



Some new results on products of Apostol–Bernoulli and Apostol–Euler polynomials



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ARTICLE INFO

Article history:

Received 21 July 2014

Available online 27 May 2015

Submitted by B.C. Berndt

Keywords:

Apostol–Bernoulli polynomials and numbers

Apostol–Euler polynomials and numbers

Convolution formulae

Recurrence formulae

ABSTRACT

We perform a further investigation for the Apostol–Bernoulli and Apostol–Euler polynomials and numbers. By making use of an elementary idea used by Euler in the discovery of his famous Pentagonal Number Theorem, we establish some new formulae for products of an arbitrary number of Apostol–Bernoulli and Apostol–Euler polynomials and numbers. These results are the corresponding generalizations of some known formulae including the higher-order convolution ones discovered by Agoh and Dilcher (2014) [5] on the classical Bernoulli and Euler polynomials.

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1. Introduction

The classical Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ are usually defined by means of the following generating functions:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi) \quad \text{and} \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (|t| < \pi). \quad (1.1)$$

In particular, the rational numbers $B_n = B_n(0)$ and integers $E_n = 2^n E_n(1/2)$ are called the classical Bernoulli numbers and Euler numbers, respectively.

As is well known, the classical Bernoulli and Euler polynomials and numbers play important roles in different areas of mathematics such as number theory, combinatorics, special functions and analysis. Numerous interesting properties for them have been well investigated by many authors; see, for example, [13,24,27,29]. This paper is primarily concerned with the Apostol–Bernoulli and Apostol–Euler polynomials and numbers. We establish some new formulae for products of an arbitrary number of Apostol–Bernoulli and Apostol–Euler polynomials and numbers by making use of an elementary idea used by Euler in the discovery of his

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famous Pentagonal Number Theorem. These results are the corresponding generalizations of some known formulae including the higher-order convolution ones recently discovered by Agoh and Dilcher [5] on the classical Bernoulli and Euler polynomials.

We now turn to the Apostol–Bernoulli polynomials $\mathcal{B}_n(x; \lambda)$ and Apostol–Euler polynomials $\mathcal{E}_n(x; \lambda)$. For arbitrary real or complex parameter λ , the Apostol–Bernoulli and Apostol–Euler polynomials are usually defined by means of the following generating functions (see, e.g., [18,20,21]):

$$\frac{te^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda) \frac{t^n}{n!} \quad (|t + \log \lambda| < 2\pi), \quad (1.2)$$

and

$$\frac{2e^{xt}}{\lambda e^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x; \lambda) \frac{t^n}{n!} \quad (|t + \log \lambda| < \pi). \quad (1.3)$$

Moreover, $\mathcal{B}_n(\lambda) = \mathcal{B}_n(0; \lambda)$ and $\mathcal{E}_n(\lambda) = 2^n \mathcal{E}_n(1/2; \lambda)$ are called the Apostol–Bernoulli numbers and Apostol–Euler numbers, respectively. Clearly, $\mathcal{B}_n(x; \lambda)$ and $\mathcal{E}_n(x; \lambda)$ reduce to $B_n(x)$ and $E_n(x)$ when $\lambda = 1$. It is worth mentioning that the Apostol–Bernoulli polynomials were firstly introduced by Apostol [8] (see also Srivastava [28] for a systematic study) in order to evaluate the value of the Hurwitz–Lerch zeta function. For some elegant results and nice methods on these polynomials and numbers, we refer the reader to [9,19,21,23,26].

This paper is organized as follows. In the second section, we recall the elementary and beautiful idea that Euler used to discover his famous Pentagonal Number Theorem, and give some new formulae for products of an arbitrary number of Apostol–Bernoulli and Apostol–Euler polynomials and numbers. In the third section, we establish some higher-order convolution formulae for Apostol–Bernoulli and Apostol–Euler polynomials, which are the corresponding generalizations of the ones of Agoh and Dilcher [5] for the classical Bernoulli and Euler polynomials.

2. Formulae of products of Apostol–Bernoulli and Apostol–Euler polynomials

In this section, we show some new formulae for products of an arbitrary number of Apostol–Bernoulli and Apostol–Euler polynomials and numbers. For convenience, in the following we always denote by $[t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}] f(t_1, t_2, \dots, t_n)$ the coefficients of $t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}$ in the power series expansion of $f(t_1, t_2, \dots, t_n)$. We first recall Euler’s famous Pentagonal Number Theorem: for $|x| < 1$,

$$\begin{aligned} (1-x)(1-x^2)(1-x^3) \cdots &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots \\ &= 1 + \sum_{n=1}^{\infty} (-1)^n \{x^{\frac{1}{2}n(3n-1)} + x^{\frac{1}{2}n(3n+1)}\}, \end{aligned} \quad (2.1)$$

which, in addition to being a striking combinatorial interpretation, can be used effectively for the calculation of the number of partitions of n (see, e.g., [6,15]). In his original proof of (2.1), Euler used the following beautiful idea (see, e.g., [7,10]):

$$(1+x_1)(1+x_2)(1+x_3) \cdots = (1+x_1) + x_2(1+x_1) + x_3(1+x_1)(1+x_2) + \cdots \quad (2.2)$$

Obviously, the finite form of (2.2) can be expressed as

$$\begin{aligned} (1+x_1)(1+x_2) \cdots (1+x_n) &= (1+x_1) + x_2(1+x_1) + \cdots \\ &\quad + x_n(1+x_1)(1+x_2) \cdots (1+x_{n-1}), \end{aligned} \quad (2.3)$$

which has been studied in great detail by Bhatnagar [11] and used to give alternate proofs of all the key summation theorems for terminating hypergeometric series and basic hypergeometric series. In what follows, we shall make use of (2.3) to establish some new formulae for products of an arbitrary number of Apostol–Bernoulli and Apostol–Euler polynomials and numbers. Clearly, for $1 \leq r \leq n$, substituting $x_r - 1$ for x_r in (2.3) gives

$$x_1 x_2 \cdots x_n - 1 = \sum_{r=1}^n (x_r - 1) x_1 x_2 \cdots x_{r-1}, \quad (2.4)$$

where the product $x_1 x_2 \cdots x_{r-1}$ is considered to be equal to 1 when $r = 1$. If we take $x_r = \lambda_r e^{t_r}$ for $1 \leq r \leq n$ in (2.4) then we have

$$\lambda_1 \lambda_2 \cdots \lambda_n e^{t_1+t_2+\cdots+t_n} - 1 = \sum_{r=1}^n (\lambda_r e^{t_r} - 1) \prod_{k=1}^{r-1} \lambda_k e^{t_k}. \quad (2.5)$$

