



Extremes of $\alpha(t)$ -locally stationary Gaussian processes with non-constant variances



Long Bai

Department of Actuarial Science, University of Lausanne, UNIL-Dorigny, 1015 Lausanne, Switzerland

ARTICLE INFO

Article history:

Received 23 June 2016

Available online 1 September 2016

Submitted by U. Stadtmueller

Keywords:

Fractional Brownian motion

$\alpha(t)$ -locally stationary

Pickands constants

Gaussian process

ABSTRACT

With motivation from [9], in this paper we derive the exact tail asymptotics of $\alpha(t)$ -locally stationary Gaussian processes with non-constant variance functions. We show that some certain variance functions lead to qualitatively new results.

© 2016 Elsevier Inc. All rights reserved.

1. Introduction and main result

For $X(t)$, $t \in [0, T]$, $T > 0$ a centered stationary Gaussian process with unit variance and continuous sample paths Pickands derived in [20] that

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim T \mathcal{H}_\alpha a^{1/\alpha} u^{2/\alpha} \mathbb{P} \{X(0) > u\}, \quad u \rightarrow \infty, \quad (1)$$

provided that the correlation function r satisfies

$$1 - r(t) \sim a |t|^\alpha, \quad t \downarrow 0, \quad a > 0, \quad \text{and } r(t) < 1, \quad \forall t \neq 0, \quad (2)$$

with $\alpha \in (0, 2]$ (\sim means asymptotic equivalence when the argument tends to 0 or ∞). Here the classical Pickands constant \mathcal{H}_α is defined by

$$\mathcal{H}_\alpha = \lim_{T \rightarrow \infty} T^{-1} \mathbb{E} \left\{ \sup_{t \in [0, T]} e^{\sqrt{2} B_\alpha(t) - t^\alpha} \right\},$$

E-mail address: Long.Bai@unil.ch.

where $B_\alpha(t), t \geq 0$ is a standard fractional Brownian motion with Hurst index $\alpha/2 \in (0, 1]$, see [20,21,5,12,10,14,7,23,11,13,6,15] for various properties of \mathcal{H}_α .

The deep contribution [3] introduced the class of locally stationary Gaussian processes with index α , i.e., a centered Gaussian process $X(t), t \in [0, T]$ with a constant variance function, say equal to 1, and correlation function satisfying

$$r(t, t+h) = 1 - a(t)|h|^\alpha + o(|t|^\alpha), \quad h \rightarrow 0,$$

uniformly with respect to $t \in [0, T]$, where $\alpha \in (0, 2]$ and $a(t)$ is a bounded, strictly positive and continuous function.

Clearly, the class of locally stationary Gaussian processes includes the stationary ones. It allows for some minor fluctuations of dependence at t and at the same time keeps stationary structure at the local scale. See [3,4,18] for studies on the locally stationary Gaussian processes with index α .

In [9] the tail asymptotics of the supremum of $\alpha(t)$ -locally stationary Gaussian processes are investigated. Such processes and random fields are of interest in various applications, see [9] and the recent contributions [2,16,17]. Following the definition in [9], a centered Gaussian process $X(t), t \in [0, T]$ with continuous sample paths and unit variance is $\alpha(t)$ -locally stationary if the correlation function $r(\cdot, \cdot)$ satisfies the following conditions:

- (i) $\alpha(t) \in C([0, T])$ and $\alpha(t) \in (0, 2]$ for all $t \in [0, T]$;
- (ii) $a(t) \in C([0, T])$ and $0 < \inf\{a(t) : t \in [0, T]\} \leq \sup\{a(t) : t \in [0, T]\} < \infty$;
- (iii) uniformly for $t \in [0, T]$

$$1 - r(t, t+h) = a(t)|h|^{\alpha(t)} + o(|h|^{\alpha(t)}), \quad h \rightarrow 0,$$

where $f(t) \in C(\mathcal{T})$ means that $f(t)$ is continuous on $\mathcal{T} \subset \mathbb{R}$.

In this paper, we shall consider the case that the variance function $\sigma^2(t) = \text{Var}(X(t))$ is not constant, assuming instead that:

- (iv) $\sigma(t)$ attains its maximum equal to 1 over $[0, T]$ at the unique point $t_0 \in [0, T]$ and for some constants $c, \gamma > 0$,

$$\frac{1}{\sigma(t)} = 1 + ce^{-|t-t_0|^{-\gamma}}(1 + o(1)), \quad t \rightarrow t_0.$$

A crucial assumption in our result is that similar to the variance function, the function $\alpha(t)$ has a certain behavior around the extreme point t_0 . Specifically, as in [9] we shall assume:

- (v) there exist $\beta, \delta, b > 0$ such that

$$\alpha(t+t_0) = \alpha(t_0) + b|t|^\beta + o(|t|^{\beta+\delta}), \quad t \rightarrow 0.$$

Remark 1.1. We remark that t_0 does not need to be the unique point such that $\alpha(t)$ is minimal on $[0, T]$, which is different from [9]. For instance, $[0, T] = [0, 2\pi]$, $t_0 = 0$ and $\alpha(t) = 1 + \frac{1}{2}\sin(t)$, then 0 is not the minimum point of $\alpha(t)$ over $[0, 2\pi]$ which means assumptions about $\alpha(t)$ in [9] are not satisfied but assumption (v) here is satisfied with

$$\alpha(t) = 1 + \frac{1}{2}|t| + o(|t|^{\frac{3}{2}}), \quad t \rightarrow 0.$$

Below we set $\alpha := \alpha(t_0)$, $a := a(t_0)$ and write Ψ for the survival function of an $N(0, 1)$ random variable. Further, define $0^a = \infty$ for $a < 0$. Our main result is stated in the next theorem.

Theorem 1.2. If a centered Gaussian process $X(t), t \in [0, T]$ with continuous sample paths is such that the assumptions (i)–(v) are valid, then we have as $u \rightarrow \infty$

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \widehat{I} a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} (\ln u)^{-\frac{1}{\gamma \wedge \beta}} \Psi(u) \begin{cases} 2^{-1/\gamma}, & \text{if } \gamma < \beta, \\ \int_0^{2^{-1/\gamma}} e^{\frac{-bx^\beta}{\alpha^2}} dx, & \text{if } \gamma = \beta, \\ \int_0^\infty e^{\frac{-bx^\beta}{\alpha^2}} dx, & \text{if } \gamma > \beta, \end{cases}$$

where $\gamma \wedge \beta = \min(\gamma, \beta)$ and

$$\widehat{I} = \begin{cases} 1, & \text{if } t_0 = 0 \text{ or } t_0 = T, \\ 2, & \text{if } t_0 \in (0, T). \end{cases}$$

Remark 1.3. i) If $\alpha(t) \equiv \alpha$ for all t in a small neighborhood of t_0 , the asymptotic of $\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\}$ is the same as in the case of $\gamma < \beta$ in Theorem 1.2.

ii) The result of case $\gamma > \beta$ in Theorem 1.2 is the same as the $\alpha(t)$ -locally stationary scenario in [9], which means that $\sigma(t)$ varies so slow in a small neighborhood of t_0 that $X(t)$ can be considered as $\alpha(t)$ -locally stationary in this small neighborhood.

The following example is a straightforward application of Theorem 1.2.

