



The lack of exponential stability of the hybrid Bresse system



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ABSTRACT

In this paper we consider the dissipative Bresse system with an attached hollow-tip body that contains granular material, the resulting model being called hybrid system. Our main result is to show that the corresponding model is not exponentially stable. Moreover, we show that the solution decays polynomially to zero as $(1+t)^{-1/2}$ when $t \rightarrow \infty$.

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1. Introduction

In this paper we consider the Bresse system with an attached hollow-tip body that contains granular material. Standard models from the theory of elasticity for the vibration of structures of this type involve hybrid systems of coupled partial and ordinary differential equations which describe the motion of the beam and tip bodies, respectively (Fig. 1). The Bresse system or the circular arc problem is modeled by the following system

$$\rho_1 \varphi_{tt} = Q_x + lN + F_1 \quad (1.1)$$

$$\rho_2 \psi_{tt} = M_x - Q + F_2 \quad (1.2)$$

$$\rho_1 w_{tt} = N_x - lQ + F_3 \quad (1.3)$$

where

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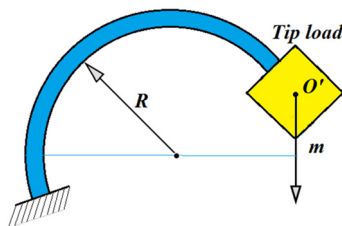


Fig. 1. Bresse beam with tip body.

$$Q = \kappa(\varphi_x + lw + \psi), \quad M = b\psi_x, \quad N = \kappa_0(w_x - l\varphi), \quad (1.4)$$

We use Q , M and N , to denote the shear force, the bending moment and the axial force, where w , φ and ψ are the longitudinal, vertical and shear angle displacements. Therefore the evolution equations are

$$\rho_1 \varphi_{tt} - \kappa(\varphi_x + \psi + lw)_x - \kappa_0 l(w_x - l\varphi) = 0 \quad \text{in }]0, L[\times]0, \infty[\quad (1.5)$$

$$\rho_2 \psi_{tt} - b\psi_{xx} + \kappa(\varphi_x + \psi + lw) = 0 \quad \text{in }]0, L[\times]0, \infty[\quad (1.6)$$

$$\rho_1 w_{tt} - \kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x + \psi + lw) = 0 \quad \text{in }]0, L[\times]0, \infty[\quad (1.7)$$

Here, $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\kappa_0 = EA$, $\kappa = k'GA$, $b = EI$ and $l = R^{-1}$. By ρ we denote the density, E is the elastic modulus, G is the shear modulus, k' is the shear factor, A is the cross-sectional area, I is the second moment of area of the cross-section and R is the radius of curvature of the beam. Here we assume that all the above coefficients are positive constants. When the curvature l is zero, the beam is configured over an interval and this model reduces to the well-known Timoshenko beam system. Therefore we can consider the Bresse system as an extension of the Timoshenko's model. To define a unique solution we prescribe the initial conditions

$$\begin{aligned} \varphi(x, 0) &= \varphi_0(x), & \psi(x, 0) &= \psi_0(x), & w(x, 0) &= w_0(x) \\ \varphi_t(x, 0) &= \varphi_1(x), & \psi_t(x, 0) &= \psi_1(x), & w_t(x, 0) &= w_1(x). \end{aligned} \quad (1.8)$$

The boundary condition are given by

$$\varphi(0, t) = \psi(0, t) = w(0, t) = 0 \quad \text{in }]0, \infty[\quad (1.9)$$

For the right end, we assume that the container is rigidly attached at $x = L$ with mass m and a center of mass O' located at distance d from the beam end. We assume that the damping effect of the internal granular material can be represented by damping coefficients τ_1 , τ_2 and τ_3 for φ , ψ and w respectively. Thus, the force balance at the end $x = L$ is

$$m_1 \varphi_{tt}(L, t) + \tau_1 \varphi_t(L, t) + \kappa[\varphi_x(L, t) + \psi(L, t) + lw(L, t)] = 0 \quad \text{in }]0, \infty[\quad (1.10)$$

$$I_m \psi_{tt}(L, t) + \tau_2 \psi_t(L, t) + b\psi_x(L, t) = 0 \quad \text{in }]0, \infty[\quad (1.11)$$

$$m_2 w_{tt}(L, t) + \tau_3 w_t(L, t) + \kappa_0[w_x(L, t) - l\varphi(L, t)] = 0 \quad \text{in }]0, \infty[\quad (1.12)$$

The energy associated to the above system $E(t) = E(\varphi, \psi, w; t)$ is given by

$$\begin{aligned} E(t) &= \frac{1}{2} \int_0^L \left[\rho_1 |\varphi_t|^2 + \rho_2 |\psi_t|^2 + b |\psi_x|^2 + \rho_1 |w_t|^2 + \kappa |\varphi_x + \psi + lw|^2 + \kappa_0 |w_x - l\varphi|^2 \right] dx \\ &\quad + m |\varphi_t(L, t)|^2 + I_m |\psi_t(L, t)|^2 + \rho_2 |w_t(L, t)|^2. \end{aligned} \quad (1.13)$$

