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Doubling measures on uniform Cantor sets ☆

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ABSTRACT

We obtain a complete description for a probability measure to be doubling on an arbitrarily given uniform Cantor set. The question of which doubling measures on such a Cantor set can be extended to a doubling measure on $[0, 1]$ is also considered.

1. Introduction

It is well known that every complete doubling metric space carries a doubling measure [7,11,16] and that doubling measures are rich when the underlying doubling metric space has no dense isolated points [8]. In the Euclidean n-space, a measure, defined by integrating an A_∞ weight, is doubling [4]. These measures form a subclass of absolutely continuous doubling measures on $\mathbb{R}^n$. Examples of singular doubling measures can be obtained from self-similar measures on the unit square [3,20]. The existence of singular doubling measures have also been studied extensively in general metric spaces [8,15,19]. Recall that a Borel regular measure μ on a metric space X is doubling, if there is a constant $C \geq 1$ such that

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r)) < \infty \text{ for all } x \in X \text{ and } r > 0. \tag{1}$$

Doubling measures naturally arise in the study of the gradients of convex functions on the Euclidean n-space. See [9,10], where it is showed that, in many cases, the gradient of a convex function is a quasisymmetric self-homeomorphism of $\mathbb{R}^n$, by which the pullback measure of the Lebesgue measure is doubling. Doubling measures can be applied to classify subsets of a metric space, in which, for example, a subset is called very thin, if it is null for all doubling measures on the metric space. This leads to the study on fat

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and thin sets [1,5,12]. Doubling measures can also be used to construct quasisymmetric deformations of the underlying metric space, so that the objective space is in some sense better than the original space, for example, it can be proved that, if a metric space is uniformly perfect and carries a doubling measure, then it is quasisymmetrically equivalent to an Ahlfors regular space [6].

Though examples of doubling measures are rich, a complete description for doubling measures on a given metric space is usually difficult. It is clear that a measure on the real line $\mathbb{R}$ is doubling if and only if it is a pullback measure of a quasisymmetric self-homeomorphism of $\mathbb{R}$. However, this is not true for higher dimensional Euclidean space. In the present paper, we study doubling measures on uniform Cantor sets. We shall give a sufficient and necessary condition for the doubling property of a measure on an arbitrarily given uniform Cantor set. The question of which doubling measure on such a Cantor set can be extended to a doubling measure on $[0, 1]$ is also discussed. For the related papers on Cantor sets and doubling measures on $[0, 1]$, we refer to [2,13,14,17].

We begin with uniform Cantor sets. Let $\mathbf{n} = \{n_k\}_{k=1}^{\infty}$ be a sequence of integers, where $n_k \geq 2$. Let $\mathbf{c} = \{c_k\}_{k=1}^{\infty}$ be a sequence of real numbers in $(0, 1)$ such that $(n_k - 1)c_k < 1$ for all $k \geq 1$. The uniform Cantor set $E(\mathbf{n}, \mathbf{c})$ is defined by

$$E(\mathbf{n}, \mathbf{c}) = \bigcap_{k=0}^{\infty} E_k, \tag{2}$$

where $\{E_k\}$ is a sequence of nested compact sets in $[0, 1]$, $E_0 = [0, 1]$, and E_k is obtained by deleting from every component I of E_{k-1} $(n_k - 1)$ open intervals of equal length $c_k|I|$, such that the remaining n_k closed intervals are of equal length. A component of E_k will be called a component of level k and a component of $E_{k-1} \setminus E_k$ will be called a gap of level k. Denote by N_k, δ_k, and ε_k, respectively, the number of components of level k, the length of a component of level k, and the length of a gap of level k. Then, from the definition

$$N_k = \prod_{i=1}^{k} n_i, \quad \delta_k = \prod_{i=1}^{k} \frac{1 - (n_i - 1)c_i}{n_i}, \quad \varepsilon_k = c_k \delta_{k-1}. \tag{3}$$

Note that the uniform Cantor set $E(\mathbf{n}, \mathbf{c})$ is of Lebesgue measure

$$\mathcal{L}(E(\mathbf{n}, \mathbf{c})) = \prod_{i=1}^{\infty} (1 - (n_i - 1)c_i).$$

Therefore

$$\mathcal{L}(E(\mathbf{n}, \mathbf{c})) > 0 \Leftrightarrow \{n_k c_k\} \in \ell^1, \tag{4}$$

where ℓ^p denotes the set of real sequences $\{a_k\}_{k=1}^{\infty}$ with $\sum_{k=1}^{\infty} |a_k|^p < \infty$.

For a uniform Cantor set $E(\mathbf{n}, \mathbf{c})$ we denote by $\mathcal{I}_k$ the family of its components of level k and by $\mathcal{G}_k$ the family of its gaps of level k. Then $\sharp\mathcal{I}_k = N_k$ and $\sharp\mathcal{G}_k = N_{k-1}(n_k - 1)$, where $\sharp$ denotes the cardinality. Let $\mathcal{I} = \cup_{k=1}^{\infty} \mathcal{I}_k$ and $\mathcal{G} = \cup_{k=1}^{\infty} \mathcal{G}_k$. To label these components and gaps, we relate a word to each of them as follows: Let

$$W_k = \{i_1 i_2 \cdots i_k : 1 \leq i_j \leq n_j, 1 \leq j \leq k\} \text{ and } W = \cup_{k=1}^{\infty} W_k. \tag{5}$$

A member in W_k is called a word of length k and a member in W is called a word of finite length. Let the words in W be ordered in the lexicographic order. Then the first n_1 words are $1, 2, \cdots, n_1$. We denote, from left to right, by $I_1, I_2, \cdots, I_{n_1}$ the n_1 components in $\mathcal{I}_1$ and by $G_1, G_2, \cdots, G_{n_1-1}$ the $n_1 - 1$ gaps in $\mathcal{G}_1$.

Inductively, supposing that I_w has been defined for a word $w \in W_{k-1}$, we denote, from left to right, by $I_{w1}, I_{w2}, \cdots, I_{wn_k}$ the n_k members of $\mathcal{I}_k$ in I_w and by $G_{w1}, G_{w2}, \cdots, G_{w(n_k-1)}$ the $n_k - 1$ members of $\mathcal{G}_k$ in I_w. With the above notation, once a word $w \in W_k$ is specified, the relative position of the component I_w of level k is determined. For example, the minimal word $11\cdots1$ of length k corresponds to the leftmost component in $\mathcal{I}_k$ and the maximal word $n_1n_2\cdots n_k$ of length k corresponds to the rightmost component in $\mathcal{I}_k$. We say that two components I_w and I_u are adjacent, if $w, u \in W$ are adjacent in the sense of the lexicographic order. For example, given $w \in W_{k-1}$ and $1 \le i < n_k$, the maximal word $win_{k+1}\cdots n_{k+t}$ of length $k+t$ with prefix wi and the minimal word $w(i+1)1\cdots1$ of length $k+t$ with prefix $w(i+1)$ are adjacent. They correspond to two adjacent components in $\mathcal{I}_{k+t}$. Note that

$$E(\mathbf{n}, \mathbf{c}) = \bigcap_{k=1}^{\infty} \bigcup_{\omega \in W_k} I_\omega$$

and that

$$I_\omega = I_{w1} \cup G_{w1} \cup I_{w2} \cup G_{w2} \cup \cdots \cup I_{w(n_k-1)} \cup G_{w(n_k-1)} \cup I_{wn_k}$$

for all $w \in W_{k-1}$, where, if $k = 1$, then $I_w = [0, 1]$ and $wi = i$.