Example 1.4. Here we consider a multifractional Brownian motion $B_{H(t)}(t)$, $t \geq 0$, i.e., a centered Gaussian process with covariance function

$$\mathbb{E} \{ B_{H(t)}(t) B_{H(s)}(s) \} = \frac{1}{2} D(H(s) + H(t)) \left[|s|^{H(s)+H(t)} + |t|^{H(s)+H(t)} - |t-s|^{H(s)+H(t)} \right],$$

where $D(x) = \frac{2\pi}{\Gamma(x+1) \sin(\frac{\pi x}{2})}$ and $H(t)$ is a Hölder function of exponent λ such that $0 < H(t) < \min(1, \lambda)$ for $t \in [0, \infty)$. For constants T_1, T_2 with $0 < T_1 < T_2$, define

$$\overline{B}_{H(t)}(t) := \frac{B_{H(t)}(t)}{\sqrt{\text{Var}(B_{H(t)}(t))}}, \quad t \in [T_1, T_2],$$

and

$$\sigma(t) := 1 - e^{-|t-t_0|^{-\gamma}}, \quad t \in [T_1, T_2],$$

with some $t_0 \in (T_1, T_2)$ and $\gamma > 0$.

By [9], $\overline{B}_{H(t)}(t)$, $t \in [T_1, T_2]$, is a $2H(t)$ -locally stationary Gaussian process with correlation function

$$r(t, t+h) = 1 - \frac{1}{2} t^{-2H(t)} |h|^{2H(t)} + o(|h|^{2H(t)}), \quad h \rightarrow 0.$$

Further, we assume that there exist $\beta, \delta, b > 0$ such that $H(t+t_0) = H(t_0) + bt^\beta + o(t^{\beta+\delta})$, as $t \rightarrow 0$. Then

$$\mathbb{P} \left\{ \sup_{t \in [T_1, T_2]} \sigma(t) \overline{B}_{H(t)}(t) > u \right\} \sim 2^{1-1/2H} \frac{\mathcal{H}_{2H}}{t_0} u^{1/H} (\ln u)^{-\frac{1}{\gamma \wedge \beta}} \Psi(u) \begin{cases} 2^{-1/\gamma}, & \text{if } \gamma < \beta, \\ \int_0^{2^{-1/\gamma}} e^{\frac{-bx^\beta}{H^2}} dx, & \text{if } \gamma = \beta, \\ \int_0^\infty e^{\frac{-bx^\beta}{H^2}} dx, & \text{if } \gamma > \beta, \end{cases} \quad u \rightarrow 0,$$

with $H := H(t_0)$.

2. Proofs

In the rest of the paper, we focus on the case when $t_0 = 0$. The complementary scenario when $t_0 \in (0, T]$ follows by analogous argumentation. Recall that

$$\mathcal{H}_\alpha = \lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{H}_\alpha[0, T], \quad \text{with } \mathcal{H}_\alpha[-S_1, S_2] = \mathbb{E} \left\{ \sup_{t \in [-S_1, S_2]} e^{\sqrt{2}B_\alpha(t) - |t|^\alpha} \right\} \in (0, \infty),$$

where $S_1, S_2 \in [0, \infty)$ with $\max(S_1, S_2) > 0$ are some constants.

Lemma 2.1. *Under the assumptions of Theorem 1.2 we have*

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \mathbb{P} \left\{ \sup_{t \in [0, \delta_1(u)]} X(t) > u \right\}, \quad u \rightarrow \infty. \quad (3)$$

Moreover, there exists a constant $C > 0$ such that for all sufficiently large u

$$\mathbb{P} \left\{ \sup_{t \in [\delta_2(u), T]} X(t) > u \right\} \leq CTu^{2/\alpha} (\ln u)^{-4/3\beta} \Psi(u), \quad (4)$$

where for some constant $q > 1$

$$\delta_1(u) = \left(\frac{1}{2 \ln u - q \ln \ln u} \right)^{1/\gamma} \quad \text{and} \quad \delta_2(u) = \left(\frac{\alpha^2 (\ln(\ln u))}{\beta (\ln u)} \right)^{1/\beta}. \quad (5)$$

By (4), in the proof of Theorem 1.2, we derive that, as $u \rightarrow \infty$,

$$\mathbb{P} \left\{ \sup_{t \in [\delta_2(u), T]} X(t) > u \right\} = o \left(\mathbb{P} \left\{ \sup_{t \in [0, \delta_2(u)]} X(t) > u \right\} \right). \quad (6)$$

Since $\delta_1(u) \rightarrow 0, \delta_2(u) \rightarrow 0$ as $u \rightarrow \infty$ and $a(t)$ is continuous, without loss of generality, we may assume that $a(t) \equiv a(0) = a$ for $t \in ([0, \delta_1(u)] \cup [0, \delta_2(u)])$. Moreover, by assumption (iv), we know that $\sigma(t) > 0$ for $t \in ([0, \delta_1(u)] \cup [0, \delta_2(u)])$. Below we use notation $\overline{X}(t) = \frac{X(t)}{\sigma(t)}$ for all t such that $\sigma(t)$ is positive.

Proof of Theorem 1.2. First we derive the asymptotic of

$$\pi(u) := \mathbb{P} \left\{ \sup_{t \in \Delta(u)} X(t) > u \right\},$$

as $u \rightarrow \infty$, where $\Delta(u) = [0, \delta(u)]$ and

$$\delta(u) = \begin{cases} \delta_1(u), & \text{if } \gamma \leq \beta, \\ \delta_2(u), & \text{if } \gamma > \beta, \end{cases}$$

with $\delta_1(u)$ and $\delta_2(u)$ in (5), which combined with Lemma 2.1 finally shows that

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \pi(u). \quad (7)$$

In the following \mathbb{Q}_i , $i \in \mathbb{N}$, are some positive constants. For some $S > 0$, let $Y_{\nu,u}(t), t \in [0, S]$ be a family of centered stationary Gaussian processes with

$$\text{Cov}(Y_{\nu,u}(s), Y_{\nu,u}(t)) = 1 - (1 - \nu)au^{-2}|s - t|^{\alpha+2b\delta^\beta(u)},$$

for $\nu \in (0, 1), u > 0$ such that $\alpha + 2b\delta^\beta(u) \leq 2$ and $s, t \in [0, S]$. Further, let $Z_{\nu,u}(t), t \in [0, S]$ be another family of centered stationary Gaussian processes with

$$\text{Cov}(Z_{\nu,u}(s), Z_{\nu,u}(t)) = 1 - (1 + \nu)au^{-2}|s - t|^\alpha,$$

for $\nu \in (0, 1), u > 0$ and $s, t \in [0, S]$. Due to assumptions (i) and (v), α is strictly smaller than 2, which guarantees that covariance function of $Y_{\nu,u}(t), t \in [0, S]$ and $Z_{\nu,u}(t), t \in [0, S]$ are positive-definite. Hence the introduced families of Gaussian processes exist.

By assumption (iv), for any small $\varepsilon \in (0, 1)$

$$1 + (1 - \varepsilon)ce^{-|t|^{-\gamma}} \leq \frac{1}{\sigma(t)} \leq 1 + (1 + \varepsilon)ce^{-|t|^{-\gamma}}, \quad (8)$$

holds for $t \in [0, \delta(u)]$.

