$\mathcal{T}_\eta(v_\eta) \rightharpoonup v \quad \textit{in } L^q\big(\Omega, W^{1,q}_{per}\big(Y, \mathbb{R}^k\big)\big).$$

Proof. See [17, Corollary 3.2, Corollary 3.3]. □

Proposition 3.4. *Let q belong to $]1, \infty[$. Let v_η converge weakly in $W^{1,q}(\Omega, \mathbb{R}^k)$ to some v. Then, up to a subsequence, there exists some $\hat{v} \in L^q(\Omega, W^{1,q}_{per}(Y, \mathbb{R}^k))$ such that*

$$\mathcal{T}_\eta(\nabla v_\eta) \rightharpoonup \nabla v + \nabla_y \hat{v} \quad \textit{in } L^q\big(\Omega \times Y, \mathbb{R}^k\big).$$

Proof. See [17, Theorem 3.5]. □

The last proposition can be generalized to $W^{m,q}(\Omega, \mathbb{R}^k)$-spaces with $m \geq 1$.

Proposition 3.5. *Let q belong to $]1, \infty[$ and $m \geq 1$. Let v_η converge weakly in $W^{m,q}(\Omega, \mathbb{R}^k)$ to some v. Then, up to a subsequence, there exists some $\hat{v} \in L^q(\Omega, W^{m,q}_{per}(Y, \mathbb{R}^k))$ such that*

$$\mathcal{T}_\eta\big(D^l v_\eta\big) \rightharpoonup D^l v \quad \text{in } L^q\big(\Omega, W^{m-l,q}\big(Y, \mathbb{R}^k\big)\big) \text{ for } |l| \leq m-1,$$
$$\mathcal{T}_\eta\big(D^l v_\eta\big) \rightharpoonup D^l v + D^l_y \hat{v} \quad \text{in } L^q\big(\Omega \times Y, \mathbb{R}^k\big) \text{ for } |l| = m$$

Proof. See [17, Theorem 3.6]. □

For a multi-valued function $h \in \mathbb{M}(\Omega, \mathbb{R}^k, \alpha, m)$[5] we define the unfolding operator as follows.

Definition 3.6. Let Y be a periodicity cell, η be a positive number and a map $h \in \mathbb{M}(\Omega, \mathbb{R}^k, p, \alpha, m)$. The unfolding operator $\mathcal{T}_\eta(h) : \Omega \times Y \times \mathbb{R}^k \to 2^{\mathbb{R}^k}$ is defined by

$$\mathcal{T}_\eta(h)(x, y, v) := \begin{cases} h\big(\eta\big[\frac{x}{\eta}\big]_Y + \eta y, v\big), & \text{a.e. } (x, y) \in \hat{\Omega}_\eta \times Y,\ v \in \mathbb{R}^k, \\ |v|^{p-2} v, & \text{a.e. } (x, y) \in \Lambda_\eta \times Y,\ v \in \mathbb{R}^k. \end{cases}$$

Obviously, by its definition the unfolding operator of a multi-valued function from $\mathbb{M}(\Omega, \mathbb{R}^k, \alpha, m)$ belongs to the set $\mathbb{M}(\Omega \times Y, \mathbb{R}^k, \alpha, m)$.

We note that the periodic unfolding method described above is an alternative to the two-scale convergence method introduced in [51] and further developed in [6]. More precisely, the two-scale convergence of a bounded sequence v_η in $L^p(\Omega, \mathbb{R}^k)$ is equivalent to the weak convergence of the corresponding unfolded sequence $\mathcal{T}_\eta(v_\eta)$ in $L^p(\Omega \times Y, \mathbb{R}^k)$ (see [17, Proposition 2.14] or [24,25,36]).

4. Unfolding the Curl Curl-operator

Our method is based on the Helmholtz–Weyl decomposition for vector fields in general L^q-spaces over a domain Ω with a C^2-boundary $\partial\Omega$. It turns out (see [34, Theorem 2.1(2)]) that the following theorem holds.

Theorem 4.1. *Let $1 < q < \infty$. Every $v \in L^q(\Omega, \mathbb{R}^3)$ can be uniquely decompose as*

$$v = h + \operatorname{Curl} w + \nabla z, \tag{19}$$

where $h \in X^q_{har}(\Omega, \mathbb{R}^3)$, $w \in V^q_\sigma(\Omega, \mathbb{R}^3)$ and $z \in W^{1,q}(\Omega, \mathbb{R}^3)$, and the triple (h, w, z) satisfies the inequality

$$\|h\|_q + \|w\|_{1,q,\Omega} + \|z\|_{1,q,\Omega} \leq C\|v\|_q, \tag{20}$$

where C is a constant depending on Ω and q. If there is another triple of functions $(\tilde{h}, \tilde{w}, \tilde{z})$ such that v can be written in the form

$$v = \tilde{h} + \operatorname{Curl} \tilde{w} + \nabla \tilde{z},$$

with $\tilde{h} \in X^q_{har}(\Omega, \mathbb{R}^3)$, $\tilde{w} \in V^q_\sigma(\Omega, \mathbb{R}^3)$ and $\tilde{z} \in W^{1,q}(\Omega, \mathbb{R}^3)$, then it holds

$$h = \tilde{h}, \qquad \operatorname{Curl} w = \operatorname{Curl} \tilde{w}, \qquad \nabla z = \nabla \tilde{z}.$$

[5] The class of functions $h \in \mathbb{M}(\Omega, \mathbb{R}^k, \alpha, m)$ is defined in Definition 5.1.

Remark 4.2. If L denotes the dimension of $V_{har}(\Omega,\mathbb{R}^3)$, i.e. $\dim V_{har}(\Omega,\mathbb{R}^3)=L$, and $\{\phi_1,...,\phi_L\}$ is a basis of $V_{har}(\Omega,\mathbb{R}^3)$, then it holds $V^q(\Omega,\mathbb{R}^3)\subset W^{1,q}(\Omega,\mathbb{R}^3)$ with the estimate

$$\|v\|_q+\|\nabla v\|_q\le C\left(\|\operatorname{Curl} v\|_q+\|\operatorname{div} v\|_q+\sum_{i=1}^{L}\big|(v,\phi_i)\big|\right)$$

for all $v\in V^q(\Omega,\mathbb{R}^3)$, where $C=C(\Omega,q)$ ([34, Theorem 2.4(2)]). The proof of the above inequality with $\sum_{i=1}^{L}|(v,\phi_i)|$ replaced by $\|v\|_q$ is performed in [34, Lemma 4.5] (for $q=2$ it can be found in [20, Theorem VII.6.1]). If we assume that the boundary $\partial\Omega$ has $L+1$ smooth connected components $\Gamma_0,\Gamma_1,...,\Gamma_L$ such that $\Gamma_1,...,\Gamma_L$ lie inside Γ_0 with $\Gamma_i\cap\Gamma_j=\emptyset$ for $i\ne j$ and

$$\partial\Omega=\bigcup_{i=0}^{L}\Gamma_i,$$

then it holds ([34, Appendix A])

$$\dim V_{har}\big(\Omega,\mathbb{R}^3\big)=L.$$

If the function v in (19) is more regular, then the function w can be chosen from a better space as the next theorem shows.

Theorem 4.3. *Let $1<q<\infty$. Assume that decomposition (19) holds. If, additionally $v\in Z^q_{\operatorname{Curl}}(\Omega,\mathbb{R}^3)$, then w in (19) can be chosen from $W^{3,q}(\Omega,\mathbb{R}^3)\cap V^q_\sigma(\Omega,\mathbb{R}^3)$ satisfying the estimate*

$$\|w\|_{3,q,\Omega}\le C\big(\|\operatorname{Curl} v\|_{1,q,\Omega}+\|v\|_q\big), \tag{21}$$

where C is a constant depending on Ω and q.

