



Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa



On the wave-breaking phenomena and global existence for the periodic rotation-two-component Camassa–Holm system[☆]

Byungsoo Moon

Department of Mathematics, Incheon National University, Incheon 22012, Republic of Korea

ARTICLE INFO

Article history:

Received 4 November 2016

Available online xxxx

Submitted by H. Liu

Keywords:

Rotation-two-component

Camassa–Holm system

Blow-up

Wave-breaking

Global existence

ABSTRACT

Considered herein is the periodic rotation-two-component Camassa–Holm system, which can be derived from the f -plane governing equations for the geophysical water waves with a constant underlying current. The nonlocal nonlinearities on blow-up criteria and wave-breaking phenomena are established. Finally, a sufficient condition for global solutions is obtained by using a method of the Lyapunov function.

© 2017 Elsevier Inc. All rights reserved.

1. Introduction

In this paper, we are concerned with the Cauchy problem of the periodic rotation-two-component Camassa–Holm system

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x \\ \quad = \sigma(2u_x u_{xx} + uu_{xxx}) - \mu u_{xxx} - (1 - 2\Omega A)\rho\rho_x + 2\Omega\rho(\rho u)_x, & t > 0, \quad x \in \mathbb{R}, \\ \rho_t + (\rho u)_x = 0, & t > 0, \quad x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), \quad \rho(t, x+1) = \rho(t, x), & t \geq 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x), & x \in \mathbb{R}. \end{cases} \quad (1.1)$$

The rotation-two-component Camassa–Holm system was recently derived in [14], following Ivanov's asymptotic perturbation analysis for the governing equation of two-dimensional rotational gravity water waves [19]. Here, the function $u(t, x)$ represents the fluid velocity in the x -direction, $\rho(t, x)$ is related to the free surfaced elevation from equilibrium, the free parameter A characterizes a linear underlying shear flow, the real

[☆] This work was supported by Incheon National University Research Grant 2015-1707.

E-mail address: bsmoon@inu.ac.kr.

dimensionless constant σ is a new parameter which provides the competition, or balance, in fluid convection between nonlinear steepening and amplification due to stretching, μ is a non-dimensional parameter and Ω characterizes the constant rotational speed of the Earth. The boundary assumptions associated with (1.1) are $u \rightarrow 0, \rho \rightarrow 1$ as $|x| \rightarrow \infty$. System (1.1) has at least three conservation laws as following:

$$E(u, \rho) = \frac{1}{2} \int_{\mathbb{S}} (u^2 + u_x^2 + (1 - 2\Omega A)(\rho - 1)^2) dx, \quad (1.2)$$

$$H_1(u, \rho) = \int_{\mathbb{S}} (u + \Omega(\rho - 1)^2) dx, \quad (1.3)$$

$$H_2(u, \rho) = \int_{\mathbb{S}} (\rho - 1) dx. \quad (1.4)$$

When $\Omega = 0$ it recovers the generalized two-component Dullin–Gottwald–Holm (gDGH2) system [4,18]:

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x = \sigma(2u_x u_{xx} + uu_{xxx}) - \mu u_{xxx} - \rho \rho_x, \\ \rho_t + (\rho u)_x = 0. \end{cases} \quad (1.5)$$

The system (1.5) has the following two Hamiltonians [5,18]:

$$F_1(u, \rho) = \frac{1}{2} \int_{\mathbb{S}} (u^2 + u_x^2 + (\rho - 1)^2) dx, \quad (1.6)$$

$$F_2(u, \rho) = \frac{1}{2} \int_{\mathbb{S}} (u^3 + \sigma uu_x^2 - Au^2 - \mu u_x^2 + 2u(\rho - 1) + u(\rho - 1)^2) dx. \quad (1.7)$$

The Cauchy problem of the gDGH2 system (1.5) on the line (non-periodic case) [14] and on the circle (periodic case) [5] have been recently investigated. The precise blow-up scenario and several blow-up results of strong solutions to the gDGH2 system (1.5) on the line [14] and on the circle [5] were presented. Moreover, the classification of all traveling wave solutions to the gDGH2 system (1.5) was established in [14].