Case 1: $\gamma < \beta$. Set for any $\varepsilon \in (0, 1)$ and all u large

$$N(0) = N(u, 0) := \left\lfloor \frac{\delta_1(u)u^{2/\alpha}}{S} \right\rfloor, \quad N_\varepsilon(u) = \left\lfloor (1 - \varepsilon) \frac{\delta_1(u)u^{2/\alpha}}{S} \right\rfloor = \left\lfloor \frac{(1 - \varepsilon)u^{2/\alpha}}{(2 \ln u - q \ln \ln u)^{1/\gamma} S} \right\rfloor,$$

$$B_j(u) = B_{j,0}(u) = \left[j \frac{S}{u^{2/\alpha}}, (j+1) \frac{S}{u^{2/\alpha}} \right], \quad j \in \mathbb{N}, \quad \mathcal{G}_u^{\pm\varepsilon} = u \left(1 + (1 \pm \varepsilon)ce^{-((1-\varepsilon)\delta_1(u))^{-\gamma}} \right).$$

We notice the fact that

$$\Psi(\mathcal{G}_u^{\pm\varepsilon}) \sim \Psi(u), \quad u \rightarrow \infty,$$

and

$$I_1(u) \leq \pi(u) \leq I_1(u) + I_2(u), \quad (9)$$

where

$$I_1(u) = \mathbb{P} \left\{ \sup_{t \in [0, (1-\varepsilon)\delta_1(u)]} X(t) > u \right\}, \quad I_2(u) = \mathbb{P} \left\{ \sup_{t \in [(1-\varepsilon)\delta_1(u), \delta_1(u)]} X(t) > u \right\}.$$

Then by Bonferroni's inequality, (8), Lemma 3.1 with $k = 0$ and Lemma 3.2

$$\begin{aligned} I_1(u) &\leq \sum_{j=0}^{N_\varepsilon(u)} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u \right\} \\ &\leq \sum_{j=0}^{N_\varepsilon(u)} \mathbb{P} \left\{ \sup_{t \in B_j(u)} \bar{X}(t) > \mathcal{G}_u^{-\varepsilon} \right\} \\ &\leq \sum_{j=0}^{N_\varepsilon(u)} \mathbb{P} \left\{ \sup_{t \in [jS, (j+1)S]} \bar{X}(tu^{-2/\alpha}) > \mathcal{G}_u^{-\varepsilon} \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=0}^{N_\epsilon(u)} \mathbb{P} \left\{ \sup_{t \in [0, S]} Z_{u, \nu}(t) > \mathcal{G}_u^{-\epsilon} \right\} \\
&\sim \sum_{j=0}^{N_\epsilon(u)} \mathcal{H}_\alpha \left[0, S((1+\nu)a)^{1/\alpha} \right] \Psi(\mathcal{G}_u^{-\epsilon}) \\
&\sim \sum_{j=0}^{N_\epsilon(u)} \mathcal{H}_\alpha \left[0, S((1+\nu)a)^{1/\alpha} \right] \Psi(u) \\
&\sim (1-\epsilon) u^{2/\alpha} \delta_1(u) \frac{\mathcal{H}_\alpha \left[0, S((1+\nu)a)^{1/\alpha} \right]}{S} \Psi(u) \\
&\sim (1-\epsilon)((1+\nu)a)^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} \delta_1(u) \Psi(u), \quad u \rightarrow \infty, \quad S \rightarrow \infty.
\end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned}
\sum_{j=0}^{N_\epsilon(u)-1} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u \right\} &\geq \sum_{j=0}^{N_\epsilon(u)-1} \mathbb{P} \left\{ \sup_{t \in [0, S]} Y_{u, \nu}(t) > \mathcal{G}_u^{+\epsilon} \right\} \\
&\sim (1-\epsilon)((1-\nu)a)^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} \delta_1(u) \Psi(u), \quad u \rightarrow \infty, \quad S \rightarrow \infty.
\end{aligned} \tag{11}$$

Since

$$I_1(u) \geq \sum_{j=0}^{N_\epsilon(u)-1} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u \right\} - \sum_{0 \leq j < k \leq N_\epsilon(u)} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u, \sup_{t \in B_k(u)} X(t) > u \right\}, \tag{12}$$

and by [9][Lemma 4.5]

$$\begin{aligned}
\sum_{0 \leq j < k \leq N_\epsilon(u)} \mathbb{P} \left\{ \sup_{t \in B_j(u)} X(t) > u, \sup_{t \in B_k(u)} X(t) > u \right\} &\leq \sum_{0 \leq j < k \leq N_\epsilon(u)} \mathbb{P} \left\{ \sup_{t \in B_j(u)} \bar{X}(t) > u, \sup_{t \in B_k(u)} \bar{X}(t) > u \right\} \\
&= o \left(u^{2/\alpha} \delta_1(u) \Psi(u) \right), \quad u \rightarrow \infty, \quad S \rightarrow \infty, \quad \epsilon \rightarrow 0.
\end{aligned} \tag{13}$$

Thus inserting (11) and (13) into (12), we have

$$\lim_{u \rightarrow \infty} \frac{I_1(u)(2 \ln u - q \ln \ln u)^{1/\gamma}}{u^{2/\alpha} \Psi(u)} \geq (1-\epsilon)((1-\nu)a)^{1/\alpha} \mathcal{H}_\alpha,$$

which combined with (10) gives that

$$I_1(u) \sim \frac{a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha}}{(2 \ln u - q \ln \ln u)^{1/\gamma}} \Psi(u), \quad u \rightarrow \infty, \quad \nu \rightarrow 0, \quad \epsilon \rightarrow 0. \tag{14}$$

By (iii) and (v), we have for all u large

$$\mathbb{E} \{ (\bar{X}(t) - \bar{X}(s))^2 \} = 2 - 2r(s, t) \leq \mathbb{Q}_1 |s - t|^\alpha,$$

uniformly holds for $s, t \in [(1-\epsilon)\delta_1(u), \delta_1(u)]$. By Piterbarg inequality for u large enough, see e.g., [22][Theorem 8.1] or an extension in [8][Lemma 5.1]

$$I_2(u) \leq \mathbb{P} \left\{ \sup_{t \in [(1-\epsilon)\delta_1(u), \delta_1(u)]} \overline{X}(t) > u \right\} \leq \mathbb{Q}_2 \epsilon \delta_1(u) u^{2/\alpha} \Psi(u), \quad (15)$$

which implies

$$\lim_{\epsilon \rightarrow 0} \lim_{u \rightarrow \infty} \frac{I_2(u)(2 \ln u - q \ln \ln u)^{1/\gamma}}{u^{2/\alpha} \Psi(u)} = 0.$$

Combining this equation with (9) and (14), we get

$$\pi(u) \sim \frac{a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha}}{(2 \ln u - q \ln \ln u)^{1/\gamma}} \Psi(u) \sim a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} (2 \ln u)^{-1/\gamma} \Psi(u), \quad u \rightarrow \infty.$$

Case 2: $\gamma = \beta$. Set

$$d_k = d_k(u) := \left(\frac{k}{\ln(u)(\ln \ln(u))^{1/\beta}} \right)^{1/\beta}, \quad A_k = A_k(u) := [d_k, d_{k+1}].$$