It follows from (2.5) that

$$\begin{aligned} \prod_{i=1}^n \frac{t_i e^{x_i t_i}}{\lambda_i e^{t_i} - 1} &= \frac{1}{\lambda_1 \lambda_2 \cdots \lambda_n e^{t_1+t_2+\cdots+t_n} - 1} \\ &\quad \times \sum_{r=1}^n (\lambda_r e^{t_r} - 1) \prod_{k=1}^{r-1} \lambda_k e^{t_k} \prod_{i=1}^n \frac{t_i e^{x_i t_i}}{\lambda_i e^{t_i} - 1}. \end{aligned} \quad (2.6)$$

Observe that

$$\begin{aligned} &(\lambda_r e^{t_r} - 1) \prod_{k=1}^{r-1} \lambda_k e^{t_k} \prod_{i=1}^n \frac{t_i e^{x_i t_i}}{\lambda_i e^{t_i} - 1} \\ &= t_r e^{x_r(t_1+t_2+\cdots+t_n)} \prod_{i=1}^{r-1} \lambda_i \frac{t_i e^{(x_i-x_r+1)t_i}}{\lambda_i e^{t_i} - 1} \prod_{i=r+1}^n \frac{t_i e^{(x_i-x_r)t_i}}{\lambda_i e^{t_i} - 1}. \end{aligned} \quad (2.7)$$

Hence, by applying (2.7) to (2.6), we obtain

$$\begin{aligned} \prod_{i=1}^n \frac{t_i e^{x_i t_i}}{\lambda_i e^{t_i} - 1} &= \sum_{r=1}^n \frac{t_r e^{x_r(t_1+t_2+\cdots+t_n)}}{\lambda_1 \lambda_2 \cdots \lambda_n e^{t_1+t_2+\cdots+t_n} - 1} \\ &\quad \times \prod_{i=1}^{r-1} \lambda_i \frac{t_i e^{(x_i-x_r+1)t_i}}{\lambda_i e^{t_i} - 1} \prod_{i=r+1}^n \frac{t_i e^{(x_i-x_r)t_i}}{\lambda_i e^{t_i} - 1}. \end{aligned} \quad (2.8)$$

Let $m_1, m_2, \dots, m_n$ be n positive integers. It follows from (1.2) that we obtain from the left hand side of (2.8),

$$\left[\frac{t_1^{m_1}}{m_1!} \frac{t_2^{m_2}}{m_2!} \cdots \frac{t_n^{m_n}}{m_n!} \right] \left(\prod_{i=1}^n \frac{t_i e^{x_i t_i}}{\lambda_i e^{t_i} - 1} \right) = \prod_{i=1}^n \mathcal{B}_{m_i}(x_i; \lambda_i). \quad (2.9)$$

Since $\mathcal{B}_0(x; \lambda) = 0$ when $\lambda \neq 1$ (see, e.g., [20]), then by (1.2) we have

$$\frac{e^{xt}}{\lambda e^t - 1} = \sum_{n=0}^{\infty} \frac{\mathcal{B}_{n+1}(x; \lambda)}{n+1} \frac{t^n}{n!} \quad (\lambda \neq 1). \quad (2.10)$$

Note that for non-negative integer N (see, e.g., [14,30]),

$$(t_1 + t_2 + \cdots + t_n)^N = \sum_{\substack{k_1 + \cdots + k_n = N \\ k_1, \dots, k_n \geq 0}} \binom{N}{k_1, k_2, \dots, k_n} t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n}, \quad (2.11)$$

where $\binom{n}{r_1, r_2, \dots, r_k}$ denotes by the multinomial coefficient given by

$$\binom{n}{r_1, r_2, \dots, r_k} = \frac{n!}{r_1! \cdot r_2! \cdots r_k!} \quad (n, r_1, r_2, \dots, r_k \geq 0). \quad (2.12)$$

If we apply (2.11) to (2.10), then we get for $\lambda_1 \lambda_2 \cdots \lambda_n \neq 1$,

$$\begin{aligned} & \frac{e^{x_r(t_1+t_2+\cdots+t_n)}}{\lambda_1 \lambda_2 \cdots \lambda_n e^{t_1+t_2+\cdots+t_n} - 1} \\ &= \sum_{N=0}^{\infty} \frac{\mathcal{B}_{N+1}(x; (x_r; \lambda_1 \lambda_2 \cdots \lambda_n))}{N+1} \sum_{\substack{k_1+k_2+\cdots+k_n=N \\ k_1, k_2, \dots, k_n \geq 0}} \frac{t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n}}{k_1! k_2! \cdots k_n!}. \end{aligned} \quad (2.13)$$