The energy identity is given by

$$\frac{d}{dt}E(t) = -\tau_1|\varphi_t(L, t)|^2 - \tau_2|\psi_t(L, t)|^2 - \tau_3|w_t(L, t)|^2.$$

Several works have studied the exponential stability of the Timoshenko's system. In the seminal paper [18], Kim and Renardy used two right end control by time derivatives of φ and ψ to get the exponential stability. This result was improved in [4] using one boundary control in the rotation-angle equation, provided the wave speed of the system are equals. Recently, Han and Xu [15] have extended this result for time delay boundary feedbacks. For other types of dissipations see [30,39,40]. Also other types of controls have been used for obtaining exponential stabilities: Energy dissipation by thermo-viscoelastic damping [25], pointwise controls [38], memory dissipation [2,11,22,26] among others. Moreover, general types of stabilities can be obtained by internal or boundary feedbacks [23], thermoelastic system [16], weakly nonlinear dissipations [24], nonuniform beams [3], and weakly damped systems [27].

Concerning hybrid Timoshenko system M. Grobbelaar-Van Dalsen [13], proved the exponential stability for a class of solutions verifying the so called *condition Z*. The point is that such class of solutions are very particular one. We believe that in general such condition is not true. To a numerical approach for calculating the eigenvalues see [41].

Since the pioneer work of Bresse [6], in contrast to Timoshenko system, the number of works related this model is not so large and almost all about the stationary problem and numerical analysis. For models involving evolutions problems of the Bresse system the number of articles is even less, especially such dealing with qualitative properties. We have for example [1,7,9,10,19,21,28,33,34,36]. In none of the above references was considered hybrid dissipative boundary condition.

Concerning the lack of exponential stability Russell [32], showed that if the dissipative mechanism is defined by a compact operator then the decay cannot be exponentially, see also [12,14]. Triggiani [35] improve the above result by showing that if the semigroup is strongly stable but not exponentially stable, then any compact perturbation of the infinitesimal generator cannot generate and exponentially stable semigroup. Finally, Prüss [29] gives a fully characterization to contractions semigroups over Hilbert spaces. That is to say: *A contraction semigroup e^{At} is exponentially stable if and only if the resolvent operator of A is uniformly bounded over the whole imaginary axes.* Even so, such characterization, in practice is not easy to verify in general. The main method used to achieve the Prüss hypotheses is to find exact solutions of the resolvent system and to show that they are not uniformly bounded over the imaginary axes, see for example [1,9,11]. Prüss' result cannot be applied to our model, therefore we use another approach, based on the Weyl's Theorem.

The main result of this paper is to prove that the system (1.5)–(1.12) is not exponentially stable. This result has important consequences to the optimal design problem (that is given two materials at our disposal with different elastic characteristic and different density, the problem consists of finding the best distributions of the two initial materials in a rod to minimize the vibration energy in the structure under periodic loading of the driving frequency). Our result in particular implies that inserting a tip body into the material does not produce a fast stabilization.

In addition to the lack of exponential stability of the hybrid Bresse system, we show that the solution decays polynomially to zero, with rates that depends on the regularity of the initial data. That is the semigroup $S(t)$ associated to system (1.5)–(1.12) decays polynomially as

$$\|S(t)U_0\|_{\mathcal{H}} \leq ct^{-1/2}\|S(t)U_0\|_{D(\mathcal{A})}.$$

Using standard semigroups arguments the above inequality can be extended to

$$\|S(t)U_0\|_{\mathcal{H}} \leq ct^{-k/2}\|S(t)U_0\|_{D(\mathcal{A}^k)},$$

for any $k \in \mathbb{N}$. In other words, the more regularity the initial data, the faster the energy decays. Let us denote by $\omega(S)$, $\omega_\sigma(\mathcal{A})$ and $\omega_{ess}(S)$ the type of the semigroup $S(t)$, the spectrum upper bound of the infinitesimal generator \mathcal{A} of $S(t)$ and the essential type of the semigroup $S(t)$ respectively, then we have

$$\omega(S) = \max \{ \omega_\sigma(A), \omega_{ess}(S) \}. \quad (1.14)$$

(See [8], Corollary 2.11, p. 258.) Since the infinitesimal generator A is dissipative we have that $\omega_\sigma(A) \leq 0$. Therefore by (1.14), to prove the lack of exponential stability of S we only need to show that $\omega_{ess}(S) = 0$. To this purpose we use the Weyl's Theorem (Theorem XIII.14, [31]). This theorem establish that *If the difference of two operators is compact, then the essential spectrum radii are the same*. More precisely

Theorem 1.1. *Let S and T be two continuous operator over a Banach space X , if $S - T$ is a compact operator then S and T have the same essential spectrum radii.*

This theorem was proved by Weyl [37] to self adjoint operators. We refer to Kato's book [17] (Theorem 5.35, p. 244) for details of the proof.

On the other hand, to show the polynomial decay we use the theorem due to Borichev and Tomilov [5] (see Theorem 4.1 stated below).