Proof. For $v\in L^q_{\operatorname{Curl}}(\Omega,\mathbb{R}^3)$ this result is proved in [33]. For $v\in Z^q_{\operatorname{Curl}}(\Omega,\mathbb{R}^3)$ the proof runs the same lines. We repeat them.

As it is shown in [34, Lemma 4.2(2)], we can choose the function $w\in V^q_\sigma(\Omega,\mathbb{R}^3)$ satisfying the equation

$$(\operatorname{Curl} w,\operatorname{Curl}\psi)_\Omega=(v,\operatorname{Curl}\psi)_\Omega,\quad \text{for all } \psi\in V^{q^*}_\sigma\big(\Omega,\mathbb{R}^3\big) \tag{22}$$

with the estimate

$$\|w\|_{1,q,\Omega}\le C\|v\|_q, \tag{23}$$

where C depends only on Ω and q. Since $\operatorname{div} w=0$ in Ω and $v\in Z^q_{\operatorname{Curl}}(\Omega,\mathbb{R}^3)$, it follows from (22) that $-\Delta w=\operatorname{Curl} v$ in the sense of distributions, and we may regard w as a weak solution of the following boundary value problem

$$-\Delta w=\operatorname{Curl} v,\quad \text{in } \Omega, \tag{24}$$

$$\operatorname{div} w=0,\quad \text{on } \partial\Omega, \tag{25}$$

$$w\cdot n=0,\quad \text{on } \partial\Omega. \tag{26}$$

Since $\operatorname{Curl} v\in W^{1,q}(\Omega,\mathbb{R}^3)$, it follows from [34, Lemma 4.3(1)] and the classical theory of Agmon, Douglas and Nirenberg [1] that the solution w of the homogeneous boundary value problem (24) belongs to $W^{3,q}(\Omega,\mathbb{R}^3)$ and the estimate

$$\|w\|_{3,q,\Omega}\le C\big(\|\operatorname{Curl} v\|_{1,q,\Omega}+\|w\|_q\big), \tag{27}$$

is valid with the constant C dependent of Ω and q. Due to (23), the estimate (27) implies (21). This completes the proof. □

Now we can state the main result of this section.

Theorem 4.4. *Let $1 < q < \infty$. Suppose that sequence v_η is weakly compact in $Z^q_{\mathrm{Curl}}(\Omega, \mathbb{R}^3)$. Then there exist*

$$v \in Z^q_{\mathrm{Curl}}\big(\Omega, \mathbb{R}^3\big), \qquad v_0 \in L^q\big(\Omega \times Y, \mathbb{R}^3\big) \quad \textit{with } \operatorname{Curl}_y v_0 = 0,$$
$$v_1 \in L^q\big(\Omega, W^{2,q}_{per}\big(Y, \mathbb{R}^3\big)\big) \quad \textit{with } \operatorname{div}_y v_1 = 0,$$

such that

$$v_\eta \rightharpoonup v \quad \textit{in } Z^q_{\mathrm{Curl}}\big(\Omega, \mathbb{R}^3\big), \tag{28}$$
$$\mathcal{T}_\eta(v_\eta) \rightharpoonup v_0 \quad \textit{in } L^q\big(\Omega \times Y, \mathbb{R}^3\big), \tag{29}$$
$$\mathcal{T}_\eta(\operatorname{Curl} v_\eta) \rightharpoonup \operatorname{Curl} v \quad \textit{in } L^q\big(\Omega, W^{1,q}_{per}\big(Y, \mathbb{R}^3\big)\big), \tag{30}$$
$$\mathcal{T}_\eta(\operatorname{Curl}\operatorname{Curl} v_\eta) \rightharpoonup \operatorname{Curl}\operatorname{Curl} v + \operatorname{Curl}_y \operatorname{Curl}_y v_1 \quad \textit{in } L^q\big(\Omega \times Y, \mathbb{R}^3\big). \tag{31}$$

Moreover, $v(x) = \int_Y v_0(x, y)dy$.

Proof. Convergence (29) and the last statement of the theorem follow from Proposition 3.2(c). Convergence (28) is obvious. Next, we prove convergences (30) and (31). According to Theorem 4.1, there exist $h_\eta \in X^q_{har}(\Omega, \mathbb{R}^3)$, $w_\eta \in V^q_\sigma(\Omega, \mathbb{R}^3)$ and $z_\eta \in W^{1,q}(\Omega, \mathbb{R}^3)$ satisfying the inequality

$$\|h_\eta\|_q + \|w_\eta\|_{1,q,\Omega} + \|z_\eta\|_{1,q,\Omega} \le C\|v_\eta\|_q \tag{32}$$

with the constant C independent of η, and such that

$$v_\eta = h_\eta + \operatorname{Curl} w_\eta + \nabla z_\eta. \tag{33}$$

Moreover, due to Theorem 4.3, w_η in (33) enjoys the inequality

$$\|w_\eta\|_{3,q,\Omega} \le C\big(\|\operatorname{Curl} v_\eta\|_{1,q,\Omega} + \|v_\eta\|_q\big) \tag{34}$$

with the constant C independent of η. Therefore, the weak compactness of v_η in $Z^q_{\mathrm{Curl}}(\Omega, \mathbb{R}^3)$ and (34) imply that w_η is weakly compact in $W^{3,q}(\Omega, \mathbb{R}^3)$. Thus, in virtue of Proposition 3.5 we conclude that there exist

$$w \in W^{3,q}\big(\Omega, \mathbb{R}^3\big) \quad \text{and} \quad w_1 \in L^q\big(\Omega, W^{3,q}_{per}\big(Y, \mathbb{R}^3\big)\big)$$

such that

$$\mathcal{T}_\eta\big(D^l w_\eta\big) \rightharpoonup D^l w \quad \text{in } L^q\big(\Omega, W^{3-l,q}\big(Y, \mathbb{R}^k\big)\big) \text{ for } |l| \le 2, \tag{35}$$
$$\mathcal{T}_\eta\big(D^l w_\eta\big) \rightharpoonup D^l w + D^l_y w_1 \quad \text{in } L^q\big(\Omega \times Y, \mathbb{R}^k\big) \text{ for } |l| = 3. \tag{36}$$

Since $\operatorname{Curl} v_\eta = \operatorname{Curl}\operatorname{Curl} w_\eta$ and $\operatorname{Curl} v = \operatorname{Curl}\operatorname{Curl} w$, we get that

$$\mathcal{T}_\eta(\operatorname{Curl} v_\eta) \rightharpoonup \operatorname{Curl} v \quad \text{in } L^q\big(\Omega, W^{1,q}_{per}\big(Y, \mathbb{R}^3\big)\big),$$
$$\mathcal{T}_\eta(\operatorname{Curl}\operatorname{Curl} v_\eta) \rightharpoonup \operatorname{Curl}\operatorname{Curl} v + \operatorname{Curl}_y \operatorname{Curl}_y v_1 \quad \text{in } L^q\big(\Omega \times Y, \mathbb{R}^3\big).$$

It is left to prove that the condition $\operatorname{Curl}_y v_0 = 0$ is valid.[6] To this end, we note first that the additive decomposition (33) implies

$$\mathcal{T}_\eta(v_\eta) = \mathcal{T}_\eta(h_\eta) + \mathcal{T}_\eta(\operatorname{Curl} w_\eta) + \mathcal{T}_\eta(\nabla z_\eta). \tag{37}$$

Since the function h_η belongs to the space of smooth functions $X^q_{har}(\Omega, \mathbb{R}^3)$, up to a subsequence, the sequence h_η converges strongly to a function h in $L^q(\Omega, \mathbb{R}^3)$. This provides that

$$\mathcal{T}_\eta(h_\eta) \to h \quad \text{in } L^q\big(\Omega \times Y, \mathbb{R}^3\big).$$