Notice that

$$\begin{aligned} & \left[\frac{t_1^{m_1}}{m_1!} \frac{t_2^{m_2}}{m_2!} \cdots \frac{t_n^{m_n}}{m_n!} \right] f(t_1, t_2, \dots, t_n) \\ &= (m_1! \cdot m_2! \cdots m_n!) [t_1^{m_1} t_2^{m_2} \cdots t_n^{m_n}] f(t_1, t_2, \dots, t_n), \end{aligned} \quad (2.14)$$

which together with (2.8) yields

$$\begin{aligned} & \left[\frac{t_1^{m_1}}{m_1!} \frac{t_2^{m_2}}{m_2!} \cdots \frac{t_n^{m_n}}{m_n!} \right] \left(\prod_{i=1}^n \frac{t_i e^{x_i t_i}}{\lambda_i e^{t_i} - 1} \right) \\ &= m_1! \cdot m_2! \cdots m_n! \sum_{r=1}^n [t_1^{m_1} \cdots t_{r-1}^{m_{r-1}} t_r^{m_r-1} t_{r+1}^{m_{r+1}} \cdots t_n^{m_n}] \left(\frac{e^{x_r(t_1+t_2+\cdots+t_n)}}{\lambda_1 \lambda_2 \cdots \lambda_n e^{t_1+t_2+\cdots+t_n} - 1} \right. \\ & \quad \left. \times \prod_{i=1}^{r-1} \lambda_i \frac{t_i e^{(x_i-x_r+1)t_i}}{\lambda_i e^{t_i} - 1} \prod_{i=r+1}^n \frac{t_i e^{(x_i-x_r)t_i}}{\lambda_i e^{t_i} - 1} \right). \end{aligned} \quad (2.15)$$

It follows from (1.2), (2.9), (2.13) and (2.15) that for positive integers $m_1, m_2, \dots, m_n$ and $\lambda_1 \lambda_2 \cdots \lambda_n \neq 1$,

$$\begin{aligned} & \mathcal{B}_{m_1}(x_1; \lambda_1) \mathcal{B}_{m_2}(x_2; \lambda_2) \cdots \mathcal{B}_{m_n}(x_n; \lambda_n) \\ &= \sum_{r=1}^n \sum_{\substack{k_1, \dots, k_{r-1}, k_{r+1}, \dots, k_n \geq 0}} \frac{m_1! \cdot m_2! \cdots m_n!}{k_1! \cdots k_{r-1}! \cdot (m_r - 1)! \cdot k_{r+1}! \cdots k_n!} \\ & \quad \times \frac{\mathcal{B}_{k_1+\cdots+k_{r-1}+(m_r-1)+k_{r+1}+\cdots+k_n+1}(x_r; \lambda_1 \lambda_2 \cdots \lambda_n)}{k_1 + \cdots + k_{r-1} + (m_r - 1) + k_{r+1} + \cdots + k_n + 1} \\ & \quad \times \prod_{i=1}^{r-1} \lambda_i \frac{\mathcal{B}_{m_i-k_i}(x_i - x_r + 1; \lambda_i)}{(m_i - k_i)!} \prod_{i=r+1}^n \frac{\mathcal{B}_{m_i-k_i}(x_i - x_r; \lambda_i)}{(m_i - k_i)!}. \end{aligned} \quad (2.16)$$

Thus, by replacing k_i by $m_i - k_i$ for $i \neq r$ in (2.16), we obtain the following formula for products of an arbitrary number of Apostol–Bernoulli polynomials.

Theorem 2.1. *Let $m_1, m_2, \dots, m_n$ be n positive integers. Then for $\lambda_1 \lambda_2 \cdots \lambda_n \neq 1$,*

$$\begin{aligned} & \mathcal{B}_{m_1}(x_1; \lambda_1) \mathcal{B}_{m_2}(x_2; \lambda_2) \cdots \mathcal{B}_{m_n}(x_n; \lambda_n) \\ &= \sum_{\substack{k_1+k_2+\cdots+k_n=m_1+m_2+\cdots+m_n \\ k_1, k_2, \dots, k_n \geq 0}} \sum_{r=1}^n \frac{m_r}{k_r} \mathcal{B}_{k_r}(x_r; \lambda_1 \lambda_2 \cdots \lambda_n) \\ & \quad \times \prod_{i=1}^{r-1} \binom{m_i}{k_i} \lambda_i \mathcal{B}_{k_i}(x_i - x_r + 1; \lambda_i) \prod_{i=r+1}^n \binom{m_i}{k_i} \mathcal{B}_{k_i}(x_i - x_r; \lambda_i). \end{aligned} \quad (2.17)$$

It becomes obvious that by setting $n = 2$ in Theorem 2.1 and applying the symmetry relation for the Apostol–Bernoulli polynomials $\lambda \mathcal{B}_n(1-x; \lambda) = (-1)^n \mathcal{B}_n(x; 1/\lambda)$ for non-negative integer n , we get

$$\begin{aligned} \mathcal{B}_m(x; \lambda) \mathcal{B}_n(y; \mu) &= n \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \mathcal{B}_{m-k}(y-x; 1/\lambda) \frac{\mathcal{B}_{n+k}(y; \lambda \mu)}{n+k} \\ & \quad + m \sum_{k=0}^n \binom{n}{k} \mathcal{B}_{n-k}(y-x; \mu) \frac{\mathcal{B}_{m+k}(x; \lambda \mu)}{m+k}, \end{aligned} \quad (2.18)$$

where m, n are positive integers and $\lambda \mu \neq 1$.

We next give some formulae for products of an arbitrary number of Apostol–Euler polynomials and numbers. By taking $x_r = -\lambda_r e^{t_r}$ for $1 \leq r \leq n$ in (2.4), we have

$$(-1)^n \lambda_1 \lambda_2 \cdots \lambda_n e^{t_1+t_2+\cdots+t_n} - 1 = \sum_{r=1}^n (-1)^r (\lambda_r e^{t_r} + 1) \prod_{k=1}^{r-1} \lambda_k e^{t_k}. \quad (2.19)$$

It follows from (2.19) that

$$\begin{aligned} \prod_{i=1}^n \frac{2e^{x_i t_i}}{\lambda_i e^{t_i} + 1} &= \frac{1}{(-1)^n \lambda_1 \lambda_2 \cdots \lambda_n e^{t_1+t_2+\cdots+t_n} - 1} \\ & \quad \times \sum_{r=1}^n (-1)^r (\lambda_r e^{t_r} + 1) \prod_{k=1}^{r-1} \lambda_k e^{t_k} \prod_{i=1}^n \frac{2e^{x_i t_i}}{\lambda_i e^{t_i} + 1}. \end{aligned} \quad (2.20)$$