We also need some terminology on vectors. A vector $P = (p_1, p_2, \cdots, p_k)$ is called a positive probability, if its components are all positive and $\sum_{i=1}^{k} p_i = 1$. A positive probability vector $P = (p_1, p_2, \cdots, p_k)$ is called C-uniform, where $C \ge 1$ is a constant, if

$$C^{-1} \le \frac{p_{i+1} + \cdots + p_{i+l}}{p_{j+1} + \cdots + p_{j+l}} \le C \text{ whenever } 0 \le i \le j \le i + l \le j + l \le k. \tag{6}$$

The condition means that two sums of consecutive l components of P are comparable, if they are adjacent. Clearly, the vector P is 1-uniform if and only if $p_1 = p_2 = \cdots = p_k$. We say that P is (C, s)-uniform, where $1 \le s \le k$, if the condition (6) holds only for $s \le l \le k$. Thus, the $(C, 1)$-uniformness and the C-uniformness are the same. We say that a sequence $\mathbb{P} = \{P_k\}$ of positive probability vectors is C-uniform, if each P_k is C-uniform. We say that $\mathbb{P}$ is ultimately C-uniform, if there is an integer $k_0 \ge 1$ such that P_k is C-uniform for each $k \ge k_0$.

Let $E(\mathbf{n}, \mathbf{c})$ be a uniform Cantor set. Let $\mathbb{P} = \{P_k\}_{k=1}^{\infty}$ be a sequence of positive probability vectors, where

$$P_k = (p_{k,1}, p_{k,2}, \cdots, p_{k,n_k})$$

has n_k components. Such a sequence of vectors will be called $\mathbf{n}$-matching. Given an $\mathbf{n}$-matching sequence $\mathbb{P}$ of vectors, there is a unique probability measure on $E(\mathbf{n}, \mathbf{c})$, denoted by $\mu_{\mathbb{P}}$, satisfying

$$\mu_{\mathbb{P}}(I_{wi}) = p_{k,i}\mu_{\mathbb{P}}(I_w) \text{ for all } w \in W_{k-1} \text{ and } 1 \le i \le n_k. \tag{7}$$

The positivity assumption of probability vectors is reasonable. Otherwise, the corresponding measure would support on a proper compact subset of $E(\mathbf{n}, \mathbf{c})$.

The following data of the uniform Cantor set $E(\mathbf{n}, \mathbf{c})$ are crucial in stating our results. Denote

$$\Lambda = \{k \ge 1 : \varepsilon_k < \delta_k\}. \tag{8}$$

Thus, a gap in $\mathcal{G}_k$ is shorter than a component in $\mathcal{I}_k$ at every level $k \in \Lambda$. For each $k \in \Lambda$ let m_k be the unique positive integer such that

$$\delta_{k+m_k} \le \varepsilon_k < \delta_{k+m_k-1}, \tag{9}$$

and let s_k be the unique positive integer such that

$$s_k\delta_{k+m_k} + (s_k-1)\varepsilon_{k+m_k} \le \varepsilon_k < (s_k+1)\delta_{k+m_k} + s_k\varepsilon_{k+m_k}. \tag{10}$$

Write $E(\mathbf{n}, \mathbf{c}, \Lambda, \{m_k\}_{k\in\Lambda}, \{s_k\}_{k\in\Lambda})$ for $E(\mathbf{n}, \mathbf{c})$, when these related data are emphasized.

Theorem 1. *Let $E = E(\mathbf{n}, \mathbf{c}, \Lambda, \{m_k\}_{k\in\Lambda}, \{s_k\}_{k\in\Lambda})$ be a uniform Cantor set. Let $\mu_{\mathbb{P}}$ be a measure determined by an $\mathbf{n}$-matching sequence $\mathbb{P}$ of positive probability vectors. Then $\mu_{\mathbb{P}}$ is doubling on E if and only if there is a constant $C > 1$ such that $\mathbb{P}$ is C-uniform and satisfies, for each $k \in \Lambda$, the following conditions:*

(a) (P_{k+t}, P_{k+t}) *is C-uniform and*

$$C^{-1}\prod_{j=1}^{t} p_{k+j,1} \le \prod_{j=1}^{t} p_{k+j,n_{k+j}} \le C\prod_{j=1}^{t} p_{k+j,1} \text{ for all } 1 \le t < m_k. \tag{11}$$

(b) (P_{k+m_k}, P_{k+m_k}) *is (C, s_k)-uniform.*

Hereafter, if $P = (p_1, p_2, \cdots, p_j)$, then $(P, P) = (p_1, p_2, \cdots, p_j, p_1, p_2, \cdots, p_j)$.

From Theorem 1 some easier conditions for the doubling property of the measure $\mu_{\mathbb{P}}$ on E can be formulated.

Corollary 1. *If $\sup_{k\ge1} n_k < \infty$, then the measure $\mu_{\mathbb{P}}$ is doubling on E if and only if there is a constant $C > 1$ such that*

(a) $C^{-1}p_{k,i} \le p_{k,i+1} \le Cp_{k,i}$ for all $1 \le i < n_k$ and for all k, and
(b) the condition (11) is satisfied for all $1 \le t < m_k$ and for all $k \in \Lambda$.

Corollary 2. *If $\sup_{k\in\Lambda} m_k < \infty$, then the measure $\mu_{\mathbb{P}}$ is doubling on E if and only if there is a constant $C > 1$ such that*

(a) $\mathbb{P}$ is C-uniform, and
(b) for every integer $k \in \Lambda$, (P_{k+t}, P_{k+t}) is C-uniform for all $1 \le t < m_k$ and (P_{k+m_k}, P_{k+m_k}) is (C, s_k)-uniform.

Corollary 3. *If $\mathbb{P}$ is an ultimately 1-uniform, then $\mu_{\mathbb{P}}$ is doubling on E.*

Theorem 1 can be generalized. Let μ be a Borel probability measure on E with $\mu(I_w) > 0$ for all $w \in W$. For each $k \ge 1$ and for each $w \in W_{k-1}$ let

$$p_{wi} = \frac{\mu(I_{wi})}{\mu(I_w)},\ 1 \le i \le n_k. \tag{12}$$

Then there is a one-to-one correspondence between the set of Borel probability measures μ on E, with $\mu(I_w) > 0$ for all $w \in W$, and the set of sequences $\{p_w\}_{w\in W}$, with $\sum_{i=1}^{n_k} p_{wi} = 1$ and $0 < p_{wi} < 1$ for all $w \in W_{k-1}$ and $k \ge 1$. Write

$$P_w = (p_{w1}, p_{w2}, \cdots, p_{wn_k}) \tag{13}$$

for each $w \in W_{k-1}$. Then μ is a measure considered in Theorem 1, provided that for every fixed integer $k \geq 1$, the vectors P_w are the same for all $w \in W_{k-1}$. Write wiu_t for the maximal word in W_{k+t} with prefix wi and $wi1^t$ the minimal word in W_{k+t} with prefix wi. Our next theorem gives a complete description of doubling measures on a uniform Cantor set.

Theorem 2. *Let $E = E(\mathbf{n}, \mathbf{c}, \Lambda, \{m_k\}_{k\in\Lambda}, \{s_k\}_{k\in\Lambda})$ be a uniform Cantor set. Let μ be a Borel probability measure on E with $\mu(I_w) > 0$ for all $w \in W$. Let $\{p_w\}_{w\in W}$ be the corresponding sequence. Then μ is doubling on E if and only if there is a constant $C > 1$ such that P_w is C-uniform for all $w \in W$ and satisfies, for each $k \in \Lambda$, the following conditions:*

(a) $(P_{wiu_{t-1}}, P_{w(i+1)1^{t-1}})$ *is C-uniform and*

$$C^{-1}\prod_{j=1}^{t} p_{wiu_j} \leq \prod_{j=1}^{t} p_{w(i+1)1^j} \leq C\prod_{j=1}^{t} p_{wiu_j} \tag{14}$$

for all $w \in W_{k-1}$, $1 \leq i < n_k$, and $1 \leq t < m_k$.