Further let $M_\epsilon(u) = \max(k \in \mathbb{N} : d_k \leq (1-\epsilon)\delta_1(u))$ for some $\epsilon \in (0, 1)$, then $M_\epsilon(u) \rightarrow \infty$, $u \rightarrow \infty$. Clearly

$$\bigcup_{k=0}^{M_\epsilon(u)-1} A_k \subset [0, (1-\epsilon)\delta_1(u)] \subset \bigcup_{k=0}^{M_\epsilon(u)} A_k.$$

We divide each interval A_k into subintervals of length $S/u^{2/\alpha(d_k)}$, i.e.,

$$B_{j,k} = B_{j,k}(u) := \left[d_k + j \frac{S}{u^{2/\alpha(d_k)}}, d_k + (j+1) \frac{S}{u^{2/\alpha(d_k)}} \right]$$

for $j = 0, 1, \dots, N(k)$, where $N(k) = N(k, u) := \left\lfloor \frac{d_{k+1}-d_k}{S} u^{2/\alpha(d_k)} \right\rfloor$. Notice that

$$\bigcup_{k=0}^{N(k)-1} B_{j,k} \subset A_k \subset \bigcup_{k=0}^{N(k)} B_{j,k}.$$

We have

$$I_1(u) \leq \pi(u) \leq I_1(u) + I_2(u), \quad (16)$$

where

$$I_1(u) = \mathbb{P} \left\{ \sup_{t \in [0, (1-\epsilon)\delta_1(u)]} X(t) > u \right\}, \quad I_2(u) = \mathbb{P} \left\{ \sup_{t \in [(1-\epsilon)\delta_1(u), \delta_1(u)]} X(t) > u \right\}.$$

Then by Bonferroni's inequality

$$\begin{aligned} I_1(u) &\geq \sum_{k=0}^{M_\epsilon(u)-1} \sum_{j=0}^{N(k)-1} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u \right\} - \sum_{\substack{(j,k), (j',k') \in \mathcal{L} \\ (j,k) \prec (j',k')}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u, \sup_{t \in B_{j',k'}} X(t) > u \right\} \\ &=: J_1(u) - J_2(u), \end{aligned} \quad (17)$$

where $\mathcal{L} = \{(j, k) : 0 \leq k \leq M_\epsilon(u) - 1, 0 \leq j \leq N(k) - 1\}$ and

$$(j, k) \prec (j', k') \text{ iff } (k < k') \vee (k = k' \wedge j < j'),$$

and by (8), Lemma 3.1 and Lemma 3.2

$$\begin{aligned} I_1(u) &\leq \sum_{k=0}^{M_\epsilon(u)} \sum_{j=0}^{N(k)} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u \right\} \\ &\leq \sum_{k=0}^{M_\epsilon(u)} \sum_{j=0}^{N(k)} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} \bar{X}(t) > \mathcal{G}_u^{-\epsilon} \right\} \\ &\leq \sum_{k=0}^{M_\epsilon(u)} \sum_{j=0}^{N(k)} \mathbb{P} \left\{ \sup_{t \in [0, S]} Z_{\nu, u}(t) > \mathcal{G}_u^{-\epsilon} \right\} \\ &\sim \sum_{k=0}^{M_\epsilon(u)} \sum_{j=0}^{N(k)} \mathcal{H}_\alpha \left[0, S((1+\nu)a)^{1/\alpha} \right] \Psi(\mathcal{G}_u^{-\epsilon}) \\ &\sim \sum_{k=0}^{M_\epsilon(u)} \frac{d_{k+1} - d_k}{S} u^{2/\alpha(d_k)} \mathcal{H}_\alpha \left[0, S((1+\nu)a)^{1/\alpha} \right] \Psi(u) \\ &= \frac{\mathcal{H}_\alpha \left[0, S((1+\nu)a)^{1/\alpha} \right]}{S} \frac{u^{2/\alpha}}{(\ln u)^{1/\beta}} \Psi(u) \sum_{k=0}^{M_\epsilon(u)} (\ln u)^{1/\beta} (d_{k+1} - d_k) e^{(\ln u) \left(\frac{2(\alpha - \alpha(d_k))}{\alpha \alpha(d_k)} \right)} \\ &\leq \frac{\mathcal{H}_\alpha \left[0, S((1+\nu)a)^{1/\alpha} \right]}{S} \frac{u^{2/\alpha}}{(\ln u)^{1/\beta}} \Psi(u) \sum_{k=0}^{M_\epsilon(u)} (\ln u)^{1/\beta} (d_{k+1} - d_k) e^{\frac{-2(1-\epsilon_1)(\ln u)(bd_k^\beta - d_k^{\beta+\delta})}{\alpha^2}} \\ &\leq \frac{\mathcal{H}_\alpha \left[0, S((1+\nu)a)^{1/\alpha} \right]}{S} \frac{u^{2/\alpha}}{(\ln u)^{1/\beta}} \Psi(u) \\ &\quad \times \sum_{k=0}^{M_\epsilon(u)} (\ln u)^{1/\beta} (d_{k+1} - d_k) e^{\frac{-2(1-\epsilon_1)b((\ln u)^{1/\beta} d_k)^\beta}{\alpha^2}} e^{\frac{2(1-\epsilon_1)(\ln u) d_k^{\beta+\delta}}{\alpha^2 M_\epsilon(u)+1}}, \end{aligned}$$

as $u \rightarrow \infty$, where $\epsilon_1 \in (0, 1)$ is a small constant.

Moreover, using that $d_{M_\epsilon(u)} \leq (1-\epsilon)\delta_1(u)$ and $\lim_{u \rightarrow \infty} (\ln u)\delta_1(u)^{\beta+\delta} = 0$, we observe that

$$\lim_{u \rightarrow \infty} e^{\frac{2(1-\epsilon_1)(\ln u) d_k^{\beta+\delta}}{\alpha^2 M_\epsilon(u)+1}} = 1.$$

Finally, since

$$\lim_{u \rightarrow \infty} \sup_{k=0, \dots, M_\epsilon(u)} (\ln u)^{1/\beta} (d_{k+1} - d_k) = 0$$

and

$$\lim_{u \rightarrow \infty} (\ln u)^{1/\beta} d_{M_\epsilon(u)+1} = (1-\epsilon) \left(\frac{1}{2} \right)^{1/\beta},$$

we obtain

$$\lim_{u \rightarrow \infty} \sum_{k=0}^{M_\epsilon(u)} (\ln u)^{1/\beta} (d_{k+1} - d_k) e^{\frac{-2(1-\epsilon_1)b((\ln u)^{1/\beta} d_k)^\beta}{\alpha^2}} = \int_0^{(1-\epsilon)(\frac{1}{2})^{1/\beta}} e^{\frac{-2(1-\epsilon_1)bx^\beta}{\alpha^2}} dx.$$

Thus

$$\lim_{u \rightarrow \infty} \frac{I_1(u)(\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} \leq \frac{\mathcal{H}_\alpha[0, S((1+\nu)a)^{1/\alpha}]}{S} \int_0^{(1-\epsilon)(\frac{1}{2})^{1/\beta}} e^{\frac{-2(1-\epsilon_1)bx^\beta}{\alpha^2}} dx, \quad (18)$$

and letting $S \rightarrow \infty, \epsilon_1, \nu \rightarrow 0$, and $\epsilon \rightarrow 0$, we get the upper bound. Similarly, we derive that

$$\lim_{\epsilon \rightarrow 0} \lim_{S \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{J_1(u)(\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} \geq a^{1/\alpha} \mathcal{H}_\alpha \int_0^{(\frac{1}{2})^{1/\beta}} e^{\frac{-2bx^\beta}{\alpha^2}} dx. \quad (19)$$