It is easy to see that

$$\begin{aligned} & (\lambda_r e^{t_r} + 1) \prod_{k=1}^{r-1} \lambda_k e^{t_k} \prod_{i=1}^n \frac{2e^{x_i t_i}}{\lambda_i e^{t_i} + 1} \\ &= 2e^{x_r(t_1+t_2+\cdots+t_n)} \prod_{i=1}^{r-1} \lambda_i \frac{2e^{(x_i-x_r+1)t_i}}{\lambda_i e^{t_i} + 1} \prod_{i=r+1}^n \frac{2e^{(x_i-x_r)t_i}}{\lambda_i e^{t_i} + 1}. \end{aligned} \quad (2.21)$$

If we apply (2.21) to (2.20), we obtain for non-negative integers $m_1, m_2, \dots, m_n$,

$$\begin{aligned}
& \left[\frac{t_1^{m_1}}{m_1!} \frac{t_2^{m_2}}{m_2!} \cdots \frac{t_n^{m_n}}{m_n!} \right] \left(\prod_{i=1}^n \frac{2e^{x_i t_i}}{\lambda_i e^{t_i} + 1} \right) \\
&= \sum_{r=1}^n (-1)^r \left[\frac{t_1^{m_1}}{m_1!} \frac{t_2^{m_2}}{m_2!} \cdots \frac{t_n^{m_n}}{m_n!} \right] \left(\frac{2e^{x_r(t_1+t_2+\cdots+t_n)}}{(-1)^n \lambda_1 \lambda_2 \cdots \lambda_n e^{t_1+t_2+\cdots+t_n} - 1} \right. \\
&\quad \times \prod_{i=1}^{r-1} \lambda_i \frac{2e^{(x_i-x_r+1)t_i}}{\lambda_i e^{t_i} + 1} \prod_{i=r+1}^n \frac{2e^{(x_i-x_r)t_i}}{\lambda_i e^{t_i} + 1} \Big). \tag{2.22}
\end{aligned}$$

We now split two cases to discuss (2.22). We first consider the case where n is an even integer. Clearly, by applying (1.2) and (2.13) to (2.22), we get for non-negative integers $m_1, m_2, \dots, m_n$ with n being an even integer and $\lambda_1 \lambda_2 \cdots \lambda_n \neq 1$,

$$\begin{aligned}
& \mathcal{E}_{m_1}(x_1; \lambda_1) \mathcal{E}_{m_2}(x_2; \lambda_2) \cdots \mathcal{E}_{m_n}(x_n; \lambda_n) \\
&= 2 \cdot m_1! \cdot m_2! \cdots m_n! \sum_{r=1}^n (-1)^r \frac{1}{k_1! \cdots k_{r-1}! \cdot m_r! \cdot k_{r+1}! \cdots k_n!} \\
&\quad \times \sum_{k_1, \dots, k_{r-1}, k_{r+1}, \dots, k_n \geq 0} \frac{\mathcal{B}_{k_1+\cdots+k_{r-1}+m_r+k_{r+1}+\cdots+k_n+1}(x_r; \lambda_1 \cdots \lambda_n)}{k_1 + \cdots + k_{r-1} + m_r + k_{r+1} + \cdots + k_n + 1} \\
&\quad \times \prod_{i=1}^{r-1} \lambda_i \frac{\mathcal{E}_{m_i-k_i}(x_i - x_r + 1; \lambda_i)}{(m_i - k_i)!} \prod_{i=r+1}^n \frac{\mathcal{E}_{m_i-k_i}(x_i - x_r; \lambda_i)}{(m_i - k_i)!}. \tag{2.23}
\end{aligned}$$

If n is an odd integer then from (1.3) and (2.11) we have

$$\begin{aligned}
& \frac{2e^{x_r(t_1+t_2+\cdots+t_n)}}{(-1)^n \lambda_1 \lambda_2 \cdots \lambda_n e^{t_1+t_2+\cdots+t_n} - 1} \\
&= - \sum_{N=0}^{\infty} \mathcal{E}_N(x_r; \lambda_1 \lambda_2 \cdots \lambda_n) \sum_{\substack{k_1+k_2+\cdots+k_n=N \\ k_1, k_2, \dots, k_n \geq 0}} \frac{t_1^{k_1}}{k_1!} \frac{t_2^{k_2}}{k_2!} \cdots \frac{t_n^{k_n}}{k_n!}. \tag{2.24}
\end{aligned}$$

It follows from (1.3), (2.14), (2.22) and (2.24) that for non-negative integers $m_1, m_2, \dots, m_n$ with n being an odd integer,

$$\begin{aligned}
& \mathcal{E}_{m_1}(x_1; \lambda_1) \mathcal{E}_{m_2}(x_2; \lambda_2) \cdots \mathcal{E}_{m_n}(x_n; \lambda_n) \\
&= m_1! \cdot m_2! \cdots m_n! \sum_{r=1}^n (-1)^{r-1} \\
&\quad \times \sum_{k_1, \dots, k_{r-1}, k_{r+1}, \dots, k_n \geq 0} \frac{\mathcal{E}_{k_1+\cdots+k_{r-1}+m_r+k_{r+1}+\cdots+k_n}(x_r; \lambda_1 \cdots \lambda_n)}{k_1! \cdots k_{r-1}! \cdot m_r! \cdot k_{r+1}! \cdots k_n!} \\
&\quad \times \prod_{i=1}^{r-1} \lambda_i \frac{\mathcal{E}_{m_i-k_i}(x_i - x_r + 1; \lambda_i)}{(m_i - k_i)!} \prod_{i=r+1}^n \frac{\mathcal{E}_{m_i-k_i}(x_i - x_r; \lambda_i)}{(m_i - k_i)!}. \tag{2.25}
\end{aligned}$$

Thus, by replacing k_i by $m_i - k_i$ for $i \neq r$ in (2.23) and (2.25), we obtain the following formulae for products of an arbitrary number of Apostol–Euler polynomials.