Moreover, when $\sigma = 1$ and $\mu = 0$, the gDGH2 system (1.5) becomes the standard two-component integrable Camassa–Holm system [10,19,21]:

$$\begin{cases} u_t - u_{txx} - Au_x + 3uu_x + \rho \rho_x = 2u_x u_{xx} + uu_{xxx}, \\ \rho_t + (\rho u)_x = 0. \end{cases} \quad (1.8)$$

The system (1.8) is completely integrable [10,19] as it can be written as a compatibility condition of two linear system (Lax pair) with a spectral parameter ζ :

$$\begin{aligned} \Psi_{xx} &= \left(-\zeta^2 \rho^2 + \zeta \left(m - \frac{A}{2} \right) + \frac{1}{4} \right) \Psi \\ \Psi_t &= \left(\frac{1}{2\zeta} - u \right) \Psi_x + \frac{1}{2} u_x \Psi, \quad m = u - u_{xx}, \end{aligned}$$

and has a bi-Hamiltonian structure corresponding to the Hamiltonian (1.6) and (1.7) with $\sigma = 1$ and $\mu = 0$. Compared with the other integrable multicomponent Camassa–Holm-type systems, the system (1.8) has caught a large amount of attention, after Constantin and Ivanov [10] derived it in the context of shallow water regime.

Furthermore, in the case of $\rho = 0$, the gDGH2 system (1.5) and two-component Camassa–Holm system (1.8) recover the DGH equation [11] and Camassa–Holm equation [1], respectively. The main significance for seeking and studying the gDGH2 system lies in capturing the two nonlinear properties of CH equation, which is the presence of multi-soliton or infinite-soliton solutions consisting of a train of peaked solitary waves of ‘peakons’ [1–3] and the occurrence of wave-breaking phenomena (i.e. a solution that remains bounded while its slope becomes unbounded in finite time) [6–9], one can refer to [4,10,12,13,15–18,23] etc. and references therein for details.

Recently, the Cauchy problem of the rotation-two-component Camassa–Holm system (1.1) on the line (non-periodic case) has been discussed in [14]. However, the system (1.1) on the circle (periodic case) seems not yet to have been studied. The goal of the present paper is to investigate the conditions to guarantee the occurrence of the wave-breaking phenomena and the permanent waves for our system (1.1) on the circle. There are two main difficulties in achieving these goals as described follow. The first main difficulty comes from estimating the cubic-order nonlinear term $2\Omega\rho(\rho u)_x$ in (1.1) which should be solved in order to establish wave-breaking phenomena. To overcome this higher-order nonlinear estimates, we use the method of characteristics together with the use of the conservation laws to obtain blow-up criteria and wave-breaking phenomena, as in [14]. On the other hand, the second main difficulty is originated from estimating the second term $2\Omega\gamma(0)\gamma(t)\bar{m}(t)\left(u - \sqrt{\frac{2(e+1)}{e-1}E(u_0, \rho_0 - 1)}\right)$ on the right hand side of (4.5) for the global existence of solutions to our system (1.1) due to appearance of the Earth’s rotation (i.e. $\Omega = 0$). Without the consideration of the Earth’s rotation, it can be possible for ensuring the existence of global solutions to use the Gronwall inequality and the property of the Lyapunov function $\bar{w}(t)$ to contradiction that $|u_x| < \infty$ from (4.5). To deal with this second main difficulty, we use the method of Lyapunov functions introduced in [10,14]. We then find a sufficient condition for global solutions which is determined by a nonzero profile to the free surface component ρ of the system (1.1) and the condition in (4.2), while the condition in (4.2) is unnecessary when the effect of the Earth’s rotation is ignored in the water waves.

The rest of this paper is organized as follows. In Section 2, we briefly give some needed results including the local well-posedness of system (1.1) and some useful lemmas to study wave-breaking phenomena and global existence. In Section 3, we give the precise blow-up scenarios and blow-up criteria, which exhibit that system (1.1) has blow-up solutions modeling wave breaking. In the final section, Section 4, we address the global existence of system (1.1).

