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A unified framework for parabolic equations with mixed boundary conditions and diffusion on interfaces ☆

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ABSTRACT

In this paper we consider scalar parabolic equations in a general non-smooth setting emphasizing interface conditions and mixed boundary conditions. In particular, we study dynamics and diffusion on a Lipschitz interface and on the boundary, where the diffusion coefficients are only assumed to be bounded, measurable and positive semidefinite. In the bulk, we consider diffusion coefficients which may degenerate towards a Lipschitz surface. For this problem class, we introduce a unified functional analytic framework based on sesquilinear forms and show maximal L^p-regularity and bounded H^∞-calculus for the corresponding operator, providing well-posedness for a large class of initial conditions and external forces.

1. Introduction

This paper presents a unified framework for a general class of linear inhomogeneous mixed initial–boundary value problems of the form

$$\zeta\partial_t u - \operatorname{div}(\mu_\Omega \nabla u) = f_\Omega \qquad \text{in } J \times (\Omega \setminus \Sigma), \tag{1.1}$$

$$u = 0 \qquad \text{on } J \times \Gamma_D, \tag{1.2}$$

$$\nu \cdot \mu_\Omega \nabla u = 0 \qquad \text{on } J \times \Gamma_N, \tag{1.3}$$

$$\zeta\partial_t u - \operatorname{div}_{\Gamma_\mathrm{d}}(\mu_{\Gamma_\mathrm{d}} \nabla_{\Gamma_\mathrm{d}} u) + \nu \cdot \mu_\Omega \nabla u = f_{\Gamma_\mathrm{d}} \qquad \text{on } J \times \Gamma_\mathrm{d}, \tag{1.4}$$

$$\zeta\partial_t u - \operatorname{div}_\Sigma(\mu_\Sigma \nabla_\Sigma u) + [\nu_\Sigma \cdot \mu_\Omega \nabla u] = f_\Sigma \qquad \text{on } J \times \Sigma, \tag{1.5}$$

$$u(0) = u_0 \qquad \text{in } (\Omega \setminus \Sigma) \times \Gamma_\mathrm{d} \times \Sigma. \tag{1.6}$$

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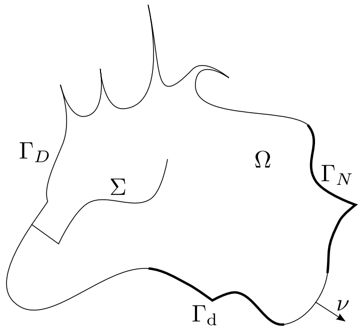


Fig. 1. Example of a domain Ω with interface Σ and boundary $\partial\Omega = \Gamma_D \dot\cup \Gamma_N \dot\cup \Gamma_{\mathrm{d}}$.

Here $J = (0, T)$ is a time interval and $\Omega \subset \mathbb{R}^d$ is a bounded domain with boundary $\partial\Omega$ and with outer unit normal vector field ν. The boundary is disjointly decomposed into a closed Dirichlet part Γ_D, a Neumann part Γ_N and a dynamic part Γ_{d} (see Fig. 1), i.e.,

$$\partial\Omega = \Gamma_D \dot\cup \Gamma_N \dot\cup \Gamma_{\mathrm{d}}.$$

Moreover, $\Sigma \subset \Omega$ is a $(d-1)$-dimensional hypersurface with unit normal vector field ν_Σ, on which a further dynamic condition is imposed, and $[\nu_\Sigma \cdot \mu_\Omega \nabla u]$ denotes the jump of $\nu_\Sigma \cdot \mu_\Omega \nabla u$ across Σ. The surface gradients on Γ_{d} and on Σ are denoted by $\nabla_{\Gamma_{\mathrm{d}}}$ and ∇_Σ. Accordingly, we write $\mathrm{div}_{\Gamma_{\mathrm{d}}}$ and div_Σ for the surface divergences, such that $\Delta_{\Gamma_{\mathrm{d}}} = \mathrm{div}_{\Gamma_{\mathrm{d}}} \nabla_{\Gamma_{\mathrm{d}}}$ and $\Delta_\Sigma = \mathrm{div}_\Sigma \nabla_\Sigma$ are the Laplace–Beltrami operators. The diffusion coefficients μ_Ω, $\mu_{\Gamma_{\mathrm{d}}}$ and μ_Σ are matrix-valued, and the relaxation coefficient ζ is positive, bounded, and bounded away from zero. The external forces f_Ω, $f_{\Gamma_{\mathrm{d}}}$ and f_Σ as well as the initial data u_0 are assumed to be given. Initial data have to be prescribed at $\Omega \setminus \Sigma$, Γ_{d} and Σ due to the corresponding dynamic equations on these sets.

Well-posedness and qualitative properties of parabolic problems with dynamic boundary conditions are well-studied, see for example [3,5,6,10,11,14,16,17,23–25,35,44,46,47]. Here, mostly the case of a smooth boundary is considered. Nonlinear degeneracy in the diffusion is treated in [3,16,25]. Mixed boundary conditions on non-smooth domains and dynamical Robin conditions are also treated in [38,39], in a setting which may include inhomogeneities in the Neumann or Dirichlet parts. Mixed Dirichlet–Wentzell boundary conditions with a smooth Wentzell boundary are treated in [47].

Here, we define and study surface diffusion on Lipschitz boundaries and interfaces with diffusion coefficients which may degenerate arbitrarily, and establish a framework in which we allow the bulk diffusion coefficients to degenerate moderately towards another Lipschitz hypersurface. In addition, we take into account mixed boundary conditions nonsmooth diffusion and relaxation coefficients. In particular, we generalize the results in [10]. Inhomogeneous Neumann boundary conditions, as well as boundary parts and interfaces evolving in time are not included in our approach, compare [38].

We say that diffusion is “degenerate”, if the coefficient matrices μ_Ω, $\mu_{\Gamma_{\mathrm{d}}}$, μ_Σ are not strongly elliptic. In fact, we only require $\mu_{\Gamma_{\mathrm{d}}}$, μ_Σ to be non-negative, and thus surface diffusion may be absent or degenerate in a very general sense. The bulk coefficient matrix μ_Ω may also degenerate but must still imply bulk regularity of the solution which allows for a trace function at Γ_{d}, Σ. Examples of this situation are given below.

We present a unified setting based on recent abstract results for sesquilinear forms from [4], which handles all these nonsmooth scenarios and their combinations at once.

Let us give more details on the assumptions for the geometry and the coefficients. The boundary parts Γ_D, Γ_N and Γ_{d} are allowed to meet, and also the interface Σ may meet any of the boundary parts Γ_D, Γ_N, Γ_{d}. Except at points close to the remainder of $\partial\Omega$, no conditions on the Dirichlet part Γ_D are imposed.

The diffusion coefficients μ_Ω, $\mu_{\Gamma_{\mathrm{d}}}$ and μ_Σ do not have to be symmetric and are assumed to be measurable, bounded and non-negative. To describe their degeneracies in a precise way, we assume pointwise estimates of the form

$$(\mu(x)\xi,\xi) \geq c_1\mu^*(x)|\xi|^2, \quad \xi \in \mathbb{R}^d, \qquad \|\mu(x)\|_{\mathcal{L}(\mathbb{R}^d)} \leq c_2\mu^*(x),$$

where μ stands for μ_Ω, μ_{Γ_d} or μ_Σ, respectively, and μ^* is in each case a measurable, bounded and non-negative function. Regarding surface diffusion, we may allow for *arbitrary* supports of $\mu^*_{\Gamma_\mathrm{d}}$ and μ^*_Σ. This is to be expected as the well-posedness of equations (1.1)–(1.6) should not depend on the presence of surface diffusion. However, it is a considerable part of our work to give a suitable definition of surface gradients which captures the exact presence of diffusion on arbitrary subsets of the surface, which may still yield regularization where diffusion is present and which still allows us to show maximal regularity of the abstract Cauchy problem.

Concerning bulk diffusion, our setting is naturally more restrictive and we only consider a class of examples of degenerate diffusion. For the function μ^*_Ω, we assume that

$$\mu^*_\Omega(x) = \operatorname{dist}(x,S)^\gamma, \qquad x \in \Omega, \tag{1.7}$$

where $S \subset \overline{\Omega}$ is an arbitrary $(d-k)$-dimensional Lipschitz submanifold of $\mathbb{R}^d$, $1 \leq k \leq d$, and the exponent is in the range $0 < \gamma < k$, which makes μ^*_Ω a Muckenhoupt weight of class $\mathcal{A}_2$. We are motivated by the study of damaged materials or materials with cracks and hope that our assumptions on diffusion coefficients and the general setting may be helpful in this context. Of particular interest is the case when $S \cap (\overline{\Gamma_\mathrm{d} \cup \Sigma}) \neq \emptyset$, i.e., when diffusion degenerates towards Γ_d or Σ, but may or may not occur along Γ_d or Σ. In general, in this case we will have to assume that $\gamma < 1$.

We describe the setting in which (1.1)–(1.6) is realized. The basis of the approach is the sesquilinear form

$$\mathfrak{t}(u,v) = \int_\Omega (\mu_\Omega \nabla u, \overline{\nabla v})\,dx + \int_{\Gamma_\mathrm{d}} \big(\mu_{\Gamma_\mathrm{d}} \nabla_{\Gamma_\mathrm{d}} u, \overline{\nabla_{\Gamma_\mathrm{d}} v}\big)\,d\mathcal{H}_{d-1} + \int_\Sigma \big(\mu_\Sigma \nabla_\Sigma u, \overline{\nabla_\Sigma v}\big)\,d\mathcal{H}_{d-1}, \tag{1.8}$$

where $\mathcal{H}_{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure. The surface gradients $\nabla_{\Gamma_\mathrm{d}}$ and ∇_Σ on the Lipschitz surfaces Γ_d and Σ are introduced in a simple, straightforward way in terms of local coordinates, such that the definitions coincide with the corresponding well-known objects in a smooth situation (see Section 3). In order to obtain a suitable weak formulation of (1.1)–(1.6), we define the domain of the form $\mathfrak{t}$ as the completion of

$$C^\infty_D(\Omega) := \big\{u|_\Omega : u \in C^\infty_c(\mathbb{R}^d),\ (\operatorname{supp} u) \cap \Gamma_D = \emptyset\big\},$$

with respect to

$$\|u\|^2_{\mathrm{Dom}(\mathfrak{t})} := \|u\|^2_{W^{1,2}(\Omega,\mu^*_\Omega)} + \|\nabla_{\Gamma_\mathrm{d}} u\|^2_{L^2(\Gamma_\mathrm{d},\mu^*_{\Gamma_\mathrm{d}})} + \|\nabla_\Sigma u\|^2_{L^2(\Sigma,\mu^*_\Sigma)}.$$

Here, $W^{1,2}(\Omega,\mu^*_\Omega)$ is a Sobolev space with weight μ^*_Ω in the gradient norm, and $L^2(\Gamma_\mathrm{d},\mu^*_{\Gamma_\mathrm{d}})$ and $L^2(\Sigma,\mu^*_\Sigma)$ are Lebesgue spaces equipped with the weights $\mu^*_{\Gamma_\mathrm{d}}$ and μ^*_Σ.

Based on the results of [4], to the form $\mathfrak{t}$, we associate an operator A_2 on the Lebesgue space

$$\mathbb{L}^2 = L^2\big((\Omega \setminus \Sigma) \cup \Gamma_\mathrm{d} \cup \Sigma, (dx + d\mathcal{H}_{d-1})\big) = L^2(\Omega \setminus \Sigma) \oplus L^2(\Gamma_\mathrm{d}) \oplus L^2(\Sigma).$$

In order to realize this setting, one must make sure that for every $v \in \mathrm{Dom}(\mathfrak{t})$, there are traces $\operatorname{tr}_\Sigma v \in L^2(\Sigma)$ and $\operatorname{tr}_{\Gamma_\mathrm{d}} v \in L^2(\Gamma_\mathrm{d})$ such that we obtain a triple $(v, v_\Sigma, v_{\Gamma_\mathrm{d}}) \in \mathbb{L}^2$, where here and in the following, we often use the notation $v_\mathcal{M}$ to indicate the restriction or trace of v on a set $\mathcal{M}$ if it is well-defined. The constitutive relation for $A_2 u$ is then given by

$$\langle A_2 u, (\phi_\Omega, \phi_\Sigma, \phi_{\Gamma_\mathrm{d}})\rangle_{\mathbb{L}^2} = \mathfrak{t}(u,\phi), \tag{1.9}$$

for all test functions $\phi \in C^\infty_D(\Omega)$.

