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Global strong solutions to the incompressible Navier–Stokes equations with density-dependent viscosity

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A B S T R A C T

This paper is concerned with the 3D incompressible Navier–Stokes equations with density-dependent viscosity in a smooth bounded domain. The global well-posedness of strong solutions is established for the case when the bound of density is suitably small, or when the total mass is small with large oscillations. The vacuum is allowed in both cases.

1. Introduction

The present paper is devoted to the study of the incompressible Navier–Stokes equations in $\mathbb{R}^3$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla \Pi = \operatorname{div}(\mu(\rho)\nabla u), \\ \operatorname{div} u = 0, \end{cases} \tag{1.1}$$

with the initial-boundary conditions:

$$(\rho, u)|_{t=0} = (\rho_0, u_0)(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Omega \times (0, T). \tag{1.2}$$

Here, ρ, u, and Π denote the density, velocity and pressure of the fluid, respectively. $\mu(\rho)$ is the viscosity coefficient assumed to satisfy

$$\mu(\xi) \in C^1[0, \infty) \quad \text{and} \quad 0 < \underline{\mu} \leq \mu(\xi) \leq \bar{\mu} \quad \text{for} \quad \forall\, \xi \in [0, \infty). \tag{1.3}$$

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The mathematical study of nonhomogeneous incompressible fluids was initiated by Kazhikov, who proved the global existence of weak solutions as well as strong ones when $\mu(\rho)$ is a constant and ρ_0 has a positive lower bound, see [3,4,14]. The unique solvability of (1.1) is first addressed by Ladyzenskaja and Solonnikov [16]. In particular, they proved the global existence of weak solutions and local existence of strong ones of the initial/initial-boundary value problem of (1.1) with large data in dimension $N \geq 2$. It is also well known that the local strong solution is indeed a global one in two dimensions or three dimensions with small data. The global well-posedness for initial data belonging to certain scale invariant spaces, see for example [1,2,8,18].

For the case when the initial data may contain vacuum and the viscosity coefficient $\mu(\rho)$ is still a positive constant, Simon [19] constructed the global weak solutions. By imposing some compatibility condition, Choe–Kim [6] established the local existence of strong solutions. Huang–Wang [11] showed that the local strong solution obtained in [6] is indeed a global one in dimension two. For the three-dimensional case, Kim [15] proved that if $\|\nabla u_0\|_{L^2}$ is sufficiently small, then (1.1) has a unique strong solution, which was generalized by Craig et al. [7] by requiring $\|u_0\|_{\dot{H}^{1/2}}$ small.

When the viscosity coefficient $\mu(\rho)$ depends on ρ, Lions [17] derived the global existence of weak solutions. Later, Desjardins [9] proved the global weak solution with more regularity for the two-dimensional case provided that $\mu(\rho)$ is a small perturbation of a positive constant in L^∞-norm. As for strong solutions away from vacuum, Gui–Zhang [10] obtained the global well-posedness in the case when the initial density ρ_0 is a small perturbation around a positive state in H^s with $s \geq 2$. To overcome the difficulties caused by the presence of vacuum, analogously to [6], Choe–Kim [5] proposed a compatibility condition:

$$-\mathrm{div}(\mu(\rho_0)\nabla u_0) + \nabla\Pi_0 = \rho_0^{1/2} g \quad \text{for some } (\nabla\Pi_0, g) \in L^2, \tag{1.4}$$

and established the local existence of strong solutions. Recently, Huang–Wang [12] obtained the global strong solutions in dimension two, provided $\|\nabla\mu(\rho_0)\|_{L^q}$ $(q > 2)$ is small enough, which had been generalized to the 3D case by Zhang [20] and Huang–Wang [13] when $\|\nabla u_0\|_{L^2}$ is small.

The main result in this paper is to establish global strong solutions under the assumption that the mass or the bound of density is suitably small, which reads as follows.

Theorem 1.1. *For some $q > 3$, assume that the initial data (ρ_0, u_0) satisfies*

$$\begin{cases} 0 \leq \inf\rho_0 \leq \rho_0 \leq \sup\rho_0 \leq \bar{\rho} < \infty, \quad \|\rho_0\|_{L^1} = m, \\ \rho_0 \in W^{1,q}, \quad \|\nabla\mu(\rho_0)\|_{L^q} \leq M, \quad u_0 \in H^1_{0,\sigma} \cap H^2, \end{cases} \tag{1.5}$$

and that the compatibility condition (1.4) holds for some $(\nabla\Pi_0, g) \in L^2$. Then there exist positive constants ϵ and $\widetilde{C}$, depending only on Ω, $\bar{\mu}$, q, $\underline{\mu}$, M, g and $\|\nabla u_0\|_{L^2}$, such that the initial-boundary value problem (1.1)–(1.4) has a global strong solution on $\Omega \times (0, T)$, satisfying