Notation. Throughout this paper, all spaces of periodic functions with function spaces are assumed to be over the unit circle \mathbb{S} in \mathbb{R}^2 , i.e. $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. Since all space of functions are over \mathbb{S} , for simplicity, we drop \mathbb{S} in our notation of function spaces if there is no ambiguity. The norm of the Lebesgue space $L^p(\mathbb{S})$, $1 \leq p \leq \infty$, is denoted by $\|\cdot\|_{L^p}$ and the Sobolev space $H^s(\mathbb{S})$, $s \in \mathbb{R}$, by $\|\cdot\|_{H^s}$. $a \lesssim b$ means that there is a uniform constant C that may be different on different lines, such that $a \leq Cb$. All of different positive constants might be denoted by the uniform constant C .

2. Preliminaries

In this section, we briefly give the needed results to pursue our goal. We first present the local well-posedness for the Cauchy problem of system (1.1) in $H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s > 3/2$ with $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. Denote the Fourier transform of a function f in the torus \mathbb{S} by $\hat{f}(k)$ with the frequency $k \in \mathbb{Z}$. Then we have

$$((1 - \widehat{\partial_x^2})^{-1}f)(k) = (1 + k^2)^{-1}\hat{f}(k) = \widehat{G} \cdot \hat{f} = \widehat{(G * f)}(k),$$

where

$$G(x) := \frac{\cosh(x - [x] - \frac{1}{2})}{2\sinh(\frac{1}{2})}, \quad x \in \mathbb{R}$$

$[x]$ stands for the greatest integer part of $x \in \mathbb{R}$ and $\widehat{G}(k) = (1 + k^2)^{-1}$. Hence $(1 - \partial_x^2)^{-1}f = G * f = \int_{\mathbb{S}} G(x - y)f(y)dy$ for all $f \in L^2(\mathbb{S})$ and $G * m = u$. We can rewrite our system (1.1) as the following “transport” type

$$\begin{cases} u_t + (\sigma u - \mu)u_x = -\partial_x G * \left((\mu - A)u + \frac{3 - \sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1 - 2\Omega A}{2}\rho^2 - \Omega\rho^2 u \right) \\ \quad \quad \quad + \Omega G * (\rho^2 u_x), & t > 0, \quad x \in \mathbb{R}, \\ \rho_t + u\rho_x = -u_x\rho, & t > 0, \quad x \in \mathbb{R}, \\ u(t, x + 1) = u(t, x), \quad \rho(t, x + 1) = \rho(t, x), & t \geq 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), \quad \rho(0, x) = \rho_0(x), & x \in \mathbb{R}. \end{cases} \quad (2.1)$$

Applying Kato’s theory [20] (see also [13]), we obtain the following local well-posedness result for the system (1.1).

Theorem 1. *Given any $X_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$, $s > 3/2$, there exist a maximal $T = T(X_0) > 0$, and a unique solution $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.1) such that $X = X(\cdot, X_0) \in C([0, T]; H^s \times H^{s-1}) \cap C^1([0, T]; H^{s-1} \times H^{s-2})$. Moreover, the solution depends continuously on the initial data, i.e., the mapping $X_0 \mapsto X(\cdot, X_0) : H^s \times H^{s-1} \rightarrow C([0, T]; H^s \times H^{s-1}) \cap C^1([0, T]; H^{s-1} \times H^{s-2})$ is continuous.*

Now, consider the initial value problem for the Lagrangian flow map:

$$\begin{cases} \frac{\partial \varphi}{\partial t} = u(t, \varphi(t, x)), & t \in [0, T), \\ \varphi(0, x) = x, & x \in \mathbb{R}, \end{cases} \quad (2.2)$$

where $u \in C^1([0, T], H^{s-1})$ denotes the first component of the solution $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.1) with initial data $X_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$ ($s > 3/2$), and $T > 0$ is the maximal time of existence. By a direct calculation, we have

$$\varphi_{tx}(t, x) = u_x(t, \varphi(t, x))\varphi_x(t, x).$$

Then,

$$\varphi_x(t, x) = e^{\int_0^t u_x(\tau, \varphi(\tau, x))d\tau} > 0, \quad (t, x) \in [0, T) \times \mathbb{R},$$

which implies that the map $\varphi(t, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing diffeomorphism of the line for every $t \in [0, T)$. Consequently, the L^∞ -norm of any function $v(t, \cdot) \in L^\infty$, $t \in [0, T)$ is preserved under the family of diffeomorphisms $\varphi(t, \cdot)$ with $t \in [0, T)$, that is,

$$\|v(t, \cdot)\|_{L^\infty(\mathbb{S})} = \|v(t, \varphi(t, \cdot))\|_{L^\infty(\mathbb{S})}, \quad t \in [0, T).$$