Next, the weak compactness of w_η in $W^{3,q}(\Omega, \mathbb{R}^3)$ together with the convergence (35) and Rellich's theorem guarantee that

$$\mathcal{T}_\eta(\operatorname{Curl} w_\eta) \to \operatorname{Curl} w \quad \text{in } L^q\big(\Omega \times Y, \mathbb{R}^3\big).$$

Proposition 3.5 applied to the gradient of z_η implies that there exist functions $z \in W^{1,q}(\Omega, \mathbb{R}^3)$ and $z_1 \in L^q(\Omega, W^{1,q}_{per}(Y, \mathbb{R}^3))$ such that

$$\mathcal{T}_\eta(\nabla z_\eta) \rightharpoonup \nabla z + \nabla_y z_1 \quad \text{in } L^q\big(\Omega \times Y, \mathcal{M}^3\big).$$

Passage to the weak limit in (37) yields now that

$$v_0 = h + \operatorname{Curl} w + \nabla z + \nabla_y z_1,$$

where on the right hand side the function z_1 depends on the variable y only. Therefore, we get that

$$\operatorname{Curl}_y v_0 = \operatorname{Curl}_y \nabla_y z_1 = 0.$$

The proof of Theorem 4.4 is complete. □

5. Homogenized system of equations

Main result First, we define a class of maximal monotone functions we deal with in this work.

Definition 5.1. For $m \in L^1(\Omega, \mathbb{R})$, $\alpha \in \mathbb{R}_+$ and $q > 1$, $\mathbb{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$ is the set of multi-valued functions $h : \Omega \times \mathbb{R}^k \to 2^{\mathbb{R}^k}$ with the following properties

- $v \mapsto h(x, v)$ is maximal monotone for almost all $x \in \Omega$,
- the mapping $x \mapsto j_\lambda(x, v) : \Omega \to \mathbb{R}^k$ is measurable for all $\lambda > 0$, where $j_\lambda(x, v)$ is the inverse of $v \mapsto v + \lambda h(x, v)$,
- for a.e. $x \in \Omega$ and every $v^* \in h(x, v)$

$$\alpha\left(\frac{|v|^q}{q} + \frac{|v^*|^{q^*}}{q^*}\right) \le \big(v, v^*\big) + m(x), \tag{38}$$

  where $1/q + 1/q^* = 1$.

[6] The proof of this result is due to an unknown reviewer of the manuscript.

Remark 5.2. We note that the condition (38) is equivalent to the following two inequalities

$$|v^*|^{q^*} \leq m_1(x) + \alpha_1 |v|^q, \tag{39}$$

$$(v, v^*) \geq m_2(x) + \alpha_2 |v|^q, \tag{40}$$

for a.e. $x \in \Omega$ and every $v^* \in h(x, v)$ and with suitable functions $m_1, m_2 \in L^1(\Omega, \mathbb{R})$ and numbers $\alpha_1, \alpha_2 \in \mathbb{R}_+$.

Remark 5.3. Visco-plasticity is typically included in the former conditions by choosing the function g to be in Norton–Hoff form, i.e.

$$g(\Sigma) = \big[|\Sigma| - \sigma_{\mathrm{y}}\big]_+^r \frac{\Sigma}{|\Sigma|}, \quad \Sigma \in \mathcal{M}^3,$$

where σ_{y} is the flow stress and r is some parameter together with $[x]_+ := \max(x, 0)$. If $g : \mathcal{M}^3 \mapsto \mathcal{S}^3$ then the flow is called irrotational (no plastic spin).

The main properties of the class $\mathbb{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$ are collected in the following proposition.

Proposition 5.4. *Let $\mathcal{H}$ be a canonical extension of a function $h : \mathbb{R}^k \to 2^{\mathbb{R}^k}$, which belongs to $\mathbb{M}(\Omega, \mathbb{R}^k, q, \alpha, m)$. Then $\mathcal{H}$ is maximal monotone, surjective and $D(\mathcal{H}) = L^p(\Omega, \mathbb{R}^k)$.*

Proof. See Corollary 2.15 in [18]. □

In linear elasticity theory it is well known (see [60, Theorem 4.2]) that a Dirichlet boundary value problem formed by the equations

$$-\operatorname{div}_x \sigma_\eta(x) = \hat{b}(x), \qquad x \in \Omega, \tag{41}$$

$$\sigma_\eta(x) = \mathbb{C}[x/\eta]\big(\operatorname{sym}\big(\nabla_x u_\eta(x)\big) - \hat{\varepsilon}_\eta(x)\big), \quad x \in \Omega, \tag{42}$$

$$u_\eta(x) = 0, \qquad x \in \partial\Omega, \tag{43}$$

to given $\hat{b} \in H^{-1}(\Omega, \mathbb{R}^3)$ and $\hat{\varepsilon}_\eta \in L^2(\Omega, \mathcal{S}^3)$ has a unique weak solution $(u_\eta, \sigma_\eta) \in H_0^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)$.

Next, we define the notion of strong solutions for the initial boundary value problem (4)–(9).

Definition 5.5 *(Strong solutions).* A function $(u_\eta, \sigma_\eta, p_\eta)$ such that

$$(u_\eta, \sigma_\eta) \in H^1\big(0, T_e; H_0^1\big(\Omega, \mathbb{R}^3\big) \times L^2\big(\Omega, \mathcal{S}^3\big)\big), \qquad \Sigma_\eta^{\mathrm{lin}} \in L^q\big(\Omega_{T_e}, \mathcal{M}^3\big),$$
$$p_\eta \in H^1\big(0, T_e; L^2_{\mathrm{Curl}}\big(\Omega, \mathcal{M}^3\big)\big) \cap L^2\big(0, T_e; Z^2_{\mathrm{Curl}}\big(\Omega, \mathcal{M}^3\big)\big)$$

is called a *strong solution* of the initial boundary value problem (4)–(9), if for every $t \in [0, T_e]$ the function $(u_\eta(t), \sigma_\eta(t))$ is a weak solution of the boundary value problem (41)–(43) with $\hat{\varepsilon}_p = \operatorname{sym} p_\eta(t)$ and $\hat{b} = b(t)$, the evolution inclusion (6) and the initial condition (7) are satisfied pointwise.

Next, we state the existence result (see [50]).

Theorem 5.6. *Suppose that $1 < q^* \leq 2 \leq q < \infty$. Assume that $\Omega \subset \mathbb{R}^3$ is a sliceable domain with a C^2-boundary, $C_1 \in L^\infty(\Omega, \mathbb{R})$ and $\mathbb{C} \in L^\infty(\Omega, \mathcal{S}^3)$ satisfying (10) and (11), respectively. Let the functions $b \in W^{1,q}(0, T_e; L^q(\Omega, \mathbb{R}^3))$ be given and $g \in \mathbb{M}(\Omega, \mathcal{M}^3, q, \alpha, m)$. Suppose that for a.e. $x \in \Omega$ the relation*

$$0 \in g\big(x/\eta, \sigma^{(0)}(x)\big) \tag{44}$$

holds, where the function $\sigma^{(0)} \in L^2(\Omega, \mathcal{S}^3)$ *is determined by equations (41)–(43) for* $\hat{\varepsilon}_p = 0$ *and* $\hat{b} = b(0)$. *Then there exists a strong unique solution* $(u_\eta, \sigma_\eta, p_\eta)$ *of the initial boundary value problem (4)–(9).*

Now we can formulate the main result of this work.