Theorem 2.2. Let $m_1, m_2, \dots, m_n$ be n non-negative integers. If n is an even integer, then

$$\begin{aligned} & \mathcal{E}_{m_1}(x_1; \lambda_1) \mathcal{E}_{m_2}(x_2; \lambda_2) \cdots \mathcal{E}_{m_n}(x_n; \lambda_n) \\ &= 2 \sum_{\substack{k_1+k_2+\dots+k_n=m_1+m_2+\dots+m_n \\ k_1, k_2, \dots, k_n \geq 0}} \sum_{r=1}^n (-1)^r \frac{\mathcal{B}_{k_r+1}(x_r; \lambda_1 \lambda_2 \cdots \lambda_n)}{k_r + 1} \\ & \quad \times \prod_{i=1}^{r-1} \binom{m_i}{k_i} \lambda_i \mathcal{E}_{k_i}(x_i - x_r + 1; \lambda_i) \\ & \quad \times \prod_{i=r+1}^n \binom{m_i}{k_i} \mathcal{E}_{k_i}(x_i - x_r; \lambda_i) \quad (\lambda_1 \lambda_2 \cdots \lambda_n \neq 1). \end{aligned} \quad (2.26)$$

If n is an odd integer, then

$$\begin{aligned} & \mathcal{E}_{m_1}(x_1; \lambda_1) \mathcal{E}_{m_2}(x_2; \lambda_2) \cdots \mathcal{E}_{m_n}(x_n; \lambda_n) \\ &= \sum_{\substack{k_1+k_2+\dots+k_n=m_1+m_2+\dots+m_n \\ k_1, k_2, \dots, k_n \geq 0}} \sum_{r=1}^n (-1)^{r-1} \mathcal{E}_{k_r}(x_r; \lambda_1 \lambda_2 \cdots \lambda_n) \\ & \quad \times \prod_{i=1}^{r-1} \binom{m_i}{k_i} \lambda_i \mathcal{E}_{k_i}(x_i - x_r + 1; \lambda_i) \prod_{i=r+1}^n \binom{m_i}{k_i} \mathcal{E}_{k_i}(x_i - x_r; \lambda_i). \end{aligned} \quad (2.27)$$

We now show some special cases of Theorem 2.2. Since the Apostol–Euler polynomials satisfy the symmetry relation $\lambda \mathcal{E}_n(1-x; \lambda) = (-1)^n \mathcal{E}_n(x; 1/\lambda)$ for non-negative integer n (see, e.g., [18]), by taking $n = 2$ in (2.26), we get for non-negative integers m, n and $\lambda \mu \neq 1$,

$$\begin{aligned} \mathcal{E}_m(x; \lambda) \mathcal{E}_n(y; \mu) &= 2 \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \mathcal{E}_{m-k}(y-x; 1/\lambda) \frac{\mathcal{B}_{n+k+1}(y; \lambda \mu)}{n+k+1} \\ & \quad - 2 \sum_{k=0}^n \binom{n}{k} \mathcal{E}_{n-k}(y-x; \mu) \frac{\mathcal{B}_{m+k+1}(x; \lambda \mu)}{m+k+1}. \end{aligned} \quad (2.28)$$

The identities (2.18) and (2.28) can be also found, for instance, in [16,17,32] where they were further generalized following the work of Carlitz [12], Agoh and Dilcher [3,4]. If we take $n = 3$ in (2.27), we obtain

$$\begin{aligned} & \mathcal{E}_{m_1}(x_1; \lambda_1) \mathcal{E}_{m_2}(x_2; \lambda_2) \mathcal{E}_{m_3}(x_3; \lambda_3) \\ &= \sum_{\substack{k_1+k_2+k_3=m_1+m_2+m_3 \\ k_1, k_2, k_3 \geq 0}} \left\{ \binom{m_2}{k_2} \binom{m_3}{k_3} \mathcal{E}_{k_1}(x_1; \mu) \mathcal{E}_{k_2}(x_2 - x_1; \lambda_2) \mathcal{E}_{k_3}(x_3 - x_1; \lambda_3) \right. \\ & \quad - \binom{m_1}{k_1} \binom{m_3}{k_3} \mathcal{E}_{k_2}(x_2; \mu) (-1)^{k_1} \mathcal{E}_{k_1}(x_2 - x_1; 1/\lambda_1) \mathcal{E}_{k_3}(x_3 - x_2; \lambda_3) \\ & \quad + \binom{m_1}{k_1} \binom{m_2}{k_2} \mathcal{E}_{k_3}(x_3; \mu) (-1)^{k_1} \mathcal{E}_{k_1}(x_3 - x_1; 1/\lambda_1) \\ & \quad \left. \times (-1)^{k_2} \mathcal{E}_{k_2}(x_3 - x_2; 1/\lambda_2) \right\}, \end{aligned} \quad (2.29)$$

where m_1, m_2, m_3 are non-negative integers and $\mu = \lambda_1 \lambda_2 \lambda_3$.

Remark 2.3. Note that (2.27) does not require the condition $\lambda_1 \lambda_2 \cdots \lambda_n \neq 1$. However, we were unable to obtain formulae analogous to (2.17) and (2.26) in the case $\lambda_1 \lambda_2 \cdots \lambda_n = 1$.