Similarly, we have

$$\inf_{x \in \mathbb{S}} v(t, x) = \inf_{x \in \mathbb{S}} v(t, \varphi(t, x)), \quad \sup_{x \in \mathbb{S}} v(t, x) = \sup_{x \in \mathbb{S}} v(t, \varphi(t, x)), \quad t \in [0, T). \quad (2.3)$$

We may use the following proposition derived in [16] to study the regularity property of solution to (1.1).

Proposition 1. [16] Let $0 < s < 1$. Suppose that $f_0 \in H^s$, $g \in L^1([0, T]; H^s)$, $v, \partial_x v \in L^1([0, T]; L^\infty)$ and that $f \in L^\infty([0, T]; H^s) \cap C([0, T]; S')$ solves the 1-dimensional linear transport equation

$$(T) \quad \begin{cases} \partial_t f + v \cdot \partial_x f = g, \\ f|_{t=0} = f_0. \end{cases}$$

Then $f \in C([0, T]; H^s)$. More precisely, there exists a constant C depending only on s such that the following statement holds:

$$\|f\|_{H^s} \leq \|f_0\|_{H^s} + C \int_0^t \|g(\tau)\|_{H^s} d\tau + C \int_0^t \|f(\tau)\|_{H^s} V'(\tau) d\tau, \quad (2.4)$$

or hence,

$$\|f\|_{H^s} \leq e^{CV(t)} \left(\|f_0\|_{H^s} + C \int_0^t \|g(\tau)\|_{H^s} d\tau \right), \quad (2.5)$$

with $V(t) = \int_0^t (\|v(\tau)\|_{L^\infty} + \|\partial_x v(\tau)\|_{L^\infty}) d\tau$.

The above proposition was proved in [16] using Littlewood–Paley analysis for the transport equation and Moser-type estimates. Using this result and performing the same argument, we can obtain the following blow-up criterion.

Theorem 2. Let $X_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$ with $s > 3/2$, and $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ be the corresponding solution to (1.1). Assume $T > 0$ is the maximal time of existence. Then

$$T < \infty \quad \implies \quad \int_0^T \|\partial_x u(\tau)\|_{L^\infty} d\tau = \infty.$$

We then give several useful conservation laws of strong solutions to (1.1).

Lemma 1. Let $X_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$, $s > 3/2$, and let T be the maximal existence time of the solution $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.1) with initial data X_0 . Then for all $t \in [0, T)$, we have

$$\begin{aligned} & \frac{1}{2} \int_{\mathbb{S}} \{u^2(t, x) + u_x^2(t, x) + (1 - 2\Omega A)(\rho - 1)^2(t, x)\} dx \\ &= \frac{1}{2} \int_{\mathbb{S}} \{u_0^2(x) + u_{0,x}^2(x) + (1 - 2\Omega A)(\rho_0 - 1)^2(x)\} dx. \end{aligned} \quad (2.6)$$

Proof. Multiplying the first equation of (1.1) by u and integrating by parts, in view of the periodicity of u and ρ , we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{S}} \{u^2(t, x) + u_x^2(t, x)\} dx = -(1 - 2\Omega A) \int_{\mathbb{S}} u \rho \rho_x dx.$$

Notice that the second equation of (1.1) can be written as

$$(\rho - 1)_t + (u(\rho - 1))_x + u_x = 0. \quad (2.7)$$

On the other hand, multiplying the equation in (2.7) by $(1 - 2\Omega A)(\rho - 1)$ and integrating by parts, in view of the periodicity of u and ρ , we get

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{S}} (1 - 2\Omega A)(\rho - 1)^2(t, x) dx = (1 - 2\Omega A) \int_{\mathbb{S}} u \rho \rho_x dx.$$