If bulk diffusion degenerates towards Γ_{d} or Σ as in (1.7), we rely on the weighted Sobolev embedding

$$W^{1,2}(\mathbb{R}^d, \mathrm{dist}(\cdot, S)^\gamma) \subset W^{\theta,q}(\mathbb{R}^d), \qquad 1 - \frac{d+\gamma}{2} \geq \theta - \frac{d}{q}, \qquad q \geq 2,$$

which seems to be new in this explicit form and is deduced from the very general embedding results in [20] (see Proposition 5.3 and [1,42] for related results about traces of Muckenhoupt weighted spaces). Here, $W^{\theta,q}(\mathbb{R}^d)$ denotes the usual Slobodetskii space.

It turns out that $-A_2$ generates an analytic C_0-semigroup $T_2(\cdot)$ of contractions on $\mathbb{L}^2$, see Proposition 4.7. This already yields the solvability of our realization of (1.1)–(1.6) for external forces $(f_\Omega, f_{\Gamma_{\mathrm{d}}}, f_\Sigma)$ in $L^2(J; \mathbb{L}^2)$ and initial data $u_0 \in \mathbb{L}^2$. We emphasize that the components of the initial data need not be related, but that the semigroup regularizes to $u(t) \in \mathrm{Dom}(\mathfrak{t})$ for all $t > 0$.

In order to treat semilinear problems, $\mathbb{L}^2$-estimates of the solution will in general not be sufficient, due to the lack of embeddings for the fractional power domains of A_2 into spaces of bounded functions. Thus, we first extend the definition of A_2 consistently to the whole $\mathbb{L}^p$-scale, $p \in [1, \infty]$. This is achieved by showing that $T_2(\cdot)$ is $\mathbb{L}^\infty$-contractive (see Proposition 4.8), which implies the existence of a consistent contraction semigroup $T_p(\cdot)$ on $\mathbb{L}^p$ by interpolation and duality. For $p \in (1, \infty)$, the negative generator A_p of the analytic semigroup $T_p(\cdot)$ is then the desired consistent extension/restriction of A_2 to $\mathbb{L}^p$. The analyticity of $T_p(\cdot)$ for $p \in (1, \infty)$ together with the contractivity of $T_p(\cdot)$ for $p \in [1, \infty]$ now allows us to apply a deep result from harmonic analysis due to [9,28,31,32,48] (see also [33, Proposition 2.2]) to conclude that A_p admits a bounded holomorphic functional calculus and maximal Lebesgue regularity (see [8,30,41] for surveys on these topics).

Hence, from an abstract point of view, the realization is as good as it can be, despite of the variety of nonsmooth effects it takes into account. The precise formulation is given in Theorems 4.11 and 5.7. Employing again that A_p is given on a scalar L^p-space, we show that the multiplication with the inverse relaxation coefficient ζ^{-1} does not change the described properties. Finally, embeddings of the type

$$\mathrm{Dom}(A_p^\theta) \subset \mathbb{L}^\infty, \tag{1.10}$$

for $p > 2$ sufficiently large and θ sufficiently close to 1 are obtained in Section 6 from semigroup estimates and an integral formula for negative fractional powers of A_p. We can quantify how the presence of surface diffusion may improve (1.10), whereas degeneracy in the bulk diffusion may clearly decrease the integrability exponent. It is an advantage of our unified framework that we can see how these effects may interact locally. In essence, we restrict our considerations to the linear case in this paper, and refer e.g. to [22, Ch. 2], [34] for results on how embeddings of type (1.10) quantify the solvability of related semilinear problems.

This paper is organized as follows. We start in Section 2 with a heuristic of how our functional analytic setting is related to (1.1)–(1.6). In Section 3, we introduce tangent spaces and the surface gradient for Lipschitz hypersurfaces in graph representation. In order to separate technical difficulties, in Section 4 we consider the case of nondegenerate bulk diffusion only, while in Section 5 we treat degenerate bulk diffusion. In Section 6, embeddings of fractional power domains into spaces of bounded functions are investigated.

Notation. Generic positive constants are denoted by C or c. By $\mathcal{L}(\mathbb{R}^d)$ we designate the space of linear operators on $\mathbb{R}^d$, which we may identify with the set of $(d \times d)$-matrices via the canonical basis. The Euclidian scalar product of $x, y \in \mathbb{R}^d$ is denoted by $x \cdot y$ or (x, y). For $p \in [1, \infty]$, the usual complex Lebesgue space is denoted by $L^p(\Omega)$.

2. Heuristics

Since the form method in [4] is very recent and presently not commonly known we give a detailed heuristics why the definition of the form $\mathfrak{t}$, together with the relation (1.9) provides the adequate functional analytic setting for equations (1.1)–(1.6). This is closely related to the classical arguments for weak formulations of boundary value problems, cf. for example [15, Ch. II.2]. In this section, we make additional regularity assumptions. Let Ω be a smooth domain and let Σ be extendible to a Lipschitz hypersurface $\Lambda = \overline{\Sigma} \cup (\Lambda \setminus \Sigma)$ which cuts Ω into two Lipschitz subdomains $\Omega = \Omega_+ \cup \Lambda \cup \Omega_-$. Let ν_Σ denote the outer normal vector field of Ω_+ at all of Λ. Assume that the equation

$$A_2 u = f \tag{2.1}$$

is satisfied in $\mathbb{L}^2$ and let $\phi \in C_D^\infty(\Omega)$ with the canonical embedding $(\phi_\Omega, \phi_\Sigma, \phi_{\Gamma_\mathrm{d}}) \in \mathbb{L}^2$. Then by definition,

$$\langle f, \phi \rangle_{\mathbb{L}^2} = \int_\Omega f \overline{\phi}\, dx + \int_\Sigma f_\Sigma \overline{\phi_\Sigma}\, d\mathcal{H}_{d-1} + \int_{\Gamma_\mathrm{d}} f_{\Gamma_\mathrm{d}} \overline{\phi_{\Gamma_\mathrm{d}}}\, d\mathcal{H}_{d-1}, \tag{2.2}$$

and

$$\langle A_2 u, \phi \rangle_{\mathbb{L}^2} = \int_\Omega (\mu_\Omega \nabla u, \overline{\nabla \phi})\, dx + \int_{\Gamma_\mathrm{d}} \big(\mu_{\Gamma_\mathrm{d}} \nabla_{\Gamma_\mathrm{d}} u, \overline{\nabla_{\Gamma_\mathrm{d}} \phi}\big)\, d\mathcal{H}_{d-1} + \int_\Sigma \big(\mu_\Sigma \nabla_\Sigma u, \overline{\nabla_\Sigma \phi}\big)\, d\mathcal{H}_{d-1}. \tag{2.3}$$

Now we additionally assume that the restrictions u_+ and u_- of u to Ω_+ and Ω_- satisfy $u_+ \in C^1(\overline{\Omega_+})$ and $u_- \in C^1(\overline{\Omega_-})$ and that on $\Omega \setminus \Sigma$, we have $u \in C^2(\Omega \setminus \Sigma)$. We note that

$$\int_\Omega (\mu_\Omega \nabla u, \overline{\nabla v})\, dx = \int_{\Omega_+} (\mu_\Omega \nabla u_+, \overline{\nabla \phi_+})\, dx + \int_{\Omega_-} (\mu_\Omega \nabla u_-, \overline{\nabla \phi_-})\, dx$$

and apply Gauss' Theorem to each of these terms to get

$$\begin{aligned}
\int_\Omega (\mu_\Omega \nabla u, \overline{\nabla v})\, dx = {} & \int_{\Omega_+} -\mathrm{div}(\mu_\Omega \nabla u_+) \overline{\nabla \phi_+}\, dx + \int_{\Omega_-} -\mathrm{div}(\mu_\Omega \nabla u_-) \overline{\phi_-}\, dx \\
& + \int_{\Gamma_N} (\nu \cdot \mu_\Omega \nabla u) \overline{\phi_{\Gamma_N}}\, d\mathcal{H}_{d-1} + \int_{\Gamma_\mathrm{d}} (\nu \cdot \mu_\Omega \nabla u) \overline{\phi_{\Gamma_d}}\, d\mathcal{H}_{d-1} \\
& + \int_\Sigma [\nu_\Sigma \cdot \mu_\Omega \nabla u] \overline{\phi_\Sigma}\, d\mathcal{H}_{d-1} + \int_{\Lambda \setminus \Sigma} [\nu_\Sigma \cdot \mu_\Omega \nabla u] \overline{\phi_{\Lambda \setminus \Sigma}}\, d\mathcal{H}_{d-1},
\end{aligned}$$

where it follows from the regularity assumptions on u that the last term vanishes. Additionally applying the manifold Gauss Theorem, cf. [36] for a non-smooth version, to the last two integrals in (2.3), we derive the expression

$$\begin{aligned}
\langle A_2 u, \phi) \rangle_{\mathbb{L}^2} = {} & \int_\Omega -\mathrm{div}(\mu_\Omega \nabla u) \overline{\phi}\, dx + \int_{\Gamma_N} (\nu \cdot \mu_\Omega \nabla u) \overline{\phi_{\Gamma_N}}\, d\mathcal{H}_{d-1} \\
& + \int_{\Gamma_\mathrm{d}} (\nu \cdot \mu_\Omega \nabla u) \overline{\phi_{\Gamma_\mathrm{d}}}\, d\mathcal{H}_{d-1} + \int_\Sigma [\nu_\Sigma \cdot \mu_\Omega \nabla u] \overline{\phi_\Sigma}\, d\mathcal{H}_{d-1} \\
& + \int_{\Gamma_\mathrm{d}} -\mathrm{div}_{\Gamma_\mathrm{d}} (\mu_{\Gamma_\mathrm{d}} \nabla_{\Gamma_\mathrm{d}} u) \overline{\phi_{\Gamma_\mathrm{d}}}\, d\mathcal{H}_{d-1} + \int_\Sigma -\mathrm{div}_\Sigma (\mu_\Sigma \nabla_\Sigma u) \overline{\phi_\Sigma}\, d\mathcal{H}_{d-1} \\
& + \int_{\partial \Gamma_\mathrm{d}} (\nu_{\partial \Gamma_\mathrm{d}} \cdot \mu_{\Gamma_\mathrm{d}} \nabla_{\Gamma_\mathrm{d}} u_{\Gamma_\mathrm{d}}) \overline{\phi_{\partial \Gamma_\mathrm{d}}}\, d\mathcal{H}_{d-2} + \int_{\partial \Sigma} (\nu_{\partial \Sigma} \cdot \mu_\Sigma \nabla_\Sigma u_\Sigma) \overline{\phi_{\partial \Sigma}}\, d\mathcal{H}_{d-2}
\end{aligned} \tag{2.4}$$

to be balanced with (2.2).

Choosing $\phi \in C_c^\infty(\Omega)$ yields

$$f_\Omega = -\mathrm{div}(\mu_\Omega \nabla u) \in L^2(\Omega).$$

The Neumann and Dirichlet boundary conditions on Γ_N and Γ_D follow, for example, as in [15, Ch. II.2], using that each Neumann part of the boundary of Ω satisfies an extension property. The remaining equalities

$$f_{\Gamma_\mathrm{d}} = -\mathrm{div}_{\Gamma_\mathrm{d}}(\mu_{\Gamma_\mathrm{d}} \nabla_{\Gamma_\mathrm{d}} u_{\Gamma_\mathrm{d}}) + \nu \cdot \mu_\Omega \nabla u \in L^2(\Gamma_\mathrm{d})$$

and

$$f_\Sigma = -\mathrm{div}_\Sigma(\mu_\Sigma \nabla_\Sigma u_\Sigma) + [\nu_\Sigma \cdot \mu_\Omega \nabla u] \in L^2(\Sigma)$$

are then identified accordingly. The last two terms in (2.4) require some more explanation. If $\partial\Gamma_\mathrm{d} \cup \partial\Sigma \subset \overline{\Omega} \setminus \Gamma_D$, we consider them to be enforcing (generalized) homogeneous Neumann boundary conditions on $\partial\Gamma_\mathrm{d}$ and $\partial\Sigma$. At points where $\partial\Gamma_\mathrm{d}$ or $\partial\Sigma$ and Γ_D intersect, we assign homogeneous Dirichlet boundary conditions. In particular, in the definition of $C_D^\infty(\Omega)$, any subset of points in $\partial\Gamma_\mathrm{d}$ and $\partial\Sigma$ may be included to enforce these Dirichlet conditions. We did not include these conditions in equations (1.1)–(1.6) to keep the presentation simple and because in general, our regularity assumptions on Γ_d and Σ are insufficient to deduce them in the usual way.

