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On reciprocal transformation of a 3-component Camassa–Holm type system

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ABSTRACT

We construct a reciprocal transformation to connect a 3-component Camassa–Holm type system proposed by Geng and Xue with the first negative flow in a generalized MKdV hierarchy. We discuss the Hamiltonian pair and infinitely many conserved quantities for the generalized MKdV hierarchy.

1. Introduction

Since the celebrated Camassa–Holm (CH) equation was derived as a model for unidirectional motion of dispersive shallow-water by Camassa and Holm in 1993 [4,5], integrable PDEs admitting peakons (CH type equations) have attracted much attention in recent years. Two of the most famous equations are the CH equation and the Degasperis–Procesi (DP) equation, which are contained in the following one parameter family of PDEs [8]:

$$m_t + um_x + bu_x m = 0, \quad m = u - u_{xx}, \tag{1}$$

where b is an arbitrary constant. When $b = 2$, the equation is the CH, which is completely integrable with a Lax pair and a bi-Hamiltonian structure [4,5]. The CH equation is solvable by inverse scattering transform [6] and possesses algebro-geometric solutions [16]. And explicit formulas for the multi-peakon solutions of the equation are studied by the inverse spectral methods [2,3]. Moreover, it is connected to the negative KdV equation by a transformation of reciprocal type [11,17,21]. When $b = 3$, the system (1) is just the DP equation, which is discovered by the method of asymptotic integrability to isolate integrable third-order equations [7]. Indeed it can be obtained by the theory of shallow water [9]. The DP equation is connected to a negative flow in the Kaup–Kupershmidt hierarchy via a reciprocal transformation. With the aid of the

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transformation, a Lax pair for the DP equation is constructed. Besides the bi-Hamiltonian structure and the integrability of the finite-dimensional peakon dynamics for the equation are discussed [8]. Furthermore, explicit formulas for the multi-peakon solutions of the DP equation are researched in Refs. [28,29].

In the past two decades, many other CH type equations are proposed and studied (see e.g. [13,18,26] and references). For example, the Novikov equation [31]

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx} \tag{2}$$

is discovered by the method of symmetry classification. The bi-Hamiltonian structure and a Lax pair for the Novikov equation are given and there is a reciprocal transformation to connect it with a negative flow in the Sawada–Kotera hierarchy [19]. In addition, explicit formulas for multi-peakon solutions of it are obtained in Ref. [20]. The Geng–Xue equation [14]

$$m_t + 3u_x vm + uvm_x = 0, \tag{3}$$

$$n_t + 3uv_x n + uvn_x = 0, \tag{4}$$

$$m = u - u_{xx}, \quad n = v - v_{xx} \tag{5}$$

is proposed as a generalization of the Novikov equation. It is also completely integrable with a Lax pair and associated bi-Hamiltonian structure as well as infinitely many conserved quantities [14,23]. Furthermore, a reciprocal transformation is constructed to connect it with the first negative flow in the modified Boussinesq hierarchy [25], and explicit formulas for the multi-peakon of the equation are considered in Ref. [30].

The subject of this paper is a 3-component CH type system proposed by Geng and Xue [15], i.e.

$$\begin{aligned} u_t &= -vp_x + u_x q + \frac{3}{2}uq_x - \frac{3}{2}u(p_x r_x - pr), \\ v_t &= 2vq_x + v_x q, \\ w_t &= vr_x + w_x q + \frac{3}{2}wq_x + \frac{3}{2}w(p_x r_x - pr), \end{aligned} \tag{6}$$

where

$$\begin{aligned} u &= p - p_{xx}, \\ v &= \frac{1}{2}(q_{xx} - 4q + p_{xx}r_x - r_{xx}p_x + 3p_x r - 3pr_x), \\ w &= r_{xx} - r, \end{aligned}$$

which admits the following spectral problem

$$\psi_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 + \lambda v & 0 & u \\ \lambda w & 0 & 0 \end{pmatrix} \psi. \tag{7}$$

The bi-Hamiltonian structure as well as infinite many conserved quantities and the dynamical system of N-peakon solutions for this system are obtained in Refs. [15,24]. It is interesting that the spectral problem (7) for this 3-component model may be reduced to that of the CH equation and that of the Geng–Xue equation, the same is true for the associated bi-Hamiltonian structures. Both of these will be done below. There exists another reason why we are intrigued about reciprocal transformations of the system (6). Note that the spectral problem (7) is gauge equivalent to that of another 3-component CH type system constructed by us in Ref. [26]. However, detailed calculations show that the reciprocal transformation on our 3-component CH type system [22] can't be applied to the system (6).