(b) $(P_{wiu_{m_k-1}}, P_{w(i+1)1^{m_k-1}})$ *is (C, s_k)-uniform.*

For a uniform Cantor set $E(\mathbf{n}, \mathbf{c})$ it is known that $\nu(E(\mathbf{n}, \mathbf{c})) = 0$ for all doubling measures ν on $[0, 1]$ if and only if $\{n_k c_k\}_{k=1}^{\infty} \notin \cup_{q\geq 1}\ell^q$ [5]. As a consequence, if $E(\mathbf{n}, \mathbf{c})$ carries a doubling measure which can be extended to a doubling measure on $[0, 1]$, then $\{n_k c_k\}_{k=1}^{\infty} \in \cup_{q\geq 1}\ell^q$. Our main result on the extension question of doubling measures on a uniform Cantor set is the following theorem.

Theorem 3. *Let $E = E(\mathbf{n}, \mathbf{c})$ be a uniform Cantor set. Let $\mu_{\mathbb{P}}$ be a measure determined by an ultimately 1-uniform $\mathbf{n}$-matching sequence $\mathbb{P}$ of positive probability vectors. Then $\mu_{\mathbb{P}}$ can be extended to a doubling measure on $[0, 1]$ if and only if $\{n_k c_k\}_{k=1}^{\infty} \in \ell^1$.*

2. Proof of Theorem 1

In what follows C will denote a constant depending only on the constants in question and it may be different in every appearance. Write $A \sim B$ for $C^{-1}B \leq A \leq CB$, and $A \preceq B$ for $A \leq CB$, when A, B are quantities. For an interval I, denote by $x^-(I)$ and $x^+(I)$, respectively, the left and the right endpoints of I.

Proof of the 'only if' part. Let $E = E(\mathbf{n}, \mathbf{c}, \Lambda, \{m_k\}_{k\in\Lambda}, \{s_k\}_{k\in\Lambda})$ be a uniform Cantor set. Let $\mu_{\mathbb{P}}$ be a measure determined by to an $\mathbf{n}$-matching sequence $\mathbb{P}$ of positive probability vectors. Suppose that $\mu_{\mathbb{P}}$ is doubling on E. We are going to show that the sequence $\mathbb{P}$ satisfies the conditions of Theorem 1. The doubling property of the measure $\mu_{\mathbb{P}}$ will be used frequently in the following equivalent form:

$$\mu(B(x, r)) \sim \mu(B(y, t)) \text{ for all } x, y \in E \text{ and } r, s > 0 \text{ with } |x - y| \preceq r \sim s.$$

Note that, for all $k \geq 1$, the endpoints of components of level k of E belong to E. The proof consists of the following claims.

Claim 1. *$p_{k,2} \leq Cp_{k,1}$ and $p_{k,n_k-1} \leq Cp_{k,n_k}$ for all $k \geq 1$.*

Proof of Claim 1. Let $w \in W_{k-1}$ be the minimal word of length $k - 1$. Then the corresponding component I_w is the leftmost component of level $k - 1$. We easily see that

$$I_{w1} = E_k \cap B(x^-(I_{w1}), \delta_k + \varepsilon_k) \text{ and } I_{w2} \subseteq B(x^-(I_{w2}), \delta_k + \varepsilon_k).$$

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Observing that $x^-(I_{w1}), x^-(I_{w2}) \in E$ and $|x^-(I_{w1}) - x^-(I_{w2})| = \delta_k + \varepsilon_k$, it follows from the doubling property of $\mu_{\mathbb{P}}$ that

$$\mu_{\mathbb{P}}(I_{w2}) \le \mu_{\mathbb{P}}(B(x^-(I_{w2}), \delta_k + \varepsilon_k)) \le C\mu_{\mathbb{P}}(B(x^-(I_{w1}), \delta_k + \varepsilon_k)) = C\mu_{\mathbb{P}}(I_{w1}),$$

giving $p_{k,2} \le Cp_{k,1}$ by (7). Similarly, we have $p_{k,n_k-1} \le Cp_{k,n_k}$ by considering the maximal word $w \in W_{k-1}$ and the rightmost two components of level k. This proves Claim 1. □

Claim 2. *$p_{k,i} \sim p_{k,i+1}$ for all $1 \le i < n_k$ and $k \ge 1$.*

Proof of Claim 2. Let k be given. When $n_k = 2$, Claim 2 follows from Claim 1 directly. Next we assume that $n_k > 2$. Consider the components I_{wi} and $I_{w(i+1)}$ of level k, where $w \in W_{k-1}$ and $1 \le i < n_k$. Then G_{wi} is the gap of level k between them. Three possible cases may happen.

Case 1. $\varepsilon_k \ge \delta_k$.

Since $\varepsilon_k \ge \delta_k$, we easily see that

$$I_{wi} = E_k \cap B(x^+(I_{wi}), \varepsilon_k) \text{ and } I_{w(i+1)} = E_k \cap B(x^-(I_{w(i+1)}), \varepsilon_k) \tag{15}$$

for $1 < i < n_k - 1$. Observing that $x^+(I_{wi}), x^-(I_{w(i+1)}) \in E$ and $|x^+(I_{wi}) - x^-(I_{w(i+1)})| = \varepsilon_k$, we get from doubling property of $\mu_{\mathbb{P}}$ that

$$\mu_{\mathbb{P}}(I_{wi}) \sim \mu_{\mathbb{P}}(I_{w(i+1)}),$$

which yields $p_{k,i} \sim p_{k,i+1}$ by (7). For $i = 1$, letting w be the minimal word in W_{k-1}, the relationship (15) remains to be true, and so $p_{k,1} \sim p_{k,2}$. For $i = n_k - 1$, considering the maximal word in W_{k-1}, we get $p_{k,n_k-1} \sim p_{k,n_k}$ in the same way.

Case 2. $\varepsilon_k < \delta_k$ and $n_{k+1} > 2$.

Consider balls $B(x, r)$ and $B(y, r)$, where

$$x = x^-(G_{wi}) - r,\ y = x^+(G_{wi}) + r,\ r = [\frac{n_{k+1} - 1}{2}](\delta_{k+1} + \varepsilon_{k+1}).$$

We see that x is an endpoint of a component of level $k+1$ in I_{wi} and that y is an endpoint of a component of level $k+1$ in $I_{w(i+1)}$, so $x, y \in E$. Note also that

$$B(x, r) \subseteq I_{wi} \text{ and } B(y, r) \subseteq I_{w(i+1)}.$$

Since $n_{k+1} \ge 3$, we have $4[\frac{n_{k+1}-1}{2}] \ge n_{k+1}$, so $4r > \delta_k$, giving

$$I_{wi} \subseteq B(x, 4r) \text{ and } I_{w(i+1)} \subseteq B(y, 4r).$$

Therefore

$$\mu_{\mathbb{P}}(I_{wi}) \sim \mu_{\mathbb{P}}(B(x, r)) \text{ and } \mu_{\mathbb{P}}(I_{w(i+1)}) \sim \mu_{\mathbb{P}}(B(y, r)).$$

Additionally, since $\varepsilon_k < \delta_k$ has been assumed, we have $|x - y| = 2r + \varepsilon_k \le 6r$, so $\mu_{\mathbb{P}}(B(x, r)) \sim \mu_{\mathbb{P}}(B(y, r))$ by the doubling property of $\mu_{\mathbb{P}}$. It then follows that $\mu_{\mathbb{P}}(I_{wi}) \sim \mu_{\mathbb{P}}(I_{w(i+1)})$, giving $p_{k,i} \sim p_{k,i+1}$ by (7).

Case 3. $\varepsilon_k < \delta_k$ and $n_{k+1} = 2$.