3. The surface gradient on Lipschitz hypersurfaces

In order to define surface diffusion on Σ and Γ_d, in this section we introduce tangent spaces and the surface gradient for a Lipschitz hypersurface $\mathcal{S}$ in graph representation in an elementary way. The idea is that Lipschitz coordinates are differentiable almost everywhere, which allows us to give definitions in coordinates analogous to the smooth case. Hence for smooth $\mathcal{S}$ we automatically recover the standard notions, see [2, Chapter VII] and [21,27] for basic accounts. For Lipschitz surfaces we also refer to [12,19,37,43].

3.1. Lipschitz hypersurfaces

Let $\mathcal{S} \subset \mathbb{R}^d$ be a *Lipschitz hypersurface in graph representation*. This means that for each $x \in \mathcal{S}$ there are *Lipschitz-graph coordinates* (g, U) and an open neighbourhood V of x in $\mathbb{R}^d$ such that $U \subset \mathbb{R}^{d-1}$ is open and $\mathrm{g} : U \to \mathcal{S} \cap V$ is bijective and of the form

$$\mathrm{g}(y) = Q \begin{pmatrix} y \\ h(y) \end{pmatrix} + x^*, \qquad y \in U,$$

where $Q \in \mathcal{L}(\mathbb{R}^d)$ is orthogonal, $x^* \in \mathbb{R}^d$ is a fixed vector and $h : U \to \mathbb{R}$ is Lipschitz continuous. For this and equivalent definitions we refer to [37, Section 2]. We endow $\mathcal{S}$ with the Hausdorff measure $\mathcal{H}_{d-1}$. Employing that the topology of $\mathbb{R}^d$ has a countable basis, standard arguments show that there is an at most countable number of Lipschitz graph coordinates $(\mathrm{g}_\alpha, U_\alpha)$ such that $\mathcal{S} \subseteq \bigcup_\alpha \mathrm{g}_\alpha(U_\alpha)$, see the proof of [37, Theorem 2.15].

By Rademacher's Theorem (see [12, Theorem 3.1.2]), Lipschitz coordinates g are almost everywhere differentiable on U in the classical sense and one has $g \in W^{1,\infty}(U, \mathbb{R}^d)$, where

$$\mathrm{g}'(y) = Q \begin{pmatrix} \mathrm{id}_{d-1} \\ h'(y) \end{pmatrix} \in \mathcal{L}(\mathbb{R}^{d-1}, \mathbb{R}^d)$$

at points $y \in U$ where g is differentiable. Observe that $\mathrm{g}'(y)$ is injective and has rank $d-1$. Hence the corresponding *metric tensor* $\mathrm{G} : U \to \mathcal{L}(\mathbb{R}^{d-1})$, defined by

$$\mathrm{G}(y) = \mathrm{g}'(y)^T \mathrm{g}'(y) = \big((\partial_i \mathrm{g}(y), \partial_j \mathrm{g}(y))\big)_{ij},$$

is for almost all $y \in U$ symmetric and positive definite. With the usual abuse of notation we write $\mathrm{G} = (\mathrm{g}_{ij})_{ij}$, and $\mathrm{G}^{-1} = (\mathrm{g}^{ij})_{ij}$ for the pointwise inverse of G.

We call Lipschitz-graph coordinates g *regular* for $x \in \mathcal{S}$ if g is differentiable at $y = \mathrm{g}^{-1}(x)$. If such regular coordinates exist, we call x regular.

Lemma 3.1. *Let $\mathcal{S}$ be a Lipschitz hypersurface in graph representation. Then $\mathcal{H}_{d-1}$-almost every point $x \in \mathcal{S}$ is regular.*

Proof. Let $N \subset \mathcal{S}$ be the set of points which are not regular. Take at most countable many coordinates $(\mathrm{g}_\alpha, U_\alpha)$ such that $\mathcal{S} \subseteq \bigcup_\alpha V_\alpha$ for $V_\alpha = \mathrm{g}_\alpha(U_\alpha)$. Then $\mathcal{H}_{d-1}(N) \le \sum_\alpha \mathcal{H}_{d-1}(N \cap V_\alpha)$. Let further $N_\alpha \subset U_\alpha$ be the set of points where g_α is not differentiable. Then $\mathcal{H}_{d-1}(N_\alpha) = 0$ by Rademacher's Theorem. Using $N \cap V_\alpha \subseteq \mathrm{g}_\alpha(N_\alpha)$ and [12, Theorem 2.4.1/1], for each α we obtain

$$\mathcal{H}_{d-1}(N \cap V_\alpha) \le \mathcal{H}_{d-1}(\mathrm{g}_\alpha(N_\alpha)) \le \mathrm{Lip}(\mathrm{g}_\alpha)^{d-1} \mathcal{H}_{d-1}(N_\alpha) = 0,$$

where $\mathrm{Lip}(\mathrm{g}_\alpha)$ is the Lipschitz constant of g_α. This shows $\mathcal{H}_{d-1}(N) = 0$. □

As another preparation we consider the properties of *transition maps.*

Lemma 3.2. *Let $(\mathrm{g}_\alpha, U_\alpha)$ and $(\mathrm{g}_\beta, U_\beta)$ be Lipschitz-graph coordinates for $\mathcal{S}$ which are both regular for $x \in \mathcal{S}$. Set $y_\alpha = \mathrm{g}_\alpha^{-1}(x) \in U_\alpha$ and $y_\beta = \mathrm{g}_\beta^{-1}(x) \in U_\beta$. Then the following assertions hold true.*

(a) *The transition map $\mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha$ is differentiable at y_α. The derivative $(\mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha)'(y_\alpha) \in \mathcal{L}(\mathbb{R}^{d-1})$ is invertible with inverse $(\mathrm{g}_\alpha^{-1} \circ \mathrm{g}_\beta)'(y_\beta)$.*
(b) *The derivatives $\mathrm{g}'_\alpha(y_\alpha)$ and $\mathrm{g}'_\beta(y_\beta)$ have the same images in $\mathbb{R}^d$. We have $v = \mathrm{g}'_\alpha(y_\alpha)\xi_\alpha$ for $\xi_\alpha \in \mathbb{R}^{d-1}$ if and only if $v = \mathrm{g}'_\beta(y_\beta)\xi_\beta$ for $\xi_\beta = (\mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha)'(y_\alpha)\xi_\alpha$.*
(c) *For the metric tensors G_α and G_β corresponding to g_α and g_β we have*

$$\mathrm{G}_\alpha(y_\alpha) = (\mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha)'(y_\alpha)^T \mathrm{G}_\beta(y_\beta) (\mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha)'(y_\alpha).$$

Proof. We write $\Phi = \mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha$ for the transition map. Observe that Φ is a homeomorphism on a neighbourhood of y_α with inverse $\Phi^{-1} = \mathrm{g}_\alpha^{-1} \circ \mathrm{g}_\beta$.

(a) The form of g_β shows that $\Phi(y)$ is given by the first $d-1$ entries of $Q_\beta^T(\mathrm{g}_\alpha(y) - x_\beta^*)$. Hence Φ is differentiable at y_α. In the same way we obtain the differentiability of Φ^{-1} at y_β. Therefore $\Phi'(y_\alpha)$ is invertible with inverse as asserted.

(b) This follows from $\mathrm{g}'_\alpha(y_\alpha) = \mathrm{g}'_\beta(y_\beta)\Phi'(y_\alpha)$ and the invertibility of $\Phi'(y_\alpha)$.

(c) We can repeat the short argument from [27, Section 1.4]. For arbitrary $\xi_\alpha, \eta_\alpha \in \mathbb{R}^{d-1}$ we use (b) to obtain

$$\begin{aligned}
(\mathrm{G}_\alpha(y_\alpha)\xi_\alpha, \eta_\alpha) &= \big(\mathrm{g}'_\alpha(y_\alpha)\xi_\alpha, \mathrm{g}'_\alpha(y_\alpha)\eta_\alpha\big) \\
&= \big(\mathrm{g}'_\beta(y_\beta)\Phi'(y_\alpha)\xi_\alpha, \mathrm{g}'_\beta(y_\beta)\Phi'(y_\alpha)\eta_\alpha\big) \\
&= (\Phi'(y_\alpha)^T \mathrm{G}_\beta(y_\beta)\Phi'(y_\alpha)\xi_\alpha, \eta_\alpha).
\end{aligned}$$

This implies the asserted formula. □

3.2. Tangent space and surface gradient

Now we can introduce the following notions.

Definition 3.3. Let $\mathcal{S}$ be a Lipschitz hypersurface in graph representation.

(a) Let $x \in \mathcal{S}$ be regular with Lipschitz graph coordinates (g, U). The *tangent space* at x is

$$T_x\mathcal{S} = \{v \in \mathbb{R}^d : \text{there is } \xi \in \mathbb{R}^{d-1} \text{ with } v = \mathrm{g}'(\mathrm{g}^{-1}(x))\xi\}.$$

(b) A function $u \in L^1_{\mathrm{loc}}(\mathcal{S})$ is called *weakly differentiable*, if for all Lipschitz graph coordinates (g, U) for $\mathcal{S}$ the function $u \circ \mathrm{g}$ is weakly differentiable on $U \subset \mathbb{R}^{d-1}$.

(c) Let $u \in L^1_{\mathrm{loc}}(\mathcal{S})$ be weakly differentiable. Then for a regular point $x \in \mathcal{S}$ the *surface gradient* $\nabla_{\mathcal{S}} u(x) \in T_x\mathcal{S}$ is given by

$$\nabla_{\mathcal{S}} u(x) = \mathrm{g}'(y)\mathrm{G}^{-1}(y)\nabla(u \circ \mathrm{g})(y) = \sum_{i,j=1}^{d-1} \mathrm{g}^{ij}(y)\partial_j(u \circ \mathrm{g})(y)\partial_i \mathrm{g}(y),$$

where (g, U) are arbitrary regular Lipschitz graph coordinates for x and $y = \mathrm{g}^{-1}(x)$.

These notions coincide with the usual ones if $\mathcal{S}$ is smooth, see, e.g., [2, Remark VII.10.11] for the representation of the surface gradient in coordinates. As in the smooth case one shows that these notions are well-defined.

Lemma 3.4. *At a regular point $x \in \mathcal{S}$, the tangent space as well as the surface gradient of a weakly differentiable function are independent of the chosen regular graph coordinates.*

Proof. The assertion for the tangent space follows from Lemma 3.2(b). For the surface gradient we let g_α and g_β be regular for x, set $y_\alpha = \mathrm{g}_\alpha^{-1}(x)$, $y_\beta = \mathrm{g}_\beta^{-1}(x)$ and

$$v_\alpha = \mathrm{g}'_\alpha(y_\alpha)\mathrm{G}_\alpha^{-1}(y_\alpha)\nabla(u \circ \mathrm{g}_\alpha)(y_\alpha), \qquad v_\beta = \mathrm{g}'_\beta(y_\beta)\mathrm{G}_\beta^{-1}(y_\beta)\nabla(u \circ \mathrm{g}_\beta)(y_\beta).$$

As above we write $\Phi = \mathrm{g}_\beta^{-1} \circ \mathrm{g}_\alpha$ for the transition map. By Lemma 3.2(b) we have $v_\alpha = v_\beta$ if and only if

$$\mathrm{G}_\beta^{-1}(y_\beta)\nabla(u \circ \mathrm{g}_\beta)(y_\beta) = \Phi'(y_\alpha)\mathrm{G}_\alpha^{-1}(y_\alpha)\nabla(u \circ \mathrm{g}_\alpha)(y_\alpha).$$

But this is a consequence of the identities

$$\nabla(u \circ \mathrm{g}_\alpha)(y_\alpha) = \Phi'(y_\alpha)^T\nabla(u \circ \mathrm{g}_\beta)(y_\beta), \qquad \mathrm{G}_\beta^{-1}(y_\beta) = \Phi'(y_\alpha)\mathrm{G}_\alpha^{-1}(y_\alpha)\Phi'(y_\alpha)^T,$$

where the latter follows from Lemma 3.2(c). □

4. Non-degenerate bulk diffusion

In this section we consider (1.1)–(1.6) with a uniformly elliptic diffusion coefficient μ_Ω in the bulk. The case when μ_Ω degenerates towards a compact Lipschitz surface is investigated in the next section.