The aim of this paper is to construct a reciprocal transformation for the 3-component CH type system (6). And the results show that the transformed system is linked to a negative flow in a 3-component extension to the MKdV hierarchy. Then the integrability of the generalized MKdV hierarchy is also considered.

2. Reductions of the spectral problem (7) and associated Hamiltonian pair

It is easy to find that the spectral problem (7) is reduced to that of the CH equation under $u = w = 0$. In addition, when $v = 0$ the reduced spectral problem for (7) is equivalent to the following spectral problem of the Geng–Xue equation [14]

$$\varphi_x = \begin{pmatrix} 0 & \lambda^{\frac{1}{2}} u & 1 \\ 0 & 0 & \lambda^{\frac{1}{2}} w \\ 1 & 0 & 0 \end{pmatrix} \varphi \tag{8}$$

by a simple gauge transformation

$$\varphi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \lambda^{-\frac{1}{2}} \\ 1 & 0 & 0 \end{pmatrix} \psi.$$

Therefore the spectral problem (7) is reduced to that of Geng–Xue equation as $v = 0$.

As pointed out in Ref. [15], the 3-component CH type system (6) possesses the following bi-Hamiltonian structure

$$\begin{pmatrix} u \\ w \\ v \end{pmatrix}_t = \mathcal{J}_1 \begin{pmatrix} \frac{\delta H_1}{\delta u} \\ \frac{\delta H_1}{\delta w} \\ \frac{\delta H_1}{\delta v} \end{pmatrix} = \mathcal{J}_2 \begin{pmatrix} \frac{\delta H_0}{\delta u} \\ \frac{\delta H_0}{\delta w} \\ \frac{\delta H_0}{\delta v} \end{pmatrix}, \tag{9}$$

where

$$\mathcal{J}_1 = \begin{pmatrix} 0 & \partial^2 - 1 & 0 \\ 1 - \partial^2 & 0 & 0 \\ 0 & 0 & -\partial v - v\partial \end{pmatrix},$$

$$\mathcal{J}_2 = \begin{pmatrix} \frac{3}{2} u \partial^{-1} u & -v - \frac{3}{2} u \partial^{-1} w & 0 \\ v - \frac{3}{2} w \partial^{-1} u & \frac{3}{2} w \partial^{-1} w & 0 \\ 0 & 0 & 0 \end{pmatrix} - 2\mathcal{N}(\partial^3 - 4\partial)^{-1}\mathcal{N}^*$$

with $\mathcal{N} = (\frac{3}{2}u\partial + u_x, \frac{3}{2}w\partial + w_x, v\partial + \partial v)^T$ and

$$H_1 = \frac{1}{4} \int [4q^2 - qq_{xx} - p_x^2 r_x^2 + 6pp_x r r_x + 3p^2 r^2] dx,$$

$$H_0 = \int [v + ur_x] dx.$$

Then direct calculations show that the compatible Hamiltonian operators for the CH and the Geng–Xue may be obtained by reductions of the Hamiltonian pair in (9). For example, when $u = w = 0$, the compatible Hamiltonian operators $\mathcal{J}_1$ and $\mathcal{J}_2$ are reduced to

$$\bar{\mathcal{J}}_1 = -v\partial - \partial v, \qquad \bar{\mathcal{J}}_2 = 2(v\partial + \partial v)(\partial^3 - 4\partial)^{-1}(v\partial + \partial v),$$

which are just $\bar{\mathcal{J}}_1$, $2\bar{\mathcal{J}}_1\bar{\mathcal{J}}_3^{-1}\bar{\mathcal{J}}_1$ comparing to the famous Hamiltonian pair

$$\bar{\mathcal{J}}_1 = -v\partial - \partial v, \qquad \bar{\mathcal{J}}_3 = \partial^3 - 4\partial.$$

Therefore a Hamiltonian pair for the CH equation is obtained. Moreover, it is not hard to show that a Hamiltonian pair and a spectral problem for the Novikov hierarchy may also be obtained by reducing those of the system (6) under the constraint $u = w$, $v = 0$.