3. Higher-order convolution formulae

In this section, we give some higher-order convolution formulae for Apostol–Bernoulli and Apostol–Euler polynomials, which are the corresponding generalizations of Agoh and Dilcher’s results on the classical Bernoulli and Euler polynomials described in [5]. The proofs depend on (2.8) and (2.22). Obviously, substituting k for n and $u_i t$ for t_i with $u_1 + u_2 + \cdots + u_k = 1$ in (2.8) gives

$$\begin{aligned} \prod_{i=1}^k \frac{u_i t e^{x_i u_i t}}{\lambda_i e^{u_i t} - 1} &= \sum_{r=1}^k \frac{u_r t e^{x_r t}}{\lambda_1 \lambda_2 \cdots \lambda_k e^t - 1} \prod_{i=1}^{r-1} \lambda_i \frac{u_i t e^{(x_i - x_r + 1) u_i t}}{\lambda_i e^{u_i t} - 1} \\ &\quad \times \prod_{i=r+1}^k \frac{u_i t e^{(x_i - x_r) u_i t}}{\lambda_i e^{u_i t} - 1}. \end{aligned} \quad (3.1)$$

It is easy to see from (1.2) that

$$\begin{aligned} \left[\frac{t^n}{n!} \right] \left(\prod_{i=1}^k \frac{u_i t e^{x_i u_i t}}{\lambda_i e^{u_i t} - 1} \right) &= \sum_{\substack{j_1 + j_2 + \cdots + j_k = n \\ j_1, j_2, \dots, j_k \geq 0}} \frac{n! \cdot u_1^{j_1} u_2^{j_2} \cdots u_k^{j_k}}{j_1! \cdot j_2! \cdots j_k!} \\ &\quad \times \mathcal{B}_{j_1}(x_1; \lambda_1) \mathcal{B}_{j_2}(x_2; \lambda_2) \cdots \mathcal{B}_{j_k}(x_k; \lambda_k). \end{aligned} \quad (3.2)$$

On the other hand, from (3.1) we get

$$\begin{aligned} \left[\frac{t^n}{n!} \right] \left(\prod_{i=1}^k \frac{u_i t e^{x_i u_i t}}{\lambda_i e^{u_i t} - 1} \right) &= \sum_{r=1}^k \sum_{\substack{j_1 + j_2 + \cdots + j_k = n \\ j_1, j_2, \dots, j_k \geq 0}} \frac{n! \cdot u_1^{j_1} u_2^{j_2} \cdots u_{r-1}^{j_{r-1}} u_r^{j_r + 1} u_{r+1}^{j_{r+1}} \cdots u_k^{j_k}}{j_1! \cdot j_2! \cdots j_k!} \\ &\quad \times \mathcal{B}_{j_r}(x_r; \lambda_1 \lambda_2 \cdots \lambda_k) \prod_{i=1}^{r-1} \lambda_i \mathcal{B}_{j_i}(x_i - x_r + 1; \lambda_i) \prod_{i=r+1}^k \mathcal{B}_{j_i}(x_i - x_r; \lambda_i). \end{aligned} \quad (3.3)$$

Notice that for positive integers $k \geq 2$ and complex numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ with $\operatorname{Re}(\alpha_j) > -1$ for $j = 1, 2, \dots, k$ (see, e.g., [31])

$$\begin{aligned} &\int_0^1 \int_0^{1-u_1} \cdots \int_0^{1-u_1-\cdots-u_{k-2}} u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_k^{\alpha_k} du_1 du_2 \cdots du_{k-1} \\ &= \frac{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1) \cdots \Gamma(\alpha_k + 1)}{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_k + k)} \quad (u_1 + u_2 + \cdots + u_k = 1). \end{aligned} \quad (3.4)$$

Hence, by equating (3.2) and (3.3), with the help of (3.4), we obtain

$$\begin{aligned} &(n+k) \sum_{\substack{j_1 + j_2 + \cdots + j_k = n \\ j_1, j_2, \dots, j_k \geq 0}} \mathcal{B}_{j_1}(x_1; \lambda_1) \mathcal{B}_{j_2}(x_2; \lambda_2) \cdots \mathcal{B}_{j_k}(x_k; \lambda_k) \\ &= \sum_{r=1}^k \sum_{\substack{j_1 + j_2 + \cdots + j_k = n \\ j_1, j_2, \dots, j_k \geq 0}} \binom{n+k}{j_r} \mathcal{B}_{j_r}(x_r; \lambda_1 \lambda_2 \cdots \lambda_k) \\ &\quad \times \prod_{i=1}^{r-1} \lambda_i \mathcal{B}_{j_i}(x_i - x_r + 1; \lambda_i) \prod_{i=r+1}^k \mathcal{B}_{j_i}(x_i - x_r; \lambda_i). \end{aligned} \quad (3.5)$$

Since the Apostol–Bernoulli polynomials satisfy the difference equation $\lambda \mathcal{B}_n(x+1; \lambda) - \mathcal{B}_n(x; \lambda) = nx^{n-1}$ for non-negative integer n (see, e.g., [20]), then we get

$$\prod_{i=1}^{r-1} \lambda_i \mathcal{B}_{j_i}(x_i - x_r + 1; \lambda_i) = \sum_{T \subseteq \{1, \dots, r-1\}} \prod_{i \in T} \mathcal{B}_{j_i}(x_i - x_r; \lambda_i) \prod_{i \in \bar{T}} j_i(x_i - x_r)^{j_i-1}. \quad (3.6)$$

Thus, by taking $x_1 = x_2 = \dots = x_k = x$ in (3.5), the following higher-order convolution formula for Apostol–Bernoulli polynomials follows immediately from (3.6).

Theorem 3.1. *Let k be a positive integer. Then for non-negative integers $n \geq k-1$,*

$$\begin{aligned} (n+k) & \sum_{\substack{j_1+j_2+\dots+j_k=n \\ j_1, j_2, \dots, j_k \geq 0}} \mathcal{B}_{j_1}(x; \lambda_1) \mathcal{B}_{j_2}(x; \lambda_2) \cdots \mathcal{B}_{j_k}(x; \lambda_k) \\ &= \sum_{r=1}^k \binom{k}{r-1} \sum_{\substack{j_1+j_2+\dots+j_r=n-k+r \\ j_1, j_2, \dots, j_r \geq 0}} \binom{n+k}{j_r} \mathcal{B}_{j_r}(x; \lambda_1 \lambda_2 \cdots \lambda_k) \\ & \quad \times \mathcal{B}_{j_1}(\lambda_1) \mathcal{B}_{j_2}(\lambda_2) \cdots \mathcal{B}_{j_{r-1}}(\lambda_{r-1}). \end{aligned} \quad (3.7)$$