$$\begin{cases} 0 \leq \rho(x,t) \leq \bar{\rho}, \quad \|\nabla\mu(\rho)\|_{L^q} \leq 2M, \quad \forall\ (x,t) \in \Omega \times [0,\infty), \\ (\rho, \mu(\rho)) \in C([0,\infty); W^{1,q}), \quad (\nabla u, \Pi) \in C([0,\infty); H^1) \cap L^1(0,\infty; W^{1,r}), \\ \rho_t \in C([0,\infty); L^q), \quad \sqrt{\rho}u_t \in L^\infty(0,\infty; L^2), \quad u_t \in L^2(0,\infty; H^1_0), \end{cases}$$

for $3 < r < \min\{q, 6\}$, provided

$$\Lambda \triangleq (\bar{\rho}m^2)^{\frac{1}{6}} \bar{\rho}^{\frac{5r-6}{4r}} \max\left\{ \bar{\rho}^{\frac{7r+6}{4r}} (1+\bar{\rho})(\bar{\rho}m^2)^{\frac{1}{6}},\ \Lambda_1,\ (\bar{\rho}m^2)^{\frac{1}{6}}\Lambda_2 \right\} \leq \epsilon,$$

where

$$\Lambda_1 \triangleq (1+\bar{\rho}^2)\exp\left\{\widetilde{C}\bar{\rho}^2\left[\bar{\rho}(\bar{\rho}m^2)^{\frac{1}{3}}+1\right]\right\}+\bar{\rho}^{\frac{15r-18}{4r}}(\bar{\rho}m^2)^{\frac{1}{6}},$$
$$\Lambda_2 \triangleq (1+\bar{\rho}^2)\exp\left\{\frac{3\widetilde{C}}{2}\bar{\rho}^2\left[\bar{\rho}(\bar{\rho}m^2)^{\frac{1}{3}}+1\right]\right\}+\bar{\rho}^{\frac{15r-18}{4r}}.$$

The main result in Theorem 1.1 is obtained by some delicate analysis for the bound of density $\bar{\rho}$ and the mass m. Inspired by [13,20], we first assume that $\|\nabla\mu(\rho)\|_{L^q} \le 2M$. Then, we can make use of the regularity results of Stokes equations (cf. Lemma 2.1) and energy estimate (cf. Lemma 3.2) to deduce the desired bound for $\|\nabla u\|_{L^2}$ under the assumption that $\bar{\rho}$ or m is suitably small. Based on some t-weighted estimates, similarly to [13], we get time independent bound for $\|\nabla u\|_{L^1(0,T;L^\infty)}$, which is actually controlled by both $\bar{\rho}$ and m (cf. Lemma 3.5). With the smallness of $\|\nabla u\|_{L^1(0,T;L^\infty)}$ at hand, we can show that the quantity $\|\nabla\mu(\rho)\|_{L^q}$ is in fact strictly less than $2M$.

2. Preliminaries

Throughout this paper, we assume Ω is a smooth bounded domain in $\mathbb{R}^3$. For simplicity, we denote

$$\int f dx \triangleq \int_\Omega f dx.$$

For $1 \le r \le \infty$ and $k \in \mathbb{N}$, the Sobolev spaces are defined in a standard way.

$$L^r \triangleq L^r(\Omega), \quad W^{k,r} \triangleq \{f \in L^r : D^\alpha f \in L^r, \ \ |\alpha| \le k\}, \quad H^k \triangleq W^{k,2}.$$

Moreover, H_0^1 and $H_{0,\sigma}^1$ represent the closure of C_0^∞ and $C_{0,\sigma}^\infty \triangleq \{f \in C_0^\infty : \mathrm{div} f = 0\}$ in H^1, respectively.

The derivations of high-order estimates rely on the following regularity results for density-dependent Stokes equations.

Lemma 2.1. *[13,20] Assume that $\rho \in W^{1,q}$ for some $3 < q < \infty$, and $0 \le \rho \le \bar{\rho}$. Let (u, Π) be the unique weak solution to the boundary value problem:*

$$-\mathrm{div}(\mu(\rho)\nabla u) + \nabla\Pi = F, \quad \mathrm{div} u = 0 \ \ in\ \Omega, \quad \int \Pi/\mu(\rho) dx = 0,$$

where

$$\mu \in C^1[0,\infty), \quad \underline{\mu} \le \mu(\rho) \le \bar{\mu} \ \ on\ [0,\bar{\rho}].$$

Then we have the following results:
(1) If $F \in L^2$, then $(u,\Pi) \in H^2 \times H^1$ and

$$\|u\|_{H^2} + \|\Pi/\mu(\rho)\|_{H^1} \le C\|F\|_{L^2}(1+\|\nabla\mu(\rho)\|_{L^q})^{\frac{q}{q-3}}. \tag{2.1}$$

(2) If $F \in L^r$ for some $r \in (3,q)$, then $(u,\Pi) \in W^{2,r} \times W^{1,r}$ and

$$\|u\|_{W^{2,r}} + \|\Pi/\mu(\rho)\|_{W^{1,r}} \le C\|F\|_{L^r}(1+\|\nabla\mu(\rho)\|_{L^q})^{\frac{qr}{2(q-r)}}. \tag{2.2}$$

Here, the constant C depends on Ω, q, r, $\bar{\mu}$ and $\underline{\mu}$.