The remaining part of this paper is organized as follows. In section 2, we show the well posedness of the model. In section 3, we show the lack of exponential stability and finally, in section 4 we show the polynomial decay of the semigroup associated to the Bresse's model with tip.

2. Well posedness

Here we consider the phase space given by

$$\mathcal{H} = H_L^1(0, L) \times L^2(0, L) \times H_L^1(0, L) \times L^2(0, L) \times H_L^1(0, L) \times L^2(0, L) \times \mathbb{C}^3$$

where

$$H_L^1(0, L) = \{f \in H^1(0, L); f(0) = 0\},$$

with norm given by

$$\begin{aligned} \|U\|_{\mathcal{H}}^2 = & \int_0^L \left[\rho_1 |\Phi|^2 + \rho_2 |\Psi|^2 + \rho_1 |W|^2 + \kappa |\varphi_x + \psi + lw|^2 + b |\psi_x|^2 + \kappa_0 |w_x - l\varphi|^2 \right. \\ & \left. + m_1 |u|^2 + I_m |v|^2 + m_2 |z|^2 \right] dx. \end{aligned} \quad (2.15)$$

For any U given by

$$U^t = (\varphi, \Phi, \psi, \Psi, w, W, u, v, w).$$

The operator \mathcal{A} will be given by

$$\mathcal{A} = \begin{pmatrix} A & \mathbf{0} \\ B & C \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ k/\rho_1 \partial_x^2 - k_0 l^2 I/\rho_1 & 0 & k/\rho_1 \partial_x & 0 & (k + k_0)/\rho_1 l \partial_x & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -k/\rho_2 \partial_x & 0 & b/\rho_2 \partial_x^2 - k/\rho_2 I & 0 & -kl/\rho_2 I & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -(k_0 + k)l/\rho_1 \partial_x & 0 & -lkI/\rho_1 & 0 & k_0/\rho_1 \partial_x^2 - l^2 kI & 0 \end{pmatrix}. \quad (2.16)$$

$$B = - \begin{pmatrix} \frac{k}{m_1} \gamma_1 & 0 & \frac{k}{m_1} \gamma_0 & 0 & \frac{kl}{m_1} \gamma_0 & 0 \\ 0 & 0 & \frac{b}{I_m} \gamma_1 & 0 & 0 & 0 \\ -\frac{l\kappa_0}{m_2} \gamma_0 & 0 & 0 & 0 & \frac{\kappa_0}{m_2} \gamma_0 & 0 \end{pmatrix}, \quad C = - \begin{pmatrix} \frac{\tau_1}{m_1} & 0 & 0 \\ 0 & \frac{\tau_2}{m_2} & 0 \\ 0 & 0 & \frac{\tau_3}{m_3} \end{pmatrix}, \quad \mathbf{0} = \mathbf{0}_{3 \times 6} \quad (2.17)$$

with domain defined as

$$D(\mathcal{A}) = \{U \in \mathcal{V} \cap \mathcal{H}; \quad \Phi(L) = u, \quad \Psi(L) = v, \quad W(L) = z\}$$

and

$$\mathcal{V} = H^2(0, L) \times H_L^1(0, L) \times H^2(0, L) \times H_L^1(0, L) \times H^2(0, L) \times H_L^1(0, L) \times \mathbb{C}^3.$$

Under this notations the system (1.5)–(1.12) can be written as

$$U_t = \mathcal{A}U, \quad U(0) = U_0.$$

Note that \mathcal{A} is a dissipative operator, in fact

$$\operatorname{Re}(\mathcal{A}U, U)_{\mathcal{H}} = -m_1|u|^2 - I_m|v|^2 - m_2|z|^2. \quad (2.18)$$

Now we are able to show the well posedness of the model.

Theorem 2.1. *The operator \mathcal{A} is the infinitesimal generator of a C_0 semigroup of contractions.*

Proof. Since \mathcal{A} is a dissipative operator, it is enough to show that $0 \in \varrho(\mathcal{A})$, the resolvent set of \mathcal{A} (see [20]). This is equivalent to show that for any $F \in \mathcal{H}$, there exists only one solution $U \in D(\mathcal{A})$ such that

$$i\lambda U - \mathcal{A}U = F,$$

where

$$F^t = (f_\varphi, f_\Phi, f_\psi, f_\Psi, f_w, f_W, f_u, f_v, f_z),$$

for $\lambda = 0$. In terms of its component we have

$$i\lambda\rho_1\varphi - \Phi = f_\varphi. \quad (2.19)$$

$$i\lambda\Phi - \kappa(\varphi_x + \psi + lw)_x - \kappa_0 l(w_x - l\varphi) = f_\Phi. \quad (2.20)$$

$$i\lambda\psi - \Psi = f_\psi. \quad (2.21)$$

$$i\lambda\rho_2\Psi - b\psi_{xx} - \kappa(\varphi_x + \psi + lw) = f_\Psi. \quad (2.22)$$