3. A reciprocal transformation

It is shown in Ref. [15] that a Lax pair for the 3-component system (6) is given by the spectral problem (7) and associated auxiliary problem

$$\psi_t = \begin{pmatrix} -\frac{1}{2}(q_x + p_x r_x - pr) & \frac{1}{\lambda} + q & \frac{p_x}{\lambda} \\ \frac{1}{\lambda} - pr_x + p_x r - q + \lambda qv & \frac{1}{2}(q_x - p_x r_x + pr) & \frac{p}{\lambda} + qu \\ -r + \lambda qw & r_x & p_x r_x - pr \end{pmatrix} \psi. \tag{10}$$

Based on this Lax pair, infinitely many conservation laws for the 3-component system (6) have been constructed [15]. Specially, one of them is

$$(v^{\frac{1}{2}})_t = (v^{\frac{1}{2}} q)_x,$$

which defines a reciprocal transformation

$$dy = v^{\frac{1}{2}} dx + v^{\frac{1}{2}} q dt, \qquad d\tau = dt. \tag{11}$$

Writing the column vector ψ in components as $\psi = (\psi_1, \psi_2, \psi_3)^T$ and eliminating ψ_2 in the spectral problem (7), we have

$$\psi_{1xx} = (1 + \lambda v)\psi_1 + u\psi_3, \qquad \psi_{3x} = \lambda w \psi_1.$$

Then under the reciprocal transformation (11) and after a gauge transformation $\psi_1 = v^{-\frac{1}{4}}\phi_1$, the above spectral problem is transformed to

$$\phi_{1yy} + (\frac{3v_y^2}{16v^2} - \frac{v_{yy}}{4v} - \frac{1}{v})\phi_1 = \lambda\phi_1 + uv^{-\frac{3}{4}}\psi_3, \quad \psi_{3y} = \lambda w v^{-\frac{3}{4}}\phi_1.$$

Using the method of factorizing Lax operator [10] and setting $\mu = \lambda^{\frac{1}{2}}$, we arrive at the following spectral problem:

$$\phi_y = \begin{pmatrix} Q_1 & \mu & 0 \\ \mu & -Q_1 & Q_2 \\ \mu Q_3 & 0 & 0 \end{pmatrix} \phi, \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \tag{12}$$

where $\phi_2 = \frac{1}{\mu}(\partial_y - Q_1)\phi_1$, $\phi_3 = \frac{1}{\mu}\psi_3$ and

$$Q_1 = \frac{v_y}{4v} + v^{-\frac{1}{2}}, \qquad Q_2 = uv^{-\frac{3}{4}}, \qquad Q_3 = wv^{-\frac{3}{4}}. \tag{13}$$

It is worth to note that many choices for the transformed spectral problems have been tried here but failed, since the variable q is lost in the time part of the Lax pairs corresponding to the transformed spectral problems.

Furthermore, through tedious calculations, we get the auxiliary problem corresponding to the spectral problem (12), that is

$$\phi_\tau = \begin{pmatrix} \frac{1}{\mu^2} - \frac{1}{2}f & \frac{1}{\mu}g & \frac{(a_y - aQ_1)g + a}{\mu} \\ \frac{(c - g^{-1}\partial^{-1}(aQ_3 + bQ_2))}{\mu} & -\frac{1}{\mu^2} - \frac{1}{2}f & \frac{aQ_1 - a_y}{\mu^2} \\ \frac{b_y - bQ_1}{\mu} & (b_y - bQ_1)g + b & f \end{pmatrix} \phi, \tag{14}$$

where

$$\begin{aligned} &a = pv^{\frac{1}{4}}, \qquad b = rv^{\frac{1}{4}}, \qquad c = q_y - 2qv^{-\frac{1}{2}}, \\ &f = \partial_y^{-1}[Q_3(a_y - aQ_1)g - Q_2(b_y - bQ_1)g + aQ_3 - bQ_2], \\ &g = -(\partial_y - 2Q_1)^{-1}2. \end{aligned}$$

Then the compatible condition of the Lax pair (12)–(14) yields just the transformed system