4.1. Assumptions on the geometry and the coefficients

In case $1 \le k \le d-1$, we say that the set $\mathcal{S}$ is a $(d-k)$-dimensional *Lipschitz submanifold* if for all $x \in \mathcal{S}$ there is an open neighbourhood V of x in $\mathbb{R}^d$ and a bi-Lipschitz mapping φ from V to $\mathbb{R}^d$ such that $\varphi(\mathcal{S} \cap V) = \,]0,1[^{d-k} \times \{0_{\mathbb{R}^k}\}$. By a compact 0-dimensional Lipschitz submanifold $\mathcal{S}$ we mean a finite union of points. Throughout the paper, we impose the following.

Assumption 4.1.

(a) $\Omega \subset \mathbb{R}^d$ is a bounded domain, $d \ge 2$.
(b) $\Gamma_{\mathrm{d}} \subset \partial\Omega$ and $\Sigma \subset \Omega$ are Lipschitz hypersurfaces in graph representation. They are endowed with the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}_{d-1}$.
(c) Γ_N is a $(d-1)$-dimensional Lipschitz submanifold of $\mathbb{R}^d$.
(d) Additionally, the closures $\overline{\Gamma_N}$, $\overline{\Gamma_{\mathrm{d}}}$ and $\overline{\Sigma}$ are contained in $(d-1)$-dimensional Lipschitz submanifolds, respectively.

We emphasize that for the Dirichlet part Γ_D, there are only assumptions in a neighbourhood of points where Γ_D meets Γ_N or Γ_{d}. In particular, in the pure Dirichlet case $\Gamma_D = \partial\Omega$ there are no assumptions on the boundary. It is not excluded that one or more of the sets Γ_D, Γ_N, Γ_{d} or Σ are empty.

Assumption 4.2.

(a) The coefficient $\mu_\Omega : \Omega \to \mathcal{L}(\mathbb{R}^d)$ is measurable, bounded and there is a constant $\mu_\Omega^* > 0$ such that

$$\big(\mu_\Omega(x)\xi, \xi\big) \ge \mu_\Omega^* |\xi|^2, \qquad x \in \Omega, \qquad \xi \in \mathbb{R}^d.$$

(b) Let $\mathcal{S}$ be either Γ_{d} or Σ. Then $\mu_{\mathcal{S}} : \mathcal{S} \to \mathcal{L}(\mathbb{R}^d)$ is measurable, and there are a measurable, bounded, nonnegative function $\mu_{\mathcal{S}}^* : \mathcal{S} \to \mathbb{R}$ and constants $c_1, c_2 > 0$ such that

$$\big(\mu_{\mathcal{S}}(x)\xi, \xi\big) \ge c_1 \mu_{\mathcal{S}}^*(x)|\xi|^2, \qquad \|\mu_{\mathcal{S}}(x)\|_{\mathcal{L}(\mathbb{R}^d)} \le c_2 \mu_{\mathcal{S}}^*(x), \qquad x \in \mathcal{S}, \quad \xi \in T_x\mathcal{S}.$$

(c) The relaxation coefficient $\zeta : \Omega \cup \Gamma_{\mathrm{d}} \to \mathbb{R}$ is measurable, bounded and there is a constant $c > 0$ such that $\zeta(x) \ge c$ for all $x \in \Omega \cup \Gamma_{\mathrm{d}} \cup \Sigma$.

The functions $\mu_{\Gamma_{\mathrm{d}}}^*$ and μ_Σ^* describe where diffusion takes place on Γ_{d} and Σ, and where diffusion degenerates. There are no restrictions on the support of these functions. An example we have in mind is $\mu_{\mathcal{S}}^*(x) = \operatorname{dist}(x, M)^\gamma$ for a subset $M \subset \mathcal{S}$ and $\gamma > 0$, which indicates that diffusion degenerates towards M and is impossible along and across M.

Remark 4.3. The above assumptions cover a large class of nonsmooth scenarios. However, our realization of (1.1)–(1.6) developed below also works under more general conditions. For instance, the interface Σ must only be a Lipschitz hypersurface in graph representation in a neighbourhood of the support of μ_Σ^*. Away from the support, as in [10] it suffices that Σ is a $(d-1)$-set (see [26, Section VII.1.1]). To avoid too many technical difficulties we do not take these issues into account.

4.2. The realization on $\mathbb{L}^2$

We construct the operator A_2 which yields a realization of the elliptic part of (1.1)–(1.6) on a suitable L^2-space, cf. Section 2. The approach based on sesquilinear forms is similar to the one used in [10].

For $p \in (1, \infty)$ we denote by $W^{1,p}(\Omega)$ the usual complex Sobolev space over Ω. We further define $W^{1,p}_D(\Omega)$ as the closure in $W^{1,p}(\Omega)$ of

$$C^\infty_D(\Omega) = \{u|_\Omega : u \in C^\infty_c(\mathbb{R}^d),\ (\operatorname{supp} u) \cap \Gamma_D = \emptyset\}.$$

Roughly speaking, elements of $W^{1,p}_D(\Omega)$ vanish on the Dirichlet part Γ_D of $\partial\Omega$.

Let $\mathrm{tr}_{\Gamma_\mathrm{d}}$ and tr_Σ be the trace operators for Γ_d and Σ. Then [10, Proposition 2.8] implies the continuity of

$$\mathrm{tr}_{\Gamma_\mathrm{d}} : W^{1,2}_D(\Omega) \to L^2(\Gamma_\mathrm{d}), \qquad \mathrm{tr}_\Sigma : W^{1,2}_D(\Omega) \to L^2(\Sigma). \tag{4.1}$$

As in the Introduction and Heuristics sections, we use the notation $u_{\Gamma_\mathrm{d}} = \mathrm{tr}_{\Gamma_\mathrm{d}} u$ and $u_\Sigma = \mathrm{tr}_\Sigma u$ for the traces, and sometimes write only u for u_{Γ_d} or u_Σ.

Definition 4.4.

(a) On $C^\infty_D(\Omega)$ we introduce the scalar product $(\cdot,\cdot)_{\mathrm{Dom}(\mathfrak{t})}$ by

$$(u, v)_{\mathrm{Dom}(\mathfrak{t})} = (u, v)_{W^{1,2}(\Omega)} + \int_{\Gamma_\mathrm{d}} \big(\nabla_{\Gamma_\mathrm{d}} u, \overline{\nabla_{\Gamma_\mathrm{d}} v}\big)\, \mu^*_\Gamma \, d\mathcal{H}_{d-1} + \int_\Sigma \big(\nabla_\Sigma u, \overline{\nabla_\Sigma v}\big)\, \mu^*_\Sigma \, d\mathcal{H}_{d-1},$$

where $(\cdot,\cdot)_{W^{1,2}(\Omega)}$ is the usual scalar product on $W^{1,2}(\Omega)$. The corresponding Hilbert norm is denoted by $\|\cdot\|_{\mathrm{Dom}(\mathfrak{t})}$.

(b) The Hilbert space $\mathrm{Dom}(\mathfrak{t})$ is defined by

$$\mathrm{Dom}(\mathfrak{t}) = \text{completion of } C^\infty_D(\Omega) \text{ with respect to } \|\cdot\|_{\mathrm{Dom}(\mathfrak{t})}.$$

(c) For $p \in [1, \infty]$ we set $\mathbb{L}^p = L^p\big((\Omega \setminus \Sigma) \cup \Gamma_\mathrm{d} \cup \Sigma, (dx + \mathcal{H}_{d-1})\big)$.

(d) The map $\mathfrak{J} : \mathrm{Dom}(\mathfrak{t}) \to \mathbb{L}^2$ is given by $\mathfrak{J}(u) = (u, u_{\Gamma_\mathrm{d}}, u_\Sigma)$.

For $\mathcal{S} \in \{\Gamma_\mathrm{d}, \Sigma\}$ we will also write

$$\|f\|^2_{L^2(\mathcal{S},\mu^*_\mathcal{S})} = \int_\mathcal{S} |f|^2 \, \mu^*_\mathcal{S} \, d\mathcal{H}_{d-1},$$

such that the Hilbert norm may be expressed as

$$\|u\|^2_{\mathrm{Dom}(\mathfrak{t})} = \|u\|^2_{W^{1,2}(\Omega)} + \|\nabla_{\Gamma_\mathrm{d}} u\|^2_{L^2(\Gamma_\mathrm{d},\mu^*_{\Gamma_\mathrm{d}})} + \|\nabla_\Sigma u\|^2_{L^2(\Sigma,\mu^*_\Sigma)}. \tag{4.2}$$

In view of $\mathrm{Dom}(\mathfrak{t}) \subseteq W^{1,2}_D(\Omega)$ and the continuity of the traces (4.1), the map $\mathfrak{J}$ is indeed well-defined. The space $\mathbb{L}^p$ can be identified as

$$\mathbb{L}^p = L^p(\Omega \setminus \Sigma) \oplus L^p(\Gamma_\mathrm{d}) \oplus L^p(\Sigma).$$

Remark 4.5. The space $\mathrm{Dom}(\mathfrak{t})$ includes an implicit definition of a weak surface gradient, even if $\mu^*_{\Gamma_\mathrm{d}}$, μ^*_Σ are only non-negative, as the operator

$$\nabla_\Sigma : \{\psi|_\Sigma : \psi \in C^\infty_D(\Omega)\} \to L^2(\mu^*_\Sigma, \Sigma)$$

continuously extends to $\mathrm{Dom}(\mathfrak{t})$ by density (analogously for Γ_d). This implies our concept of degenerate diffusion on Σ and Γ_d. The regularity of elements u of $\mathrm{Dom}(\mathfrak{t})$ on Γ_d and Σ is determined by the supports of $\mu^*_{\Gamma_\mathrm{d}}$ and μ^*_Σ. On subsets where these are strictly positive, u_{Γ_d} and u_Σ have square integrable weak surface gradients in the sense of Section 3.

The operator A_2 will be derived from the sesquilinear form

$$\mathfrak{t}(u,v) = \int_\Omega (\mu_\Omega \nabla u, \overline{\nabla v})\, dx + \int_{\Gamma_\mathrm{d}} (\mu_{\Gamma_\mathrm{d}} \nabla_{\Gamma_\mathrm{d}} u, \overline{\nabla_{\Gamma_\mathrm{d}} v})\, d\mathcal{H}_{d-1} + \int_\Sigma (\mu_\Sigma \nabla_\Sigma u, \overline{\nabla_\Sigma v})\, d\mathcal{H}_{d-1},$$

which is originally defined for $u, v \in C^\infty_D(\Omega)$.

Lemma 4.6. *The form* $\mathfrak{t}$ *extends continuously to a sesquilinear form on* $\mathrm{Dom}(\mathfrak{t})$*. It is* $\mathfrak{J}$*-elliptic, i.e., there is* $c > 0$ *such that*

$$\operatorname{Re}\mathfrak{t}(u,u) + \|\mathfrak{J}u\|^2_{\mathbb{L}^2} \geq c\|u\|^2_{\mathrm{Dom}(\mathfrak{t})}, \qquad u \in \mathrm{Dom}(\mathfrak{t}).$$

Moreover, the map $\mathfrak{J} : \mathrm{Dom}(\mathfrak{t}) \to \mathbb{L}^2$ *has dense range and is continuous and compact.*

Proof. The continuity and the compactness of $\mathfrak{J}$ follow from $\mathrm{Dom}(\mathfrak{t}) \subseteq W^{1,2}_D(\Omega)$ and [10, Lemma 2.10]. The proof in [10] also shows that $\mathfrak{J}C^\infty_D(\Omega)$ is dense in $\mathbb{L}^2$, hence $\mathfrak{J}\,\mathrm{Dom}(\mathfrak{t})$ is dense since $C^\infty_D(\Omega) \subset \mathrm{Dom}(\mathfrak{t})$.