Theorem 1.1 will be proved by combining the global a priori estimates with the following local existence results [5].

Lemma 2.2. *For $q > 3$, assume that the initial data (ρ_0, u_0) satisfies*

$$0 \le \rho_0 \le \bar{\rho}, \quad \rho_0 \in W^{1,q}, \quad \|\nabla\mu(\rho_0)\|_{L^q} \le 2M, \quad u_0 \in H^1_{0,\sigma} \cap H^2,$$

and that (1.4) holds for some $(\nabla\Pi_0, g) \in L^2$. Then there exists a positive small time T^* and a unique strong solution (ρ, u, Π) of (1.1)–(1.3) such that for $3 < r < \min\{q, 6\}$,

$$\begin{cases} 0 \le \rho(x,t) \le \bar{\rho}, \quad \|\nabla\mu(\rho)\|_{L^q} \le 2M, \quad \forall\ (x,t) \in \Omega \times [0,T^*], \\ (\rho, \mu(\rho)) \in C([0,T^*]; W^{1,q}), \quad (\nabla u, \Pi) \in C([0,T^*]; H^1) \cap L^1(0,T^*; W^{1,r}), \\ \rho_t \in C([0,T^*]; L^q), \quad \sqrt{\rho}u_t \in L^\infty(0,T^*; L^2), \quad u_t \in L^2(0,T^*; H^1_0). \end{cases}$$

3. Proof of Theorem 1.1

In this section, we establish some time-weighted a priori estimates, which together with the local existence (cf. Lemma 2.2) will complete the proof of Theorem 1.1. To this end, we assume that the following a priori hypothesis holds for some $T > 0$:

$$\sup_{0\le t\le T} \|\nabla\mu(\rho)\|_{L^q} \le 2M \quad \text{for } q > 3. \tag{3.1}$$

For simplicity, we shall use the same letter C to denote some positive constant which maybe dependent on Ω, $\bar{\mu}$, q, $\underline{\mu}$, u_0, g and M, but is independent of m, $\bar{\rho}$ and T.

Let (ρ, u, Π) be a strong solution of (1.1)–(1.5) on $\Omega \times (0,T)$.

We begin with the upper bound of density, which is an easy consequence of the transport equation $(1.1)_1$.

Lemma 3.1. *It holds that*

$$0 \le \rho \le \bar{\rho}, \tag{3.2}$$

for all $(x,t) \in \Omega \times (0,T)$.

Note that $\|\sqrt{\rho_0}u_0\|_{L^2} \le \|\sqrt{\rho_0}\|_{L^3}\|u_0\|_{L^6}$. The following lemma comes easily from the basic energy estimate and (3.2).

Lemma 3.2. *It holds that*

$$\sup_{0\le t\le T} \|\sqrt{\rho}u\|^2_{L^2} + \underline{\mu}\int_0^T \|\nabla u\|^2_{L^2} dt \le C(\bar{\rho}m^2)^{\frac{1}{3}}\|\nabla u_0\|^2_{L^2}. \tag{3.3}$$

We now give the upper bound for the L^2-norm of the gradient of velocity, which does not to be small under the small assumption on $\bar{\rho}$ or m.

Lemma 3.3. *Assume (3.1) holds and*

$$\underline{\mu} \sup_{0\le t\le T} \|\nabla u\|^2_{L^2} \le 3\|\sqrt{\mu(\rho_0)}\nabla u_0\|^2_{L^2}. \tag{3.4}$$

There exists an absolute positive constant ϵ_1 depending Ω, $\bar{\mu}$, $\underline{\mu}$, q, M and $\|\nabla u_0\|_{L^2}$, such that if

$$\bar{\rho}^3(1+\bar{\rho})(\bar{\rho}m^2)^{\frac{1}{3}} \le \epsilon_1,$$

then

$$\underline{\mu} \sup_{0\le t\le T} \|\nabla u\|_{L^2}^2 + \int_0^T \|\sqrt{\rho} u_t\|_{L^2}^2 dt \le 2\|\sqrt{\mu(\rho_0)}\nabla u_0\|_{L^2}^2. \tag{3.5}$$

Moreover,

$$\underline{\mu} \sup_{0\le t\le T} t\|\nabla u\|_{L^2}^2 + \int_0^T t\|\sqrt{\rho} u_t\|_{L^2}^2 dt \le C(\bar{\rho} m^2)^{\frac{1}{3}}. \tag{3.6}$$