In the case of $\varepsilon_{k+1} \geq \delta_{k+1}$, consider $B(x^-(I_{wi2}), r)$ and $B(x^+(I_{w(i+1)1}), r)$, where $r = \delta_{k+1} + \frac{\varepsilon_k}{3}$. Since $r < \delta_{k+1} + \frac{\delta_k}{3} \leq \delta_{k+1} + \varepsilon_{k+1}$, we see that

$$I_{wi2} \subseteq E_k \cap B(x^-(I_{wi2}), r) \subseteq I_{wi}$$

and

$$I_{w(i+1)1} = E_k \cap B(x^+(I_{w(i+1)1}), r) \subseteq I_{w(i+1)}.$$

Observing that $|x^+(I_{w(i+1)1} - x^-(I_{wi2})| = 2\delta_{k+1} + \varepsilon_k \leq 3r$, we get from the doubling property of $\mu_{\mathbb{P}}$ that

$$\mu_{\mathbb{P}}(B(x^-(I_{wi2}), r)) \sim \mu_{\mathbb{P}}(B(x^+(I_{w(i+1)1}), r)).$$

Since $n_{k+1} = 2$, we have $\mu_{\mathbb{P}}(I_{wi2}) \sim \mu_{\mathbb{P}}(I_{wi})$ and $\mu_{\mathbb{P}}(I_{w(i+1)1}) \sim \mu_{\mathbb{P}}(I_{w(i+1)})$ by Claim 1. Therefore $\mu_{\mathbb{P}}(I_{wi}) \sim \mu_{\mathbb{P}}(I_{w(i+1)})$, and so $p_{k,i} \sim p_{k,i+1}$ by (7).

In the case of $\varepsilon_{k+1} < \delta_{k+1}$, since $n_{k+1} = 2$, we have

$$B(x^-(I_{wi2}), \delta_{k+1}) \subseteq I_{wi} \subseteq B(x^-(I_{wi2}), 3\delta_{k+1})$$

and

$$B(x^+(I_{w(i+1)1}), \delta_{k+1}) \subseteq I_{w(i+1)} \subseteq B(x^+(I_{w(i+1)1}), 3\delta_{k+1}).$$

Since $\varepsilon_k < \delta_k$, we get $|x^+(I_{w(i+1)1} - x^-(I_{wi2})| = 2\delta_{k+1} + \varepsilon_k \leq 5\delta_{k+1}$. Then $\mu_{\mathbb{P}}(I_{wi}) \sim \mu_{\mathbb{P}}(I_{w(i+1)})$ by the doubling property of $\mu_{\mathbb{P}}$, giving $p_{k,i} \sim p_{k,i+1}$. This completes the proof of Claim 2. □

Claim 3. $\mathbb{P}$ *is* C*-uniform.*

Proof of Claim 3. Since the C-uniformness of $\mathbb{P}$ is equivalent to that every P_k satisfies

$$p_{k,i+1} + \cdots + p_{k,i+l} \sim p_{k,i+l+1} + \cdots + p_{k,i+2l} \tag{16}$$

for all $i \geq 0$ and $l \geq 1$ with $i+2l \leq n_k$, it suffices to show (16). Fix k. When $l = 1$ or $n_k \leq 3$, the relationship (16) follows from Claim 2 directly. Next we assume that $l \geq 2$ and $n_k > 3$. Let $w \in W_{k-1}$ and $i \geq 0$ be given such that $i + 2l \leq n_k$. Observing that there are balls $B(x, r)$ and $B(y, r)$ with $x, y \in E$ and $|x - y| \leq 3r$ such that

$$E_k \cap B(x, r) \subseteq I_{w(i+1)} \cup \cdots \cup I_{w(i+j)} \subseteq B(x, 3r)$$

and

$$E_k \cap B(y, r) \subseteq I_{w(i+j+1)} \cup \cdots \cup I_{w(i+2j)} \subseteq B(y, 3r),$$

we immediately get

$$\mu_{\mathbb{P}}(I_{w(i+1)} \cup \cdots \cup I_{w(i+j)}) \sim \mu_{\mathbb{P}}(I_{w(i+j+1)} \cup \cdots \cup I_{w(i+2j)}),$$

giving the relationship (16) by (7). This completes the proof of Claim 3. □

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Claim 4. *$\prod_{j=1}^{t} p_{k+j,1} \sim \prod_{j=1}^{t} p_{k+j,n_{k+j}}$ for all $k \in \Lambda$ and $1 \le t < m_k$.*

Proof of Claim 4. Let $k \in \Lambda$ be fixed. Without loss of generality, assume that $m_k > 1$. Let $1 \le t < m_k$ be given. Then $\varepsilon_k < \delta_{k+t}$ by the definition of m_k. Let $w \in W_{k-1}$ and $1 \le i < n_k$ be given. Let $wiu \in W_{k+t}$ be the maximal word with prefix wi and $w(i+1)v \in W_{k+t}$ be the minimal word with prefix $w(i+1)$. Then, by (7)

$$\mu_{\mathbb{P}}(I_{wiu}) = \mu_{\mathbb{P}}(I_w) p_{k,i} \prod_{j=1}^{t} p_{k+j,n_{k+j}}$$

and

$$\mu_{\mathbb{P}}(I_{w(i+1)v}) = \mu_{\mathbb{P}}(I_w) p_{k,i+1} \prod_{j=1}^{t} p_{k+j,1}.$$

Note that the gap between I_{wiu} and $I_{w(i+1)v}$ is G_{wi}. Since $\varepsilon_k < \delta_{k+t}$, one has $|G_{wi}| < |I_{wiu}| = |I_{w(i+1)v}|$. To complete this proof, we consider two cases:

Case 1. $n_{k+t+1} > 2$.

Arguing as Case 2 of Claim 2, we have two balls, $B(x,r)$ and $B(y,r)$, with $x, y \in E$ and $|x-y| \le 6r$, such that

$$B(x,r) \subseteq I_{wiu} \subseteq B(x,4r) \text{ and } B(y,r) \subseteq I_{w(i+1)v} \subseteq B(y,4r),$$

which, combined with the doubling property of $\mu_{\mathbb{P}}$, yields $\mu_{\mathbb{P}}(I_{wiu}) \sim \mu_{\mathbb{P}}(I_{w(i+1)v})$.

Case 2. $n_{k+t+1} = 2$.

Consider two subcases: $\varepsilon_{k+t+1} \ge \delta_{k+t+1}$ and $\varepsilon_{k+t+1} < \delta_{k+t+1}$. In the first subcase, arguing as Case 3 of Claim 2, we have two balls, $B(x,r)$ and $B(y,r)$, with $x, y \in E$ and $|x-y| \le 3r$, such that

$$I_{wiu2} \subseteq E_{k+t} \cap B(x,r) \subseteq I_{wiu} \text{ and } I_{w(i+1)v1} = E_{k+t} \cap B(y,r) \subseteq I_{w(i+1)v}.$$

In the second subcase, arguing as Case 3 of Claim 2, we have two balls, $B(x,r)$ and $B(y,r)$, with $x, y \in E$ and $|x-y| \le 5r$, such that

$$B(x,r) \subseteq I_{wiu} \subseteq B(x,3r) \text{ and } B(y,r) \subseteq I_{w(i+1)v} \subseteq B(y,3r).$$

Therefore, in both subcases, we have $\mu_{\mathbb{P}}(I_{wiu}) \sim \mu_{\mathbb{P}}(I_{w(i+1)v})$.