Theorem 5.7. *Suppose that all assumptions of Theorem 5.6 are fulfilled. Then there exists*

$$
\begin{aligned}
&u_0 \in H^1\big(0,T_e;H_0^1\big(\Omega,\mathbb{R}^3\big)\big), \qquad u_1 \in H^1\big(0,T_e;L^2\big(\Omega,H^1_{per}\big(Y,\mathbb{R}^3\big)\big)\big),\\
&\sigma_0 \in L^\infty\big(0,T_e;L^2\big(\Omega\times Y,\mathcal{S}^3\big)\big), \qquad \sigma \in L^\infty\big(0,T_e;L^2\big(\Omega,\mathcal{S}^3\big)\big),\\
&p \in H^1\big(0,T_e;L^2\big(\Omega,\mathcal{M}^3\big)\big)\cap L^2\big(0,T_e;Z^2_{\mathrm{Curl}}\big(\Omega,\mathcal{M}^3\big)\big),\\
&p_0 \in H^1\big(0,T_e;L^2\big(\Omega\times Y,\mathcal{M}^3\big)\big) \quad \textit{with } \operatorname{Curl}_y p_0 = 0,
\end{aligned}
$$

and

$$
p_1 \in L^2\big(0,T_e;L^2\big(\Omega,W^{2,q^*}_{per}\big(Y,\mathcal{M}^3\big)\big)\big) \quad \textit{with } \operatorname{div}_y p_1 = 0,
$$

such that

$$
\begin{aligned}
&u_\eta \rightharpoonup u_0 \quad \textit{in } H^1\big(0,T_e;H_0^1\big(\Omega,\mathbb{R}^3\big)\big), && (45)\\
&p_\eta \rightharpoonup p \quad \textit{in } H^1\big(0,T_e;L^2\big(\Omega,\mathcal{M}^3\big)\big)\cap L^2\big(0,T_e;Z^2_{\mathrm{Curl}}\big(\Omega,\mathcal{M}^3\big)\big), && (46)\\
&\mathcal{T}_\eta(\nabla u_\eta) \rightharpoonup \nabla u_0 + \nabla_y u_1 \quad \textit{in } H^1\big(0,T_e;L^2\big(\Omega\times Y,\mathbb{R}^3\big)\big), && (47)\\
&\sigma_\eta \overset{*}{\rightharpoonup} \sigma \quad \textit{in } L^\infty\big(0,T_e;L^2\big(\Omega,\mathcal{S}^3\big)\big), && (48)\\
&\mathcal{T}_\eta(\sigma_\eta) \overset{*}{\rightharpoonup} \sigma_0 \quad \textit{in } L^\infty\big(0,T_e;L^2\big(\Omega\times Y,\mathcal{S}^3\big)\big), && (49)\\
&\mathcal{T}_\eta(p_\eta) \rightharpoonup p_0 \quad \textit{in } L^2\big(0,T_e;L^2\big(\Omega\times Y,\mathcal{M}^3\big)\big), && (50)\\
&\mathcal{T}_\eta(\partial_t p_\eta) \rightharpoonup \partial_t p_0 \quad \textit{in } L^2\big(0,T_e;L^2\big(\Omega\times Y,\mathcal{M}^3\big)\big), && (51)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathcal{T}_\eta(\operatorname{Curl} p_\eta) \rightharpoonup \operatorname{Curl} p \quad \textit{in } L^2\big(0,T_e;L^2\big(\Omega,H^1_{per}\big(Y,\mathcal{M}^3\big)\big)\big), && (52)\\
&\mathcal{T}_\eta(\operatorname{dev}\operatorname{sym} p_\eta) \rightharpoonup \operatorname{dev}\operatorname{sym} p_0 \quad \textit{in } L^2\big(\Omega_{T_e}\times Y,\mathcal{M}^3\big), && (53)\\
&\mathcal{T}_\eta(\operatorname{Curl}\operatorname{Curl} p_\eta) \rightharpoonup \tilde{p} \quad \textit{in } L^2\big(\Omega_{T_e}\times Y,\mathcal{M}^3\big), && (54)\\
&\mathcal{T}_\eta\big(\Sigma^{\mathrm{lin}}_\eta\big) \rightharpoonup \Sigma^{\mathrm{lin}}_0 \quad \textit{in } L^q\big(\Omega_{T_e}\times Y,\mathcal{M}^3\big), && (55)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{p} &:= \operatorname{Curl}\operatorname{Curl} p + \operatorname{Curl}_y \operatorname{Curl}_y p_1,\\
\Sigma^{\mathrm{lin}}_0 &:= \sigma_0 - C_1[y]\operatorname{dev}\operatorname{sym} p_0 - C_2\tilde{p},
\end{aligned}
$$

and $(u_0, u_1, \sigma, \sigma_0, p, p_0, p_1)$ *is a solution of the following system of equations:*

$$
\begin{aligned}
-\operatorname{div}_x \sigma(x,t) &= b(x,t), && (56)\\
-\operatorname{div}_y \sigma_0(x,y,t) &= 0, && (57)\\
\sigma_0(x,y,t) &= \mathbb{C}[y]\big(\operatorname{sym}\big(\nabla_x u_0(x,t) + \nabla_y u_1(x,y,t) - p_0(x,y,t)\big)\big), && (58)\\
\partial_t p_0(x,y,t) &\in g\big(y,\Sigma^{\mathrm{lin}}_0(x,y,t)\big), && (59)
\end{aligned}
$$

which holds for $(x, y, t) \in \Omega \times \mathbb{R}^3 \times [0, T_e]$, *and the initial condition and boundary condition*

$$p_0(x, y, 0) = 0, \quad x \in \Omega, \tag{60}$$

$$p(x, t) \times n(x) = 0, \quad (x, t) \in \partial\Omega \times [0, T_e), \tag{61}$$

$$u_0(x, t) = 0, \quad (x, t) \in \partial\Omega \times [0, T_e). \tag{62}$$

The functions σ *and* p *are related to* σ_0 *and* p_0 *in the following ways*

$$\sigma(x, t) = \int_Y \sigma_0(x, y, t)dy, \qquad p(x, t) = \int_Y p_0(x, y, t)dy.$$

The proof of Theorem 5.7 is divided into two parts. In the next lemma we derive the uniform estimates for $(u_\eta, \sigma_\eta, p_\eta)$ and then, based on these estimates, we show the convergence result.

5.1. Uniform estimates

First, we show that the sequence of solutions $(u_\eta, \sigma_\eta, p_\eta)$ is weakly compact.

Lemma 5.8. *Let all assumptions of Theorem 5.7 be satisfied. Then the sequence of solutions* (u_η, σ_η) *is weakly compact in* $H^1(0, T_e; H_0^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3))$ *and* p_η *is weakly compact in* $H^1(0, T_e; L^2(\Omega, \mathcal{M}^3)) \cap L^2(0, T_e, Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3))$.

Proof. To prove the lemma we recall the basic steps in the proof of the existence result (Theorem 5.6). For more details the reader is referred to [50]. The time-discretized problem for (4)–(9) is introduced as follows:

Let us fix any $m \in \mathbb{N}$ and set

$$h := \frac{T_e}{2^m}, \qquad p^0_{\eta,m} := 0 \qquad b^n_m := \frac{1}{h} \int_{(n-1)h}^{nh} b(s)ds \in L^q\big(\Omega, \mathbb{R}^3\big), \quad n = 1, ..., 2^m.$$