By [9][Lemma 4.5]

$$\begin{aligned} J_2(u) &= \sum_{\substack{(j,k), (j',k') \in \mathcal{L} \\ (j,k) \prec (j',k')}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u, \sup_{t \in B_{j',k'}} X(t) > u \right\} \\ &\leq \sum_{\substack{(j,k), (j',k') \in \mathcal{L} \\ (j,k) \prec (j',k')}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} \bar{X}(t) > u, \sup_{t \in B_{j',k'}} \bar{X}(t) > u \right\} \\ &= o\left(u^{2/\alpha} (\ln u)^{-1/\beta} \Psi(u)\right), \quad u \rightarrow \infty, S \rightarrow \infty, \epsilon \rightarrow 0. \end{aligned} \quad (20)$$

Thus inserting (19) and (20) into (17), we get

$$\lim_{\epsilon \rightarrow 0} \lim_{S \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{I_1(u)(\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} \geq a^{1/\alpha} \mathcal{H}_\alpha \int_0^{(\frac{1}{2})^{1/\beta}} e^{\frac{-2(1-\epsilon_1)bx^\beta}{\alpha^2}} dx. \quad (21)$$

By (15)

$$\lim_{\epsilon \rightarrow 0} \lim_{u \rightarrow \infty} \frac{I_2(u)(\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} = 0. \quad (22)$$

Hence according to (16), (18), (21), and (22), we have

$$\pi(u) \sim a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} (\ln u)^{-1/\beta} \Psi(u) \int_0^{(\frac{1}{2})^{1/\beta}} e^{\frac{-2bx^\beta}{\alpha^2}} dx, \quad u \rightarrow \infty.$$

Case 3: $\gamma > \beta$. We consider $\pi(u) = \mathbb{P} \left\{ \sup_{t \in [0, \delta_2(u)]} X(t) > u \right\}$ with

$$\delta_2(u) = \left(\frac{\alpha^2 (\ln(\ln u))}{\beta (\ln u)} \right)^{1/\beta}.$$

Set for some $\epsilon > 0$

$$\mathcal{F}_u^{\pm\epsilon} = u \left(1 + (1 \pm \epsilon) c e^{-(\delta_2(u))^{-\gamma}} \right), \quad \mathcal{K} = \{t \in [0, T] : \sigma(t) \neq 0\},$$

and we observe that

$$\Psi(\mathcal{F}_u^{\pm\epsilon}) \sim \Psi(u), \quad u \rightarrow \infty.$$

By [9][Theorem 2.1]

$$\begin{aligned} \pi(u) &\leq \mathbb{P} \left\{ \sup_{t \in [0, \delta_2(u)]} \overline{X}(t) > u \right\} \\ &\leq \mathbb{P} \left\{ \sup_{t \in \mathcal{K}} \overline{X}(t) > u \right\} \\ &\sim a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} (\ln u)^{-\frac{1}{\beta}} \int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}} dx \Psi(u), \quad u \rightarrow \infty. \end{aligned} \quad (23)$$

Let $d_k, A_k, B_{j,k}, N(k)$ be the same as in **Case 2** and $M(u) = \max\{k \in \mathbb{N} : d_k \leq \delta_2(u)\}$. Clearly

$$\bigcup_{k=0}^{M(u)-1} A_k \subset [0, \delta_2(u)] \subset \bigcup_{k=0}^{M(u)} A_k, \quad \bigcup_{k=0}^{N(k)-1} B_{j,k} \subset A_k \subset \bigcup_{k=0}^{N(k)} B_{j,k},$$

and by Bonferroni's inequality

$$\begin{aligned} \pi(u) &\geq \sum_{k=0}^{M(u)-1} \sum_{j=0}^{N(k)-1} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u \right\} - \sum_{\substack{(j,k), (j',k') \in \mathcal{L}' \\ (j,k) \prec (j',k')}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u, \sup_{t \in B_{j',k'}} X(t) > u \right\} \\ &=: J'_1(u) - J'_2(u), \end{aligned} \quad (24)$$

where $\mathcal{L}' = \{(j, k) : 0 \leq k \leq M(u) - 1, 0 \leq j \leq N(k) - 1\}$.

By (8), Lemma 3.1, Lemma 3.2 and similar argumentation as (19) with $\mathcal{G}_u^{\pm\epsilon}$ replaced by $\mathcal{F}_u^{\pm\epsilon}$ and the fact that $(\ln u)^{1/\beta} d_{M(u)+1} \rightarrow \infty, u \rightarrow \infty$, we get

$$\lim_{S \rightarrow \infty} \lim_{u \rightarrow \infty} \frac{J'_1(u) (\ln u)^{1/\beta}}{u^{2/\alpha} \Psi(u)} \geq a^{1/\alpha} \mathcal{H}_\alpha \int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}} dx. \quad (25)$$

By [9][Lemma 4.5]

$$\begin{aligned} J'_2(u) &= \sum_{\substack{(j,k), (j',k') \in \mathcal{L}' \\ (j,k) \prec (j',k')}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} X(t) > u, \sup_{t \in B_{j',k'}} X(t) > u \right\} \\ &\leq \sum_{\substack{(j,k), (j',k') \in \mathcal{L}' \\ (j,k) \prec (j',k')}} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} \overline{X}(t) > u, \sup_{t \in B_{j',k'}} \overline{X}(t) > u \right\} \\ &= o \left(u^{2/\alpha} (\ln u)^{-1/\beta} \Psi(u) \right), \quad u \rightarrow \infty. \end{aligned} \quad (26)$$

Hence inserting (25) and (26) into (24), we have

$$\lim_{u \rightarrow \infty} \frac{\pi(u)(\ln u)^{1/\beta}}{u^{2/\alpha}\Psi(u)} \geq a^{1/\alpha} \mathcal{H}_\alpha \int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}} dx,$$

which combined with (23) gives that

$$\pi(u) \sim a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} (\ln u)^{-1/\beta} \Psi(u) \int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}} dx, \quad u \rightarrow \infty.$$

Consequently, according to Lemma 2.1 and

$$\pi(u) \leq \mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \leq \pi(u) + \mathbb{P} \left\{ \sup_{t \in [\delta(u), T]} X(t) > u \right\},$$

(7) is proved and all claims follow. \square

3. Appendix

In this section we present the proofs of the lemmas used in the proof of Theorem 1.2.

Proof of Lemma 2.1. Below \mathbb{Q}_k , $k = 0, 1, 2, \dots$, are some positive constants.

Step 1: First we prove (3). By the continuity of $\sigma(t)$ in $[0, T]$, for any small enough constant $0 < \theta < 1$

$$\sup_{t \in [\theta, T]} \sigma(t) =: \rho(\theta) < \sigma(t_0) = \sigma(0) = 1.$$

Then by Borell inequality in [1]

$$\mathbb{P} \left\{ \sup_{t \in [\theta, T]} X(t) > u \right\} \leq \exp \left(-\frac{(u - \mathbb{Q}_0)^2}{2\rho^2(\theta)} \right) = o(\Psi(u)),$$

as $u \rightarrow \infty$, where $\mathbb{Q}_0 = \mathbb{E} \left\{ \sup_{t \in [0, T]} X(t) \right\} < \infty$.