Proof. Note that

$$\mu(\rho)_t + u \cdot \nabla\mu(\rho) = 0,$$

due to $(1.1)_1$. Multiplying $(1.1)_2$ by u_t, we have after integration by parts that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \mu(\rho)|\nabla u|^2 dx + \int \rho|u_t|^2 dx \\
&\le \left|\frac{1}{2}\int (2\rho u\cdot\nabla u\cdot u_t + u\cdot\nabla\mu(\rho)|\nabla u|^2)dx\right| \\
&\le \|\sqrt{\rho}u_t\|_{L^2}\|\sqrt{\rho}u\|_{L^6}\|\nabla u\|_{L^3} + C\|\nabla\mu(\rho)\|_{L^3}\|u\|_{L^6}\|\nabla u\|_{L^4}^2 \\
&\le \frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\bar{\rho}\|\nabla u\|_{L^2}^3\|\nabla u\|_{H^1} + C\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{3/2} \\
&\le \frac{1}{4}\|\sqrt{\rho}u_t\|_{L^2}^2 + \delta\|\nabla u\|_{H^1}^2 + C(\delta^{-1}\bar{\rho}^2 + \delta^{-3})\|\nabla u\|_{L^2}^6,
\end{aligned} \tag{3.7}$$

where we have used the Young and Gagliardo–Nirenberg inequalities. Applying Lemma 2.1 with $F = \rho u_t + \rho u\cdot\nabla u$, we derive that

$$\begin{aligned}
\|\nabla u\|_{H^1} &\le C\|F\|_{L^2}(1+\|\nabla\mu(\rho)\|_{L^q})^{\frac{q}{q-3}} \\
&\le C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2} + C\bar{\rho}\|u\|_{L^6}\|\nabla u\|_{L^3} \\
&\le C\bar{\rho}^{\frac{1}{2}}\|\sqrt{\rho}u_t\|_{L^2} + C\bar{\rho}\|\nabla u\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{H^1}^{\frac{1}{2}},
\end{aligned}$$

which implies

$$\|\nabla u\|_{H^1}^2 \le C_1\bar{\rho}\|\sqrt{\rho}u_t\|_{L^2}^2 + C\bar{\rho}^4\|\nabla u\|_{L^2}^6. \tag{3.8}$$

By choosing $\delta = (4C_1\bar{\rho})^{-1}$, we obtain from (3.7) and (3.8) that

$$\frac{d}{dt}\int \mu(\rho)|\nabla u|^2 dx + \int \rho|u_t|^2 dx \le C\bar{\rho}^3(1+\bar{\rho})\|\nabla u\|_{L^2}^6. \tag{3.9}$$

From the Gronwall inequality, (3.3) and (3.4), we infer

$$\begin{aligned}
&\underline{\mu}\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2 + \int_0^T \|\sqrt{\rho}u_t\|_{L^2}^2 dt \\
&\le \|\sqrt{\mu(\rho_0)}\nabla u_0\|_{L^2}^2 \exp\left\{C_2\|\nabla u_0\|_{L^2}^4\bar{\rho}^3(1+\bar{\rho})(\bar{\rho}m^2)^{\frac{1}{3}}\right\},
\end{aligned}$$

which deduces (3.5), provided $\bar\rho^3(1+\bar\rho)(\bar\rho m^2)^{\frac13} \le \epsilon_1 \triangleq \min\{1, C_2^{-1}\|\nabla u_0\|_{L^2}^{-4}\ln 2\}$. Furthermore, multiplying (3.9) by t, we similarly obtain

$$\begin{aligned}
&\underline{\mu}\sup_{0\le t\le T} t\|\nabla u\|_{L^2}^2 + \int_0^T t\|\sqrt{\rho}u_t\|_{L^2}^2 dt \\
&\le C\exp\left\{C_2\|\nabla u_0\|_{L^2}^4\bar\rho^3(1+\bar\rho)(\bar\rho m^2)^{\frac13}\right\}\int_0^T \|\nabla u\|_{L^2}^2 dt \\
&\le C(\bar\rho m^2)^{\frac13},
\end{aligned}$$

where (3.3) has been used. □

Lemma 3.4. *Let (3.1) hold. There exists a positive constant $\widetilde{C}$ as described in Theorem 1.1, such that*

$$\sup_{0\le t\le T} t\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^T t\|\nabla u_t\|_{L^2}^2 dt \le C(\bar\rho m^2)^{\frac13}(1+\bar\rho^4)\exp\left\{2\widetilde{C}\bar\rho^2\left[\bar\rho(\bar\rho m^2)^{\frac13}+1\right]\right\}, \tag{3.10}$$

$$\sup_{0\le t\le T} t^2\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^T t^2\|\nabla u_t\|_{L^2}^2 dt \le C(\bar\rho m^2)^{\frac13}(1+\bar\rho^4)\exp\left\{3\widetilde{C}\bar\rho^2\left[\bar\rho(\bar\rho m^2)^{\frac13}+1\right]\right\}, \tag{3.11}$$

provided $\bar\rho^3(1+\bar\rho)(\bar\rho m^2)^{\frac13} \le \epsilon_1$.