Now $\mu_{\mathbb{P}}(I_{wiu}) \sim \mu_{\mathbb{P}}(I_{w(i+1)v})$ is proved. Since $p_{k,i} \sim p_{k,i+1}$ has been proved in Claim 2, we get $\prod_{j=1}^{t} p_{k+j,1} \sim \prod_{j=1}^{t} p_{k+j,n_{k+j}}$. This completes the proof of Claim 4. □

Claim 5. *(P_{k+t}, P_{k+t}) is C-uniform for all $k \in \Lambda$ and $1 \le t < m_k$.*

Proof of Claim 5. Since $\mathbb{P}$ has been proved to be C-uniform, it suffices to show that the sum of the first j terms of P_{k+t} is comparable to the sum of the last j terms of P_{k+t} for all $k \in \Lambda$, $1 \le t < m_k$, and $1 \le j \le n_{k+t}$. Let these integers be given. Let $w \in W_{k-1}$ and $1 \le i < n_k$ be given. Let $u \in W_{k+t-1}$ be the maximal word with prefix wi and $v \in W_{k+t-1}$ be the minimal word with prefix $w(i+1)$. Then G_{wi} is the gap between I_u and I_v. Consider components $I_{un_{k+t}}$, $I_{u(n_{k+t}-1)}$, $\cdots$, $I_{u(n_{k+t}+1-j)}$ and I_{v1}, I_{v2}, $\cdots$, I_{vj} of level $k+t$. Since $\varepsilon_k < \delta_{k+t}$, by the same argument as that of Claim 2 we get

$$\mu_{\mathbb{P}}(I_{un_{k+t}} \cup \cdots \cup I_{u(n_{k+t}+1-j)}) \sim \mu_{\mathbb{P}}(I_{v1} \cup \cdots \cup I_{vj}),$$

which, together with Claim 2 and Claim 4, yields

$$p_{k+t,n_{k+t}} + \cdots + p_{k+t,n_{k+t}+1-j} \sim p_{k+t,1} + \cdots + p_{k+t,j}.$$

This completes the proof of Claim 5. □

Claim 6. (P_{k+m_k}, P_{k+m_k}) *is* (C, s_k)*-uniform for all* $k \in \Lambda$*.*

Proof of Claim 6. Since $\mathbb{P}$ has been proved to be C-uniform, it suffices to show that the sum of the first j terms of P_{k+m_k} is comparable to the sum of the last j terms of P_{k+m_k} for all $k \in \Lambda$ and $s_k \le j \le n_{k+m_k}$, where, by the definition, s_k satisfies

$$s_k\delta_{k+m_k} + (s_k - 1)\varepsilon_{k+m_k} \le \varepsilon_k < (s_k + 1)\delta_{k+m_k} + s_k\varepsilon_{k+m_k}.$$

Let k and j be given. Let $w \in W_{k-1}$ and $1 \le i < n_k$. Let $u \in W_{k+m_k-1}$ be the maximal word with prefix wi and let $v \in W_{k+m_k-1}$ be the minimal word with prefix $w(i+1)$. Then G_{wi} is the gap between $I_{un_{k+m_k}}$ and I_{v1}.

Case 1. $s_k = j = 1$. In this case, $\delta_{k+m_k} \le \varepsilon_k \le 2\delta_{k+m_k} + \varepsilon_{k+m_k}$, so we have

$$I_{un_{k+m_k}} \subseteq E \cap B(x^-(G_{wi}), \varepsilon_k) \subseteq I_{un_{k+m_k}} \cup I_{u(n_{k+m_k}-1)}$$

and

$$I_{v1} \subseteq E \cap B(x^+(G_{wi}), \varepsilon_k) \subseteq I_{v1} \cup I_{v2}.$$

From Claim 2 and Claim 4 we get

$$\mu_{\mathbb{P}}(I_{un_{k+m_k}}) \sim \mu_{\mathbb{P}}(I_{un_{k+m_k}} \cup I_{u(n_{k+m_k}-1)}) \text{ and } \mu_{\mathbb{P}}(I_{v1}) \sim \mu_{\mathbb{P}}(I_{v1} \cup I_{v2)}).$$

Since $x^-(G_{wi}), x^+(G_{wi}) \in E$ and $|x^-(G_{wi}) - x^+(G_{wi})| = \varepsilon_k$, it follows from the doubling property of $\mu_{\mathbb{P}}$ that $\mu_{\mathbb{P}}(B(x^-(G_{wi}), \varepsilon_k)) \sim \mu_{\mathbb{P}}(B(x^+(G_{wi}), \varepsilon_k))$, so

$$\mu_{\mathbb{P}}(I_{un_{k+m_k}}) \sim \mu_{\mathbb{P}}(I_{v1}),$$

which, together with Claim 4, yields $p_{k+m_k,n_{k+m_k}} \sim p_{k+m_k,1}$.

Case 2. $s_k > 1$ or $j > 1$. In this case, arguing as we did in Claim 3, we get

$$\mu_{\mathbb{P}}(I_{un_{k+m_k}} \cup \cdots \cup I_{u(n_{k+m_k}+1-j)}) \sim \mu_{\mathbb{P}}(I_{v1} \cup \cdots \cup I_{vj}),$$

which, combined with Claim 4, yields

$$p_{k+m_k,n_{k+m_k}} + \cdots + p_{k+m_k,n_{k+m_k}+1-j} \sim p_{k+m_k,1} + \cdots + p_{k+m_k,j}.$$

This completes the proof of Claim 6 and the 'only if' part is thus proved. □

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Proof of the 'if' part. Suppose that $\mathbb{P}$ meets the conditions of Theorem 1. We are going to show that $\mu_{\mathbb{P}}$ is doubling on E.

Let $B(x,r)$ be a ball of the real line with $x \in E$ and $r \in (0,1)$. Let I be the smallest component such that $I \supseteq E \cap B(x,2r)$. Suppose that I is at the level $k-1$. Then $B(x,2r)$ intersects at least two components of level k, so $2r \geq \varepsilon_k$. Denote

$$\mathcal{A} = \{L : L \in \mathcal{I}_k,\ L \cap B(x,2r) \neq \emptyset\}.$$

Then $\cup_{L\in\mathcal{A}} L \subseteq I$ and $4r \geq (\sharp\mathcal{A}-2)\delta_k + (\sharp\mathcal{A}-1)\varepsilon_k$, where $\sharp$ denotes the cardinality. We consider four cases as follows.

Case 1. $\sharp\mathcal{A} \geq 4$.

In this case, we see that $\sharp\mathcal{A} \sim \sharp\{L \in \mathcal{A} : L \subseteq B(x,r)\}$. Since P_k is C-uniform, we get $\mu_{\mathbb{P}}(B(x,r)) \sim \mu_{\mathbb{P}}(B(x,2r))$.

Case 2. $\sharp\mathcal{A} = 3$.

In this case, $4r > \delta_k$. When $n_{k+1} \geq 4$, we see that $B(x,r)$ contains at least $[\frac{n_{k+1}}{4}]$ components of level $k+1$ in $L(x)$, where $L(x) \in \mathcal{A}$ is the component of level k containing x. Since P_{k+1} and P_k are C-uniform, we have

$$\mu_{\mathbb{P}}(B(x,r)) \sim \mu_{\mathbb{P}}(L(x)) \sim \mu_{\mathbb{P}}(B(x,2r)).$$

When $n_{k+1} = 2$ or 3, we see from $\sharp\mathcal{A} = 3$ that $B(x,r)$ contains at least a component of level $k+1$. Therefore the last relationship remains true.

Case 3. $\sharp\mathcal{A} = 2$ and $\varepsilon_k \geq \delta_k$.

In this case, since $2r \geq \varepsilon_k$, we have $2r \geq \delta_k$. By the same argument as that of Case 2 we get $\mu_{\mathbb{P}}(B(x,r)) \sim \mu_{\mathbb{P}}(B(x,2r))$.

Case 4. $\sharp\mathcal{A} = 2$ and $\varepsilon_k < \delta_k$.