Then we are looking for functions $u^n_{\eta,m} \in H^1(\Omega, \mathbb{R}^3)$, $\sigma^n_{\eta,m} \in L^2(\Omega, \mathcal{S}^3)$ and $p^n_{\eta,m} \in Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3)$ with $p^n_{\eta,m}(x) \in \mathfrak{sl}(3)$ for a.e. $x \in \Omega$ and

$$\Sigma^{\mathrm{lin}}_{n,m} := \sigma^n_{\eta,m} - C_1[x/\eta] \operatorname{dev} \operatorname{sym} p^n_{\eta,m} - \frac{1}{m} p^n_{\eta,m} - C_2 \operatorname{Curl} \operatorname{Curl} p^n_{\eta,m} \in L^q\big(\Omega, \mathcal{M}^3\big)$$

solving the following problem

$$-\operatorname{div}_x \sigma^n_{\eta,m}(x) = b^n_m(x), \tag{63}$$

$$\sigma^n_{\eta,m}(x) = \mathbb{C}[x/\eta]\big(\operatorname{sym}\big(\nabla_x u^n_{\eta,m}(x) - p^n_{\eta,m}(x)\big)\big) \tag{64}$$

$$\frac{p^n_{\eta,m}(x) - p^{n-1}_{\eta,m}(x)}{h} \in g\big(x/\eta, \Sigma^{\mathrm{lin}}_{n,m}(x)\big), \tag{65}$$

together with the boundary conditions

$$p^n_{\eta,m}(x) \times n(x) = 0, \quad x \in \partial\Omega, \tag{66}$$

$$u^n_{\eta,m}(x) = 0, \quad x \in \partial\Omega. \tag{67}$$

Such functions $(u^n_{\eta,m}, \sigma^n_{\eta,m}, p^n_{\eta,m})$ exist and satisfy the following estimate

$$\begin{aligned}
&\frac{1}{2}\Big(\big\|\mathbb{B}^{1/2}\sigma^l_{\eta,m}\big\|_2^2 + \alpha_1\big\|\operatorname{dev}\operatorname{sym} p^l_{\eta,m}\big\|_2^2 + \frac{1}{m}\big\|p^l_{\eta,m}\big\|_2^2 + C_2\big\|\operatorname{Curl} p^l_{\eta,m}\big\|_2^2\Big) \\
&\quad + h\hat{C}\sum_{n=1}^{l}\Big(\big\|\Sigma^{\text{lin}}_{n,m}\big\|_q^q + \Big\|\frac{p^n_{\eta,m} - p^{n-1}_{\eta,m}}{h}\Big\|_{q^*}^{q^*}\Big) \\
&\le C^{(0)} + \int_\Omega m(x)dx + h\tilde{C}\sum_{n=1}^{l}\big(\big\|b^n_m\big\|_q^q + \big\|\big(b^n_m - b^{n-1}_m\big)/h\big\|_2^2\big)
\end{aligned} \tag{68}$$

for any fixed $l \in [1, 2^m]$, where (here $\mathbb{B} := \mathbb{C}^{-1}$)

$$2C^{(0)} := \big\|\mathbb{B}^{1/2}\sigma^{(0)}\big\|_2^2$$

and $\tilde{C}$, $\hat{C}$ are some positive constants independent of η (see [50] for details). To proceed further we introduce the Rothe approximation functions.

Rothe approximation functions For any family $\{\xi^n_m\}_{n=0,\dots,2m}$ of functions in a reflexive Banach space X, we define *the piecewise affine interpolant* $\xi_m \in C([0, T_e], X)$ by

$$\xi_m(t) := \Big(\frac{t}{h} - (n-1)\Big)\xi^n_m + \Big(n - \frac{t}{h}\Big)\xi^{n-1}_m \quad \text{for } (n-1)h \le t \le nh \tag{69}$$

and *the piecewise constant interpolant* $\bar{\xi}_m \in L^\infty(0, T_e; X)$ by

$$\bar{\xi}_m(t) := \xi^n_m \quad \text{for } (n-1)h < t \le nh,\ n = 1, \dots, 2^m, \quad \text{and} \quad \bar{\xi}_m(0) := \xi^0_m. \tag{70}$$

For the further analysis we recall the following property of $\bar{\xi}_m$ and ξ_m:

$$\|\xi_m\|_{L^q(0,T_e;X)} \le \|\bar{\xi}_m\|_{L^q(-h,T_e;X)} \le \big(h\big\|\xi^0_m\big\|_X^q + \|\bar{\xi}_m\|^q_{L^q(0,T_e;X)}\big)^{1/q}, \tag{71}$$

where $\bar{\xi}_m$ is formally extended to $t \le 0$ by ξ^0_m and $1 \le q \le \infty$ (see [56]).

Now, from (68) we get immediately that

$$\begin{aligned}
&\bar{C}\big\|\bar{\sigma}_{\eta,m}(t)\big\|_\Omega^2 + \alpha_1\big\|\operatorname{dev}\operatorname{sym}\bar{p}_{\eta,m}(t)\big\|_2^2 + \frac{1}{m}\big\|\bar{p}_{\eta,m}(t)\big\|_2^2 + C_2\big\|\operatorname{Curl}\bar{p}_{\eta,m}(t)\big\|_2^2 \\
&\quad + 2\hat{C}\big(\|\partial_t p_{\eta,m}\|^{q^*}_{q^*,\Omega\times(0,T_e)} + \big\|\bar{\Sigma}^{\text{lin}}_m\big\|^q_{q,\Omega\times(0,T_e)}\big) \\
&\le 2C^{(0)} + 2\|m\|_{1,\Omega} + 2\tilde{C}\|b\|^q_{W^{1,q}(0,T_e;L^q(\Omega,\mathcal{S}^3))},
\end{aligned} \tag{72}$$

where $\bar{C}$ is some other constant independent of η. In [50] it is shown that the Rothe approximation functions $(u_{\eta,m}, \sigma_{\eta,m}, p_{\eta,m})$ and $(\bar{u}_{\eta,m}, \bar{\sigma}_{\eta,m}, \bar{p}_{\eta,m})$ converge to the same limit $(u_\eta, \sigma_\eta, p_\eta)$. Due to the lower semi-continuity of the norm and (72) this convergence is uniform with respect to η. Therefore, estimate (72) provides that

$$\{\sigma_\eta\}_\eta \text{ is uniformly bounded in } L^\infty\big(0, T_e; L^2\big(\Omega, \mathcal{S}^3\big)\big), \tag{73}$$

$$\{\operatorname{dev}\operatorname{sym} p_\eta\}_\eta \text{ is uniformly bounded in } L^\infty\big(0, T_e; L^2\big(\Omega, \mathcal{M}^3\big)\big), \tag{74}$$

$$\{\operatorname{Curl} p_\eta\}_\eta \text{ is uniformly bounded in } L^\infty\big(0, T_e; L^2\big(\Omega, \mathcal{M}^3\big)\big), \tag{75}$$

$$\{p_\eta\}_\eta \text{ is uniformly bounded in } W^{1,q^*}\big(0,T_e;L^{q^*}\big(\Omega,\mathcal{M}^3\big)\big), \tag{76}$$

$$\big\{\Sigma^{\text{lin}}_\eta\big\}_\eta \text{ is uniformly bounded in } L^q\big(\Omega_{T_e},\mathcal{M}^3\big). \tag{77}$$

Furthermore, from estimates (3), (73)–(77) we obtain easily that

$$\{u_\eta\}_\eta \text{ is uniformly bounded in } L^2\big(0,T_e;H^1_0\big(\Omega,\mathbb{R}^3\big)\big), \tag{78}$$

$$\{p_\eta\}_\eta \text{ is uniformly bounded in } L^2\big(0,T_e;Z^2_{\text{Curl}}\big(\Omega,\mathcal{M}^3\big)\big). \tag{79}$$