It is clear that $\mathfrak{t} : C^\infty_D(\Omega) \times C^\infty_D(\Omega) \to \mathbb{C}$ is sesquilinear. Given $u, v \in C^\infty_D(\Omega)$ we use the assumption $\|\mu_\mathcal{S}(x)\|_{\mathcal{L}(\mathbb{R}^d)} \leq c_2\mu^*_\mathcal{S}(x)$ for $\mathcal{S} \in \{\Gamma_\mathrm{d}, \Sigma\}$, Hölder's inequality and (4.2) to estimate

$$\begin{aligned}|\mathfrak{t}(u,v)| &\leq \|\mu_\Omega\|_\infty \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \\ &\quad + c_2\|\nabla_{\Gamma_\mathrm{d}} u\|_{L^2(\Gamma_\mathrm{d},\mu^*_{\Gamma_\mathrm{d}})} \|\nabla_{\Gamma_\mathrm{d}} v\|_{L^2(\Gamma_\mathrm{d},\mu^*_{\Gamma_\mathrm{d}})} + c_2\|\nabla_\Sigma u\|_{L^2(\Sigma,\mu^*_\Sigma)} \|\nabla_\Sigma v\|_{L^2(\Sigma,\mu^*_\Sigma)} \\ &\leq C\|u\|_{\mathrm{Dom}(\mathfrak{t})}\|v\|_{\mathrm{Dom}(\mathfrak{t})}.\end{aligned}$$

Hence $\mathfrak{t}$ extends continuously to a sesquilinear form on $\mathrm{Dom}(\mathfrak{t})$. To show its $\mathfrak{J}$-ellipticity, for $u \in C^\infty_D(\Omega)$ we use the assumption $(\mu_\mathcal{S}\xi, \xi) \geq c_1\mu^*_\mathcal{S}|\xi|^2$ for $\mathcal{S} \in \{\Gamma_\mathrm{d}, \Sigma\}$ to get

$$\begin{aligned}\operatorname{Re}\mathfrak{t}(u,u) + \|\mathfrak{J}u\|^2_{\mathbb{L}^2} &\geq \mu^*_\Omega\|\nabla u\|^2_{L^2(\Omega)} + c_1\|\nabla_{\Gamma_\mathrm{d}} u\|^2_{L^2(\Gamma_\mathrm{d},\mu^*_{\Gamma_\mathrm{d}})} + c_1\|\nabla_\Sigma u\|^2_{L^2(\Sigma,\mu^*_\Sigma)} + \|u\|^2_{L^2(\Omega)} \\ &\geq c\|u\|^2_{\mathrm{Dom}(\mathfrak{t})}.\end{aligned}$$

This inequality carries over to all $u \in \mathrm{Dom}(\mathfrak{t})$ by density and the continuity of $\mathfrak{J}$. □

Now the operator A_2 can be derived from $\mathfrak{t}$ as follows.

Proposition 4.7. *There is a closed, densely defined operator* A_2 *on* $\mathbb{L}^2$ *associated with the form* $\mathfrak{t}$*. For* $\varphi, \psi \in \mathbb{L}^2$ *we have* $\varphi \in \mathrm{Dom}(A_2)$ *and* $A_2\varphi = \psi$ *if and only if there is* $u \in \mathrm{Dom}(\mathfrak{t})$ *such that* $\varphi = \mathfrak{J}u$ *and*

$$(\psi, \mathfrak{J}v)_{\mathbb{L}^2} = \mathfrak{t}(u,v) \qquad \text{for all } v \in \mathrm{Dom}(\mathfrak{t}).$$

The operator $-A_2$ *generates an analytic* C_0*-semigroup*

$$T_2(\cdot) = (T_2(t))_{t\geq 0}$$

of contractions on $\mathbb{L}^2$*. Furthermore,* A_2 *has compact resolvent.*

Proof. All assertions except the contraction property are a consequence of Lemma 4.6 and the general results of [4, Theorem 2.1, Lemma 2.7]. For the contractivity we observe that for $\varphi \in \mathrm{Dom}(A_2)$ with $\varphi = \mathfrak{J}u$ for $u \in \mathrm{Dom}(\mathfrak{t})$ we have $\mathrm{Re}\,(A_2\varphi, \varphi) = \mathrm{Re}\,\mathfrak{t}(u, u) \geq 0$. Hence the vertex of A_2 is zero and the contractivity of the semigroup follows from [29, Theorem IX.1.24]. □

4.3. Properties of A_2 and extension to $\mathbb{L}^p$

The key to the extension of A_2 to all $\mathbb{L}^p$-spaces is the $\mathbb{L}^\infty$-contractivity of the semigroup $T_2(\cdot)$. For the contractivity we will employ that A_2 is associated with the form $\mathfrak{t}$. In this situation, suitable invariance criteria for closed convex sets are available.

By $\mathbb{L}^2_\mathbb{R}$ we denote the subspace of real-valued elements of $\mathbb{L}^2$.

Proposition 4.8. *The semigroup $T_2(\cdot)$ generated by $-A_2$ leaves $\mathbb{L}^2_\mathbb{R}$ invariant, it is $\mathbb{L}^\infty$-contractive and positive.*

Proof. The set $\mathbb{L}^2_\mathbb{R}$ is closed and convex, and $\varphi \mapsto \mathrm{Re}\,\varphi$ is the orthogonal projection onto $\mathbb{L}^2_\mathbb{R}$. For $u \in C^\infty_D(\Omega)$ we have $\mathrm{Re}\,\mathfrak{t}(u, u - \mathrm{Re}\,u) \geq 0$, and this inequality carries over to all $u \in \mathrm{Dom}(\mathfrak{t})$ by density. Hence each $T_2(t)$ leaves $\mathbb{L}^2_\mathbb{R}$ invariant by [4, Proposition 2.9(iii)].

For the $\mathbb{L}^\infty$-contractivity and the positivity, as in [10, Prop. 2.16] it suffices to show that $T_2(\cdot)$ leaves the closed and convex set $\mathcal{C} = \{\varphi \in \mathbb{L}^2_\mathbb{R} : \varphi \leq \mathbb{1}\}$ invariant. Again, we apply a criterion from [4], on a dense subset of $\mathrm{Dom}(\mathfrak{t})$.

For a real-valued function u we define $u \wedge \mathbb{1}$ by $(u \wedge \mathbb{1})(x) = \min(u(x), 1)$. The orthogonal projection P of $\mathbb{L}^2$ onto $\mathcal{C}$ is given by $P\varphi = (\mathrm{Re}\,\varphi) \wedge \mathbb{1}$. Moreover, for $u \in C^\infty_D(\Omega)$ one has $P\mathfrak{J}u = \mathfrak{J}((\mathrm{Re}\,u) \wedge \mathbb{1})$ and

$$\mathrm{Re}\,\mathfrak{t}((\mathrm{Re}\,u) \wedge \mathbb{1}, u - (\mathrm{Re}\,u) \wedge \mathbb{1}) = 0.$$

Hence, [4, Proposition 2.9(iv)] yields the invariance of $\mathcal{C}$. □

Now standard interpolation and duality arguments together with [40, Proposition 3.12] allow to extend $T_2(\cdot)$ to the entire $\mathbb{L}^p$-scale as follows.

Proposition 4.9. *For all $p \in [1, \infty]$ the semigroup $T_2(\cdot)$ generated by $-A_2$ extends consistently to a contraction semigroup $T_p(\cdot)$ on $\mathbb{L}^p$, which is strongly continuous for $p \in [1, \infty)$ and analytic for $p \in (1, \infty)$.*

We define

$$A_p \text{ is the negative generator of } T_p(\cdot).$$

Then A_p coincides with A_2 on $\mathrm{Dom}(A_p) \cap \mathrm{Dom}(A_2)$. Let the relaxation coefficient $\zeta \in \mathbb{L}^\infty$ be as in Assumption 4.1. Rescaling in measure as in the proof of [10, Theorem 2.21] and using [10, Proposition 2.20], we obtain that the operators $-\zeta^{-1}A_p$ generate consistent contractive semigroups on (the rescaled) $\mathbb{L}^p$ for $p \in [1, \infty]$, which are analytic for $p \in (1, \infty)$.

From the contractivity of the semigroup $T_p(\cdot)$ for any $p \in [1, \infty]$, it follows that for any $t \geq 0$, $T_p(t)$ is contractively regular, see [33, Section 2]. We quote the following abstract result from [33, Proposition 2.2].

Proposition 4.10. *Let (Π, τ) be a measure space and let $T(\cdot)$ be a bounded analytic semigroup on $L^p(\Pi)$, $p \in (1, \infty)$. Let $-A$ be the generator of $T(\cdot)$. Assume that for any $t \geq 0$, $T(t)$ is contractively regular. Then A admits a bounded holomorphic functional calculus with angle θ strictly smaller than $\frac{\pi}{2}$.*

We can now derive the main result of this section.

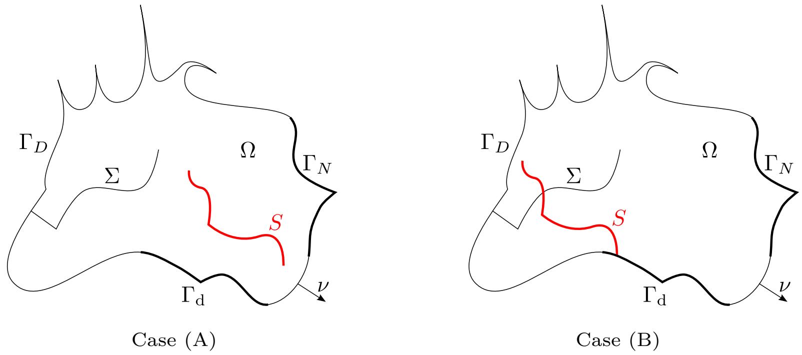


Fig. 2. Example domains with $S \cap (\overline{\Gamma_{\mathrm{d}} \cup \Sigma}) = \emptyset$, Case (A), and $S \cap (\overline{\Gamma_{\mathrm{d}} \cup \Sigma}) \neq \emptyset$, Case (B).

Theorem 4.11. *For each $p \in (1, \infty)$ the operator $\zeta^{-1} A_p$ with domain $\mathrm{Dom}(A_p)$ admits a bounded holomorphic functional calculus on $\mathbb{L}^p$, with angle strictly smaller than $\frac{\pi}{2}$. As a consequence, $\zeta^{-1} A_p$ enjoys maximal parabolic L^s-regularity for all $s \in (1, \infty)$, and $-\zeta^{-1} A_p$ generates an analytic C_0-semigroup on $\mathbb{L}^p$. Furthermore, the fractional power domains are given by complex interpolation, i.e.,*

$$\mathrm{Dom}(A_p^\theta) = [\mathbb{L}^p, \mathrm{Dom}(A_p)]_\theta, \qquad \theta \in [0, 1],$$

and the resolvent of $\zeta^{-1} A_p$ is compact.

5. Degenerate bulk diffusion

In this section we generalize the above setting and allow for degeneracies in the bulk diffusion coefficient μ_Ω. Of special interest is the case when the degeneracy takes place at the dynamic boundary part Γ_{d} or the dynamic interface Σ. In this case the continuity of the map $\mathfrak{J} : \mathrm{Dom}(\mathfrak{t}) \to \mathbb{L}^2$, which is crucial for the approach used in the last section, depends on the degeneracy of the bulk diffusion.

Throughout we keep Assumption 4.1, but we replace the uniform ellipticity of μ_Ω by the assumption that there are constants $c_1, c_2 > 0$ such that

$$\big(\mu_\Omega(x)\xi, \xi\big) \geq c_1 \mu_\Omega^*(x)|\xi|^2, \qquad \|\mu_\Omega(x)\|_{\mathcal{L}(\mathbb{R}^d)} \leq c_2 \mu_\Omega^*(x), \qquad x \in \Omega, \quad \xi \in \mathbb{R}^d, \tag{5.1}$$

where

$$\mu_\Omega^*(x) = \mathrm{dist}(x, S)^\gamma$$

for a compact $(d-k)$-dimensional Lipschitz submanifold $S \subset \overline{\Omega}$, $1 \leq k \leq d$, and $0 \leq \gamma < k$ for the distance exponent. We refer to Section 4 for a definition of a Lipschitz submanifold.

Observe that for $\gamma = 0$ we are in the nondegenerate situation of the previous section. We must distinguish the two cases

Case (A): μ_Ω degenerates at a distance from the dynamics surfaces only, $S \cap (\overline{\Gamma_{\mathrm{d}} \cup \Sigma}) = \emptyset$,
Case (B): μ_Ω degenerates directly at the dynamics surfaces, $S \cap (\overline{\Gamma_{\mathrm{d}} \cup \Sigma}) \neq \emptyset$,

see Fig. 2.