Let I_{wi} and $I_{w(i+1)}$ be members of $\mathcal{A}$, where $w \in W_{k-1}$ and $1 \leq i < n_k$. Without loss of generality, assume that $x \in I_{wi}$. Let $J(x)$ be the biggest component such that $x \in J(x) \subseteq B(x,r)$. Then $J(x) = I$ or $J(x) \subseteq I_{wi}$. It is obvious that $\mu_{\mathbb{P}}(B(x,r)) = \mu_{\mathbb{P}}(B(x,2r))$ for $J(x) = I$. Also, if $J(x) = I_{wi}$, we easily see from the C-uniformness of P_k that

$$\mu_{\mathbb{P}}(B(x,r)) \geq \mu_{\mathbb{P}}(I_{wi}) \geq C\mu_{\mathbb{P}}(I_{wi} \cup I_{w(i+1)}) \geq C\mu_{\mathbb{P}}(B(x,2r)).$$

Next assume $J(x) \neq I_{wi}$. Thus $J(x)$ is at the level $k+t$ for some $t \geq 1$. Let I_u be the component of level $k+t-1$ containing $J(x)$, where $u \in W_{k+t-1}$. Then we have $B(x,r) \subset I_u \cup G \cup I_v$ and $r < \delta_{k+t-1}$ by the maximality of $J(x)$, where I_v is the nearest component of level $k+t-1$ on the right of I_u and G is the gap between I_u and I_v. Therefore, $B(x,2r)$ meets at most four components of level $k+t-1$, that is, $\sharp\mathcal{B} \leq 4$, where

$$\mathcal{B} = \{L : L \text{ is a component of } E_{k+t-1} \text{ and } L \cap B(x,2r) \neq \emptyset\}.$$

Clearly, $I_u \in \mathcal{B}$. We claim that $\mu_{\mathbb{P}}(L) \sim \mu_{\mathbb{P}}(I_u)$ for all $L \in \mathcal{B}$. In fact, let L and K be two adjacent components in $\mathcal{B}$, then the gap between L and K is at the level $k+j$ for some $0 \leq j \leq t-1$. When the gap is at the level $k+t-1$, we immediately get $\mu_{\mathbb{P}}(L) \sim \mu_{\mathbb{P}}(K)$ from the C-uniformness of P_{k+t-1}. The gap being at the level $k+j$ for some $1 \le j \le t-2$ is possible only when

$$\varepsilon_{k+j} \le 2r < 2\delta_{k+t-1} < \delta_{k+j},$$

and if it happened, we may use the C-uniformness of (P_{k+j}, P_{k+j}) and the condition (11) to get $\mu_{\mathbb{P}}(L) \sim \mu_{\mathbb{P}}(K)$. Finally, we consider the case where the gap is at the level k. Note that $t \le m_k + 1$, in fact, if not, we would get from the definition of m_k that $2r \le 2\delta_{k+m_k+1} < \delta_{k+m_k} \le \varepsilon_k$, contradicting $2r \ge \varepsilon_k$. Also, if $t = m_k + 1$ happened, (P_{k+m_k}, P_{k+m_k}) would have been C-uniform. In fact, if $t = m_k + 1$ then $\delta_{k+m_k} \le \varepsilon_k \le 2r \le 2\delta_{k+m_k}$ by the definition of m_k, so $s_k = 1$, and hence (P_{k+m_k}, P_{k+m_k}) is C-uniform, because it has been assumed to be (C, s_k)-uniform. Then $\mu_{\mathbb{P}}(L) \sim \mu_{\mathbb{P}}(K)$ follows from the above discussion and the assumptions of Theorem 1. This proves the claim.

Now let

$$\mathcal{C} = \{L : L \text{ is a component of } E_{k+t} \text{ and } L \subseteq I_u \cap B(x,r)\}.$$

Two possible subcases may happen.

Subcase 1. $B(x, 2r) \supseteq I_u$.

In this subcase, we have $\sharp\mathcal{C} \ge [\frac{n_{k+t}}{2}]$. It follows from the C-uniformness of P_{k+t} that

$$\mu_{\mathbb{P}}(B(x,r)) \ge C\mu_{\mathbb{P}}(I_u).$$

Since $\sharp\mathcal{B} \le 4$, we get from the above claim that $\mu_{\mathbb{P}}(B(x,r)) \sim \mu_{\mathbb{P}}(B(x,2r))$.

Subcase 2. $B(x, 2r) \subset I_u \cup G \cup I_v$.

In this subcase, it is clear that $2r < |I_u| = \delta_{k+t-1}$, so we have $t \le m_k$ by the definition of m_k. We see that $I_u \subseteq I_{wi}$ and $I_v \subseteq I_{w(i+1)}$, with $u \in W_{k+t-1}$ being the maximal word with prefix wi and $v \in W_{k+t-1}$ being the minimal word with prefix $w(i+1)$. Thus G is a gap of level k. When $1 \le t < m_k$, we use the assumption (a) to get $\mu_{\mathbb{P}}(B(x,r)) \sim \mu_{\mathbb{P}}(B(x,2r))$. When $t = m_k$, remembering that s_k has been defined to satisfy

$$s_k\delta_{k+m_k} + (s_k - 1)\varepsilon_{k+m_k} \le \varepsilon_k < (s_k + 1)\delta_{k+m_k} + s_k\varepsilon_{k+m_k},$$

which, together with $2r \ge \varepsilon_k$, yields $2r \ge s_k\delta_{k+m_k} + (s_k - 1)\varepsilon_{k+m_k}$, we may use the C-uniformness of P_{k+m_k}, the (C, s_k)-uniformness of (P_{k+m_k}, P_{k+m_k}), and the assumption (11) to get $\mu_{\mathbb{P}}(B(x,r)) \sim \mu_{\mathbb{P}}(B(x,2r))$. This completes the proof of the 'if' part. □

3. Proof of Theorem 2

The proof of Theorem 2 may go in step as that of Theorem 1. For the proof of the 'only if' part, suppose that μ is doubling on the uniform Cantor set E. We shall prove that the sequence $\{P_w\}_{w\in W}$ satisfies the conditions of Theorem 2. As we did in Theorem 1, the proof consists of the following claims.

Claim A. *$p_{w2} \le Cp_{w1}$ and $p_{w(n_k-1)} \le Cp_{wn_k}$ for all $w \in W_{k-1}$ and $k \ge 1$.*

Claim B. *$p_{wi} \sim p_{w(i+1)}$ for all $w \in W_{k-1}$, $1 \le i < n_k$, and $k \ge 1$.*

Claim C. *P_w is C-uniform for all $w \in W$.*

Claim D. *$\prod_{j=1}^{t} p_{w(i+1)1^j} \sim \prod_{j=1}^{t} p_{wiu_j}$ for all $w \in W_{k-1}$, $1 \le i < n_k$, $1 \le t < m_k$, and $k \in \Lambda$.*

Claim E. *$(P_{wiu_{t-1}}, P_{w(i+1)1^{t-1}})$ is C-uniform for all $w \in W_{k-1}$, $1 \le i < n_k$, $1 \le t < m_k$, and $k \in \Lambda$.*

Claim F. *$(P_{wiu_{m_k-1}}, P_{w(i+1)1^{m_k-1}})$ is (C, s_k)-uniform for all $k \in \Lambda$.*

If, for every fixed integer $k \ge 1$, the vectors P_w are the same for all $w \in W_{k-1}$, then the above claims are exactly those in Theorem 1. Under this condition, *Claim A* has a simpler proof, because the word $w \in W_{k-1}$ in question may be assumed to be the minimal or maximal words. Now, without assuming this condition, we are going to prove *Claim A*. Instead, we will show the following

Lemma 1. *$p_{w2} \sim p_{w1}$ and $p_{w(n_k-1)} \sim p_{wn_k}$ for all $w \in W_{k-1}$ and $k \ge 1$.*

Proof of Lemma 1. We only prove $p_{w2} \sim p_{w1}$. The proof of the other relationship is obviously similar. Let $k \ge 1$ and $w \in W_{k-1}$ be given. To prove $p_{w2} \sim p_{w1}$, it suffices to show $\mu(I_{w2}) \sim \mu(I_{w1})$. Without loss of generality, assume that $w \in W_{k-1}$ is neither minimal nor maximal.