Proof. Differentiating $(1.1)_2$ with respect to t, multiplying it by u_t in L^2 and integrating by parts, we find

$$\begin{aligned}
&\frac12\frac{d}{dt}\int \rho|u_t|^2dx + \int \mu(\rho)|\nabla u_t|^2dx \\
&= \int u\cdot\nabla\mu(\rho)\nabla u\cdot\nabla u_t dx - 2\int \rho u\cdot\nabla u_t\cdot u_t dx \\
&\quad - \int \rho u_t\cdot\nabla u\cdot u_t dx - \int \rho u\cdot\nabla(u\cdot\nabla u\cdot u_t)dx \triangleq \sum_{i=1}^4 I_i.
\end{aligned} \tag{3.12}$$

We estimate the right-hand side of (3.12) terms by term. Firstly, based on (3.1), (3.8), the Young and Sobolev inequalities, we get

$$\begin{aligned}
I_1 &\le \|u\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^3}\|\nabla u\|_{L^6}\|\nabla u_t\|_{L^2} \\
&\le C\|\nabla u_t\|_{L^2}\|\nabla u\|_{H^1}^2 \\
&\le \frac{\underline{\mu}}{8}\|\nabla u_t\|_{L^2}^2 + C\bar\rho^2\|\sqrt{\rho}u_t\|_{L^2}^4 + C\bar\rho^8\|\nabla u\|_{L^2}^{12}.
\end{aligned}$$

The Gagliardo–Nirenberg and Young inequalities together with (3.8) lead to

$$\begin{aligned}
I_2 &\le C\bar\rho^{1/2}\|\sqrt{\rho}u_t\|_{L^3}\|\nabla u_t\|_{L^2}\|u\|_{L^6} \\
&\le C\bar\rho^{1/2}\|\sqrt{\rho}u_t\|_{L^2}^{1/2}\|\sqrt{\rho}u_t\|_{L^6}^{1/2}\|\nabla u_t\|_{L^2}\|\nabla u\|_{L^2} \\
&\le \frac{\underline{\mu}}{8}\|\nabla u_t\|_{L^2}^2 + C\bar\rho^3\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4,
\end{aligned}$$

$$\begin{aligned}I_3 &\le C\|\sqrt{\rho}u_t\|_{L^4}^2\|\nabla u\|_{L^2}\\ &\le C\bar{\rho}^{\frac{3}{4}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{1}{2}}\|\nabla u_t\|_{L^2}^{\frac{3}{2}}\|\nabla u\|_{L^2}\\ &\le \frac{\mu}{8}\|\nabla u_t\|_{L^2}^2 + C\bar{\rho}^3\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4,\end{aligned}$$

and

$$\begin{aligned}I_4 &\le C\bar{\rho}\|\nabla u\|_{L^2}^2\|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2}\\ &\le \frac{\mu}{8}\|\nabla u_t\|_{L^2}^2 + C\bar{\rho}^3\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^4 + C\bar{\rho}^6\|\nabla u\|_{L^2}^{10}.\end{aligned}$$

Substituting all the above estimates into (3.12), using (3.5), we arrive at

$$\begin{aligned}&\frac{d}{dt}\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2\\ &\quad\le C\bar{\rho}^3\|\sqrt{\rho}u_t\|_{L^2}^2\|\nabla u\|_{L^2}^2 + C\bar{\rho}^2\|\sqrt{\rho}u_t\|_{L^2}^4 + C\bar{\rho}^6(1+\bar{\rho}^2)\|\nabla u\|_{L^2}^6.\end{aligned}\tag{3.13}$$

By the Gronwall inequality, we get from (3.3) and (3.5) that

$$\begin{aligned}&\sup_{0\le t\le T}\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^T\|\nabla u_t\|_{L^2}^2dt\\ &\le C\left(\|\sqrt{\rho_0}g\|_{L^2}^2 + \bar{\rho}^6(1+\bar{\rho}^2)\int_0^T\|\nabla u\|_{L^2}^2dt\right)\exp\left\{C\int_0^T(\bar{\rho}^3\|\nabla u\|_{L^2}^2 + \bar{\rho}^2\|\sqrt{\rho}u_t\|_{L^2}^2)dt\right\}\\ &\le C[1+\bar{\rho}^3(1+\bar{\rho})]\exp\left\{\widetilde{C}\left[\bar{\rho}^3(\bar{\rho}m^2)^{\frac{1}{3}} + \bar{\rho}^2\right]\right\}\\ &\le C(1+\bar{\rho}^4)\exp\left\{\widetilde{C}\bar{\rho}^2\left[\bar{\rho}(\bar{\rho}m^2)^{\frac{1}{3}}+1\right]\right\},\end{aligned}\tag{3.14}$$

where we have used $\bar{\rho}^3(1+\bar{\rho})(\bar{\rho}m^2)^{\frac{1}{3}} \le 1$. Hence, multiplying (3.13) by t and applying the Gronwall inequality, we obtain from (3.3), (3.6) and (3.14) that