$$i\lambda w - W = f_w. \quad (2.23)$$

$$i\lambda\rho_1 W - \kappa_0(w_x - l\varphi)_x - \kappa l(\varphi_x + \psi + lw) = f_W. \quad (2.24)$$

$$i\lambda m_1 u + \tau_1 u + \kappa(\varphi_x + \psi + lw)(L) = f_u. \quad (2.25)$$

$$i\lambda I_m v + \tau_2 v + b\psi_x(L) = f_v. \quad (2.26)$$

$$i\lambda m_2 z + \tau_3 z + k_0(w_x - l\varphi)(L) = f_z. \quad (2.27)$$

Taking $\lambda = 0$, we get

$$\Phi = f_\varphi, \quad \Psi = f_\psi, \quad W = f_w.$$

$$\kappa(\varphi_x + \psi + lw)_x + \kappa_0 l(w_x - l\varphi) = f_\Phi \quad \text{in }]0, L[\times]0, \infty[\quad (2.28)$$

$$b\psi_{xx} + \kappa(\varphi_x + \psi + lw) = f_\Psi \quad \text{in }]0, L[\times]0, \infty[\quad (2.29)$$

$$\kappa_0(w_x - l\varphi)_x + \kappa l(\varphi_x + \psi + lw) = f_W \quad \text{in }]0, L[\times]0, \infty[\quad (2.30)$$

satisfying

$$\varphi(0) = \psi(0) = w(0) = 0.$$

Since $u = \Phi(L)$, $v = \Psi(L)$, $z = W(L)$, using (2.19), (2.21), (2.23) we get

$$\kappa(\varphi_x + \psi + lw)_x(L) = -\tau_1 f_\varphi(L), \quad b\psi_x(L) = -\tau_2 f_\psi(L), \quad \kappa_0(w_x - l\varphi)_x(L) = -\tau_1 f_z(L)$$

Using the Lions's Stampacchia theorem we prove that the above system has only one weak solution verifying

$$(\varphi, \psi, w) \in H_L^1(0, L) \times H_L^1(0, L) \times H_L^1(0, L).$$

Finally, using the equation we conclude that $U \in D(\mathcal{A})$. Therefore $0 \in \mathcal{D}(\mathcal{A})$. \square

As a consequence of the above result we have

Theorem 2.2. *Let us denote by*

$$U_0^t = (\varphi_0, \Phi_0, \psi_0, \Psi, w, W, u_0, v_0, z_0),$$

then for any $U_0 \in \mathcal{H}$ there exists only one weak solution (mild solution) of system (1.5)–(1.12) satisfying

$$U^t = (\varphi, \varphi_t, \psi, \psi_t, w, w_t, u, v, z) \in C(0, T; \mathcal{H}).$$

Instead is $U_0 \in D(\mathcal{A})$ we have that

$$U \in C^1(0, T; \mathcal{H}) \cap C(0, T; D(\mathcal{A})).$$

We finish this section establishing an observability result to Bresse system. To do that let us introduce the following notations.

$$I_\varphi(L, t) = |\varphi_t(L, t)|^2 + |\varphi_x(L, t) + \psi(L) + lw(L, t)|^2 \quad (2.31)$$

$$I_\psi(L, t) = |\psi_t(L, t)|^2 + |\psi_x(L, t)|^2 \quad (2.32)$$

$$I_w(L, t) = |w_t(L, t)|^2 + |w_x(L, t) - l\varphi(L, t)|^2 \quad (2.33)$$

analogously

$$I_\varphi(0, t) = |\varphi_x(0, t)|^2, \quad I_\psi(0, t) = |\psi_x(0, t)|^2, \quad I_w(0, t) = |w_x(0, t)|^2. \quad (2.34)$$

The observability result is establish in the following lemma.

Lemma 2.1. *There exists a positive constant C such that*

$$\int_0^T I_\varphi(L, t) + I_\psi(L, t) + I_w(L, t) dt \leq CE(0),$$

$$\int_0^T I_\varphi(0, t) + I_\psi(0, t) + I_w(0, t) dt \leq CE(0).$$

Proof. Let us denote by $q(x) = (x - L/2)$. Multiplying equations (3.44), (3.45), (3.46) by $q(x)[\varphi_x + \psi + lw]$, $q(x)\psi_x$ and $q(x)[w_x - l\varphi]$ respectively we get

$$\underbrace{\int_0^L \rho_1 \varphi_{tt} q(x) [\varphi_x + \psi + lw] dx}_{:= J_1} - \underbrace{\kappa \int_0^L (\varphi_x + \psi + lw)_x q(x) [\varphi_x + \psi + lw] dx}_{:= J_2}$$

$$= \kappa_0 \int_0^L l(w_x - l\varphi) q(x) [\varphi_x + \psi + lw] dx. \quad (2.35)$$