Case 1. $\varepsilon_k \ge \delta_k$.

Let I_u be the component of level k next to the left of I_{w1}. Let I_v be the component of level k next to the right of I_{w2}. Let G be the gap between I_u and I_{w1}. Let G' be the gap between I_{w2} and I_v. Then G' is a gap of level k if and only if $n_k > 2$. Remember that μ is doubling on E and that $\varepsilon_k \ge \delta_k$ has been assumed.

Noting that, if $\delta_k \le |G| < \varepsilon_k$ then

$$I_{w1} = E_k \cap B(x^+(I_{w1}), |G|) \text{ and } I_u = E_k \cap B(x^-(I_u), |G|),$$

and that, if $|G| < \delta_k$ then

$$I_{w1} = E_k \cap B(x^+(I_{w1}), \delta_k) \text{ and } I_u = E_k \cap B(x^-(I_u), \delta_k),$$

we find that, if $|G| < \varepsilon_k$, then

$$\mu(I_{w1}) \sim \mu(I_u). \tag{17}$$

Arguing as we just did, if $|G'| < \varepsilon_k$, then

$$\mu(I_{w2}) \sim \mu(I_v). \tag{18}$$

Now we apply (17) and (18) to prove $\mu(I_{w1}) \sim \mu(I_{w2})$. We consider nine subcases.

Subcase 1. $|G| \ge \varepsilon_k$ and $|G'| \ge \varepsilon_k$.

Since $\varepsilon_k \ge \delta_k$ has been assumed, we have

$$I_{w1} = E_k \cap B(x^+(I_{w1}), \varepsilon_k) \text{ and } I_{w2} = E_k \cap B(x^-(I_{w2}), \varepsilon_k),$$

and so $\mu(I_{w1}) \sim \mu(I_{w2})$.

Subcase 2. $|G| < \delta_k$ and $|G'| < \delta_k$.

In view of (17) and (18), $\mu(I_u) \sim \mu(I_{w1})$ and $\mu(I_v) \sim \mu(I_{w2})$. Since $|G'| < \delta_k \leq \varepsilon_k$, we have $n_k = 2$, as we have said. We see that the gap next to the right of I_v belongs to $\mathcal{G}_k$. Then we have

$$\mu(I_{w1}) + \mu(I_u) = \mu(B(x^+(I_u), \varepsilon_k + \delta_k))$$

and

$$\mu(I_{w2}) + \mu(I_v) = \mu(B(x^-(I_v), \varepsilon_k + \delta_k)),$$

and so $\mu(I_{w1}) \sim \mu(I_{w2})$.

Subcase 3. $\delta_k \leq |G| < \varepsilon_k$ and $|G'| \geq \varepsilon_k$.

In view of (17), $\mu(I_{w1}) \sim \mu(I_u)$. Observing that

$$I_{w1} \subseteq E_k \cap B(x^+(I_{w1}), \varepsilon_k) \subseteq I_u \cup I_{w1} \text{ and } I_{w2} = E_k \cap B(x^-(I_{w2}), \varepsilon_k),$$

we get $\mu(I_{w1}) \sim \mu(I_{w2})$.

Subcase 4. $\delta_k \leq |G'| < \varepsilon_k$ and $|G| \geq \varepsilon_k$.

Arguing as we just did in Case 3, we have $\mu(I_{w1}) \sim \mu(I_{w2})$.

Subcase 5. $\delta_k \leq |G| < \varepsilon_k$ and $|G'| < \delta_k$.

We have $\mu(I_u) \sim \mu(I_{w1})$ and $\mu(I_v) \sim \mu(I_{w2})$. We also have

$$I_{w1} \subseteq E_k \cap B(x^-(I_{w1}), \varepsilon_k + \delta_k) \subseteq I_u \cup I_{w1}$$

and

$$\mu(I_{w2}) + \mu(I_v) = \mu(B(x^-(I_v), \varepsilon_k + \delta_k)),$$

and so $\mu(I_{w1}) \sim \mu(I_{w2})$.

Subcase 6. $\delta_k \leq |G'| < \varepsilon_k$ and $|G| < \delta_k$.

We get $\mu(I_{w1}) \sim \mu(I_{w2})$ by an argument analogous to Case 5.

Subcase 7. $\delta_k \leq |G| < \varepsilon_k$ and $\delta_k \leq |G'| < \varepsilon_k$.

We have $\mu(I_u) \sim \mu(I_{w1})$ and $\mu(I_v) \sim \mu(I_{w2})$. We also have

$$I_{w1} \subseteq E_k \cap B(x^-(I_{w1}), \varepsilon_k + \delta_k) \subseteq I_u \cup I_{w1}$$

and

$$I_{w2} \subseteq E_k \cap B(x^+(I_{w2}), \varepsilon_k + \delta_k) \subseteq I_v \cup I_{w2},$$

giving $\mu(I_{w1}) \sim \mu(I_{w2})$.

Subcase 8. $|G| < \delta_k$ and $|G'| \geq \varepsilon_k$.

We have $\mu(I_u) \sim \mu(I_{w1})$. Noting that

$$I_{w2} = E_k \cap B(x^-(I_{w2}), \varepsilon_k) \text{ and } I_{w1} \subseteq E_k \cap B(x^+(I_{w1}), \varepsilon_k) \subseteq I_u \cup I_{w1},$$

we get $\mu(I_{w1}) \sim \mu(I_{w2})$.

Subcase 9. $|G'| < \delta_k$ and $|G| \geq \varepsilon_k$.
Arguing as in Case 8, we have $\mu(I_{w1}) \sim \mu(I_{w2})$.

Case 2. $\varepsilon_k < \delta_k$, $n_{k+1} = 2$, and $\varepsilon_{k+1} \geq \delta_{k+1}$.

We consider two subcases: $\varepsilon_k < \delta_{k+1}$ and $\varepsilon_k \geq \delta_{k+1}$. When $\varepsilon_k < \delta_{k+1}$, one has $\mu(I_{w12}) \sim \mu(I_{w21})$ by comparing the sizes of $B(x^-(I_{w12}), \delta_{k+1})$ and $B(x^+(I_{w21}), \delta_{k+1})$ in measure μ, which, combined with an observation that $I_{w1} \subset B(x^-(I_{w12}), 2\varepsilon_{k+1})$, yields

$$\mu(I_{w1}) \leq C\mu(B(x^-(I_{w12}), \varepsilon_{k+1})) \leq C(\mu(I_{w12}) + \mu(I_{w21})) \leq C\mu(I_{w2}).$$

As $\mu(I_{w2}) \leq C\mu(I_{w1})$ may be obtained similarly, we have $\mu(I_{w1}) \sim \mu(I_{w2})$. When $\varepsilon_k \geq \delta_{k+1}$, since $3\varepsilon_{k+1} \geq \delta_k > \varepsilon_k$ by the assumptions of Case 2, one also has $\mu(I_{w12}) \sim \mu(I_{w21})$ by comparing $B(x^-(I_{w12}), r_k)$ and $B(x^+(I_{w21}), r_k)$, where $r_k = \min\{\varepsilon_k, \varepsilon_{k+1}\}$. Therefore, arguing as in the previous subcase gives $\mu(I_{w1}) \sim \mu(I_{w2})$.

Case 3. $\varepsilon_k < \delta_k$, $n_{k+1} = 2$, and $\varepsilon_{k+1} < \delta_{k+1}$.

Case 4. $\varepsilon_k < \delta_k$ and $n_{k+1} > 2$.

For Cases 3 and 4, the proof of $\mu(I_{w1}) \sim \mu(I_{w2})$ is the same as that of Claim 2 in Section 2. We omit it. □

Based on Lemma 1, *Claims B–F* can be proved in the same way as *Claims 2–6* of Section 2. This completes the proof of the 'only if' part. □

For the proof of the 'if' part, suppose that the sequence $\{P_w\}_{w\in W}$ satisfies the conditions of Theorem 2. We have to prove that μ is doubling on the uniform Cantor set E. Arguing as we did in the proof of Theorem 1, the doubling property of μ can be proved similarly, and so it is omitted here.