It is obvious that the case $\lambda_1 = \lambda_2 = \dots = \lambda_k = 1$ in Theorem 3.1 gives the corresponding result of Agoh and Dilcher [5, Theorem 1] on the classical Bernoulli polynomials, namely

$$\begin{aligned} (n+k) & \sum_{\substack{j_1+j_2+\dots+j_k=n \\ j_1, j_2, \dots, j_k \geq 0}} B_{j_1}(x) B_{j_2}(x) \cdots B_{j_k}(x) \\ &= \sum_{r=0}^{k-1} \binom{k}{r} \sum_{\substack{j_0+j_1+\dots+j_r=n-k+1+r \\ j_0, j_1, \dots, j_r \geq 0}} \binom{n+k}{j_0} B_{j_0}(x) B_{j_1} B_{j_2} \cdots B_{j_r}, \end{aligned} \quad (3.8)$$

which is a more general form of the following result due to Matiyasevich [2,22,25]

$$(n+2) \sum_{k=2}^{n-2} B_k B_{n-k} - 2 \sum_{k=2}^{n-2} \binom{n+2}{k} B_k B_{n-k} = n(n+1) B_n \quad (n \geq 4). \quad (3.9)$$

We next give two higher-order convolution formulae for Apostol–Euler polynomials. In a consideration similar to (3.1), from (2.22) we discover for $u_1 + u_2 + \dots + u_k = 1$,

$$\begin{aligned} \left[\frac{t^n}{n!} \right] \left(\prod_{i=1}^k \frac{2e^{x_i u_i t}}{\lambda_i e^{u_i t} + 1} \right) &= \sum_{r=1}^k (-1)^r \left[\frac{t^n}{n!} \right] \left(\frac{2e^{x_r t}}{(-1)^k \lambda_1 \lambda_2 \cdots \lambda_k e^t - 1} \right. \\ & \quad \left. \times \prod_{i=1}^{r-1} \lambda_i \frac{2e^{(x_i - x_r + 1) u_i t}}{\lambda_i e^{u_i t} + 1} \prod_{i=r+1}^k \frac{2e^{(x_i - x_r) u_i t}}{\lambda_i e^{u_i t} + 1} \right), \end{aligned} \quad (3.10)$$

which together with (1.2) and (1.3) means that if k is an even integer then

$$\begin{aligned}
& \sum_{\substack{j_1+j_2+\dots+j_k=n \\ j_1, j_2, \dots, j_k \geq 0}} \frac{n! \cdot u_1^{j_1} u_2^{j_2} \dots u_k^{j_k}}{j_1! \cdot j_2! \dots j_k!} \mathcal{E}_{j_1}(x_1; \lambda_1) \mathcal{E}_{j_2}(x_2; \lambda_2) \dots \mathcal{E}_{j_k}(x_k; \lambda_k) \\
&= -2 \sum_{r=1}^k \sum_{\substack{j_1+j_2+\dots+j_k=n+1 \\ j_1, j_2, \dots, j_k \geq 0}} \frac{n! \cdot u_1^{j_1} u_2^{j_2} \dots u_{r-1}^{j_{r-1}} u_r^0 u_{r+1}^{j_{r+1}} \dots u_k^{j_k}}{j_1! \cdot j_2! \dots j_k!} \\
&\quad \times \mathcal{B}_{j_r}(x_r; \lambda_1 \lambda_2 \dots \lambda_k) \prod_{i=1}^{r-1} \{-\lambda_i \mathcal{E}_{j_i}(x_i - x_r + 1; \lambda_i)\} \\
&\quad \times \prod_{i=r+1}^k \mathcal{E}_{j_i}(x_i - x_r; \lambda_i), \tag{3.11}
\end{aligned}$$

and if k is an odd integer then

$$\begin{aligned}
& \sum_{\substack{j_1+j_2+\dots+j_k=n \\ j_1, j_2, \dots, j_k \geq 0}} \frac{n! \cdot u_1^{j_1} u_2^{j_2} \dots u_k^{j_k}}{j_1! \cdot j_2! \dots j_k!} \mathcal{E}_{j_1}(x_1; \lambda_1) \mathcal{E}_{j_2}(x_2; \lambda_2) \dots \mathcal{E}_{j_k}(x_k; \lambda_k) \\
&= \sum_{r=1}^k \sum_{\substack{j_1+j_2+\dots+j_k=n \\ j_1, j_2, \dots, j_k \geq 0}} \frac{n! \cdot u_1^{j_1} u_2^{j_2} \dots u_{r-1}^{j_{r-1}} u_r^0 u_{r+1}^{j_{r+1}} \dots u_k^{j_k}}{j_1! \cdot j_2! \dots j_k!} \\
&\quad \times \mathcal{E}_{j_r}(x_r; \lambda_1 \lambda_2 \dots \lambda_k) \prod_{i=1}^{r-1} \{-\lambda_i \mathcal{E}_{j_i}(x_i - x_r + 1; \lambda_i)\} \\
&\quad \times \prod_{i=r+1}^k \mathcal{E}_{j_i}(x_i - x_r; \lambda_i). \tag{3.12}
\end{aligned}$$

Since the Apostol–Euler polynomials satisfy the difference equation $\lambda \mathcal{E}_n(x+1; \lambda) + \mathcal{E}_n(x; \lambda) = 2x^n$ for non-negative integer n (see, e.g., [18]), we obtain

$$\begin{aligned}
& \prod_{i=1}^{r-1} \{-\lambda_i \mathcal{E}_{j_i}(x_i - x_r + 1; \lambda_i)\} \\
&= \sum_{T \subseteq \{1, \dots, r-1\}} \prod_{i \in T} \mathcal{E}_{j_i}(x_i - x_r; \lambda_i) \prod_{i \in \bar{T}} \{-2(x_i - x_r)^{j_i}\}. \tag{3.13}
\end{aligned}$$

Thus, by setting $x_1 = x_2 = \dots = x_k = x$ in (3.11) and (3.12), with the help of (3.4) and (3.13), we get the following result.