5.1. Weighted function spaces

In order to incorporate the degeneracy of μ_Ω into the domain of the sesquilinear form $\mathfrak{t}$ we introduce weighted function spaces.

We define $W_D^{1,2}(\Omega,\mu_\Omega^*)$ as the closure of $C_D^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W_D^{1,2}(\Omega,\mu_\Omega^*)}$, which is given by

$$\|u\|^2_{W_D^{1,2}(\Omega,\mu_\Omega^*)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega,\mu_\Omega^*)}.$$

As before, here we write

$$\|f\|^2_{L^2(\Omega,\mu_\Omega^*)} = \int_\Omega |f|^2 \mu_\Omega^*\, dx.$$

Note that μ_Ω^* appears as a weight only in the gradient, the $L^2(\Omega)$-norm remains unweighted.

We record the following properties. For the general theory of *Muckenhoupt weights* we refer to [18, Chapter 9]. We recall that a weight $0 \le \mu \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $\mu \neq 0$ belongs to the Muckenhoupt class $\mathcal{A}_2$, if

$$\sup_{Q \text{ cube in } \mathbb{R}^n} \Big(\frac{1}{|Q|}\int_Q \mu(x)\, dx\Big)\Big(\frac{1}{|Q|}\int_Q \frac{1}{\mu(x)}\, dx\Big) < \infty.$$

Lemma 5.1.

(a) *The weight μ_Ω^* belongs to the Muckenhoupt class $\mathcal{A}_2$.*
(b) *One has the continuous embedding $W_D^{1,2}(\Omega,\mu_\Omega^*) \subset W^{1,1}(\Omega)$.*
(c) *$W_D^{1,2}(\Omega,\mu_\Omega^*)$ is a Hilbert space with scalar product*

$$(u,v)_{W_D^{1,2}(\Omega,\mu_\Omega^*)} = (u,v)_{L^2(\Omega)} + \int_\Omega (\nabla u, \nabla v)\, \mu_\Omega^*\, dx.$$

Proof. Assertion (a) follows from our assumption $0 \le \gamma < k$, see [13, Lemma 2.3]. Using Hölder's inequality, it is straightforward to check that $L^2(\Omega,\mu_\Omega^*) \subset L^1(\Omega)$ (see also [18, Exercise 9.3.6]), which yields (b). Then (c) follows from (b). □

To prove the continuity of $\mathrm{tr}_{\Gamma_\mathrm{d}}$ and tr_Σ on $W_D^{1,2}(\Omega,\mu_\Omega^*)$ we start with an extension operator of this space to $W^{1,2}(\mathbb{R}^d,\mu_\Omega^*)$. Here the norm is given by

$$\|u\|^2_{W^{1,2}(\mathbb{R}^d,\mu_\Omega^*)} = \|u\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u\|^2_{L^2(\mathbb{R}^d,\mu_\Omega^*)}.$$

Lemma 5.2. *There is a continuous extension operator $\mathcal{E} : W_D^{1,2}(\Omega,\mu_\Omega^*) \to W^{1,2}(\mathbb{R}^d,\mu_\Omega^*)$. For any $u \in W_D^{1,2}(\Omega,\mu_\Omega^*)$ we have that $\operatorname{supp}\mathcal{E}u \subset B(0,2R)$, where $R = \sup\{|x| : x \in \Omega\}$.*

Proof. *Step 1.* From Assumption 4.1 we find a finite open covering $\bigcup_{\alpha=1}^N V_\alpha \subset B(0,2R)$ of $\overline{\Omega}$ with the following properties. For $\alpha = 1,\dots,N_\Omega$ the sets V_α are strictly contained in Ω; for $\alpha = N_\Omega+1,\dots,N_D$ we have $V_\alpha \cap \Gamma_D \neq \emptyset$ and $V_\alpha \cap (\overline{\Gamma_N \cup \Gamma_\mathrm{d}}) = \emptyset$; for $\alpha = N_D+1,\dots,N$ there is a bi-Lipschitz map φ_α from V_α to the open unit cube Q in $\mathbb{R}^d$ such that

$$\varphi_\alpha(\Omega \cap V_\alpha) = Q_-, \qquad \varphi_\alpha(\partial\Omega \cap V_\alpha) = Q_0,$$

where $Q_- \subset Q$ is the open lower half-cube in $\mathbb{R}^d$ and $Q_0 = \{x \in Q : x_d = 0\}$. We further take a smooth partition of unity $(\psi_\alpha)_\alpha$ for $\overline{\Omega}$ subordinate to the cover $\bigcup_\alpha V_\alpha$, i.e., such that $\operatorname{supp}\psi_\alpha$ is contained in V_α.

Step 2. For any $u \in C_D^\infty(\Omega)$ and $\alpha = N_D+1,\dots,N$ we have that $\psi_\alpha u$ is compactly supported in $\overline{\Omega} \cap V_\alpha$. Choose an open subcube $\widetilde{Q} \subset Q$ such that $\varphi_\alpha(\operatorname{supp}\psi_\alpha) \subset \widetilde{Q}$. Then $W_\alpha = \varphi_\alpha^{-1}(\widetilde{Q}_-)$ is a domain with

Lipschitz boundary which contains $\operatorname{supp}\psi_\alpha$. Finally, take smooth cut-off functions ϕ_α such that $\phi_\alpha \equiv 1$ on $\operatorname{supp}\psi_\alpha$ and $\operatorname{supp}\phi_\alpha \subset V_\alpha$.

Step 3. Now for $u \in C_D^\infty(\Omega)$ we define $\mathcal{E}u$ by

$$\mathcal{E}u = \sum_{\alpha=1}^{N_\Omega} \psi_\alpha u + \sum_{\alpha=N_\Omega+1}^{N_D} \mathcal{E}_\alpha(\psi_\alpha u) + \sum_{\alpha=N_D+1}^{N} \phi_\alpha \mathcal{E}_\alpha(\psi_\alpha u|_{W_\alpha}),$$

where the extensions $\mathcal{E}_\alpha$ are given as follows. For $\alpha = N_\Omega + 1, \dots, N_D$ we define $\mathcal{E}_\alpha(\psi_\alpha u)$ as the trivial extension by zero of $\psi_\alpha u$ from $V_\alpha \cap \Omega$ to $\mathbb{R}^d$. Since $V_\alpha \cap (\overline{\Gamma_N \cup \Gamma_{\mathrm{d}}}) = \emptyset$ and u is supported away from Γ_D, for those α we have

$$\|\mathcal{E}_\alpha(\psi_\alpha u)\|_{W^{1,2}(\mathbb{R}^d,\mu_\Omega^*)} = \|\psi_\alpha u\|_{W^{1,2}(\Omega,\mu_\Omega^*)} \le C\|u\|_{W^{1,2}(\Omega,\mu_\Omega^*)}.$$

For $\alpha = N_D + 1, \dots N$ we let $\mathcal{E}_\alpha : W^{1,2}(W_\alpha, \mu_\Omega^*) \to W^{1,2}(\mathbb{R}^d, \mu_\Omega^*)$ be the extension operator from [7] for the Lipschitz domain W_α. Then

$$\begin{aligned}\|\phi_\alpha \mathcal{E}_\alpha(\psi_\alpha u|_{W_\alpha})\|_{W^{1,2}(\mathbb{R}^d,\mu_\Omega^*)} &\le C\|\mathcal{E}_\alpha(\psi_\alpha u|_{W_\alpha})\|_{W^{1,2}(\mathbb{R}^d,\mu_\Omega^*)} \le C\|\psi_\alpha u|_{W_\alpha}\|_{W^{1,2}(W_\alpha,\mu_\Omega^*)} \\ &= C\|\psi_\alpha u\|_{W^{1,2}(\Omega,\mu_\Omega^*)} \le C\|u\|_{W^{1,2}(\Omega,\mu_\Omega^*)}.\end{aligned}$$

Therefore $\mathcal{E}$ extends continuously from $C_D^\infty(\Omega)$ to $\mathcal{E} : W_D^{1,2}(\Omega, \mu_\Omega^*) \to W^{1,2}(\mathbb{R}^d, \mu_\Omega^*)$, which gives the desired extension operator. □

In a next step we prove Sobolev embeddings of $W^{1,2}(\mathbb{R}^d, \mu_\Omega^*)$ into unweighted Slobodetskii spaces $W^{\theta,q}(\mathbb{R}^d)$, by using the criteria derived in [20].

Proposition 5.3. *Assume $q \in [2, \infty)$ and $\theta \in (0, 1)$ are such that $1 - \frac{d+\gamma}{2} \ge \theta - \frac{d}{q}$. Then*

$$W^{1,2}(\mathbb{R}^d, \mu_\Omega^*) \subset W^{\theta,q}(\mathbb{R}^d).$$

Proof. *Step 1.* Let $B^1_{2,2}(\mathbb{R}^d, \mu_\Omega^*)$ be the Besov space with respect to the weight μ_Ω^*. Since μ_Ω^* belongs to the Muckenhoupt class $\mathcal{A}_2$ by Lemma 5.1, it follows from Remark 1.7 and Proposition 1.8 of [20] that

$$W^{1,2}(\mathbb{R}^d, \mu_\Omega^*) \subset B^1_{2,2}(\mathbb{R}^d, \mu_\Omega^*).$$

Moreover, $W^{\theta,q}(\mathbb{R}^d) = B^\theta_{q,q}(\mathbb{R}^d)$ for $\theta \in (0, 1)$ by [45, Section 2.3.1]. The asserted embedding will thus be a consequence of

$$B^1_{2,2}(\mathbb{R}^d, \mu_\Omega^*) \subset B^\theta_{q,q}(\mathbb{R}^d). \tag{5.2}$$

Step 2. We derive this embedding from the sufficient condition given in [20, Proposition 2.1(i)]. Let $Q(x, r)$ be the cube in $\mathbb{R}^d$ with edges parallel to the coordinate axes, centred at $x \in \mathbb{R}^d$ with edge length $r > 0$. According to [20], (5.2) holds true if we show that

$$\sup_{l\in\mathbb{N}_0, m\in\mathbb{Z}^d} 2^{-l(1-\theta+\frac{d}{q})} \Big(\int\limits_{Q(2^{-l}m, 2^{-l})} \operatorname{dist}(x, S)^\gamma \, dx \Big)^{-1/2} < \infty.$$

By the assumption $1 - \frac{d+\gamma}{2} \ge \theta - \frac{d}{q}$, this will be a consequence of the estimate

$$\int\limits_{Q(2^{-l}m,2^{-l})} \operatorname{dist}(x,S)^\gamma \, dx \geq c2^{-l(d+\gamma)}, \qquad l \in \mathbb{N}, \qquad m \in \mathbb{Z}^d, \tag{5.3}$$

where $c > 0$ is independent of l and m. In the sequel we prove (5.3).

Step 3. Since S is Lipschitzian, there is a tube S_κ of width $\kappa > 0$ around S such that every $Q(2^{-l}m, 2^{-l}) \subset S_\kappa$ lies in a neighbourhood V of S which is mapped to the unit cube in $\mathbb{R}^d$ by a bi-Lipschitz map ψ such that $\psi(S \cap V) = (-1,1)^{d-k} \times \{0_{\mathbb{R}^k}\}$.

Choose $l_0 \in \mathbb{N}$ such that $2^{-l_0\gamma} + 2^{-l_0} \leq \kappa$. We claim that it suffices to prove (5.3) for $l \geq l_0$ and m such that $Q(2^{-l}m, 2^{-l}) \subset S_\kappa$, where c is independent of those l and m.

Assume (5.3) is proved for those l and m. Let $l \geq l_0$ and m be such that $Q(2^{-l}m, 2^{-l})$ is not contained in S_κ. Then we trivially have

$$\int\limits_{Q(2^{-l}m,2^{-l})} \operatorname{dist}(x,S)^\gamma \, dx \geq c2^{-ld}2^{-l_0\gamma} \geq c2^{-l(d+\gamma)}.$$

This yields (5.3) for $l \geq l_0$ and arbitrary m. Let $l < l_0$. Then

$$\int\limits_{Q(2^{-l}m,2^{-l})} \operatorname{dist}(x,S)^\gamma \, dx \geq \int\limits_{Q(2^{-l_0}(2^{l_0-l}m),2^{-l_0})} \operatorname{dist}(x,S)^\gamma \, dx \geq c2^{-l_0(d+\gamma)} \geq \tilde{c}2^{-l(d+\gamma)},$$

where $\tilde{c} = 2^{-l_0(d+\gamma)}$ is independent of l and m.