$$\begin{aligned}&\sup_{0\le t\le T} t\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^T t\|\nabla u_t\|_{L^2}^2dt\\ &\le C\int_0^T(\bar{\rho}^6(1+\bar{\rho}^2)t\|\nabla u\|_{L^2}^6 + \|\sqrt{\rho}u_t\|_{L^2}^2)dt\exp\left\{\widetilde{C}\bar{\rho}^2\left[\bar{\rho}(\bar{\rho}m^2)^{\frac{1}{3}}+1\right]\right\}\\ &\le C\int_0^T(\bar{\rho}m^2)^{\frac{1}{3}}(\bar{\rho}^6(1+\bar{\rho}^2)\|\nabla u\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2)dt\exp\left\{\widetilde{C}\bar{\rho}^2\left[\bar{\rho}(\bar{\rho}m^2)^{\frac{1}{3}}+1\right]\right\}\\ &\le C(\bar{\rho}m^2)^{\frac{1}{3}}\left[(\bar{\rho}m^2)^{\frac{1}{3}}\bar{\rho}^6(1+\bar{\rho}^2) + (1+\bar{\rho}^4)\exp\left\{\widetilde{C}\bar{\rho}^2\left[\bar{\rho}(\bar{\rho}m^2)^{\frac{1}{3}}+1\right]\right\}\right]\\ &\quad\times\exp\left\{\widetilde{C}\bar{\rho}^2\left[\bar{\rho}(\bar{\rho}m^2)^{\frac{1}{3}}+1\right]\right\}\\ &\le C(\bar{\rho}m^2)^{\frac{1}{3}}(1+\bar{\rho}^4)\exp\left\{2\widetilde{C}\bar{\rho}^2\left[\bar{\rho}(\bar{\rho}m^2)^{\frac{1}{3}}+1\right]\right\}.\end{aligned}\tag{3.15}$$

In a similar manner, we deduce from (3.3), (3.6), (3.14) and (3.15) that

$$
\begin{aligned}
&\sup_{0\le t\le T} t^2\|\sqrt{\rho}u_t\|_{L^2}^2 + \int_0^T t^2\|\nabla u_t\|_{L^2}^2 dt \\
&\le C\int_0^T \left(\bar\rho^6(1+\bar\rho^2)t^2\|\nabla u\|_{L^2}^6 + t\|\sqrt{\rho}u_t\|_{L^2}^2\right) dt \exp\left\{\widetilde{C}\bar\rho^2\left[\bar\rho(\bar\rho m^2)^{\frac13}+1\right]\right\} \\
&\le C\int_0^T \left((\bar\rho m^2)^{\frac23}\bar\rho^6(1+\bar\rho^2)\|\nabla u\|_{L^2}^2 + (\bar\rho m^2)^{\frac13} t\|\nabla u_t\|_{L^2}^2\right) dt \exp\left\{\widetilde{C}\bar\rho^2\left[\bar\rho(\bar\rho m^2)^{\frac13}+1\right]\right\} \\
&\le C(\bar\rho m^2)^{\frac23}(1+\bar\rho^4)\exp\left\{3\widetilde{C}\bar\rho^2\left[\bar\rho(\bar\rho m^2)^{\frac13}+1\right]\right\}.
\end{aligned}
$$

The proof of this lemma is finished. □

With the t-weighted estimates in Lemma 3.4 at hand, we now deal with $\|\nabla u\|_{L^\infty}$, which is similar to [13].

Lemma 3.5. *Let (3.1) hold. It has for any $r\in(3,\min\{q,6\})$ that*

$$
\int_0^T \|\nabla u\|_{L^\infty} dt \le C\Lambda, \tag{3.16}
$$

provided $\bar\rho^3(1+\bar\rho)(\bar\rho m^2)^{\frac13}\le\epsilon_1$.

Proof. Let $F=\rho u_t+\rho u\cdot\nabla u$. Applying Lemma 2.1, we have

$$
\begin{aligned}
\|\nabla u\|_{W^{1,r}} &\le C\|F\|_{L^r}(1+\|\nabla\mu(\rho)\|_{L^p})^{\frac{qr}{2(q-r)}} \\
&\le C(\|\rho u_t\|_{L^r}+\|\rho u\cdot\nabla u\|_{L^r}) \\
&\le C(\|\rho u_t\|_{L^2}^{\frac{6-r}{2r}}\|\rho u_t\|_{L^6}^{\frac{3(r-2)}{2r}}+\bar\rho\|u\|_{L^6}\|\nabla u\|_{L^{\frac{6r}{6-r}}}) \\
&\le C\left(\|\rho u_t\|_{L^2}^{\frac{6-r}{2r}}(\bar\rho\|\nabla u_t\|_{L^2})^{\frac{3(r-2)}{2r}}+\bar\rho\|\nabla u\|_{L^2}^{\frac{6(r-1)}{5r-6}}\|\nabla u\|_{W^{1,r}}^{\frac{4r-6}{5r-6}}\right),
\end{aligned}
$$

which deduces

$$
\|\nabla u\|_{W^{1,r}} \le C\left(\bar\rho^{\frac{5r-6}{4r}}\|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3(r-2)}{2r}}+\bar\rho^{\frac{5r-6}{r}}\|\nabla u\|_{L^2}^{\frac{6(r-1)}{r}}\right).
$$