Theorem 3.2. *Let k, n be positive integers. If k is an even integer, then*

$$\begin{aligned}
& (n+k) \sum_{\substack{j_1+j_2+\dots+j_k=n \\ j_1, j_2, \dots, j_k \geq 0}} \mathcal{E}_{j_1}(x; \lambda_1) \mathcal{E}_{j_2}(x; \lambda_2) \dots \mathcal{E}_{j_k}(x; \lambda_k) \\
&= \sum_{r=1}^k \binom{k}{r-1} (-2)^{k-r+1} \sum_{\substack{j_1+j_2+\dots+j_r=n+1 \\ j_1, j_2, \dots, j_r \geq 0}} \binom{n+k}{j_r} \mathcal{B}_{j_r}(x; \lambda_1 \lambda_2 \dots \lambda_k) \\
&\quad \times \mathcal{E}_{j_1}(0; \lambda_1) \mathcal{E}_{j_2}(0; \lambda_2) \dots \mathcal{E}_{j_{r-1}}(0; \lambda_{r-1}). \tag{3.14}
\end{aligned}$$

If k is an odd integer, then

$$\begin{aligned} & \sum_{\substack{j_1+j_2+\dots+j_k=n \\ j_1, j_2, \dots, j_k \geq 0}} \mathcal{E}_{j_1}(x; \lambda_1) \mathcal{E}_{j_2}(x; \lambda_2) \cdots \mathcal{E}_{j_k}(x; \lambda_k) \\ &= \sum_{r=1}^k \binom{k}{r-1} (-2)^{k-r} \sum_{\substack{j_1+j_2+\dots+j_r=n \\ j_1, j_2, \dots, j_r \geq 0}} \binom{n+k-1}{j_r} \mathcal{E}_{j_r}(x; \lambda_1 \lambda_2 \cdots \lambda_k) \\ & \quad \times \mathcal{E}_{j_1}(0; \lambda_1) \mathcal{E}_{j_2}(0; \lambda_2) \cdots \mathcal{E}_{j_{r-1}}(0; \lambda_{r-1}). \end{aligned} \quad (3.15)$$

We next give some special cases of [Theorem 3.2](#). By setting $\lambda_1 = \lambda_2 = \cdots = \lambda_k = 1$ in [Theorem 3.2](#), we get the corresponding higher-order convolution formulae for the classical Euler polynomials due to Agoh and Dilcher [[5](#), [Theorem 2](#), [Theorem 3](#)]: if k is an even integer then

$$\begin{aligned} & (n+k) \sum_{\substack{j_1+j_2+\dots+j_k=n \\ j_1, j_2, \dots, j_k \geq 0}} E_{j_1}(x) E_{j_2}(x) \cdots E_{j_k}(x) \\ &= \sum_{r=0}^{k-1} (-1)^r \binom{k}{r} 2^{k-r} \sum_{\substack{j_0+j_1+\dots+j_r=n+1 \\ j_0, j_1, \dots, j_r \geq 0}} \binom{n+k}{j_0} B_{j_0}(x) \\ & \quad \times E_{j_1}(0) E_{j_2}(0) \cdots E_{j_r}(0), \end{aligned} \quad (3.16)$$

and if k is an odd integer then

$$\begin{aligned} & \sum_{\substack{j_1+j_2+\dots+j_k=n \\ j_1, j_2, \dots, j_k \geq 0}} E_{j_1}(x) E_{j_2}(x) \cdots E_{j_k}(x) \\ &= \sum_{r=0}^{k-1} (-1)^r \binom{k}{r} 2^{k-1-r} \sum_{\substack{j_0+j_1+\dots+j_r=n \\ j_0, j_1, \dots, j_r \geq 0}} \binom{n+k-1}{j_0} E_{j_0}(x) \\ & \quad \times E_{j_1}(0) E_{j_2}(0) \cdots E_{j_r}(0). \end{aligned} \quad (3.17)$$

If we take $k = 2$ in [\(3.14\)](#), we get for non-negative integer n ,

$$\begin{aligned} & (n+2) \sum_{k=0}^n \mathcal{E}_k(x; \lambda) \mathcal{E}_{n-k}(x; \mu) \\ &= 4(n+2) \mathcal{B}_{n+1}(x; \lambda \mu) - 4 \sum_{k=0}^{n+1} \binom{n+2}{k+1} \mathcal{B}_{n+1-k}(x; \lambda \mu) \mathcal{E}_k(0; \lambda). \end{aligned} \quad (3.18)$$

The case $\lambda = \mu = 1$ in [\(3.18\)](#) gives for non-negative integer n (see, e.g., [[5](#), [Corollary 2](#)])

$$(n+2) \sum_{k=0}^n E_k(x) E_{n-k}(x) = 4(n+2) B_{n+1}(x) - 4 \sum_{k=0}^{n+1} \binom{n+2}{k} B_k(x) E_k(0). \quad (3.19)$$

On the other hand, since $E_n(0) = 2(1 - 2^{n+1})B_{n+1}/(n+1)$ for non-negative integers n (see, e.g., [[1](#)]), by setting $\lambda = \mu = 1$ in [\(3.18\)](#), we get for non-negative integer n ,

$$(n+2) \sum_{k=0}^n E_k(x) E_{n-k}(x) = 8 \sum_{k=2}^{n+2} \binom{n+2}{k} (2^k - 1) \frac{B_k}{k} B_{n+2-k}(x), \quad (3.20)$$

which was discovered by Pan and Sun [25, Corollary 2.3].

Acknowledgments

The author expresses his gratitude to the anonymous referee for his/her valuable and detailed comments, which have led to a significant improvement on the presentation of this paper. This work is supported by the Foundation for Fostering Talents in Kunming University of Science and Technology (Grant No. KKSYS201307047) and the National Natural Science Foundation of China (Grant Nos. 11326050, 11071194).

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