Adding the above two equations, we obtain

$$\frac{1}{2} \int_{\mathbb{S}} \{u^2(t, x) + u_x^2(t, x) + (1 - 2\Omega A)(\rho - 1)^2(t, x)\} dx = 0.$$

This completes the proof of the Lemma 1. \square

Lemma 2. Let $X_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$, $s > 3/2$, and let T be the maximal existence time of the solution $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.1) with initial data X_0 . Then for all $t \in [0, T)$, we have

$$\int_{\mathbb{S}} \{u(t, x) + \Omega(\rho - 1)^2(t, x)\} dx = \int_{\mathbb{S}} \{u_0(x) + \Omega(\rho_0 - 1)^2(x)\} dx, \quad (2.8)$$

$$\int_{\mathbb{S}} (\rho - 1)(t, x) dx = \int_{\mathbb{S}} (\rho_0 - 1)(x) dx. \quad (2.9)$$

Proof. Integrating the first equation of (2.1) by parts, in view of the periodicity of u and G , we get

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{S}} u(t, x) dx &= \int_{\mathbb{S}} -(\sigma u - \mu) u_x dx \\ &\quad - \int_{\mathbb{S}} \partial_x G * \left((\mu - A)u + \frac{3 - \sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \frac{1 - 2\Omega A}{2} \rho^2 - \Omega \rho^2 u \right) dx + \int_{\mathbb{S}} \Omega G * (\rho^2 u_x) dx \\ &= \int_{\mathbb{S}} \Omega G * (\rho^2 u_x) dx \end{aligned}$$

Multiplying the second equation of (2.7) by $\Omega(\rho - 1)$ and integrating by parts, we have

$$\frac{d}{dt} \int_{\mathbb{S}} \Omega(\rho - 1)^2(t, x) dx = \int_{\mathbb{S}} \Omega u (\rho^2)_x dx.$$

Adding the above two equations and using the identity $G * f = f + \partial_x^2 G * f$, we obtain

$$\frac{d}{dt} \int_{\mathbb{S}} \{u(t, x) + \Omega(\rho - 1)^2(t, x)\} dx = 0.$$

On the other hand, integrating the second equation of (2.7) by parts, in view of the periodicity of u and ρ , we get

$$\frac{d}{dt} \int_{\mathbb{S}} (\rho - 1)(t, x) dx = 0$$

This completes the proof of the Lemma 2. \square

Lemma 3. [22] (i) For every $f \in H^1(\mathbb{S})$, we have

$$\max_{x \in [0, 1]} f^2(x) \leq \frac{e+1}{2(e-1)} \|f\|_{H^1(\mathbb{S})}^2,$$

where the constant $\frac{e+1}{2(e-1)}$ is sharp.

(ii) For every $f \in H^3(\mathbb{S})$, we have

$$\max_{x \in [0, 1]} f^2(x) \leq c \|f\|_{H^1(\mathbb{S})}^2,$$

with the best possible constant c lying within the range $(1, \frac{13}{12}]$. Moreover, the constant c is $\frac{e+1}{2(e-1)}$.

Lemma 4. [7] Let $T > 0$ and $v \in C^1([0, T]; H^2(\mathbb{R}))$. Then for every $t \in [0, T)$, there exists at least one point $\xi(t) \in \mathbb{R}$ with

$$m(t) := \inf_{x \in \mathbb{S}} v_x(t, x) = v_x(t, \xi(t)),$$

and the function $m(t)$ is almost everywhere differentiable on $(0, T)$ with

$$\frac{dm}{dt}(t) = v_{tx}(t, \xi(t)), \quad \text{a.e. on } (0, T).$$

3. Wave-breaking phenomenon

In this section, we establish the precise blow-up mechanism of strong solution to system (1.1). It is shown that the solution to the system (2.1) can only have singularities which correspond to wave breaking.

Theorem 3 (Wave-breaking criteria). Suppose that $1 - 2\Omega A > 0$. Let $X_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$ with $s > 3/2$, and let T be the maximal existence time of the solution $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.1) with initial data X_0 .

