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A connection between ultraspherical and pseudo-ultraspherical polynomials

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A B S T R A C T

The pseudo-ultraspherical polynomial of degree n is defined by $\tilde{C}_n^{(\lambda)}(x) = (-i)^n C_n^{(\lambda)}(ix)$ where $C_n^{(\lambda)}(x)$ is the ultraspherical polynomial. We derive a simple expression linking the polynomials $C_n^{(\lambda)}$ and $\tilde{C}_n^{(\frac{1}{2}-\lambda-n)}$ and show how to derive various properties of the zeros of $\tilde{C}_n^{(\lambda)}$ when $\lambda < 1-n$ from properties of the zeros of $C_n^{(\lambda)}$ when $\lambda > -\frac{1}{2}$.

1. Introduction

The sequence of ultraspherical polynomials belongs to the class of sequences of classical orthogonal polynomials whose orthogonality depends on the values of one or more parameters being constrained to lie within a specified range. The ultraspherical sequence $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$, given by

$$C_n^{(\lambda)}(x) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(-1)^m (\lambda)_{n-m}}{m!(n-2m)!}(2x)^{n-2m}, \tag{1.1}$$

is orthogonal on $(-1, 1)$, with respect to the weight function $(1-x^2)^{\lambda-1/2}$, provided $\lambda > -1/2$.

For values of the parameter outside the range that ensures orthogonality, the sequence of ultraspherical polynomials can be defined by the three term recurrence relation that holds when the sequence is orthogonal or by an appropriate generating function, see [2,11]. When $\lambda > -\frac{1}{2}$, $C_n^{(\lambda)}$ has n real, simple zeros which lie in the open interval $(-1, 1)$ and are symmetric about the origin. As λ decreases below $-1/2$, the zeros of $C_n^{(\lambda)}$ depart from the interval $(-1, 1)$ through the endpoints -1 and 1 with an additional two zeros leaving $[-1, 1]$

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each time λ decreases through $-k+1/2$, $k = 1, 2, \ldots, \lfloor n/2 \rfloor$. A full description of the departure of the zeros, as well as the kinematics of the collisions of zeros at ± 1, when $\lambda = -3/2, -5/2, \ldots$ can be found in [4]. An unusual phenomenon occurs as λ decreases below $-[(n+1)/2]$, namely, the zeros of $C_n^{(\lambda)}$ appear on the imaginary axis with an additional pair of zeros joining the imaginary axis each time λ decreases through successive negative integers. When $\lambda < 1-n$, all n zeros of the polynomial $C_n^{(\lambda)}(x)$ are simple and lie on the imaginary axis (see [4, Th. 3]).

Using the definition of Ismail [9, (20.1.1)] for pseudo-Jacobi polynomials, we define the pseudo-ultraspherical polynomial $\tilde{C}_n^{(\lambda)}$ by

$$\tilde{C}_n^{(\lambda)}(x) := (-i)^n C_n^{(\lambda)}(ix) = \sum_{m=0}^{\lfloor n/2 \rfloor} \frac{(\lambda)_{n-m}}{m!(n-2m)!}(2x)^{n-2m}. \tag{1.2}$$

Since it is known [3, Th. 3] that the polynomial $\tilde{C}_n^{(\lambda)}$ has n real, simple zeros for each fixed $\lambda < 1-n$, a natural question is whether the (finite) sequence $\{\tilde{C}_n^{(\lambda)}\}_{n=1}^{-\lfloor \lambda+1 \rfloor}$ is orthogonal for each fixed $\lambda < 1-n$. The three term recurrence relation satisfied by the sequence of pseudo-ultraspherical polynomials, that follows via a simple substitution from the recurrence relation satisfied by the sequence of ultraspherical polynomials $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$, is given by [5, (4.1)]:

$$(n+1)\tilde{C}_{n+1}^{(\lambda)}(x) = 2x(n+\lambda)\tilde{C}_n^{(\lambda)}(x) + (n+2\lambda-1)\tilde{C}_{n-1}^{(\lambda)}(x). \tag{1.3}$$

One might expect that since $n+2\lambda-1 < 0$ for each $n \in \mathbb{N}$, $n \geq 1$, $\lambda < 1-n$, the sequence $\{\tilde{C}_n^{(\lambda)}\}_{n=1}^{-\lfloor \lambda+1 \rfloor}$ is orthogonal with respect to the weight function $(1+x^2)^{\lambda-1/2}$ for each fixed $\lambda < 1-n$. A positive result in this direction was proved by Richard Askey, [1, (1.8)], in the context of the complex orthogonality of Jacobi polynomials. However, the Askey integral [1, (1.8)] converges for a more restricted range of λ values, namely $\lambda < -n$, and diverges when $-n < \lambda < 1-n$. Although a finite sequence of polynomials that satisfies a three term recurrence relation of appropriate type can be embedded in an infinite sequence of orthogonal polynomials, the restriction $n < 1-\lambda$ that applies in this instance does not make the process meaningful and the measure of orthogonality is not unique since there are infinitely many choices of coefficients in the three term recurrence relation that generates the polynomials of degree greater than n.