On the one hand, if $0<T\le1$, we derive from (3.3), (3.5) and (3.10) that

$$
\begin{aligned}
&\int_0^T \|\nabla u\|_{L^\infty} dt \le C\int_0^T \|\nabla u\|_{W^{1,r}} dt \\
&\le C\bar\rho^{\frac{5r-6}{4r}}\int_0^T \|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3(r-2)}{2r}} dt + C\int_0^T \bar\rho^{\frac{5r-6}{r}}\|\nabla u\|_{L^2}^{\frac{6(r-1)}{r}} dt \\
&\le C\bar\rho^{\frac{5r-6}{4r}}\left(\int_0^T (t^{\frac12}\|\sqrt{\rho}u_t\|_{L^2})^{\frac{6-r}{2r}}(t^{\frac12}\|\nabla u_t\|_{L^2})^{\frac{3(r-2)}{2r}} t^{-\frac12} dt\right) \\
&\quad + C\bar\rho^{\frac{5r-6}{r}}\|\nabla u\|_{L^2}^{\frac{4r-6}{r}}\int_0^T \|\nabla u\|_{L^2}^2 dt
\end{aligned} \tag{3.17}
$$

$$
\begin{aligned}
&\le C\bar\rho^{\frac{5r-6}{4r}}(t^{1/2}\|\sqrt{\rho}u_t\|_{L^2})^{\frac{6-r}{2r}}\left(\int_0^T t\|\nabla u_t\|_{L^2}^2 dt\right)^{\frac{3(r-2)}{4r}}\left(\int_0^T t^{-\frac{2r}{r+6}}dt\right)^{\frac{r+6}{4r}}\\
&\quad + C\bar\rho^{\frac{5r-6}{r}}(\bar\rho m^2)^{\frac13}\\
&\le C(\bar\rho m^2)^{\frac16}\bar\rho^{\frac{5r-6}{4r}}\left[(1+\bar\rho^2)\exp\left\{\widetilde{C}\bar\rho^2\left[\bar\rho(\bar\rho m^2)^{\frac13}+1\right]\right\}+\bar\rho^{\frac{15r-18}{4r}}(\bar\rho m^2)^{\frac16}\right].
\end{aligned}
$$

On the other hand, for $T>1$, (3.11) leads to

$$
\begin{aligned}
&\int_1^T \|\nabla u\|_{L^\infty}dt \le C\int_1^T \|\nabla u\|_{W^{1,r}}dt\\
&\le C\bar\rho^{\frac{5r-6}{4r}}\int_1^T \|\sqrt{\rho}u_t\|_{L^2}^{\frac{6-r}{2r}}\|\nabla u_t\|_{L^2}^{\frac{3(r-2)}{2r}}dt + C\int_1^T \bar\rho^{\frac{5r-6}{r}}\|\nabla u\|_{L^2}^{\frac{6(r-1)}{r}}dt\\
&\le C\bar\rho^{\frac{5r-6}{4r}}\int_1^T (t\|\sqrt{\rho}u_t\|_{L^2})^{\frac{6-r}{2r}}(t\|\nabla u_t\|_{L^2})^{\frac{3(r-2)}{2r}}t^{-1}dt + C\bar\rho^{\frac{5r-6}{r}}(\bar\rho m^2)^{\frac13}\\
&\le C\bar\rho^{\frac{5r-6}{4r}}(t\|\sqrt{\rho}u_t\|_{L^2})^{\frac{6-r}{2r}}\left(\int_1^T t^2\|\nabla u_t\|_{L^2}^2 dt\right)^{\frac{3(r-2)}{4r}}\left(\int_1^T t^{-\frac{4r}{r+6}}dt\right)^{\frac{r+6}{4r}}\\
&\quad + C\bar\rho^{\frac{5r-6}{r}}(\bar\rho m^2)^{\frac13}\\
&\le C(\bar\rho m^2)^{\frac13}\bar\rho^{\frac{5r-6}{4r}}\left[(1+\bar\rho^2)\exp\left\{\frac{3\widetilde{C}}{2}\bar\rho^2\left[\bar\rho(\bar\rho m^2)^{\frac13}+1\right]\right\}+\bar\rho^{\frac{15r-18}{4r}}\right].
\end{aligned}
\tag{3.18}
$$

Combining (3.17) and (3.18) leads to (3.16). The proof of Lemma 3.5 is complete. □

In view of (3.16), one can prove the norm of $\|\nabla\mu(\rho)\|_{L^q}$ with $3<q<\infty$ is strictly less than $2M$, provided $\bar\rho$ or m is small enough. This particularly finishes the proof of the a priori assumption (3.1).