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_\tau = \begin{pmatrix} \frac{1}{2}(cg)_y \\ -\frac{3}{2}Q_2 f - (a_y - aQ_1)g - a \\ \frac{3}{2}Q_3 f + (b_y - bQ_1)g + b \end{pmatrix}, \qquad \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = 0, \tag{15}$$

where

$$\begin{aligned} F_1 &= a_{yy} - a(Q_{1y} + Q_1^2) + Q_2, \\ F_2 &= Q_3 - b_{yy} + b(Q_{1y} + Q_1^2), \\ F_3 &= 2 - c_y - 2Q_1 c + Q_2(b_y - bQ_1) + Q_3(a_y - aQ_1) \\ &\quad + (g^{-1} + 2g^{-2}\partial_y^{-1})(aQ_3 + bQ_2). \end{aligned}$$

Therefore the 3-component CH type system (6) and the Lax pair (7)–(10) are transformed to (15) and (12)–(14) respectively. Since the transformed spectral problem (12) may be reduced to that of the MKdV hierarchy as $Q_2 = Q_3 = 0$, we call the hierarchy associated with (12) a generalized MKdV hierarchy.

4. The Hamiltonian pair and conserved quantities of the generalized MKdV hierarchy

4.1. The Hamiltonian pair of the generalized MKdV hierarchy

In Ref. [12] a generalized KdV hierarchy is proposed associated with a 3×3 matrix spectral problem

$$\varphi_x = \begin{pmatrix} 0 & 1 & 0 \\ \lambda + v_1 & 0 & u_1 \\ \lambda w_1 & 0 & 0 \end{pmatrix} \varphi. \tag{16}$$

It is not hard to find that the above spectral problem is equivalent to a spectral problem of Yajima–Oikawa hierarchy (see e.g. [27,32]) or a spectral problem relating to a energy-dependent Lax operator (Example 4.1 in Ref. [1]) by two gauge transformations respectively. In fact a detail calculation shows that it is also gauge equivalent to the transformed spectral problem (12). Writing the column vector φ in components and comparing it with the similar result of (12), we find the potentials in the two systems are related by

$$u_1 = Q_2, \quad v_1 = Q_{1y} + Q_1^2, \quad w_1 = Q_3. \tag{17}$$

We note that a bi-Hamiltonian structure of the generalized KdV hierarchy associated with the spectral problem (16) is obtained, but the proof is not given. Since the original expression of the Hamiltonian pair is complicated, we rewrite it as

$$\mathcal{E}_1 = \begin{pmatrix} -\frac{3}{2}u_1\partial_y^{-1}u_1 & 0 & \frac{3}{2}u_1\partial_y^{-1}w_1 + 1 \\ 0 & -\frac{1}{2}\partial_y^3 + \partial v_1 + v_1\partial & 0 \\ \frac{3}{2}w_1\partial_y^{-1}u_1 - 1 & 0 & -\frac{3}{2}w_1\partial_y^{-1}w_1 \end{pmatrix},$$
$$\mathcal{E}_2 = \begin{pmatrix} 0 & 0 & \partial_y^2 - v_1 \\ 0 & 0 & 0 \\ v_1 - \partial_y^2 & 0 & 0 \end{pmatrix} + \frac{1}{2}\Omega\partial_y^{-1}\Omega^*,$$

where

$$\Omega = (\frac{1}{2}u_1\partial_y + \partial_y u_1, -\frac{1}{2}\partial_y^3 + \partial_y v_1 + v_1\partial_y, \frac{1}{2}w_1\partial + \partial_y w)^T.$$

Then through the relation (17), we may obtain the bi-Hamiltonian structure for the generalized MKdV hierarchy admitting the spectral problem (12). The proof of this is an implied proof for the bi-Hamiltonian structure of the above generalized KdV hierarchy. Since the method of trace identity hasn't been taken, we only give the Hamiltonian pair here.