Step 4. It remains to prove (5.3) for $l \geq l_0$ and m such that $Q(2^{-l}m, 2^{-l}) \subset S_\kappa$. The integral in (5.3) transforms as

$$\int\limits_{Q(2^{-l}m,2^{-l})} \operatorname{dist}(x,S)^\gamma \, dx = \int\limits_{\psi(Q(2^{-l}m,2^{-l}))} \operatorname{dist}(\psi^{-1}(y),S)^\gamma |\det \psi'(y)|^{-1} \, dy,$$

where $|\det \psi'|^{-1} \geq c$ can be uniformly chosen by compactness of S. From the bi-Lipschitz property of ψ it follows that $\operatorname{dist}(\psi^{-1}(y), S) \simeq \operatorname{dist}(y, \psi(S))$. Since $\psi(S) \subset \mathbb{R}^{d-k} \times \{0_{\mathbb{R}^k}\}$, we thus get

$$\int\limits_{Q(2^{-l}m,2^{-l})} \operatorname{dist}(x,S)^\gamma \, dx \geq c \int\limits_{\psi(Q(2^{-l}m,2^{-l}))} \left(|y_{d-k+1}|^\gamma + \ldots + |y_d|^\gamma\right) dy.$$

Again the bi-Lipschitz property of ψ yields $\delta > 0$, independent of l and m, such that $Q(\psi(2^{-l}m), \delta 2^{-l})$ is contained in $\psi(Q(2^{-l}m, 2^{-l}))$. It therefore remains to estimate

$$\int\limits_{Q(\psi(2^{-l}m),\delta 2^{-l})} \left(|y_{d-k+1}|^\gamma + \ldots + |y_d|^\gamma\right) dy = \delta^{d-1}2^{-l(d-1)} \sum_{j=0}^{k-1} \int\limits_{\psi_j(2^{-l}m)-\delta 2^{-l}}^{\psi_j(2^{-l}m)+\delta 2^{-l}} |\tau|^\gamma \, d\tau.$$

For each j, here the integral is given by

$$\eta(s,t) := \frac{1}{\gamma+1}(\operatorname{sign}(s+t)|s+t|^{\gamma+1} - \operatorname{sign}(s-t)|s-t|^{\gamma+1}),$$

where $s = \psi_j(2^{-l}m) \in \mathbb{R}$ and $t = \delta 2^{-l} > 0$. By distinguishing the three cases $s \geq t$, $s \in (-t,t)$ and $s \leq -t$ and using the triangle inequality for the $(\gamma+1)$-norm in $\mathbb{R}^2$, we see that $\eta(s,t) \geq ct^{\gamma+1}$, where c is independent of s. We thus obtain the estimate

$$\int\limits_{Q(\psi(2^{-l}m),\delta 2^{-l})} \left(|y_{d-k+1}|^\gamma + \ldots + |y_d|^\gamma\right) dy \geq c2^{-l(d+\gamma)},$$

independently of m, and this gives (5.3). □

Remark 5.4. A scaling argument gives necessary conditions on the parameters for embedding of the type given in Proposition 5.3 to hold, at least in the model case $S = \mathbb{R}^{d-k} \times \{0_k\}$, where

$$\operatorname{dist}(x,S)^\gamma \sim |x_1|^\gamma + \ldots + |x_k|^\gamma. \tag{5.4}$$

Assuming $\|u\|_{W^{\theta,q}(\mathbb{R}^d)} \leq C\|u\|_{W^{1,2}(\mathbb{R}^d;\operatorname{dist}(\cdot,S)^\gamma)}$ for a constant C independent of u, replacing u by $u(\lambda\cdot)$ with $\lambda > 0$ and rescaling $y = \lambda x$ such that $dx = \lambda^{-d}dx$, we obtain

$$\lambda^{-\frac{d}{q}}\|u\|_{L^q(\mathbb{R}^d)} + \lambda^{\theta-\frac{d}{q}}[u]_{W^{\theta,q}(\mathbb{R}^d)} \leq C\big(\lambda^{-\frac{d}{2}}\|u\|_{L^2(\mathbb{R}^d)} + \lambda^{1-\frac{d+\gamma}{2}}\|\nabla u\|_{L^2(\mathbb{R}^d;\operatorname{dist}(\cdot,S)^\gamma dx)}\big).$$

Letting $\lambda \to \infty$, this shows that for any $\theta \in (0,1)$ and $q \geq 2$ the condition $1 - \frac{d+\gamma}{2} \geq \theta - \frac{d}{q}$ is necessary.

We combine the above results to obtain the following properties of the traces.

Proposition 5.5. *For $1 < r < \frac{2(d-1)}{d+\gamma-2}$ the trace operators $\operatorname{tr}_{\Gamma_{\mathrm{d}}}$ and tr_Σ are continuous and compact maps*

$$\operatorname{tr}_{\Gamma_{\mathrm{d}}} : W^{1,2}(\Omega,\mu_\Omega^*) \to L^r(\Gamma_{\mathrm{d}}, d\mathcal{H}_{d-1}), \qquad \operatorname{tr}_\Sigma : W^{1,2}(\Omega,\mu_\Omega^*) \to L^r(\Sigma, d\mathcal{H}_{d-1}).$$

Proof. We consider Σ, the arguments for Γ_{d} are the same. Let $\mathcal{E}$ be the extension operator for $W^{1,2}(\Omega,\mu_\Omega^*)$ from Lemma 5.2. As in the proof of [10, Proposition 2.8] one can show that $\operatorname{tr}_\Sigma = \operatorname{tr}_\Sigma\mathcal{E}$. Proposition 5.3 together with the support property of $\mathcal{E}$ implies that there is $\varepsilon > 0$ such that $\mathcal{E}$ maps $W^{1,2}(\Omega,\mu_\Omega^*)$ compactly into $W^{1/r+\varepsilon,r}(\mathbb{R}^d)$ for $r > 1$, provided $1 - \frac{d+\gamma}{2} > \frac{1-d}{r}$. Since $d \geq 2$ and $\gamma > 0$ we have $1 - \frac{d+\gamma}{2} < 0$, such that this inequality is equivalent to $r < \frac{2(d-1)}{d+\gamma-2}$. Now [10, Lemma 2.7] implies that tr_Σ maps $W^{1/r+\varepsilon,r}(\mathbb{R}^d)$ continuously into $L^r(\Sigma, d\mathcal{H}_{d-1})$ for those r. Altogether, tr_Σ is continuous and compact. □

5.2. The operators A_p on $\mathbb{L}^p$

We modify $\operatorname{Dom}(\mathfrak{t})$ from Definition 4.4 to take into account the degeneracy of the diffusion coefficient μ_Ω. We set

$$(u,v)_{\operatorname{Dom}(\mathfrak{t})} = (u,v)_{W^{1,2}(\Omega,\mu_\Omega^*)} + \int\limits_{\Gamma_{\mathrm{d}}} \big(\nabla_{\Gamma_{\mathrm{d}}}u, \overline{\nabla_{\Gamma_{\mathrm{d}}}v}\big)\, \mu_{\Gamma_{\mathrm{d}}}^*\, d\mathcal{H}_{d-1} + \int\limits_\Sigma \big(\nabla_\Sigma u, \overline{\nabla_\Sigma v}\big)\, \mu_\Sigma^*\, d\mathcal{H}_{d-1},$$

and define as before $\operatorname{Dom}(\mathfrak{t})$ as the completion of $C_D^\infty(\Omega)$ with respect to the corresponding Hilbert norm $\|\cdot\|_{\operatorname{Dom}(\mathfrak{t})}$. It is now given by

$$\|u\|^2_{\operatorname{Dom}(\mathfrak{t})} = \|u\|^2_{W^{1,2}(\Omega,\mu_\Omega^*)} + \|\nabla_{\Gamma_{\mathrm{d}}}u\|^2_{L^2(\Gamma_{\mathrm{d}},\mu^*_{\Gamma_{\mathrm{d}}})} + \|\nabla_\Sigma u\|^2_{L^2(\Sigma,\mu_\Sigma^*)}.$$

Recall that the map $\mathfrak{J}$ is for $u \in C_D^\infty(\Omega)$ given by $\mathfrak{J}u = (u, u_{\Gamma_{\mathrm{d}}}, u_\Sigma)$. In the following we distinguish between the cases when the surface S, where the bulk diffusion degenerates, is away from Γ_{d} and Σ, and where the relation between these sets is arbitrary. In the second case we have to restrict to $\gamma < 1$ for the distance exponent to obtain the continuity of $\mathfrak{J}$ into $\mathbb{L}^2$.

Lemma 5.6. *Assume either $0 < \gamma < d-k$ and $S \cap (\overline{\Gamma_{\mathrm{d}} \cup \Sigma}) = \emptyset$ (Case (A)), or assume $0 < \gamma < 1$ (Case (B)). Then $\mathfrak{J} : \operatorname{Dom}(\mathfrak{t}) \to \mathbb{L}^2$ is continuous and has dense range. If (additionally) $0 < \gamma < 2$, then $\mathfrak{J}$ is compact.*

Proof. *Step 1.* Since $\mathrm{Dom}(\mathfrak{t}) \subset W_D^{1,2}(\Omega, \mu_\Omega^*)$, for continuity and compactness it suffices to consider $\mathfrak{J}$ on $W_D^{1,2}(\Omega, \mu_\Omega^*)$ instead of $\mathrm{Dom}(\mathfrak{t})$.

By definition we have $W_D^{1,2}(\Omega, \mu_\Omega^*) \subset L^2(\Omega)$. We claim that the latter embedding is also compact if $\gamma < 2$. Decompose the embedding into the extension $\mathcal{E}$ to $W^{1,2}(\mathbb{R}^d, \mu_\Omega^*)$ from Lemma 5.2 and the restriction to Ω. By Proposition 5.3 we have $W^{1,2}(\mathbb{R}^d, \mu_\Omega^*) \subset W^{\theta,2}(\mathbb{R}^d)$ for some $\theta > 0$, provided $\gamma < 2$. The support property yields that $\mathcal{E}$ is compact if θ is chosen slightly smaller. Hence $W_D^{1,2}(\Omega, \mu_\Omega^*)$ embeds compactly into $L^2(\Omega)$ for $\gamma < 2$.

Step 2. We show that the traces at Γ_{d} and Σ are continuous and compact from $W_D^{1,2}(\Omega, \mu_\Omega^*)$ into $L^2(\Gamma_{\mathrm{d}})$ and $L^2(\Sigma)$, respectively. Assume $\gamma < 1$. Then $\frac{2(d-1)}{d+\gamma-2} > 2$, and the assertion follows from Proposition 5.5. Next assume $S \cap (\overline{\Gamma_{\mathrm{d}} \cup \Sigma}) = \emptyset$. Choose a smooth cut-off ψ such that $\psi \equiv 0$ on S and $\psi \equiv 1$ in a neighbourhood of $\overline{\Gamma_{\mathrm{d}} \cup \Sigma}$. Then $\mathrm{tr}_\Sigma u = \mathrm{tr}_\Sigma(\psi u)$ for all $u \in W_D^{1,2}(\Omega, \mu_\Omega^*)$. The multiplication with ψ is continuous from $W_D^{1,2}(\Omega, \mu_\Omega^*)$ into the unweighted space $W_D^{1,2}(\Omega)$, and tr_Σ is continuous and compact from $W_D^{1,2}(\Omega)$ to $L^2(\Sigma)$ by [10, Lemma 2.10], analogously for $\mathrm{tr}_{\Gamma_{\mathrm{d}}}$.