Additional regularity of discrete solutions In order to get the additional a priori estimates, we extend the function b to $t<0$ by setting $b(t)=b(0)$. The extended function b is in the space $W^{1,p}(-2h,T_e;W^{-1,p}(\Omega,\mathbb{R}^3))$. Then, we set $b^0_m=b^{-1}_m:=b(0)$. Let us further set

$$p^{-1}_{\eta,m}:=p^0_{\eta,m}-h\mathcal{G}_\eta\big(\Sigma^{\text{lin}}_{0,m}\big),$$

where $\mathcal{G}_\eta:L^p(\Omega,\mathcal{M}^3)\to 2^{L^q(\Omega,\mathfrak{sl}(3))}$ denotes the canonical extensions of $g(x/\eta,\cdot):\mathcal{M}^3\to 2^{\mathfrak{sl}(3)}$. The assumption (44) and the homogeneous initial condition imply that $p^{-1}_{\eta,m}=0$. Next, we define functions $(u^{-1}_{\eta,m},\sigma^{-1}_{\eta,m})$ and $(u^0_{\eta,m},\sigma^0_{\eta,m})$ as solutions of the linear elasticity problem (41)–(43) to the data $\hat b=b^{-1}_m$, $\hat\gamma=0$, $\hat\varepsilon_p=0$ and $\hat b=b^0_m$, $\hat\gamma=0$, $\hat\varepsilon_p=0$, respectively. Obviously, the following estimate holds

$$\left\{\left\|\frac{u^0_{\eta,m}-u^{-1}_{\eta,m}}{h}\right\|_2,\left\|\frac{\sigma^0_{\eta,m}-\sigma^{-1}_{\eta,m}}{h}\right\|_2\right\}\le C, \tag{80}$$

where C is some positive constant independent of m and η. Taking now the incremental ratio of (65) for $n=1,...,2^m$, we obtain[7]

$$\text{rt}\,p^n_{\eta,m}-\text{rt}\,p^{n-1}_{\eta,m}=\mathcal{G}_\eta\big(\Sigma^{\text{lin}}_{n,m}\big)-\mathcal{G}_\eta\big(\Sigma^{\text{lin}}_{(n-1),m}\big).$$

Let us now multiply the last identity by $-(\Sigma^{\text{lin}}_{n,m}-\Sigma^{\text{lin}}_{(n-1),m})/h$. Then using the monotonicity of $\mathcal{G}_\eta$ we obtain that

$$\begin{aligned}&\frac{1}{m}\big(\text{rt}\,p^n_{\eta,m}-\text{rt}\,p^{n-1}_{\eta,m},\text{rt}\,p^n_{\eta,m}\big)_\Omega+\big(\text{rt}\,p^n_{\eta,m}-\text{rt}\,p^{n-1}_{\eta,m},C_1\operatorname{dev}\operatorname{sym}\big(\text{rt}\,p^n_{\eta,m}\big)\big)_\Omega\\&\quad+\big(\text{rt}\,p^n_{\eta,m}-\text{rt}\,p^{n-1}_{\eta,m},C_2\operatorname{Curl}\operatorname{Curl}\big(\text{rt}\,p^n_{\eta,m}\big)\big)_\Omega\le\big(\text{rt}\,p^n_{\eta,m}-\text{rt}\,p^{n-1}_{\eta,m},\text{rt}\,\sigma^n_{\eta,m}\big)_\Omega.\end{aligned}$$

With (63) and (64) the previous inequality can be rewritten as follows

$$\begin{aligned}&\frac{1}{m}\big(\text{rt}\,p^n_{\eta,m}-\text{rt}\,p^{n-1}_{\eta,m},\text{rt}\,p^n_{\eta,m}\big)_\Omega+\big(\text{rt}\,p^n_{\eta,m}-\text{rt}\,p^{n-1}_{\eta,m},C_1\operatorname{dev}\operatorname{sym}\big(\text{rt}\,p^n_{\eta,m}\big)\big)_\Omega\\&\quad+\big(\text{rt}\,p^n_{\eta,m}-\text{rt}\,p^{n-1}_{\eta,m},C_2\operatorname{Curl}\operatorname{Curl}\big(\text{rt}\,p^n_{\eta,m}\big)\big)_\Omega+\big(\text{rt}\,\sigma^n_{\eta,m}-\text{rt}\,\sigma^{n-1}_{\eta,m},\mathbb{C}^{-1}\,\text{rt}\,\sigma^n_{\eta,m}\big)_\Omega\\&\le\big(\text{rt}\,u^n_{\eta,m}-\text{rt}\,u^{n-1}_{\eta,m},\text{rt}\,b^n_m\big)_\Omega.\end{aligned}$$

As in the proof of (68), multiplying the last inequality by h and summing with respect to n from 1 to l for any fixed $l\in[1,2^m]$ we get the estimate

$$\begin{aligned}&\frac{h}{m}\big\|\text{rt}\,p^l_{\eta,m}\big\|_2^2+h\alpha_1\big\|\operatorname{dev}\operatorname{sym}\text{rt}\,p^l_{\eta,m}\big\|_2^2+h\big\|\mathbb{B}^{1/2}\,\text{rt}\,\sigma^l_{\eta,m}\big\|_2^2+hC_2\big\|\operatorname{Curl}\text{rt}\,p^l_{\eta,m}\big\|_2^2\\&\quad\le 2hC^{(0)}+2h\sum_{n=1}^{l}\big(\text{rt}\,u^n_{\eta,m}-\text{rt}\,u^{n-1}_{\eta,m},\text{rt}\,b^n_m\big)_\Omega,\end{aligned} \tag{81}$$

[7] For sake of simplicity we use the following notation $\text{rt}\,\phi^n_m:=(\phi^n_m-\phi^{n-1}_m)/h$, where $\phi^0_m,\phi^1_m,...,\phi^{2m}_m$ is any family of functions.

where now $C^{(0)}$ denotes

$$2C^{(0)} := \left\|\mathbb{B}^{1/2}\,\mathrm{rt}\,\sigma^0_{\eta,m}\right\|_2^2.$$

We note that (80) yields the uniform boundness of $C^{(0)}$ with respect to m. Now, using Young's inequality with $\epsilon > 0$ in (81) and then summing the resulting inequality for $l = 1, ..., 2^m$ we derive the inequality

$$\begin{aligned}&\frac{1}{m}\|\partial_t p_m\|^2_{2,\Omega_{T_e}} + \alpha_1\left\|\mathrm{dev}\,\mathrm{sym}(\partial_t p_m)\right\|^2_{2,\Omega_{T_e}} + C_2\left\|\mathrm{Curl}(\partial_t p_m)\right\|^2_{2,\Omega_{T_e}} + C\|\partial_t\sigma_m\|^2_{2,\Omega_{T_e}}\\ &\quad\le C_\varepsilon\|\partial_t b_m\|^2_{2,\Omega_{T_e}} + 2\varepsilon\|\partial_t u_m\|^2_{2,\Omega_{T_e}},\end{aligned} \tag{82}$$

where C_ε is some positive constant independent of m and η. Using now inequality (3), the condition $\partial_t p_m(x,t) \in \mathfrak{sl}(3)$ for a.e. $(x,t) \in \Omega_{T_e}$, and the ellipticity theory of linear systems we obtain that

$$\frac{1}{m}\|\partial_t p_m\|^2_{2,\Omega_{T_e}} + C_\epsilon(\Omega)\|\partial_t p_m\|^2_{2,\Omega_{T_e}} + C\|\partial_t\sigma_m\|^2_{2,\Omega_{T_e}} \le C_\varepsilon\|\partial_t b_m\|^2_{2,\Omega_{T_e}}, \tag{83}$$

where $C_\varepsilon(\Omega)$ is some further positive constant independent of m and η. Since b_m is uniformly bounded in $W^{1,q}(\Omega_{T_e}, \mathcal{S}^3)$, estimates (82) and (83) imply

$$\{\mathrm{dev}\,\mathrm{sym}\,\partial_t p_\eta\}_\eta \text{ is uniformly bounded in } L^2\big(0,T_e; L^2\big(\Omega,\mathcal{M}^3\big)\big), \tag{84}$$

$$\{\partial_t\sigma_\eta\}_\eta \text{ is uniformly bounded in } L^2\big(0,T_e; L^2\big(\Omega,\mathcal{M}^3\big)\big), \tag{85}$$

$$\{\mathrm{Curl}\,\partial_t p_\eta\}_\eta \text{ is uniformly bounded in } L^2\big(0,T_e; L^2\big(\Omega,\mathcal{M}^3\big)\big), \tag{86}$$

$$\{p_\eta\}_\eta \text{ is uniformly bounded in } H^1\big(0,T_e; L^2_{\mathrm{Curl}}\big(\Omega,\mathcal{M}^3\big)\big). \tag{87}$$

The proof of the lemma is complete. □

5.2. *Proof of Theorem 5.7*

Now, we can prove Theorem 5.7.