By assumption (iv), for any small $\varepsilon \in (0, 1)$, when θ small enough

$$1 + (1 - \varepsilon)ce^{-|t|^{-\gamma}} \leq \frac{1}{\sigma(t)} \leq 1 + (1 + \varepsilon)ce^{-|t|^{-\gamma}},$$

holds for $t \in [0, \theta]$. Then

$$\frac{1}{\sigma(t)} \geq 1 + (1 - \varepsilon)ce^{-|t|^{-\gamma}} \geq 1 + (1 - \varepsilon)cu^{-2}(\ln u)^q$$

uniformly holds for $t \in [\delta_1(u), \theta]$.

Moreover by assumption (i) and (iii), when θ small enough

$$\begin{aligned} \mathbb{E} \{ (X(t) - X(s))^2 \} &= \mathbb{E} \{ X^2(t) \} + \mathbb{E} \{ X^2(s) \} - 2\mathbb{E} \{ X(t)X(s) \} \\ &\leq 2 - 2(1 - 2a(t)|t - s|^{\alpha(t)}) \\ &\leq \mathbb{Q}_1 |t - s|^\varsigma \end{aligned}$$

holds uniformly for $s, t \in [0, \theta]$, where $\mathbb{Q}_1 = \sup_{t \in [0, \theta]} 4a(t)$ and $\varsigma = \inf_{t \in [0, \theta]} \alpha(t) > 0$.

Then by Piterbarg inequality

$$\mathbb{P} \left\{ \sup_{t \in [\delta_1(u), \theta]} X(t) > u \right\} \leq \mathbb{Q}_2 \theta u^{2/\varsigma} \Psi(u[1 + (1 - \varepsilon)cu^{-2}(\ln u)^q]) = o(\Psi(u)), \quad u \rightarrow \infty.$$

Further, since

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \leq \mathbb{P} \left\{ \sup_{t \in [0, \delta_1(u)]} X(t) > u \right\} + \mathbb{P} \left\{ \sup_{t \in [\delta_1(u), \theta]} X(t) > u \right\} + \mathbb{P} \left\{ \sup_{t \in [\theta, T]} X(t) > u \right\},$$

and

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \geq \mathbb{P} \left\{ \sup_{t \in [0, \delta_1(u)]} X(t) > u \right\} \geq \mathbb{P}\{X(0) > u\} = \Psi(u),$$

we get

$$\mathbb{P} \left\{ \sup_{t \in [0, T]} X(t) > u \right\} \sim \mathbb{P} \left\{ \sup_{t \in [0, \delta_1(u)]} X(t) > u \right\}, \quad u \rightarrow \infty.$$

Step 2: Next we prove (4). When $\gamma \leq \beta$, since $\delta_1(u) = o(\delta_2(u))$, as $u \rightarrow \infty$ and by **Step 1**

$$\mathbb{P} \left\{ \sup_{t \in [\delta_1(u), T]} X(t) > u \right\} = o(\Psi(u)), \quad u \rightarrow \infty.$$

Then for u large enough, (4) is obvious.

When $\gamma > \beta$, for u large enough, we have $\delta_2(u) < \delta_1(u)$ and

$$\mathbb{P} \left\{ \sup_{t \in [\delta_2(u), T]} X(t) > u \right\} \leq \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} X(t) > u \right\} + \mathbb{P} \left\{ \sup_{t \in [\delta_1(u), T]} X(t) > u \right\}.$$

By **Step 1**, we know for all u large

$$\mathbb{P} \left\{ \sup_{t \in [\delta_1(u), T]} X(t) > u \right\} \leq \Psi(u),$$

and then we just need to deal with $\mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} X(t) > u \right\}$.

Since $\delta_1(u) \rightarrow 0$, $u \rightarrow \infty$, then by assumption (v)

$$\alpha(t) > \alpha + \frac{3}{4}b(\delta_2(u))^\beta$$

holds for all $t \in [\delta_2(u), \delta_1(u)]$ when u large enough.

Let $\eta_u = u^{-2/(\alpha + \frac{3}{4}b(\delta_2(u))^\beta)}$. For sufficiently large u and $s, t \in [\delta_2(u), \delta_1(u)]$, there exists a constant $\mathbb{Q}_3 > 0$ such that

$$1 - r(s, t) \leq 1 - e^{-\mathbb{Q}_3 |s-t|^{\alpha + \frac{3}{4}b(\delta_2(u))^\beta}}.$$

Let $Y_u(t), t \geq 0$ be a family of centered stationary Gaussian processes with correlation functions

$$r_Y(s, t) = e^{\mathbb{Q}_3 |s-t|^{\alpha + \frac{3}{4}b(\delta_2(u))^\beta}}.$$

Then from Slepian's inequality we get for any constant $S > 0$

$$\begin{aligned} \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} X(t) > u \right\} &\leq \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} \frac{X(t)}{\sigma(t)} > u \right\} \\ &\leq \mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} Y_u(t) > u \right\} \\ &\leq \mathbb{P} \left\{ \sup_{t \in [0, S]} Y_u(t) > u \right\} \\ &\leq \sum_{i=0}^{\lfloor S\eta_u^{-1} \rfloor + 1} \mathbb{P} \left\{ \sup_{t \in [i\eta_u, (i+1)\eta_u]} Y_u(t) > u \right\} \\ &\leq (\lfloor S\eta_u^{-1} \rfloor + 1) \mathbb{P} \left\{ \sup_{t \in [0, \eta_u]} Y_u(t) > u \right\}, \end{aligned}$$

for sufficiently large u . Notice that for each $s, t \in [0, 1]$

$$1 - r_Y(\eta_u t, \eta_u s) = \mathbb{Q}_3 u^{-2} |s - t|^{\alpha + \frac{3}{4}b(\delta_2(u))^\beta} (1 + o(1)) = \mathbb{Q}_3 u^{-2} |s - t|^\alpha (1 + o(1)), \quad u \rightarrow \infty.$$

Hence, from [22][Lemma D.1]

$$\mathbb{P} \left\{ \sup_{t \in [0, \eta_u]} Y_u(t) > u \right\} \sim \mathcal{H}_\alpha[1] \Psi(u),$$

as $u \rightarrow \infty$. Combining this with the fact that

$$\begin{aligned} \eta_u^{-1} &= u^{2/(\alpha + \frac{3}{4}\delta_2(u))} = u^{2/\alpha} u^{2/(\alpha + \frac{3}{4}\delta_2(u)) - 2/\alpha} = u^{2/\alpha} u^{-\frac{3}{2}(\delta_2(u))^\beta / (\alpha(\alpha + \frac{3}{4}(\delta_2(u))^\beta))} \\ &= u^{2/\alpha} u^{-\frac{3}{2} \frac{\alpha^2 (\ln(\ln u))}{\beta (\ln u)}} / (\alpha(\alpha + \frac{3}{4}(\delta_2(u))^\beta)) \leq u^{2/\alpha} u^{-\frac{4}{3} \frac{\ln(\ln u)}{\beta (\ln u)}} = u^{2/\alpha} (\ln u)^{-4/(3\beta)}, \end{aligned}$$

we get for some constant \mathbb{Q}_4 and all u large enough

$$\mathbb{P} \left\{ \sup_{t \in [\delta_2(u), \delta_1(u)]} X(t) > u \right\} \leq \mathbb{Q}_4 S u^{2/\alpha} (\ln u)^{-4/3\beta} \Psi(u).$$