The weight function of orthogonality for the ultraspherical sequence $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$ when $\lambda > -\frac{1}{2}$ is $(1-x^2)^{\lambda-1/2}$ while the weight function of orthogonality for the (finite) pseudo-ultraspherical sequence $\{\tilde{C}_n^{(\lambda)}\}_{n=1}^{-\lfloor \lambda+1 \rfloor}$ is $(1+x^2)^{\lambda-1/2}$ provided $\lambda < -n$ so the connection between the two weight functions is to replace x by ix. However, there is no reason to suppose that a connection exists between the zeros of $C_n^{(\lambda)}$ when $\lambda > -\frac{1}{2}$ and the zeros of $\tilde{C}_n^{(\lambda')}$ when $\lambda' < 1-n$.

Note that a proof of the orthogonality of the pseudo-ultraspherical polynomials using differential equations can be found in [5, §2.1]. A proof starting from the Rodrigues formula for ultraspherical polynomials is given in [6].

In Section 2, we derive an identity between the two ultraspherical polynomials $C_n^{(\lambda)}$ and $C_n^{(\frac{1}{2}-\lambda-n)}$. Using this identity, we find a simple and explicit algebraic relationship between the zeros of $C_n^{(\lambda)}$ and those of $\tilde{C}_n^{(\lambda')}$ where $\lambda' = \frac{1}{2} - \lambda - n$. In Sections 3 and 4, we show how our identity can be used to derive monotonicity properties of zeros of pseudo-ultraspherical polynomials from similar properties of the zeros of ultraspherical polynomials (and vice-versa).

2. A connection between $C_n^{(\lambda)}$ and $C_n^{(1/2-\lambda-n)}$

We shall make use of the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ (see, for example [10, Ch. 4] for a definition) in polynomial form with the numerator parameter $a = -n$, $n \in \mathbb{N}$.

Theorem 2.1. *Suppose $C_n^{(\lambda)}(z)$ is the ultraspherical polynomial of degree n where z is arbitrary, complex and not equal to 0 or ± 1, $\lambda \in \mathbb{R}$, $n \in \mathbb{N}$, and let $\lambda' = \frac{1}{2} - \lambda - n$. Then*

$$(\lambda + 1/2)_n z^n C_n^{(\lambda)}\left(\frac{z+1/z}{2}\right) = (2\lambda)_n (1-z^2)^n 2^{-2n} C_n^{(\lambda')}\left(\frac{z^2+1}{z^2-1}\right), \tag{2.1}$$

or, equivalently,

$$(2\lambda + n)_n C_n^{(\lambda)}(\cos\theta) = (\lambda)_n (-i2\sin\theta)^n C_n^{(\lambda')}(-i\cot\theta), \tag{2.2}$$

or

$$(2\lambda + n)_n C_n^{(\lambda)}(x) = (\lambda)_n (-2i\sqrt{1-x^2})^n C_n^{(\lambda')}(-ix/\sqrt{1-x^2}),\ |x| \le 1. \tag{2.3}$$

Proof. For each fixed λ and any complex z, $z \neq 0$, we have [3, Lemma 2]

$$(2\lambda)_n {}_2F_1(-n, \lambda; 2\lambda; 1-z^2) = n! z^n C_n^{(\lambda)}\left(\frac{z+1/z}{2}\right), \tag{2.4}$$

where ${}_2F_1$ is the Gauss hypergeometric function and $(-)_n$ is the Pochhammer symbol. The identity (2.4) was pointed out by Richard Askey [personal communication] and follows from the generating function for the ultraspherical polynomials. □