Proof. Due to Lemma 5.8, we have that the sequence of solutions (u_η, σ_η) is weakly compact in $H^1(0,T_e; H^1_0(\Omega,\mathbb{R}^3) \times L^2(\Omega,\mathcal{S}^3))$ and the sequence p_η is weakly compact in $H^1(0,T_e; L^2(\Omega,\mathcal{M}^3)) \cap L^2(0,T_e; Z^2_{\mathrm{Curl}}(\Omega,\mathcal{M}^3))$. Thus, by Proposition 3.2, Proposition 3.4 and Theorem 4.4, the uniform estimates (73)–(87) yield that there exist functions u_0, u_1, σ, σ_0, p, p_0 and p_1 with the prescribed regularities in Theorem 5.7 such that the convergences in (45)–(55) hold. Note that (47)–(50) give Eq. (58), i.e.

$$\sigma_0(x,y,t) = \mathbb{C}[y]\big(\mathrm{sym}\big(\nabla_x u_0(x,t) + \nabla_y u_1(x,y,t) - p_0(x,y,t)\big)\big), \quad \text{a.e.} \tag{88}$$

By Proposition 3.2, the weak-star limit σ of σ_η in $L^\infty(0,T_e; L^2(\Omega,\mathcal{S}^3))$ and the weak limit p of p_η in $L^2(0,T_e; L^2(\Omega,\mathcal{M}^3))$ are related to σ_0 and p_0 in the following ways

$$\sigma(x,t) = \int_Y \sigma_0(x,y,t)dy, \qquad p(x,t) = \int_Y p_0(x,y,t)dy.$$

Now, as in [19], we consider any $\phi \in C_0^\infty(\Omega,\mathbb{R}^3)$. Then, by the weak convergence of σ_η, the passage to the weak limit in (4) yields

$$\int_\Omega \big(\sigma(x,t), \nabla\phi(x)\big)dx = \int_\Omega \big(b(x,t), \phi(x)\big)dx, \tag{89}$$

i.e. $\operatorname{div}_x \sigma = b$ in the sense of distributions. Next, define $\phi_\eta(x) = \eta\phi(x)\psi(x/\eta)$, where $\phi \in C_0^\infty(\Omega, \mathbb{R}^3)$ and $\psi \in C_{per}^\infty(Y, \mathbb{R}^3)$. Then, one obtains that

$$\phi_\eta \rightharpoonup 0, \quad \text{in } H_0^1\big(\Omega, \mathbb{R}^3\big), \quad \text{and} \quad \mathcal{T}_\eta(\nabla\phi_\eta) \to \phi\nabla_y\psi, \quad \text{in } L^2\big(\Omega, H_{per}^1\big(Y, \mathbb{R}^3\big)\big).$$

Therefore, since ϕ_η has a compact support,

$$\int\limits_{\Omega\times Y} \big(\mathcal{T}_\eta\big(\sigma_\eta(t)\big), \mathcal{T}_\eta(\nabla\phi_\eta)\big)dxdy = \int\limits_{\Omega} \big(b(t), \phi_\eta\big)dx. \tag{90}$$

The passage to the limit in (90) leads to

$$\int\limits_{\Omega\times Y} \big(\sigma_0(x, y, t), \phi(x)\nabla_y\psi(y)\big)dxdy = 0.$$

Thus, in virtue of the arbitrariness of ϕ, one can conclude that

$$\int\limits_{\Omega\times Y} \big(\sigma_0(x, y, t), \nabla_y\psi(y)\big)dxdy = 0, \tag{91}$$

i.e. $\operatorname{div}_y \sigma_0(x, \cdot, t) = 0$ in the sense of distributions.

Next, let $\mathcal{T}_\eta(\mathcal{G}_\eta) : L^p(\Omega \times Y, \mathbb{R}^N) \to 2^{L^q(\Omega\times Y, \mathbb{R}^N)}$ and $\mathcal{G} : L^p(\Omega, \mathbb{R}^N) \to 2^{L^q(\Omega, \mathbb{R}^N)}$ denote the canonical extensions of $\mathcal{T}_\eta(g_\eta)(x, y) : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ and $g(y) : \mathbb{R}^N \to 2^{\mathbb{R}^N}$, respectively. Here, $g(y)$ is the pointwise limit graph of the convergent sequence of graphs $\mathcal{T}_\eta(g_\eta)(x, y)$. The existence of the limit graph for $\mathcal{T}_\eta(g_\eta)(x, y)$ guaranteed by Theorem 2.8. Indeed, the resolvent $j_\lambda^{\mathcal{T}_\eta(g_\eta)}$ converges pointwise to the resolvent j_λ^g, what follows from the periodicity of the mapping $y \to g(y, z) : Y \to 2^{\mathbb{R}^N}$ and the simple computations:

$$j_\lambda^{\mathcal{T}_\eta(g_\eta)}(x, y, z) = \mathcal{T}_\eta\big(j_\lambda^{g_\eta}\big)(x, y, z) = j_\lambda^g(y, z),$$

for a.e. $(x, y) \in \Omega \times Y$ and every $z \in \mathbb{R}^N$. Thus, by Theorem 2.8 we get that

$$\mathcal{T}_\eta(g_\eta)(x, y) \rightarrowtail g(y) \tag{92}$$

holds for a.e. $(x, y) \in \Omega \times Y$. Since $g_\eta \in \mathcal{M}(\Omega, \mathbb{R}^N, p, \alpha, m)$, by Definition 3.6 of the unfolding operator for a multi-valued function it follows that $\mathcal{T}_\eta(g_\eta) \in \mathcal{M}(\Omega \times Y, \mathbb{R}^N, p, \alpha, m)$. Therefore, due to this and convergence (92), by Proposition 5.4(b) we obtain that

$$\mathcal{T}_\eta(\mathcal{G}_\eta) \rightarrowtail \mathcal{G}. \tag{93}$$

To prove that the limit functions (σ_0, p_0) satisfy (59), we apply Theorem 2.6. Since the graph convergence is already established, we show that condition (14) is fulfilled. Using Eqs. (4) and (5), we successfully compute that

$$\begin{aligned} &\frac{1}{|Y|}\int\limits_{\Omega\times Y} \big(\mathcal{T}_\eta\big(\partial_t p_\eta(t)\big), \mathcal{T}_\eta\big(\Sigma_\eta^{\mathrm{lin}}(t)\big)\big)dxdy \\ &\quad = \frac{1}{|Y|}\int\limits_{\Omega\times Y} \big(\mathcal{T}_\eta\big(\partial_t\big(\operatorname{sym}\big(\nabla u_\eta(t)\big) - \mathbb{C}^{-1}\sigma_\eta(t)\big)\big), \mathcal{T}_\eta\big(\sigma_\eta(t)\big)\big)dxdy \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|Y|}\int\limits_{\Omega\times Y}\big(\mathcal{T}_\eta\big(\partial_t p_\eta(t)\big),\mathcal{T}_\eta\big(\Sigma^{\mathrm{lin}}_{\mathrm{sh},\eta}(t)+\Sigma^{\mathrm{lin}}_{\mathrm{curl},\eta}(t)\big)\big)dxdy\\
&=\int\limits_{\Omega}\big(b(t),\partial_t u_\eta(t)\big)dx-\frac{1}{|Y|}\int\limits_{\Omega\times Y}\big(\mathcal{T}_\eta\big(\partial_t\mathbb{C}^{-1}\sigma_\eta(t)\big),\mathcal{T}_\eta\big(\sigma_\eta(t)\big)\big)dxdy\\
&-\frac{1}{|Y|}\int\limits_{\Omega\times Y}\big(C_1\mathcal{T}_\eta\big(\partial_t\operatorname{dev}\operatorname{sym}p_\eta(t)\big),\mathcal{T}_\eta\big(\operatorname{dev}\operatorname{sym}p_\eta(t)\big)\big)dxdy\\
&-\frac{1}{|Y|}\int\limits_{\Omega\times Y}\big(C_2\mathcal{T}_\eta\big(\partial_t\operatorname{Curl}p_\eta(t)\big),\mathcal{T}_\eta\big(\operatorname{Curl}p_\eta(t)\big)\big)dxdy.
\end{aligned}
$$