Then the result follows. \square

Lemma 3.1. Under the notation in the proof of Theorem 1.2, for $(j, k) \in \mathcal{U} = \{(j, k) : 0 \leq k \leq M^*(u), 0 \leq j \leq N(k)\}$ and $\lim_{u \rightarrow \infty} \frac{f(u)}{u} = 1$, there exists u_0 such that for each $u \geq u_0$

$$\underline{1)} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} \overline{X}(t) > f(u) \right\} \geq \mathbb{P} \left\{ \sup_{t \in [0, S]} Y_{\nu, u}(t) > f(u) \right\};$$

$$\underline{2)} \mathbb{P} \left\{ \sup_{t \in B_{j,k}} \overline{X}(t) > f(u) \right\} \leq \mathbb{P} \left\{ \sup_{t \in [0, S]} Z_{\nu, u}(t) > f(u) \right\},$$

where

$$M^*(u) = \begin{cases} 0, & \text{if } \gamma < \beta, \\ M_\epsilon(u), & \text{if } \gamma = \beta, \\ M(u), & \text{if } \gamma > \beta. \end{cases}$$

Proof of Lemma 3.1. Since the proofs of scenarios $\gamma < \beta$, $\gamma = \beta$, and $\gamma > \beta$ are similar, we only present the proof of $\gamma = \beta$. Set $X_{j,k,u}(t) = \overline{X}\left(d_k + \frac{jS+t}{u^{2/\alpha(d_k)}}\right)$, then $\sup_{t \in B_{j,k}} \overline{X}(t) \stackrel{d}{=} \sup_{t \in [0,S]} X_{j,k,u}(t)$. It is enough to analyze the supremum of $X_{j,k,u}(t)$.

1) For sufficiently large u and $s, t \in [0, T]$

$$\begin{aligned} 1 - \text{Cov}(X_{j,k,u}(s), X_{j,k,u}(t)) &= 1 - \text{Cov}\left(\overline{X}\left(d_k + \frac{jS+s}{u^{2/\alpha(d_k)}}\right), \overline{X}\left(d_k + \frac{jS+t}{u^{2/\alpha(d_k)}}\right)\right) \\ &\geq (1 - \nu/2)^{1/3} a \left| u^{-2/\alpha(d_k)}(s-t) \right|^{\alpha(d_k + u^{-2/\alpha(d_k)}(jS+t))} \\ &= (1 - \nu/2)^{1/3} a u^{-2\alpha(d_k + u^{-2/\alpha(d_k)}(jS+t))/\alpha(d_k)} |(s-t)|^{\alpha(d_k + u^{-2/\alpha(d_k)}(jS+t))} \\ &= (1 - \nu/2)^{1/3} a \times I_1 \times I_2. \end{aligned} \quad (27)$$

We deal with I_1 and I_2 separately. For sufficiently large u , uniformly with respect to k ,

$$\begin{aligned} I_1 &= u^{-2\alpha(d_k + u^{-2/\alpha(d_k)}(jS+t))/\alpha(d_k)} \\ &= u^{-2} u^{2(\alpha(d_k) - \alpha(d_k + u^{-2/\alpha(d_k)}(jS+t)))/\alpha(d_k)} \\ &= u^{-2} e^{2(\ln u)(\alpha(d_k) - \alpha(d_k + u^{-2/\alpha(d_k)}(jS+t)))/\alpha(d_k)} \\ &\geq u^{-2} (1 - \nu/2)^{1/3}, \end{aligned} \quad (28)$$

where the last inequality follows from the fact that

$$\begin{aligned} (\ln u) \left| \alpha(d_k) - \alpha(d_k + u^{-2/\alpha(d_k)}(jS+t)) \right| &\leq (\ln u) \left(\left| b(d_k)^\beta - b(d_k + u^{-2/\alpha(d_k)}(jS+t))^\beta \right| + 2\delta_1^{\beta+\delta}(u) \right) \\ &\leq (\ln u) \left(\frac{b}{(\ln u)(\ln \ln u)^{1/\beta}} + 2\delta_1^{\beta+\delta}(u) \right) \\ &\leq \frac{b}{(\ln \ln u)^{1/\beta}} + 2(\ln u) \left(\frac{1}{2 \ln u - q \ln \ln u} \right)^{\frac{\beta+\delta}{\gamma}} \rightarrow 0, \quad u \rightarrow \infty. \end{aligned}$$

For I_2 , we need to prove that

$$I_2 \geq (1 - \nu/2)^{1/3} |s-t|^{\alpha+2b\delta_1^\beta(u)}. \quad (29)$$

Assumption (v) implies that

$$\alpha(d_k + u^{-2/\alpha(d_k)}(jS+t)) < \alpha + 2b\delta_1^\beta(u) \quad (30)$$

for each $(j, k) \in \mathcal{U}$. Thus if $|s-t| < 1$, then (29) holds immediately. If $1 \leq |s-t| \leq S$, then by (30)

$$\begin{aligned} I_2 &= |(s-t)|^{\alpha(d_k + u^{-2/\alpha(d_k)}(jS+t))} \\ &\geq T^{\alpha(d_k + u^{-2/\alpha(d_k)}(jS+t)) - \alpha - 2b\delta_1^\beta(u)} |s-t|^{\alpha+2b\delta_1^\beta(u)} \\ &\geq T^{-2b\delta_1^\beta(u)} |s-t|^{\alpha+2b\delta_1^\beta(u)} \\ &\geq (1 - \nu/2)^{1/3} |s-t|^{\alpha+2b\delta_1^\beta(u)} \end{aligned}$$

for sufficiently large u . The above combined with (27), (28) and (29) gives that for sufficiently large u , uniformly with respect to $(j, k) \in \mathcal{U}$,

$$1 - \text{Cov}(X_{j,k,u}(s), X_{j,k,u}(t)) \geq (1 - \nu/2)au^{-2}|s - t|^{\alpha+2b\delta_1^\beta(u)} \geq 1 - \text{Cov}(Y_{\nu,u}(s), Y_{\nu,u}(t)).$$

Thus by Slepian's inequality 1) is proved.

2) For all u large

$$\begin{aligned} 1 - \text{Cov}(X_{j,k,u}(s), X_{j,k,u}(t)) &= 1 - \text{Cov}\left(\overline{X}\left(d_k + \frac{jS + s}{u^{2/\alpha(d_k)}}\right), \overline{X}\left(d_k + \frac{jS + t}{u^{2/\alpha(d_k)}}\right)\right) \\ &\leq (1 + \nu)^{1/3} a \left| u^{-2/\alpha(d_k)}(s - t) \right|^{\alpha(d_k + u^{-2/\alpha(d_k)}(jS + t))}. \end{aligned}$$

Following the argument analogous to that for the proof of 1), we obtain that for sufficiently large u , uniformly with respect to k , and $s, t \in [0, S]$

$$1 - \text{Cov}(X_{j,k,u}(s), X_{j,k,u}(t)) \leq 1 - \text{Cov}(Z_{\nu,u}(s), Z_{\nu,u}(t)).$$