A second relationship between the ultraspherical polynomial $C_n^{(\lambda)}(z)$ and the hypergeometric polynomial ${}_2F_1(-n, \lambda; 2\lambda; z)$ follows from [3, (2)]:

$$(2\lambda)_{2n} {}_2F_1(-n, \lambda; 2\lambda; z) = (\lambda)_n n! z^n C_n^{(\lambda')}\left(1 - \frac{2}{z}\right), \tag{2.5}$$

where $\lambda' = \frac{1}{2} - \lambda - n$. This result follows also from [11, (4.7.6), second equation]. Replacing z by $1 - z^2$ in (2.5), we have

$${}_2F_1(-n, \lambda; 2\lambda; 1-z^2) = \frac{(\lambda)_n n! (1-z^2)^n}{(2\lambda)_{2n}} C_n^{(\lambda')}\left(\frac{z^2+1}{z^2-1}\right), \tag{2.6}$$

where $\lambda' = \frac{1}{2} - \lambda - n$. From (2.4) and (2.6), we obtain

$$(2\lambda + n)_n z^n C_n^{(\lambda)}\left(\frac{z+1/z}{2}\right) = (\lambda)_n (1-z^2)^n C_n^{(\lambda')}\left(\frac{z^2+1}{z^2-1}\right), \tag{2.7}$$

which is equivalent to (2.1).

Putting $z = e^{i\theta}$ in (2.1) and simplifying, we obtain (2.2) and (2.3).

3. Zeros of $C_n^{(\lambda)}$ and $\tilde{C}_n^{(\lambda')}$

In what follows we consider the nontrivial x-zeros of $C_n^{(\lambda)}(x)$ and $\tilde{C}_n^{(\lambda')}(x)$. Figures 1 and 2 illustrate the zeros as functions of λ and λ', respectively, in the case $n = 12$. We have $C_n^{(\lambda)}(x) \equiv \tilde{C}_n^{(\lambda')}(x) \equiv 0$, for $\lambda = 0, -1, \ldots, -\lfloor (n-1)/2 \rfloor$. For each such value λ^* of λ we define the zeros by a limiting process:

$$z(\lambda^*) = \lim_{\lambda \to \lambda^*} z(\lambda).$$

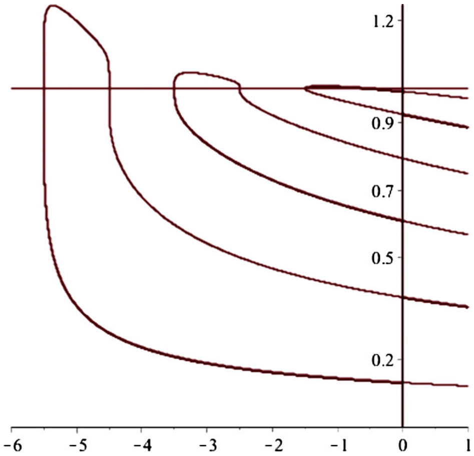

Fig. 1. The real x-zeros of $C_{12}^{(\lambda)}(x)$, for $0 < x < 1.5$, plotted (using MAPLE) as functions of λ, $-6 < \lambda < 1$. Zeros leave the interval $(0, 1)$ as λ decreases through the values $-1/2, -5/2, -9/2$. In addition, as λ decreases through the values $-3/2, -7/2, -11/2$, pairs of formerly real zeros collide at 1, becoming complex.

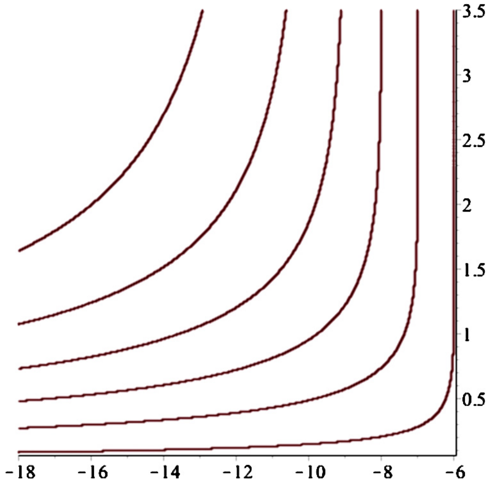

Fig. 2. The real x-zeros of $\tilde{C}_{12}^{(\lambda')}(x)$, for $0 < x < 3.5$, plotted (using MAPLE) as functions of λ', $-18 < \lambda' < -6$. As λ' increases through the values $-11, -10, -9, -8, -7, -6$ a positive zero goes to infinity. There is a similar picture for the negative zeros. For $\lambda' \geq -6$, the nonzero zeros are non-real.