Note that

$$J_1 = \int_0^L \rho_1 \varphi_{tt} q(x) [\varphi_x + \psi + lw] dx$$

$$= \frac{d}{dt} \int_0^L \rho_1 \varphi_t q(x) [\varphi_x + \psi + lw] dx - \int_0^L \rho_1 \varphi_t q(x) [\varphi_{xt} + \psi_t + lw_t] dx$$

$$= \frac{d}{dt} \int_0^L \rho_1 \varphi_t q(x) [\varphi_x + \psi + lw] dx - \frac{1}{2} \rho_1 q(x) |\varphi_t|^2 \Big|_{x=0}^{x=L} + R,$$

where

$$R = \frac{1}{2} \int_0^L \rho_1 q_x |\varphi_t|^2 dx + \int_0^L \rho_1 \varphi_t q(x) [\psi_t + lw_t] dx,$$

$$- \rho_1 q(x) |\varphi_t|^2 \Big|_{x=0}^{x=L} = -\frac{\rho_1 L}{2} |\varphi_t(L, t)|^2.$$

Note that

$$|R| \leq CE(t).$$

Estimating J_2 we get

$$J_2 = \kappa \int_0^L (\varphi_x + \psi + lw)_x q(x) [\varphi_x + \psi + lw] dx$$

$$= \frac{kL}{2} (|\varphi_x(L, t) + \psi(L, t) + lw(L, t)|^2 + |\varphi_x(0, t)|^2) + \frac{1}{2} \int_0^L \rho_1 q_x |\varphi_x + \psi + lw|^2 dx,$$

where

$$- \rho_1 q(x) |\varphi_t|^2 \Big|_{x=0}^{x=L} = -\frac{\rho_1 L}{2} |\varphi_t(L, t)|^2, \quad |R| \leq CE(t).$$

Substitution of J_1 and J_2 on (2.35) we get

$$\int_0^T I_\varphi(L, t) + I_\varphi(0, t) dt \leq CE(0).$$

Using similar arguments we conclude that

$$\begin{aligned} \int_0^T I_\psi(L, t) + I_\psi(0, t) dt &\leq CE(0), \\ \int_0^T I_w(L, t) + I_w(0, t) dt &\leq CE(0). \end{aligned}$$

From where our conclusion follows. \square

3. The lack of exponential stability

In this section we prove that the semigroup $S(t)$, defined by the Bresse system (1.5)–(1.12) is not exponentially stable, to do that let us denote by $T(t)$ the semigroup defined by a conservative Bresse system,

$$\rho_1 \tilde{\varphi}_{tt} - \kappa(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{w})_x - \kappa_0 l(\tilde{w}_x - l\tilde{\varphi}) = 0 \quad \text{in }]0, L[\times]0, \infty[\quad (3.36)$$

$$\rho_2 \tilde{\psi}_{tt} - b\tilde{\psi}_{xx} + \kappa(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{w}) = 0 \quad \text{in }]0, L[\times]0, \infty[\quad (3.37)$$

$$\rho_1 \tilde{w}_{tt} - \kappa_0(\tilde{w}_x - l\tilde{\varphi})_x + \kappa l(\tilde{\varphi}_x + \tilde{\psi} + l\tilde{w}) = 0 \quad \text{in }]0, L[\times]0, \infty[\quad (3.38)$$

$$\begin{aligned} \tilde{\varphi}(x, 0) &= \tilde{\varphi}_0(x), & \tilde{\psi}(x, 0) &= \tilde{\psi}_0(x), & \tilde{w}(x, 0) &= \tilde{w}_0(x) \\ \tilde{\varphi}_t(x, 0) &= \tilde{\varphi}_1(x), & \tilde{\psi}_t(x, 0) &= \tilde{\psi}_1(x), & \tilde{w}_t(x, 0) &= \tilde{w}_1(x). \end{aligned} \quad (3.39)$$

The boundary condition are given by

$$\tilde{\varphi}(0, t) = \tilde{\psi}(0, t) = \tilde{w}(0, t) = 0 \quad \text{in }]0, \infty[, \quad (3.40)$$

and the tip condition at L is given by

$$m\tilde{\varphi}_{tt}(L, t) + \kappa[\tilde{\varphi}_x(L, t) + \tilde{\psi}(L, t) + l\tilde{w}(L, t)] = 0 \quad \text{in }]0, \infty[\quad (3.41)$$

$$I_m \tilde{\psi}_{tt}(L, t) + b\tilde{\psi}_x(L, t) = 0 \quad \text{in }]0, \infty[\quad (3.42)$$

$$\rho_2 \tilde{w}_{tt}(L, t) + \kappa_0[\tilde{w}_x(L, t) - l\tilde{\varphi}(L, t)] = 0 \quad \text{in }]0, \infty[. \quad (3.43)$$

Where the dissipative coefficients, $\tau_i = 0$ vanishes for $i = 1, 2, 3$. Therefore, the above system is conservative, so we have that

$$\|T(t)U_0\| = \|U_0\| \quad \Rightarrow \quad \|T(t)\| = 1.$$

Here we use the Weyl [Theorem 1.1](#) of the invariance of the essential spectrum. We will show that the difference of S and T is a compact operator, then the essential spectral radius $R_{ess}(S)$, $R_{ess}(T)$ are equals. Since the T is unitary semigroup, the essential type $\omega_{ess}(S)$ is equals to zero. Using [\(1.14\)](#), we get that $\omega(S) = 0$, so the semigroup $S(t)$ is not exponentially stable. In fact let us denote by

$$\tilde{\Phi} = \varphi - \tilde{\varphi}, \quad \tilde{\Psi} = \varphi - \tilde{\varphi}, \quad \tilde{W} = w - \tilde{w}.$$