Step 3. By the proof of [10, Lemma 2.10] we have that $\mathfrak{J} C_D^\infty(\Omega)$ is dense in $\mathbb{L}^2$. Hence $\mathfrak{J} W_D^{1,2}(\Omega, \mu_\Omega^*)$ is dense since $C_D^\infty(\Omega) \subset W_D^{1,2}(\Omega, \mu_\Omega^*)$. $\square$

Now one can argue in the same way as in Lemma 4.6 to show that the sesquilinear form

$$\mathfrak{t}(u,v) = \int_\Omega (\mu_\Omega \nabla u, \overline{\nabla v})\, dx + \int_{\Gamma_{\mathrm{d}}} (\mu_{\Gamma_{\mathrm{d}}} \nabla_{\Gamma_{\mathrm{d}}} u, \overline{\nabla_{\Gamma_{\mathrm{d}}} v})\, d\mathcal{H}_{d-1} + \int_\Sigma (\mu_\Sigma \nabla_\Sigma u, \overline{\nabla_\Sigma v})\, d\mathcal{H}_{d-1}$$

extends continuously from $C_D^\infty(\Omega)$ to $\mathrm{Dom}(\mathfrak{t})$, and that it is $\mathfrak{J}$-elliptic. Therefore, as in Proposition 4.7 we obtain a closed and densely defined operator A_2 associated with $\mathfrak{t}$, which is the negative generator of an analytic C_0-semigroup $T_2(\cdot)$ on $\mathbb{L}^2$. In order to show that $T_2(\cdot)$ is $\mathbb{L}^\infty$-contractive, it suffices to see that as in the proof of Proposition 4.8, $T_2(\cdot)$ leaves $\mathbb{L}^2_{\mathbb{R}}$ and $\mathcal{C}$ invariant.

Then, as in Section 4.3, the semigroup $T_2(\cdot)$ on $\mathbb{L}^2$ extends consistently to $T_p(\cdot)$ on $\mathbb{L}^p$ for $p \in [1,\infty]$, and for the generators A_p and the relaxation coefficient ζ we obtain our main result.

Theorem 5.7. *Assume either $0 < \gamma < d - k$ and $S \cap (\overline{\Gamma_{\mathrm{d}} \cup \Sigma}) = \emptyset$ (Case (A)), or assume $0 < \gamma < 1$ (Case (B)). Then for each $p \in (1,\infty)$ the operator $\zeta^{-1} A_p$ with domain $\mathrm{Dom}(A_p)$ admits a bounded holomorphic functional calculus on $\mathbb{L}^p$, with angle strictly smaller than $\frac{\pi}{2}$. As a consequence, $\zeta^{-1} A_p$ enjoys maximal parabolic L^s-regularity for all $s \in (1,\infty)$ and $-\zeta^{-1} A_p$ generates an analytic C_0-semigroup on $\mathbb{L}^p$. Furthermore, the fractional power domains are given by complex interpolation, i.e.,*

$$\mathrm{Dom}(A_p^\theta) = [\mathbb{L}^p, \mathrm{Dom}(A_p)]_\theta, \qquad \theta \in [0,1].$$

The resolvent of $\zeta^{-1} A_p$ is compact if $\gamma < 2$.

6. Embeddings for fractional power domains

Let A_p be the operator from Theorem 4.11 or Theorem 5.7. In this section we investigate conditions on $p \in (2,\infty)$ and $\theta \in (0,1)$ such that for the domain of the fractional power A_p^θ we have

$$\mathrm{Dom}(A_p^\theta) \hookrightarrow \mathbb{L}^\infty. \tag{6.1}$$

We in particular aim to quantify the conditions in dependence on whether diffusion is degenerate or not, and where it degenerates.

Our motivation is semilinear versions of (1.1)–(1.6), i.e., where the right-hand side $(f_\Omega, f_{\Gamma_{\mathrm{d}}}, f_\Sigma)$ depends nonlinearly on the solution itself. If (6.1) holds true, then the Nemytzkii operator induced by a nonlinearity is well-defined on $\mathrm{Dom}(A_p^\theta)$ with values in $\mathbb{L}^p$, which in principle allows to apply the standard theory for semilinear parabolic equations to obtain local-in-time well-posedness (see the introduction for further references).

The key to the embedding (6.1) is the regularity of the image of $\mathfrak{J}$.

Lemma 6.1. *Let $p, r \in (2, \infty)$ and $\theta \in (0, 1)$ such that $\theta > \frac{r}{(r-2)p}$. Assume*

$$\mathfrak{J}\,\mathrm{Dom}(\mathfrak{t}) \subset \mathbb{L}^r. \tag{6.2}$$

Then $\mathrm{Dom}(A_p^\theta) \subset \mathbb{L}^\infty$.

Proof. Let $T_p(\cdot)$ be the semigroup on $\mathbb{L}^p$ generated by $-A_p$. The arguments given in the proof of [10, Lemma 2.19] show that there is $C > 0$ such that

$$\|e^{-t} T_2(t)\varphi\|_{\mathbb{L}^\infty} \le C t^{-\frac{r}{(r-2)2}} \|\varphi\|_{\mathbb{L}^2}, \qquad t > 0, \qquad \varphi \in \mathbb{L}^2.$$

Interpolating this inequality with the $\mathbb{L}^\infty$-contractivity of $T_2(\cdot)$, we obtain that

$$\|e^{-t} T_p(t)\varphi\|_{\mathbb{L}^\infty} \le C t^{-\frac{r}{(r-2)p}} \|\varphi\|_{\mathbb{L}^p}, \qquad t > 0, \qquad \varphi \in \mathbb{L}^p. \tag{6.3}$$

Since $1 + A_p$ is invertible, we have that

$$u \mapsto \|(A_p + 1)^\theta u\|_{\mathbb{L}^p}$$

defines an equivalent norm on $\mathrm{Dom}(A_p^\theta)$. For $\theta \in (0, 1)$ it is further well-known that

$$(A_p + 1)^{-\theta} = C_\theta \int_0^\infty t^{\theta-1} e^{-t} T_p(t)\, dt.$$

Using (6.3) for $t \in (0, 1)$ and the contractivity of $T_p(\cdot)$ for $t > 1$, for $u \in \mathrm{Dom}(A_p^\theta)$ we obtain

$$\|u\|_{\mathbb{L}^\infty} \le C \|u\|_{\mathrm{Dom}(A_p^\theta)} \int_0^1 t^{\theta-1-\frac{r}{(r-2)p}}\, dt + C\|u\|_{\mathrm{Dom}(A_p^\theta)} \int_1^\infty e^{-t}\, dt.$$

Here the first integral is finite if $\theta > \frac{r}{(r-2)p}$. In this case the embedding $\mathrm{Dom}(A_p^\theta) \subset \mathbb{L}^\infty$ follows. □

In the sequel we determine $r_0 > 2$ as large as possible such that (6.2) holds for all $2 < r < r_0$. Since

$$\mathbb{L}^r = L^r(\Omega) \oplus L^r(\Gamma_{\mathrm{d}}) \oplus L^r(\Sigma),$$

the number r_0 depends on how large r can be such that

$$\mathrm{Dom}(\mathfrak{t}) \subset L^r(\Omega), \qquad \mathrm{tr}_{\Gamma_{\mathrm{d}}} : \mathrm{Dom}(\mathfrak{t}) \to L^r(\Gamma_{\mathrm{d}}), \qquad \mathrm{tr}_\Sigma : \mathrm{Dom}(\mathfrak{t}) \to L^r(\Sigma),$$

are simultaneously continuous. In turn, this depends on whether the bulk diffusion degenerates or not, if it degenerates at $\overline{\Gamma_{\mathrm{d}} \cup \Sigma}$ where traces are taken, and *where* the surface diffusion on Γ_{d} and Σ degenerates.

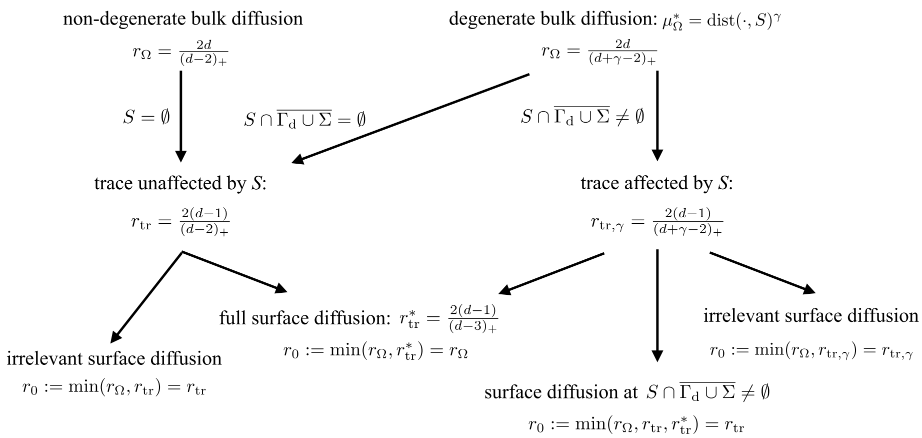
It follows from Lemma 5.2 and Proposition 5.3 that

$$\mathrm{Dom}(\mathfrak{t}) \subset W_D^{1,2}(\Omega) \subset L^r(\Omega)$$

for $r < r_\Omega := \frac{2d}{(d+\gamma-2)_+}$. If $S = \emptyset$ or $S \cap \overline{\Gamma_{\mathrm{d}} \cup \Sigma} = \emptyset$ (Case (A)), then by [10, Proposition 2.8] the traces are continuous from $\mathrm{Dom}(\mathfrak{t}) \subset W^{1,2}_D(\Omega)$ into $L^r(\Gamma_{\mathrm{d}})$ and $L^r(\Sigma)$ for all $r < r_{\mathrm{tr}} := \frac{2(d-1)}{(d-2)_+}$. In case $S \cap \overline{\Gamma_{\mathrm{d}} \cup \Sigma} \neq \emptyset$ (Case (B)), where in Theorem 5.7 it is assumed that $\gamma < 1$, Proposition 5.5 shows that the traces are continuous only for $r < r_{\mathrm{tr},\gamma} := \frac{2(d-1)}{(d+\gamma-2)_+}$.

The regularity of the traces improves if surface diffusion is present. Assume that the surface diffusion is uniformly nondegenerate, i.e., $\mu^*_{\Gamma_{\mathrm{d}}}, \mu^*_\Sigma \geq \eta > 0$. Then the traces belong to $W^{1,2}(\Gamma_{\mathrm{d}})$ and $W^{1,2}(\Sigma)$. By Sobolev embeddings, the traces are thus continuous into L^r for $r < r^*_{\mathrm{tr}} := \frac{2(d-1)}{(d-3)_+}$. Observe that $r^*_{\mathrm{tr}} > r_{\mathrm{tr}}$, which quantifies the regularity improvement obtained from surface diffusion. Finally, assume that $S \cap \overline{\Gamma_{\mathrm{d}} \cup \Sigma} \neq \emptyset$ and that $\mu^*_{\Gamma_{\mathrm{d}}}, \mu^*_\Sigma \geq \eta > 0$ in a neighbourhood of $S \cap \overline{\Gamma_{\mathrm{d}} \cup \Sigma}$. Then the traces belong to $W^{1,2}$ in this neighbourhood, such that they belong to L^r for $r < \min(r_{\mathrm{tr}}, r^*_{\mathrm{tr}}) = r_{\mathrm{tr}}$. This improves the case without surface diffusion on the critical set $S \cap \overline{\Gamma_{\mathrm{d}} \cup \Sigma}$ since $r_{\mathrm{tr}} > r_{\mathrm{tr},\gamma}$.

Now the number r_0 can be chosen as the minimum of r_Ω and r_{tr}, $r_{\mathrm{tr},\gamma}$ or r^*_{tr} according to the cases described above. The following figure gives an overview.



One can check that if $0 \leq \gamma < 1$, in any case we have $r_0 > 2$. Together with Lemma 6.1 we thus have the following result.

Theorem 6.2. *Assume $0 \leq \gamma < 1$. Then there are $\theta_0 \in (0,1)$ and $p_0 \in (2,\infty)$ such that $\mathrm{Dom}(A_p^\theta) \hookrightarrow \mathbb{L}^\infty$ for all $\theta \in (\theta_0, 1)$ and $p \in (p_0, \infty)$.*

It is interesting to note that if diffusion is nowhere degenerate, then one can take $r_0 = \frac{2d}{(d-2)_+}$. In this case, by Lemma 6.1 we have $\mathrm{Dom}(A_p^\theta) \hookrightarrow \mathbb{L}^\infty$ provided

$$2\theta > \frac{d}{p}.$$

This is precisely the optimal relation for the embedding of $H^{2\theta,p}$ into L^∞. In a smooth situation one indeed expects that $\mathrm{Dom}(A_p) \subset H^{2,p}(\Omega)$ and thus $\mathrm{Dom}(A_p^\theta) \subset H^{2\theta,p}(\Omega)$, which shows that the above considerations are optimal at least in this case.

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