Theorem 3.1. *Let $C_n^{(\lambda)}$ and $\tilde{C}_n^{(\lambda')}$ denote, respectively, the ultraspherical polynomial and pseudo-ultraspherical polynomial of degree n where $\lambda' = \frac{1}{2} - \lambda - n$. Then x is a zero of $\tilde{C}_n^{(\lambda')}$ if and only if $x/\sqrt{1+x^2}$ is a zero of $C_n^{(\lambda)}$.*

Proof. From (2.2), we see that $x = \cos\theta$ is a zero of $C_n^{(\lambda)}$ if and only if $-i\cot\theta = -ix/\sqrt{1-x^2}$ is a zero of $C_n^{(\lambda')}$. Further, from the definition (1.2), we see that x is a zero of $\tilde{C}_n^{(\lambda')}$ if and only if ix is a zero of $C_n^{(\lambda')}$ which, in turn, holds if and only if $-i(ix)/\sqrt{1-(ix)^2} = x/\sqrt{1+x^2}$ is a zero of $C_n^{(\lambda)}$, where $\lambda' = \frac{1}{2} - \lambda - n$. $\square$

We use the notation $x_{nk}(\lambda)$, $k = 1, \ldots, \lfloor n/2 \rfloor$ for the positive zeros in decreasing order of $C_n^{(\lambda)}(x)$, and $\tilde{x}_{nk}(\lambda')$, $k = 1, \ldots, \lfloor n/2 \rfloor$ for the positive zeros in decreasing order of $\tilde{C}_n^{(\lambda')}(x)$ where $\lambda' = 1/2 - \lambda - n$ (or, equivalently, $\lambda = \frac{1}{2} - \lambda' - n$).

From Theorem 3.1, we get

$$x_{nk}(\lambda) = [\tilde{x}_{nk}(\lambda')^{-2} + 1]^{-1/2}, \qquad \tilde{x}_{nk}(\lambda') = [x_{nk}(\lambda)^{-2} - 1]^{-1/2}. \tag{3.1}$$

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In [11, §6.72] there is a detailed discussion of the numbers of x-zeros of the general Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ in $(-1,1),(1,\infty)$, and $(-\infty,-1)$. Specializing [11, Theorem 6.72] to the case $\alpha=\beta=\lambda-1/2$, $-n<\lambda<1/2$, we find that the number of x-zeros of $C_n^{(\lambda)}$ on $(0,1)$ is $\lfloor (X+1)/2\rfloor$ for n even and $\lfloor X/2\rfloor$ for n odd, where $X=E(n+2\lambda)$ and the function $E:R\to R$ is defined by

$$E(u)=\begin{cases}0, & u\le 0,\\ -1-\lfloor -u\rfloor, & u>0.\end{cases} \tag{3.2}$$

This leads immediately to:

Corollary 3.2. *The number of x-zeros of $C_n^{(\lambda)}$ on $(0,1)$ is*

(i) $\lfloor n/2\rfloor$, *for* $\lambda>-1/2$;
(ii) $\lfloor n/2\rfloor-k$, *for* $-\frac12-k<\lambda<\frac12-k$, $k=1,2,\dots,\lfloor n/2\rfloor-1$, *and*
(iii) 0, *for* $\lambda<1/2-\lfloor n/2\rfloor$.

Corollary 3.3. *The number of x-zeros of $\tilde{C}_n^{(\lambda')}$ on $(0,\infty)$ is*

(i) $\lfloor n/2\rfloor$ *for* $\lambda'<1-n$,
(ii) $\lfloor n/2\rfloor-k$, *for* $-n+k<\lambda'<-n+k+1$, $k=1,2,\dots,\lfloor n/2\rfloor-1$, *and*
(iii) 0 *for* $\lambda'>-n+\lfloor n/2\rfloor$.

This agrees with [5, Theorem 3.5], obtained from the explicit representation (1.2).

Corollaries 3.2 and 3.3 show that, for $k=1,\dots,\lfloor n/2\rfloor$, $x_{n,k}(\lambda)$ is defined on the interval $1/2-k<\lambda<\infty$ and $\tilde{x}_{n,k}(\lambda')$ is defined on the interval $-\infty<\lambda'<k-n$.