So we have that $\tilde{\Phi}$, $\tilde{\Psi}$ and \tilde{W} verify,

$$\rho_1 \tilde{\Phi}_{tt} - \kappa(\tilde{\Phi}_x + \tilde{\Psi} + l\tilde{W})_x - \kappa_0 l(\tilde{W}_x - l\tilde{\Phi}) = 0 \quad \text{in }]0, L[\times]0, \infty[\quad (3.44)$$

$$\rho_2 \tilde{\Psi}_{tt} - b\tilde{\Psi}_{xx} + \kappa(\tilde{\Phi}_x + \tilde{\Psi} + l\tilde{W}) = 0 \quad \text{in }]0, L[\times]0, \infty[\quad (3.45)$$

$$\rho_1 \tilde{W}_{tt} - \kappa_0(\tilde{W}_x - l\tilde{\Phi})_x + \kappa l(\tilde{\Phi}_x + \tilde{\Psi} + l\tilde{W}) = 0 \quad \text{in }]0, L[\times]0, \infty[, \quad (3.46)$$

with the initial condition

$$\begin{aligned} \tilde{\Phi}(x, 0) &= 0, & \tilde{\Psi}(x, 0) &= 0, & \tilde{W}(x, 0) &= 0 \\ \tilde{\Phi}_t(x, 0) &= 0, & \tilde{\Psi}_t(x, 0) &= 0, & \tilde{W}_t(x, 0) &= 0. \end{aligned} \quad (3.47)$$

The boundary condition at $x = 0$ is given by

$$\tilde{\Phi}(0, t) = \tilde{\Psi}(0, t) = \tilde{W}(0, t) = 0 \quad \text{in }]0, \infty[. \quad (3.48)$$

Instead, on $x = L$ we get

$$m\tilde{\Phi}_{tt}(L, t) + \kappa[\tilde{\Phi}_x(L, t) + \tilde{\Psi}(L, t) + l\tilde{W}(L, t)] = -\tau_1 \varphi_t(L, t) \quad \text{in }]0, \infty[\quad (3.49)$$

$$I_m \tilde{\Psi}_{tt}(L, t) + b\tilde{\Psi}_x(L, t) = -\tau_2 \psi_t(L, t) \quad \text{in }]0, \infty[\quad (3.50)$$

$$\rho_2 \tilde{W}_{tt}(L, t) + \kappa_0[\tilde{W}_x(L, t) - l\tilde{\Phi}(L, t)] = -\tau_3 w_t(L, t) \quad \text{in }]0, \infty[. \quad (3.51)$$

To facilitate our analysis let denote by

$$\mathcal{E}(t) = E(\tilde{\Phi}, \tilde{\Psi}, \tilde{W}; t).$$

Now we are in condition to show the main result of this paper.

Theorem 3.1. *The semigroup S is not exponentially stable.*

Proof. Let us take a sequence of initial data bounded on the space

$$\mathcal{H} = H_L^1(0, L) \times H_L^1(0, L) \times H_L^1(0, L) \times L^2(0, L) \times L^2(0, L) \times L^2(0, L) \times \mathbb{C}^3.$$

That is

$$U_0^n = (\varphi_{0,n}, \psi_{0,n}, w_{0,n}, \varphi_{1,n}, \psi_{1,n}, w_{1,n}) \in \mathcal{H}.$$

Let us denote the corresponding solutions by

$$U^n = (\varphi_n, \psi_n, w_n, \varphi_{t,n}, \psi_{t,n}, w_{t,n}) \in \mathcal{H},$$

and

$$\tilde{U}^n = (\tilde{\varphi}_n, \tilde{\psi}_n, \tilde{w}_n, \tilde{\varphi}_{t,n}, \tilde{\psi}_{t,n}, \tilde{w}_{t,n}) \in \mathcal{H}.$$

We will prove that the difference $U^n - \tilde{U}^n = [S(t) - T(t)]U_0^n$ converges strongly in the space

$$C(0, T; \mathcal{H}).$$

For simplicity we remove the super-index in the remaining part of this prove. In fact, multiplying equations (3.44), (3.45), (3.46) by $\tilde{\Phi}_t$, $\tilde{\Psi}_t$, \tilde{W}_t respectively and summing up the product result, we get

$$\frac{d}{dt}\mathcal{E}(t) = -\tau_1\varphi_t(L, t)\tilde{\Phi}_t(L, t) - \tau_2\psi_t(L, t)\tilde{\Psi}_t(L, t) - \tau_3w_t(L, t)\tilde{W}_t(L, t).$$

Integrating over $[0, T]$ we get

$$\mathcal{E}(t) = -\int_0^T \left[\tau_1\varphi_t(L, t)\tilde{\Phi}_t(L, t) + \tau_2\psi_t(L, t)\tilde{\Psi}_t(L, t) + \tau_3w_t(L, t)\tilde{W}_t(L, t) \right] dt. \quad (3.52)$$