Again the application of Slepian's inequality completes the proof. \square

Lemma 3.2. For $S > 1$, $\nu \in (0, 1)$, and $\lim_{u \rightarrow \infty} \frac{f(u)}{u} = 1$, as $u \rightarrow \infty$, we have

$$\begin{aligned} \underline{1)} \mathbb{P}\left\{\sup_{t \in [0, S]} Y_{\nu,u}(t) > f(u)\right\} &= \mathcal{H}_\alpha[0, S((1 - \nu)a)^{1/\alpha}] \Psi(f(u))(1 + o(1)); \\ \underline{2)} \mathbb{P}\left\{\sup_{t \in [0, S]} Z_{\nu,u}(t) > f(u)\right\} &= \mathcal{H}_\alpha[0, S((1 + \nu)a)^{1/\alpha}] \Psi(f(u))(1 + o(1)). \end{aligned}$$

Proof of Lemma 3.2. We present the proof of 1) and omit the proof of 2) since it follows with similar arguments. Following the definition of $Y_{\nu,u}(t)$, for each $s, t \in [0, S]$

$$\begin{aligned} \lim_{u \rightarrow \infty} f^2(u) \left[1 - \text{Cov}\left(Y_{\nu,u}\left(t(a(1 - \nu))^{-1/\alpha}\right), Y_{\nu,u}\left(s(a(1 - \nu))^{-1/\alpha}\right)\right) \right] \\ = \lim_{u \rightarrow \infty} (a(1 - \nu))^{1 - (\alpha + 2b\delta^\beta(u))/\alpha} |s - t|^{\alpha + 2b\delta^\beta(u)} = |s - t|^\alpha. \end{aligned}$$

Moreover, for all $s, t \in [0, S]$, sufficiently large u and some constant $C > 0$

$$\begin{aligned} f^2(u) \left[1 - \text{Cov}\left(Y_{\nu,u}\left(t(a(1 - \nu))^{-1/\alpha}\right), Y_{\nu,u}\left(s(a(1 - \nu))^{-1/\alpha}\right)\right) \right] \\ \leq (a(1 - \nu))^{1 - (\alpha + 2b\delta^\beta(u))/\alpha} |s - t|^{\alpha + 2b\delta^\beta(u)} \leq CT^{2\alpha} |s - t|^\alpha, \end{aligned}$$

where the last inequality follows from the fact that

$$|s - t|^{\alpha + 2b\delta^\beta(u)} \leq |s - t|^\alpha, \text{ if } |s - t| < 1,$$

and

$$|s - t|^{\alpha + 2b\delta^\beta(u)} \leq T^{2\alpha} \leq T^{2\alpha} |s - t|^\alpha, \text{ if } 1 \leq |s - t| \leq T.$$

Hence, by [19][Lemma 7], we conclude that

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,S]}Y_{\nu,u}(t)>f(u)\right\}&=\mathbb{P}\left\{\sup_{t\in[0,((1-\nu)a)^{1/\alpha}S]}Y_{\nu,u}((a(1-\nu))^{-1/\alpha}t)>f(u)\right\}\\&=\mathcal{H}_\alpha\left[0,((1-\nu)a)^{1/\alpha}S\right]\Psi(f(u))(1+o(1)),\end{aligned}$$

as $u \rightarrow \infty$. This completes the proof. \square

Acknowledgments

Thanks to Swiss National Science Foundation grant no. 200021-166274.

References

- [1] R.J. Adler, J.E. Taylor, *Random Fields and Geometry*, Springer Monogr. Math., Springer, New York, 2007.
- [2] A. Ayache, N.-R. Shieh, Y. Xiao, Multiparameter multifractional Brownian motion: local nondeterminism and joint continuity of the local times, *Ann. Inst. Henri Poincaré Probab. Stat.* 47 (4) (2011) 1029–1054.
- [3] S.M. Berman, *Sojourns and Extremes of Stochastic Processes*, The Wadsworth & Brooks/Cole Statistics/Probability Series, Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1992.
- [4] H.U. Bräker, *High Boundary Excursions of Locally Stationary Gaussian Processes*, Universitat Bern, 1993.
- [5] K. Dębicki, Ruin probability for Gaussian integrated processes, *Stochastic Process. Appl.* 98 (1) (2002) 151–174.
- [6] K. Dębicki, S. Engelke, E. Hashorva, Generalized Pickands constants and stationary max-stable processes, <http://arxiv.org/pdf/1602.01613.pdf>, 2016.
- [7] K. Dębicki, E. Hashorva, L. Ji, Tail asymptotics of supremum of certain Gaussian processes over threshold dependent random intervals, *Extremes* 17 (3) (2014) 411–429.
- [8] K. Dębicki, E. Hashorva, P. Liu, Ruin probabilities and passage times of γ -reflected Gaussian process with stationary increments, <http://arXiv.org/abs/1511.09234>, 2015.
- [9] K. Dębicki, P. Kisowski, Asymptotics of supremum distribution of $\alpha(t)$ -locally stationary Gaussian processes, *Stochastic Process. Appl.* 118 (11) (2008) 2022–2037.
- [10] K. Dębicki, K.M. Kosiński, On the infimum attained by the reflected fractional Brownian motion, *Extremes* 17 (3) (2014) 431–446.
- [11] K. Dębicki, E. Hashorva, L. Ji, K. Tabiś, Extremes of vector-valued Gaussian processes: exact asymptotics, *Stochastic Process. Appl.* 125 (11) (2015) 4039–4065.
- [12] A.B. Dieker, Extremes of Gaussian processes over an infinite horizon, *Stochastic Process. Appl.* 115 (2) (2005) 207–248.
- [13] A.B. Dieker, T. Mikosch, Exact simulation of Brown–Resnick random fields at a finite number of locations, *Extremes* 18 (2015) 301–314.
- [14] A.B. Dieker, B. Yakir, On asymptotic constants in the theory of Gaussian processes, *Bernoulli* 20 (3) (2014) 1600–1619.
- [15] E. Hashorva, Representations of max-stable processes via exponential tilting, <https://arxiv.org/abs/1605.03208>, 2016.
- [16] E. Hashorva, L. Ji, Extremes of $\alpha(\mathbf{t})$ -locally stationary Gaussian random fields, *Trans. Amer. Math. Soc.* 368 (1) (2016) 1–26.
- [17] E. Hashorva, M. Lifshits, O. Seleznev, Approximation of a random process with variable smoothness, in: *Mathematical Statistics and Limit Theorems*, Springer, Cham, 2015, pp. 189–208.
- [18] J. Hüslér, Extreme values and high boundary crossings of locally stationary Gaussian processes, *Ann. Probab.* 18 (1990) 1141–1158.
- [19] J. Hüslér, V.I. Piterbarg, On the ruin probability for physical fractional Brownian motion, *Stochastic Process. Appl.* 113 (2) (2004) 315–332.
- [20] J. Pickands III, Upcrossing probabilities for stationary Gaussian processes, *Trans. Amer. Math. Soc.* 145 (1969) 51–73.
- [21] V.I. Piterbarg, On the paper by J. Pickands “Upcrossing probabilities for stationary Gaussian processes”, *Vestnik Moskov. Univ. Ser. I Mat. Meh.* 27 (5) (1972) 25–30.
- [22] V.I. Piterbarg, *Asymptotic Methods in the Theory of Gaussian Processes and Fields*, Transl. Math. Monogr., vol. 148, American Mathematical Society, Providence, RI, 1996.
- [23] V.I. Piterbarg, *Twenty Lectures About Gaussian Processes*, Atlantic Financial Press, London, New York, 2015.