We see that the second expression in (3.1) can be written

$$\tilde{x}_{n,k}(\lambda')^{-2}=x_{n,k}(\lambda)^{-2}-1. \tag{3.3}$$

From this it is clear that if $x_{n,k}(\lambda)$ is an increasing (decreasing) function of λ on an interval $a<\lambda<b$, then $\tilde{x}_{n,k}(\lambda')$ is a decreasing (increasing) function of λ' on the corresponding interval $1/2-b-n<\lambda'<1/2-a-n$. It follows that the well-known [11, Theorem 6.21.1] decrease of the positive zeros $x_{nk}(\lambda)$ of $C_n^{(\lambda)}$ as functions of λ, $\lambda>-1/2$ implies the increase of the positive zeros $\tilde{x}_{nk}(\lambda)$ as functions of λ, $\lambda<1-n$. This monotonicity result was proved in a different way in [5, Theorem 3.4]. A further result [5, Theorem 3.3] covers the case where only some of the zeros are real and, using (3.1), this gives the (apparently new) result that, even for $\lambda\le -\frac12$, the positive zeros $x_{nk}(\lambda)$ of $C_n^{(\lambda)}$ that lie within the interval $(0,1)$ are decreasing functions of λ.

4. Scaled monotonicity

Let $f(\lambda)$ be a decreasing function of λ and let $g(\lambda')=f(\lambda)$, $\lambda'=1/2-\lambda-n$. Then (3.3) gives

$$g(\lambda')[\tilde{x}_{nk}(\lambda')^{-2}+1]=f(\lambda)x_{nk}(\lambda)^{-2} \tag{4.1}$$

and we immediately obtain

Theorem 4.1. *If $f(\lambda)x_{nk}(\lambda)^{-2}$ is a decreasing function of λ on a subinterval of $(-1/2,\infty)$, then $g(\lambda')[\tilde{x}_{nk}(\lambda')^{-2}+1]$ is an increasing function of λ' on the corresponding subinterval of $(-\infty,1-n)$.*

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Given that $x_{n,k}(\lambda)$ is a decreasing function of λ, it is a natural question to find a function $f(\lambda)$ such that the product $f(\lambda)x_{n,k}(\lambda)$ is increasing. It was shown by Á. Elbert and P.D. Siafarikas [8] that, for $n = 2, 3, \dots,$

$$\left(\lambda + \frac{n^2+1/2}{2n+1}\right) x_{n,k}(\lambda)^2 \text{ increases with } \lambda,\ -1/2 < \lambda < \infty. \tag{4.2}$$

This monotonicity result was the culmination of a series of gradual improvements in papers published in the 1980s and 1990s (see [7, §2.2]) and an improvement on the simpler result

$$(\lambda + 1/2)x_{n,k}(\lambda)^2 \text{ increases with } \lambda,\ -1/2 < \lambda < \infty. \tag{4.3}$$

In the cases $n = 2, 3$ the explicit expression (1.1) shows that $(1 + \lambda)\ x_{21}(\lambda)^2$ and $(2 + \lambda)\ x_{31}(\lambda)^2$ are increasing functions of λ on $(-1, \infty)$ and $(-2, \infty)$, respectively, whereas (4.2) gives the weaker results that $(9/10+\lambda)\ x_{21}(\lambda)^2$ and $(19/14+\lambda)\ x_{31}(\lambda)^2$ are increasing functions of λ on $(-1/2, \infty)$. But, although it can be improved in special cases, there does not appear to be any generally stronger version of (4.2) known to be valid for all integers $n \geq 2$.

Theorem 4.1 can be used to get analogues of these results for the zeros of the corresponding pseudo-ultraspherical polynomials. If we choose

$$f(\lambda) = \left(\lambda + \frac{n^2+1/2}{2n+1}\right)^{-1},$$

the result (4.2) leads to

$$-\left(\lambda' + \frac{n^2-1}{2n+1}\right)^{-1} [\tilde{x}_{n,k}(\lambda')^2 + 1] \text{ increases with } \lambda',\ -\infty < \lambda' < 1-n. \tag{4.4}$$

Similarly, if we choose $f(\lambda) = (\lambda+1/2)^{-1}$, we have $g(\lambda') = (1-\lambda'-n)^{-1}$. Then it follows from (4.3) that $f(\lambda)x_{nk}(\lambda)^{-2}$ is an increasing function of λ on $(-1/2, \infty)$, so Theorem 4.1 gives, after some simplification, that $(1-\lambda'-n)^{-1}[\tilde{x}_{nk}(\lambda')^{-2}+1]$ is an increasing function of λ' on the interval $(-\infty, 1-n)$ where $\tilde{x}_{nk}(\lambda')^{-2}$ is decreasing.

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