Theorem 1. *The generalized MKdV hierarchy associated with the spectral problem (12) has a pair of Hamiltonian operators*

$$\mathcal{K}_1 = \begin{pmatrix} -\frac{1}{2}\partial_y & 0 & 0 \\ 0 & \frac{3}{2}Q_2\partial_y^{-1}Q_2 & -1 - \frac{3}{2}Q_2\partial_y^{-1}Q_3 \\ 0 & 1 - \frac{3}{2}Q_3\partial_y^{-1}Q_2 & \frac{3}{2}Q_3\partial_y^{-1}Q_3 \end{pmatrix},$$
$$\mathcal{K}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (\partial_y + Q_1)(\partial_y - Q_1) \\ 0 & -(\partial_y + Q_1)(\partial_y - Q_1) & 0 \end{pmatrix} + \frac{1}{2}\mathcal{F}\partial_y^{-1}\mathcal{F}^*,$$

where

$$\mathcal{F} = (-\frac{1}{2}\partial_y(\partial_y - 2Q_1), \frac{1}{2}Q_2\partial_y + \partial_y Q_2, \frac{1}{2}Q_3\partial_y + \partial_y Q_3)^T.$$

Proof. In order to prove the theorem, we need to show the two operators $\mathcal{K}_1$ and $\mathcal{K}_2$ are Hamiltonian and they constitute a compatible pair. In fact one may find that it suffices to show $a\mathcal{K}_1 + b\mathcal{K}_2$ is a Hamiltonian operator for two arbitrary constants a and b. Setting $\mathcal{L} = a\mathcal{K}_1 + b\mathcal{K}_2$, we can easily show that $\mathcal{L}$ is skew-symmetric and only need to prove

$$\mathrm{pr}\ \mathrm{v}_{\mathcal{L}\theta}\Theta_{\mathcal{L}} = 0.$$

To make expressions compact, we introduce

$$e = \partial_y^{-1}(Q_2\theta_2 - Q_3\theta_3),$$
$$h = \partial_y^{-1}(\frac{1}{2}\theta_{1yy} + Q_1\theta_{1y} + \frac{3}{2}Q_2\theta_{2y} + \frac{1}{2}Q_{2y}\theta_2 + \frac{3}{2}Q_3\theta_{3y} + \frac{1}{2}Q_{3y}\theta_3),$$

then we have

$$\mathcal{L}\theta = \begin{pmatrix} -\frac{1}{2}a\theta_{1y} + \frac{1}{4}bh_{yy} - \frac{1}{2}b(Q_1h)_y \\ \frac{3}{2}aQ_2e - a\theta_3 + b\theta_{3yy} - b(Q_{1y} + Q_1^2)\theta_3 - \frac{3}{4}bQ_2h_y - \frac{1}{2}bQ_{2y}h \\ -\frac{3}{2}aQ_3e + a\theta_2 - b\theta_{2yy} + b(Q_{1y} + Q_1^2)\theta_2 - \frac{3}{4}bQ_3h_y - \frac{1}{2}bQ_{3y}h \end{pmatrix},$$

and

$$\begin{aligned}\Theta_{\mathcal{L}} &= \frac{1}{2}\int \theta \wedge \mathcal{L}\theta dy \\ &= \int [\frac{3}{4}a(Q_2\theta_2 - Q_3\theta_3) \wedge e - \frac{1}{4}a\theta_1 \wedge \theta_{1y} - a\theta_2 \wedge \theta_3 + b\theta_2 \wedge \theta_{3yy} \\ &\quad - b(Q_{1y} + Q_1^2)\theta_2 \wedge \theta_3 + \frac{1}{4}b(\frac{1}{2}\theta_{1yy} + Q_1\theta_{1y} + \frac{3}{2}Q_2\theta_{2y} + \frac{1}{2}Q_{2y}\theta_2 \\ &\quad + \frac{3}{2}Q_3\theta_{3y} + \frac{1}{2}Q_{3y}\theta_3) \wedge h]dy.\end{aligned}$$