Using the observability result establish in Lemma 2.1 and the dynamic boundary condition (1.10), (1.11), (1.12) we conclude that

$$\begin{aligned} \varphi_t(L, \cdot), \quad \psi_t(L, \cdot), \quad w_t(L, \cdot) \quad \text{are bounded in} \quad H^1(0, T). \\ \tilde{\varphi}_t(L, \cdot), \quad \tilde{\psi}_t(L, \cdot), \quad \tilde{w}_t(L, \cdot) \quad \text{are bounded in} \quad H^1(0, T). \end{aligned}$$

Therefore, there exists a subsequence of $\varphi_t(L, \cdot)$, $\psi_t(L, \cdot)$ and $w_t(L, \cdot)$, that converges strongly in $L^2(0, T)$. Similarly for $\tilde{\varphi}_t(L, \cdot)$, $\tilde{\psi}_t(L, \cdot)$ and $\tilde{w}_t(L, \cdot)$. So we have that the right hand side of identity (3.52) is convergent. Hence, the energy $\mathcal{E}(t)$ converges strongly in $C(0, T)$, which implies that the difference of the semigroups is a compact operator over the space of initial data \mathcal{H} . That is the operator

$$S(t) - T(t) : \mathcal{H} \rightarrow \mathcal{H}$$

is a compact operator. Therefore the radius of the essential spectrum are equals, that is

$$e^{\omega_{ess}(S)} = e^{\omega_{ess}(T)} = 1.$$

The last identity holds because the system is conservative, therefore we get that $\omega_{ess}(S) = 0$. Using identity (1.14) we conclude that the type of the semigroup S is zero. This implies that S is not exponentially stable. \square

4. Polynomial rate of decay

In this section we show that the semigroup associated to the hybrid Bresse system decays polynomially to zero. The main tool we use is the following Theorem due to Borichev and Tomilov [5],

Theorem 4.1. Let $S(t)$ be a bounded C_0 -semigroup on a Hilbert space \mathcal{H} with generator A such that $i\mathbb{R} \subset \rho(A)$. Then

$$\|(i\eta I - A)^{-1}\| \leq C|\eta|^\alpha, \quad \forall \eta \in \mathcal{R} \quad \Leftrightarrow \quad \|S(t)A^{-1}\| \leq \frac{C}{t^{1/\alpha}}.$$

To do that let us introduce the following notations

$$\mathcal{I}_\varphi(L) = \rho_1 |\Phi(L)|^2 + k |\varphi_x(L) + \psi(L) + lw(L)|^2, \quad (4.53)$$

$$\mathcal{I}_\psi(L) = \rho_2 |\Psi(L)|^2 + b |\psi_x(L)|^2, \quad (4.54)$$

$$\mathcal{I}_w(L) = \rho_1 |W(L)|^2 + k_0 |w_x(L) - l\varphi(L)|^2. \quad (4.55)$$

The key idea to show the polynomial decay is summarized in the following lemma.

Lemma 4.1. There exists a positive constant C such that

$$\|U\|_{\mathcal{H}}^2 \leq C [\mathcal{I}_\varphi(L) + \mathcal{I}_\psi(L) + \mathcal{I}_w(L)].$$

Proof. To achieve the above inequality we introduce the function $q_1(x) = \int_0^x e^{sn} ds$, note that

$$q_1(x) = \int_0^x e^{ns} ds = \frac{e^{nx} - 1}{n}, \quad \text{and} \quad q_{1,x}(x) = e^{nx}.$$

Here n will be chosen large enough. Multiplying equations (2.20), (2.22), (2.24) by $q_1(x) \overline{[\varphi_x + \psi + lw]}$, $q_1(x) \overline{\psi_x}$ and $q_1(x) \overline{[w_x - l\varphi]}$ respectively we get

$$\begin{aligned} & \underbrace{\int_0^L q(x) \rho_1 i \lambda \Phi \overline{[\varphi_x + \psi + lw]} dx}_{:= J_1} - \underbrace{\kappa \int_0^L (\varphi_x + \psi + lw)_x q(x) \overline{[\varphi_x + \psi + lw]} dx}_{:= J_2} \\ &= \kappa_0 \int_0^L l(w_x - l\varphi) q(x) \overline{[\varphi_x + \psi + lw]} dx \end{aligned} \quad (4.56)$$