4. Proof of Theorem 3

Let $E = E(\mathbf{n}, \mathbf{c}, \Lambda, \{m_k\}_{k\in\Lambda}, \{s_k\}_{k\in\Lambda})$ be a uniform Cantor set. The proof of Theorem 3 is based on the following lemmas.

Lemma 2. *If E carries a doubling measure which can be extended to a doubling measure on $[0, 1]$, then $\{n_k c_k\}_{k=1}^{\infty} \in \cup_{q\geq 1} \ell^q$.*

Proof. If $\{n_k c_k\}_{k=1}^{\infty} \notin \cup_{q\geq 1} \ell^q$, then we have $\nu(E) = 0$ for all doubling measures ν on $[0, 1]$ (see [5]), a contradiction. □

Lemma 3. *Suppose that E carries a doubling measure which can be extended to a doubling measure on $[0, 1]$ and that $\sup_{k\geq 1} n_k < \infty$. Then $\sup_{k\in\Lambda} m_k = \infty$.*

Proof. If $\sup_{k\in\Lambda} m_k < \infty$, then there is an integer N such that $\varepsilon_k \geq \delta_{k+N}$ for all $k \geq 1$, i.e.

$$\prod_{i=k}^{k+N} \frac{1 - (n_i - 1)c_i}{n_i} \leq c_k.$$

It follows that

$$(\frac{1-\sup_{i\geq 1}(n_i-1)c_i}{\sup_{i\geq 1} n_i})^{N+1} \leq c_k \tag{19}$$

for all $k \geq 1$. Since E carries a doubling measure which can be extended to a doubling measure on $[0,1]$, we have from Lemma 2 that $\{n_k c_k\}_{k=1}^{\infty} \in \cup_{q\geq 1}\ell^q$. This implies $\sup_{k\geq 1}(n_k-1)c_k < 1$, which, together with (19) and the assumption $\sup_{k\geq 1} n_k < \infty$, gives $\inf_{k\geq 1} c_k > 0$. This implies that E is porous, so we have $\nu(E) = 0$ for all doubling measures ν on $[0,1]$ (see [18]), a contradiction. □

Proof of Theorem 3. Suppose $\{n_k c_k\}_{k=1}^{\infty} \in \ell^1$. Then $\mathcal{L}(E) > 0$ by (4). Since $\mathbb{P}$ is an ultimately 1-uniform **n**-matching probability sequence, there is an integer $k_0 \geq 1$ such that $p_{k,i} = \frac{1}{n_k}$ for all $k \geq k_0$ and $1 \leq i \leq n_k$. By this, we have

$$\mu_{\mathbb{P}} = \sum_{I\in\mathcal{I}_{k_0-1}} \frac{\mu_{\mathbb{P}}(I)}{\mathcal{L}(E\cap I)}\mathcal{L}|_{E\cap I}.$$

Let

$$\nu = \sum_{I\in\mathcal{I}_{k_0-1}} \frac{\mu_{\mathbb{P}}(I)}{\mathcal{L}(E\cap I)}\mathcal{L}|_I + \sum_{J\in\mathcal{G}_{k_0-1}} \mathcal{L}|_J.$$

Since $\mathcal{I}_{k_0-1} \cup \mathcal{G}_{k_0-1}$ forms a partition of $[0,1]$ by finite intervals and the restrictions of the measure ν to these intervals are Lebesgue, we easily see that ν is doubling on $[0,1]$. Moreover, since $E\cap(\cup_{G\in\mathcal{G}_{k_0-1}} G) = \emptyset$, we have $\mu_{\mathbb{P}} = \nu|_E$. This proves that $\mu_{\mathbb{P}}$ can be extended to a doubling measure on $[0,1]$.

Conversely, suppose that $\mu_{\mathbb{P}}$ can be extended to a doubling measure ν on $[0,1]$. Then we have from Lemma 2 that $\{n_k c_k\}_{k=1}^{\infty} \in \ell^q$ for some $q \geq 1$, which yields $\lim_{k\to\infty} n_k c_k = 0$. We are going to show $\{n_k c_k\}_{k=1}^{\infty} \in \ell^1$.

Since $\lim_{k\to\infty} n_k c_k = 0$ and $\mathbb{P}$ is ultimately 1-uniform, we may choose an integer $k_0 \geq 1$ such that

$$n_k c_k < 1/3 \text{ for all } k \geq k_0 \tag{20}$$

and that

$$p_{k,i} = \frac{1}{n_k} \text{ for all } k \geq k_0 \text{ and } 1 \leq i \leq n_k. \tag{21}$$

Therefore $\{k : k \geq k_0\} \subseteq \Lambda$, where Λ is as in (8). Let $k \geq k_0$ be fixed. By the definitions of integers m_k and s_k, we have

$$s_k\delta_{k+m_k} + (s_k-1)\varepsilon_{k+m_k} \leq \varepsilon_k < (s_k+1)\delta_{k+m_k} + s_k\varepsilon_{k+m_k}. \tag{22}$$

Since $\varepsilon_k = c_k\delta_{k-1}$ by the construction of the uniform Cantor set E, we see from (3) that the right-hand inequality of (22) can be rewritten as

$$c_k < \frac{(s_k+1)(1-(n_{k+m_k}-1)c_{k+m_k}) + s_k n_{k+m_k} c_{k+m_k}}{n_{k+m_k}} \prod_{i=k}^{k+m_k-1} \frac{1-(n_i-1)c_i}{n_i}.$$

Therefore

$$n_k c_k \leq \frac{3s_k}{n_{k+1}n_{k+2}\cdots n_{k+m_k}}. \tag{23}$$

On the other hand, let $w \in W_{k-1}$, $1 \leq i < n_k$, and let $G_{wi} \in \mathcal{G}_k$ be the corresponding gap of level k. Then $|G_{wi}| = \varepsilon_k$. Let $J = [x^-(G_{wi}) - \varepsilon_k, x^-(G_{wi})]$ be an interval, where $x^-(G_{wi})$ is the left endpoint. We see from the left-hand inequality of (22) that J contains at least s_k components of level $k + m_k$ in I_{wi}. Since ν is an extension of $\mu_{\mathbb{P}}$ and ν is doubling on $[0, 1]$, we get from (21) that

$$\nu(G_{wi}) \geq C\nu(J) \geq \frac{Cs_k \mu_{\mathbb{P}}(I_{wi})}{n_{k+1} \cdots n_{k+m_k}}.$$

Now, summing over all $1 \leq i < n_k$ yields

$$\nu(\bigcup_{i=1}^{n_k-1} G_{wi}) \geq \frac{Cs_k}{n_{k+1} \cdots n_{k+m_k}} \mu_{\mathbb{P}}(\bigcup_{i=1}^{n_k-1} I_{wi}). \tag{24}$$

Since $\mathbb{P}$ is ultimately 1-uniform, we have $\mu_{\mathbb{P}}(\cup_{i=1}^{n_k-1} I_{wi}) \sim \mu_{\mathbb{P}}(I_w)$. Summing over all $w \in W_{k-1}$, we get from (23) that

$$\nu(\bigcup_{J \in \mathcal{G}_k} J) \geq \frac{Cs_k}{n_{k+1} \cdots n_{k+m_k}} \geq Cn_k c_k.$$

Finally, summing over all $k \geq k_0$ yields

$$\infty > \nu([0,1]) \geq \sum_{k=k_0}^{\infty} \nu(\bigcup_{J \in \mathcal{G}_k} J) \geq C \sum_{k=k_0}^{\infty} n_k c_k.$$

This proves $\{n_k c_k\} \in \ell^1$. □

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