Let $\mathcal{L}\theta \equiv (L_1, L_2, L_3)^T$, we have therefore

$$\begin{aligned}\mathrm{pr}\ \mathrm{v}_{\mathcal{L}\theta}\Theta_{\mathcal{L}} &= \int [\frac{3}{2}a(L_2 \wedge \theta_2 - L_3 \wedge \theta_3) \wedge e - b(L_{1y} + 2Q_1L_1) \wedge \theta_2 \wedge \theta_3 \\ &\quad + \frac{1}{2}b(L_1 \wedge \theta_{1y} + \frac{3}{2}L_2 \wedge \theta_{2y} + \frac{1}{2}L_{2y} \wedge \theta_2 + \frac{3}{2}L_3 \wedge \theta_{3y} + \frac{1}{2}L_{3y} \wedge \theta_3) \wedge h]dy \\ &= \int [\frac{3}{2}aL_2 \wedge \theta_2 \wedge e - \frac{3}{2}aL_3 \wedge \theta_3 \wedge e + bL_1 \wedge (\theta_{2y} \wedge \theta_3 + \theta_2 \wedge \theta_{3y}) \\ &\quad - 2bQ_1L_1 \wedge \theta_2 \wedge \theta_3 + \frac{1}{2}bL_1 \wedge \theta_{1y} \wedge h - \frac{1}{4}bL_2 \wedge (\theta_{2y} \wedge h + \theta_2 \wedge h_y) \\ &\quad + \frac{3}{4}bL_2 \wedge \theta_{2y} \wedge h + \frac{3}{4}bL_3 \wedge \theta_{3y} \wedge h - \frac{1}{4}bL_3(\theta_{3y} \wedge h + \theta_3 \wedge h_y)]dy \\ &= \int [bL_1 \wedge ((\theta_2 \wedge \theta_3)_y - 2Q_1\theta_2 \wedge \theta_3 + \frac{1}{2}\theta_{1y} \wedge h) + L_2 \wedge (\frac{3}{2}a\theta_2 \wedge e \\ &\quad + \frac{1}{2}b\theta_{2y} \wedge h - \frac{1}{4}b\theta_2 \wedge h_y) + L_3 \wedge (\frac{1}{2}b\theta_{3y} \wedge h - \frac{3}{2}a\theta_3 \wedge e - \frac{1}{4}b\theta_3 \wedge h_y)]dy \\ &= \int [\frac{1}{8}b^2h_y \wedge h \wedge (\theta_{1yy} + 2Q_1\theta_{1y} + 3Q_2\theta_{2x} + Q_{2y}\theta_2 + 3Q_3\theta_{3y} + Q_{3y}\theta_3) \\ &\quad + ab\theta_3 \wedge \theta_2 \wedge h_y + \frac{3}{4}ab(Q_2\theta_2 - Q_3\theta_3) \wedge e_y \wedge h + ab(\frac{1}{2}\theta_{1yy} + Q_1\theta_{1y}) \wedge \theta_2 \\ &\quad \wedge \theta_3 - \frac{3}{2}ab(\theta_{2y} \wedge \theta_3 + \theta_{3y} \wedge \theta_2) \wedge e_y]dy \\ &= \int [\frac{1}{4}b^2h_y \wedge h \wedge h_y + ab(\frac{1}{2}\theta_{1yy} + Q_1\theta_{1y} + \frac{3}{2}Q_2\theta_{2y} + \frac{3}{2}Q_3\theta_{3y}) \wedge \theta_2 \wedge \theta_3 \\ &\quad + \frac{3}{4}abe_y \wedge e_y \wedge h - abh_y \wedge \theta_2 \wedge \theta_3]dy\end{aligned}$$

$$= \int [ab(\frac{1}{2}\theta_{1yy} + Q_1\theta_{1y} + \frac{3}{2}Q_2\theta_{2y} + \frac{3}{2}Q_3\theta_{3y} - h_y) \wedge \theta_2 \wedge \theta_3]dy$$
$$= 0.$$

Therefore the theorem is proven. □

4.2. Conserved quantities

The spectral problem (12) can be used to construct infinitely many conserved quantities for the associated hierarchy. Setting $\rho = (\ln\phi_1)_x$, it is not hard to find that ρ satisfies the equation

$$\rho_{yy} + 3\rho\rho_y - \frac{Q_{2y}}{Q_2}(\rho_y + \rho^2 - \mu^2) + \rho^3 - \mu^2\rho - (\rho + Q_2\partial\frac{1}{Q_2})(Q_1^2 + Q_{1y}) - \mu^2 Q_2 Q_3 = 0,$$

then we may obtain infinitely many conserved densities by expanding ρ in powers of μ. However, it is not easy to solve the above equation. Therefore a better formulation for calculation is needed.