Integrating the last identity over $(0,t)$ and using the integration-by-parts formula we get that

$$
\begin{aligned}
&\frac{1}{|Y|}\int\limits_0^t\big(\mathcal{T}_\eta\big(\partial_t p_\eta(t)\big),\mathcal{T}_\eta\big(\Sigma^{\mathrm{lin}}_\eta(t)\big)\big)_{\Omega\times Y}dt\\
&=\int\limits_0^t\big(b(t),\partial_t u_\eta(t)\big)_\Omega dt-\frac{1}{2}\big\|\mathcal{T}_\eta\big(\mathcal{B}^{1/2}\sigma_\eta(t)\big)\big\|^2_{2,\Omega\times Y}+\frac{1}{2}\big\|\mathcal{T}_\eta\big(\mathcal{B}^{1/2}\sigma_\eta(0)\big)\big\|^2_{2,\Omega\times Y}\\
&-\frac{1}{2}\big\|C_1^{1/2}\mathcal{T}_\eta\big(\operatorname{dev}\operatorname{sym}p_\eta(t)\big)\big\|^2_{2,\Omega\times Y}-\frac{1}{2}\big\|C_2^{1/2}\mathcal{T}_\eta\big(\operatorname{Curl}p_\eta(t)\big)\big\|^2_{2,\Omega\times Y},
\end{aligned}
\tag{94}
$$

where $\mathcal{B}=\mathbb{C}^{-1}$. Moreover, since $\sigma_\eta(0)$ solves the linear elasticity problem (41)–(43) with $\hat{\varepsilon}_\eta=0$ and $\hat{b}=b(t)$, by [48, Theorem 4.1], we can conclude that $\mathcal{T}_\eta(\mathcal{B}^{1/2}\sigma_\eta(0))$ converges to $\mathcal{B}^{1/2}\sigma_0(0)$ strongly in $L^2(\Omega\times Y,\mathcal{S}^3)$. Thus, by the lower semi-continuity of the norm the passing to the limit in (94) yields

$$
\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{|Y|}\int\limits_0^t\big(\mathcal{T}_\eta\big(\partial_t p_\eta(t)\big),\mathcal{T}_\eta\big(\Sigma^{\mathrm{lin}}_\eta(t)\big)\big)_{\Omega\times Y}dt\\
&\le\int\limits_0^t\big(b(t),\partial_t u_0(t)\big)_\Omega dt-\frac{1}{2}\big\|\mathcal{B}^{1/2}\sigma_0(t)\big\|^2_{2,\Omega\times Y}+\frac{1}{2}\big\|\mathcal{B}^{1/2}\sigma_0(0)\big\|^2_{2,\Omega\times Y}\\
&-\frac{1}{2}\big\|C_1^{1/2}\operatorname{dev}\operatorname{sym}p_0(t)\big\|^2_{2,\Omega\times Y}-\frac{1}{2}\big\|C_2^{1/2}\operatorname{Curl}p_0(t)\big\|^2_{2,\Omega\times Y},
\end{aligned}
$$

or

$$
\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{|Y|}\int\limits_0^t\big(\mathcal{T}_\eta\big(\partial_t p_\eta(t)\big),\mathcal{T}_\eta\big(\Sigma^{\mathrm{lin}}_\eta(t)\big)\big)_{\Omega\times Y}dt\\
&\le\int\limits_0^t\big(b(t),\partial_t u_0(t)\big)_\Omega dt-\frac{1}{|Y|}\int\limits_0^t\big(\partial_t\mathbb{C}^{-1}\sigma_0(t),\sigma_0(t)\big)_{\Omega\times Y}dt\\
&-\frac{1}{|Y|}\int\limits_0^t\big(\partial_t\operatorname{dev}\operatorname{sym}p_0(t),C_1\operatorname{dev}\operatorname{sym}p_0(t)\big)_{\Omega\times Y}dt\\
&-\frac{1}{|Y|}\int\limits_0^t\big(\partial_t\operatorname{Curl}p_0(t),C_2\operatorname{Curl}p_0(t)\big)_{\Omega\times Y}dt
\end{aligned}
\tag{95}
$$

We note that (89) and (91) imply

$$\int_{\Omega} \big(b(t), \partial_t u_0(t)\big)dx = \frac{1}{|Y|} \int_{\Omega\times Y} \big(\sigma_0(t), \partial_t \varepsilon\big(\nabla u_0(t) + \nabla_y u_1(t)\big)\big)dxdy. \tag{96}$$

And, since for almost all $(x, y, t) \in \Omega \times Y \times (0, T_e)$ one has

$$\big(\partial_t \operatorname{dev}\operatorname{sym} p_0(x, y, t), C_1[y] \operatorname{dev}\operatorname{sym} p_0(x, y, t)\big) = \big(\partial_t p_0(x, y, t), C_1[y] \operatorname{dev}\operatorname{sym} p_0(x, y, t)\big),$$

and that for almost all $t \in (0, T_e)$

$$\big(\partial_t \operatorname{Curl} p_0(t), C_2 \operatorname{Curl} p_0(t)\big)_{\Omega\times Y} = \big(\partial_t p_0(t), C_2 \operatorname{Curl}\operatorname{Curl} p_0(t)\big)_{\Omega\times Y},$$

the relations (95) and (96) together with (88) yield

$$\begin{aligned}&\limsup_{n\to\infty} \frac{1}{|Y|} \int_0^t \big(\mathcal{T}_\eta\big(\partial_t p_\eta(t)\big), \mathcal{T}_\eta\big(\Sigma_\eta^{\mathrm{lin}}(t)\big)\big)_{\Omega\times Y} dt \\ &\le \frac{1}{|Y|} \int_0^t \big(\partial_t p_0(t), \Sigma_0^{\mathrm{lin}}(t)\big)_{\Omega\times Y} dt.\end{aligned} \tag{97}$$

In virtue of convergence (93) and inequality (97), Theorem 2.6 yields that

$$\big[\Sigma_0^{\mathrm{lin}}(x, y, t), \partial_t p_0(x, y, t)\big] \in Gr\, g(y)$$

or, equivalently, that

$$\partial_t p_0(x, y, t) \in g\big(y, \Sigma_0^{\mathrm{lin}}(x, y, t)\big).$$

The initial and boundary conditions (60)–(62) for the limit functions u_0, p and p_0 are easily obtained from the weak compactness of u_η and p_η in the spaces $H^1(0, T_e; H_0^1(\Omega, \mathbb{R}^3))$ and $H^1(0, T_e; L^2(\Omega, \mathcal{M}^3)) \cap L^2(0, T_e; Z^2_{\mathrm{Curl}}(\Omega, \mathcal{M}^3))$, respectively. Therefore, summarizing everything done above, we conclude that the functions $(u_0, u_1, \sigma, \sigma_0, p, p_0, p_1)$ satisfy the homogenized initial-boundary value problem formed by the equations/inequalities (56)–(62). $\square$

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