Note that

$$\begin{aligned} \operatorname{Re} J_1 &= \operatorname{Re} \int_0^L q(x) \rho_1 i \lambda \Phi \overline{[\varphi_x + \psi + lw]} dx \\ &= -\operatorname{Re} \int_0^L q(x) \rho_1 \Phi \overline{\Phi_x} dx - \operatorname{Re} \int_0^L q(x) \rho_1 \Phi [\Psi + lw] dx + R \\ &= \frac{1}{2} \int_0^L q_{1,x}(x) \rho_1 |\Phi|^2 dx - \frac{1}{2} q_1(L) |\Phi(L)|^2 + R \end{aligned}$$

where

$$R = \operatorname{Re} \int_0^L q_1(x) \rho_1 \Phi[\Psi + lW] \, dx.$$

Estimating J_2 we get

$$\begin{aligned} \operatorname{Re} J_2 &= \operatorname{Re} \kappa \int_0^L (\varphi_x + \psi + lw)_x q(x) \overline{[\varphi_x + \psi + lw]} \, dx \\ &= \frac{kq_1(L)}{2} (|\varphi_x(L) + \psi(L) + lw(L)|^2) - \frac{1}{2} \int_0^L \rho_1 q_x |\varphi_x + \psi + lw|^2 \, dx. \end{aligned}$$

Substitution of J_1 and J_2 on (4.56) and recalling that $q_x = e^{nx}$ we get

$$\frac{1}{2} \int_0^L e^{nx} [\rho_1 |\Phi|^2 + k |\varphi_x + \psi + lw|^2] \, dx = \frac{1}{2} q_1(L) I_\varphi(L) + \operatorname{Re} \int_0^L q_1(x) \rho_1 \Phi[\Psi + lW] \, dx.$$

Note that

$$\operatorname{Re} \int_0^L q_1(x) \rho_1 \Phi[\overline{\Psi + lW}] \, dx \leq C \int_0^L \frac{e^{nx} - 1}{n} \mathcal{E}(x) \, dx,$$

where

$$\mathcal{E}(x) = \rho_1 |\Phi|^2 + k |\varphi_x + \psi + lw|^2 + \rho_2 |\Psi|^2 + b |\psi_x|^2 + \rho_1 |W|^2 + k_0 |w_x + l\varphi|^2.$$

So we have

$$\frac{1}{2} \int_0^L e^{nx} [\rho_1 |\Phi|^2 + k |\varphi_x + \psi + lw|^2] \, dx \leq \frac{1}{2} q_1(L) I_\varphi(L) + \frac{C}{n} \int_0^L e^{nx} \mathcal{E}(x) \, dx. \quad (4.57)$$

Using the same above reasoning we get

$$\frac{1}{2} \int_0^L e^{nx} [\rho_2 |\Psi|^2 + b |\psi_x|^2] \, dx \leq \frac{1}{2} q_1(L) I_\psi(L) + \frac{C}{n} \int_0^L e^{nx} \mathcal{E}(x) \, dx. \quad (4.58)$$

$$\frac{1}{2} \int_0^L e^{nx} [\rho_2 |W|^2 + k_0 |w_x + l\varphi|^2] \, dx \leq \frac{1}{2} q_1(L) I_w(L) + \frac{C}{n} \int_0^L e^{nx} \mathcal{E}(x) \, dx. \quad (4.59)$$

Using inequalities (4.57)–(4.59) we conclude that there exists a positive constant C such that

$$\int_0^L \mathcal{E}(x) \, dx \leq C I_\varphi(L) + C I_\psi(L) + C I_w(L),$$

provided n is large enough. From where our conclusion follows. \square

Now we are in condition to show the main result of this section

Theorem 4.2. *The semigroup defined by the dissipative Bresse system decays polynomially to zero. That is, there exists positive constant C such that*

$$\|U(t)\|_{\mathcal{H}} \leq \frac{C}{t^{1/2}} \|U_0\|_{D(\mathcal{A})}$$

if we take more regular initial data $U_0 \in D(\mathcal{A}^k)$ we get

$$\|S(t)U_0\|_{\mathcal{H}} \leq ct^{-k/2} \|S(t)U_0\|_{D(\mathcal{A}^k)}.$$

Proof. Let us consider the resolvent equation

$$i\lambda U - \mathcal{A}U = F.$$

Taking the inner product in \mathcal{H} with U and using inequality (2.18) we get that

$$m_1|u|^2 + I_m|v|^2 + m_2|z|^2 = \operatorname{Re}(U, F)_{\mathcal{H}}.$$

From Lemma 4.1 we get

$$\|U\|_{\mathcal{H}}^2 \leq C\mathcal{I}_{\varphi}(L) + C\mathcal{I}_{\psi}(L) + C\mathcal{I}_w(L). \quad (4.60)$$

From (2.25) we get that

$$|\varphi_x(L) + \psi(L) + lw(L)|^2 \leq c|\lambda|^2|u|^2$$

for λ large enough. Therefore we get

$$\mathcal{I}_{\varphi}(L) \leq c|\lambda|^2|u|^2.$$

Similarly, we also conclude that

$$\mathcal{I}_{\psi}(L) \leq c|\lambda|^2|v|^2, \quad \mathcal{I}_w(L) \leq c|\lambda|^2|w|^2.$$

Substitution of this inequalities into (4.60) we get that

$$\|U\|_{\mathcal{H}}^2 \leq c|\lambda|^2 \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}.$$

So we have

$$\|U\|_{\mathcal{H}} \leq c|\lambda|^2 \|F\|_{\mathcal{H}}.$$

From where our conclusion follows. \square

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