Introducing $a = \frac{\phi_2}{\phi_1}$, $b = \frac{\phi_3}{\phi_1}$, we have

$$\rho = Q_1 + \mu a. \tag{18}$$

Based on the spectral problem (12), we have two simple equations for the variables a and b, which are

$$a_y = \mu - 2aQ_1 + bQ_2 - \mu a^2, \tag{19}$$
$$b_y = \mu Q_3 - bQ_1 - \mu ab. \tag{20}$$

Then we can obtain the coefficients for the expansion of ρ by solving a from the above system, which are just conserved densities. To begin with, expanding a, b as $a = \sum_{i\geq 0} a_i\mu^{-i}$, $b = \sum_{j\geq 0} b_j\mu^{-j}$, we can get the recursive relations for a_i, b_j, which are

$$a_{m+1} = -\frac{1}{2}a_{my} - a_m Q_1 + \frac{1}{2}b_m Q_2 - \frac{1}{2}\sum_{i\neq 0, j\neq 0}^{i+j=m+1} a_i a_j, \tag{21}$$
$$b_{m+1} = -b_m Q_1 - b_{my} - a_{m+1}Q_3 - \sum_{i\neq 0, j\neq 0}^{i+j=m+1} a_i b_j, \tag{22}$$

and the first few are given by

$$\begin{aligned}
&a_0 = 1, \qquad\qquad b_0 = Q_3,\\
&a_1 = \frac{1}{2}Q_2Q_3 - Q_1, \qquad b_1 = -Q_{3y} - \frac{1}{2}Q_2Q_3^2,\\
&a_2 = -\frac{1}{4}Q_{2y}Q_3 - \frac{3}{4}Q_2Q_{3y} + \frac{1}{2}Q_{1y} + \frac{1}{2}Q_1^2 - \frac{3}{8}Q_2^2Q_3^2,\\
&b_2 = Q_{3yy} + \frac{3}{4}Q_3^2Q_{2y} + \frac{9}{4}Q_2Q_3Q_{3y} - \frac{1}{2}Q_3Q_{1y} - \frac{1}{2}Q_3Q_1^2 + \frac{5}{8}Q_2^2Q_3^3.
\end{aligned}$$

Substituting the expanding of a into (18), we get an infinitely sequence of conserved densities, then infinitely many conserve quantities may be obtained, and the first two conserved quantities are

$$\bar{\rho}_0 = \frac{1}{2}\int Q_2Q_3 dy,$$
$$\bar{\rho}_1 = \frac{1}{2}\int [Q_1^2 - Q_2Q_{3y} - \frac{3}{4}Q_2^2Q_3^2]dy.$$

Furthermore, we can also consider the cases of $\rho = \ln(\phi_k)_y$, $k = 2, 3$, which are similar to the previous approach and we don't discuss it.

5. A relation between the transformed system (15) and the generalized MKdV hierarchy

We start with the first negative flow in the generalized MKdV hierarchy associated with the spectral problem (12),

$$\mathcal{K}_2\mathcal{K}_1^{-1}\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_\tau = 0, \tag{23}$$

which may be rewritten as

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_\tau = \mathcal{K}_1\begin{pmatrix} A \\ B \\ C \end{pmatrix}, \qquad \mathcal{K}_2\begin{pmatrix} A \\ B \\ C \end{pmatrix} = 0. \tag{24}$$

In order to show the relation between the transformed system (15) and the negative flow (24), we may set

$$A = -cg, \qquad B = (b_y - bQ_1)g + b, \qquad C = (a_y - aQ_1)g + a.$$

Using $g_y = 2Q_1g - 2$, we can simplify the first negative flow for the generalized MKdV hierarchy in the form

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}_\tau = \begin{pmatrix} \frac{1}{2}(cg)_y \\ -\frac{3}{2}Q_2f - (a_y - aQ_1)g - a \\ \frac{3}{2}Q_3f + (b_y - bQ_1)g + b \end{pmatrix}, \qquad \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = 0,$$

where

$$\begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\partial_y(\partial_y - 2Q_1)\partial_y^{-1}E \\ \frac{3}{2}g_yF_1 + gF_{1y} - \frac{1}{2}(\frac{3}{2}Q_2\partial_y + Q_{2y})\partial_y^{-1}E \\ \frac{3}{2}g_yF_2 + gF_{2y} - \frac{1}{2}(\frac{3}{2}Q_3\partial_y + Q_{3y})\partial_y^{-1}E \end{pmatrix},$$

herein

$$E = (\partial_y g - \frac{1}{2}g\partial_y)F_3 + Q_3gF_1 - Q_2gF_2.$$

Then one can verify that the transformed system (15) is just a reduction of the first negative flow (23) in the generalized MKdV hierarchy.

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