Lemma 3.6. *Under the hypothesis (3.1), there exists a positive constant ϵ as described in Theorem 1.1, such that*

$$\|\nabla\mu(\rho)\|_{L^q} \le \frac{3M}{2}, \tag{3.19}$$

provided $\Lambda\le\epsilon$. Moreover,

$$\|\nabla\rho\|_{L^q} \le C\|\nabla\rho_0\|_{L^q}. \tag{3.20}$$

Proof. It follows directly from $(1.1)_1$ that

$$\frac{d}{dt}\|\nabla\mu(\rho)\|_{L^q} \le C\|\nabla u\|_{L^\infty}\|\nabla\mu(\rho)\|_{L^q}.$$

In view of (3.16), we find

$$\begin{aligned}\|\nabla\mu(\rho)\|_{L^q} &\le \|\nabla\mu(\rho_0)\|_{L^q}\exp\{C\int_0^T \|\nabla u\|_{L^\infty}dt\} \\ &\le \|\nabla\mu(\rho_0)\|_{L^q}\exp\{C_3\Lambda\} \\ &\le \frac{3M}{2},\end{aligned}$$

provided $\Lambda \le \epsilon \triangleq \min\{\epsilon_1, C_3^{-1}\ln\frac{3}{2}\}$. The proof of (3.20) is similar. □

Proof of Theorem 1.1. Thanks to all the a priori estimates established above, we now are ready to prove Theorem 1.1. In fact, this can be done in a similar manner as that in [13], we omit it here for simplicity. □

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References

[1] H. Abidi, G. Gui, P. Zhang, On the decay and stability to global solutions of the 3-D inhomogeneous Navier–Stokes equations, Comm. Pure Appl. Math. 64 (2011) 832–881.
[2] H. Abidi, G. Gui, P. Zhang, On the wellposedness of three-dimensional inhomogeneous Navier–Stokes equations in the critical spaces, Arch. Ration. Mech. Anal. 204 (1) (2012) 189–230.
[3] S.A. Antontsev, A.V. Kazhikov, Mathematical study of flows of nonhomogeneous fluids, Lectures at the University of Novosibirsk, USSR, 1973.
[4] S.N. Antontsev, A.V. Kazhiktov, V.N. Monakhov, Boundary Value Problems in Mechanics of Nonhomogeneous Fluids, Elsevier, 1989.
[5] Y. Cho, H. Kim, Unique solvability for the density-dependent Navier–Stokes equations, Nonlinear Anal. 59 (2004) 465–480.
[6] H.J. Choe, H. Kim, Strong solutions of the Navier–Stokes equations for nonhomogeneous incompressible fluids, Comm. Partial Differential Equations 28 (2003) 1183–1201.
[7] W. Craig, X. Huang, Y. Wang, Global wellposedness for the 3D inhomogeneous incompressible Navier–Stokes equations, J. Math. Fluid Mech. 15 (2013) 747–758.
[8] R. Danchin, Local and global well-posedness results for flows of inhomogeneous viscous fluids, Adv. Differential Equations 9 (2004) 353–386.
[9] B. Desjardins, Regularity results for two-dimensional flows of multiphase viscous fluids, Arch. Ration. Mech. Anal. 137 (1997) 135–158.
[10] G. Gui, P. Zhang, Global smooth solutions to the 2-D inhomogeneous Navier–Stokes equations with variable viscosity, Chin. Ann. Math. Ser. B 30 (2009) 607–630.
[11] X. Huang, Y. Wang, Global strong solution to the 2D nonhomogeneous incompressible MHD system, J. Differential Equations 254 (2013) 511–527.
[12] X. Huang, Y. Wang, Global strong solution with vacuum to the two dimensional density-dependent Navier–Stokes system, SIAM J. Math. Anal. 46 (3) (2014) 1771–1788.
[13] X. Huang, Y. Wang, Global strong solution of 3D inhomogeneous Navier–Stokes equations with density-dependent viscosity, J. Differential Equations 259 (4) (2015) 1606–1627.
[14] A.V. Kazhikov, Resolution of boundary value problems for nonhomogeneous viscous fluids, Dokl. Akad. Nauk 216 (5) (1974) 1008–1010.
[15] H. Kim, A blow-up criterion for the nonhomogeneous incompressible Navier–Stokes equations, SIAM J. Math. Anal. 37 (2006) 1417–1434.
[16] O.A. Ladyzhenskaya, V.A. Solonnikov, Unique solvability of an initial- and boundary-value problem for viscous incompressible nonhomogeneous fluids, J. Sov. Math. 9 (5) (1978) 697–749.
[17] P.L. Lions, Mathematical Topics in Fluid Mechanics, vol. I: Incompressible Models, Oxford Lecture Ser. Math. Appl., vol. 3, Oxford University Press, New York, 1996.
[18] M. Paicu, P. Zhang, Global solutions to the 3-D incompressible inhomogeneous Navier–Stokes system, J. Funct. Anal. 262 (2012) 3556–3584.
[19] J. Simon, Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure, SIAM J. Math. Anal. 21 (1990) 1093–1117.
[20] J. Zhang, Global well-posedness for the incompressible Navier–Stokes equations with density-dependent viscosity coefficient, J. Differential Equations 259 (5) (2015) 1722–1742.