Then the corresponding solution $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ blows up in finite time $T < \infty$ if and only if

$$\lim_{t \uparrow T^-} \left\{ \sup_{x \in \mathbb{S}} |u_x(t, x)| \right\} = +\infty \quad (3.1)$$

Furthermore, if $\sigma = 1$ and $\mu = 0$, then the corresponding solution $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ blows up in finite time $T < \infty$ if and only if

$$\lim_{t \uparrow T^-} \left\{ \inf_{x \in \mathbb{S}} u_x(t, x) \right\} = -\infty \quad (3.2)$$

Proof. By Theorem 1 and a simple density argument, we show the desired results are valid when $s \geq 3$, so we take $s = 3$ in the proof. We may also assume that $u_0 \neq 0$. Otherwise, the results become trivial. Let $T > 0$ be the maximal time of existence of the corresponding solution $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to system (2.1).

We first prove the case of (3.1). If (3.1) holds, the Sobolev embedding theorem $H^s(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ with $s > 1/2$ then implies that the corresponding solution blows up in finite time. Conversely, assume that $T < \infty$ and (3.1) is not valid. Then there is some positive number $C_0 > 0$, such that

$$|u_x| \leq C_0, \quad \forall (t, x) \in [0, T) \times \mathbb{S}.$$

Therefore, Theorem 2 implies that the maximal existence time $T = \infty$, which contradicts the assumption that $T < \infty$.

Next we will try to prove the blow-up criterion (3.2) in the case of $\sigma = 1$ and $\mu = 0$. Note that if $G(x) := \frac{\cosh(x - [x] - \frac{1}{2})}{2 \sinh(\frac{1}{2})}$, $x \in \mathbb{R}$, then $(1 - \partial_x^2)^{-1} f = G * f$ for all $f \in L^2(\mathbb{S})$. Differentiating the first equation in (2.1) with respect to x and using the identity $-\partial_x^2 G * f = f - G * f$, we obtain

$$\begin{aligned} u_{tx} + uu_{xx} &= -\frac{1}{2}u_x^2 + \frac{1 - 2\Omega A}{2}\rho^2 - \Omega u\rho^2 + A\partial_x^2 G * u + u^2 \\ &\quad - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1 - 2\Omega A}{2}\rho^2 - \Omega(\rho^2 u) \right) + \Omega \partial_x G * (\rho^2 u_x). \end{aligned} \quad (3.3)$$

Given $x \in \mathbb{S}$, let

$$m(t) = u_x(t, \varphi(t, x)), \quad \gamma(t) = \rho(t, \varphi(t, x)), \quad t \in [0, T), \quad (3.4)$$

where $\varphi(t, x)$ is defined by (2.2). Using these notations, Equation (3.3) and the second one of (2.1) become

$$\begin{cases} m'(t) = -\frac{1}{2}m^2(t) + \frac{1 - 2\Omega A}{2}\gamma^2(t) + f(t, \varphi(t, x)), \\ \gamma'(t) = -\gamma m, \quad \text{a.e. } t \in [0, T), \end{cases} \quad (3.5)$$

where the notation $'$ denotes the derivative with respect to t and f represents the function

$$f = u^2 - \Omega u\rho^2 + A\partial_x^2 G * u - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1 - 2\Omega A}{2}\rho^2 \right) + \Omega G * (\rho^2 u) + \Omega \partial_x G * (\rho^2 u_x). \quad (3.6)$$

In view of the definition of $m(t)$, assume that $T < \infty$ and (3.2) is not valid. Then there is some positive number $C_1 > 0$, such that

$$\inf_{x \in \mathbb{S}} u_x(t, x) \geq -C_1, \quad \forall t \in [0, T).$$

With this restriction, it then follows from (3.5) that, for each $x \in \mathbb{S}$,

$$|\rho(t, \varphi(t, x))| = |\gamma(t)| = |\gamma(0)| e^{\int_0^t -C_1(\tau) d\tau} \leq \|\rho_0\|_{L^\infty} e^{C_1 t},$$

which leads to

$$\|\rho(t, \cdot)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty} e^{C_1 t}. \quad (3.7)$$

Now we derive the upper bound for f in order to demonstrate the boundedness of $m(t)$.

$$\begin{aligned}
f &= u^2 - \Omega u \rho^2 + A \partial_x^2 G * u - G * \left(u^2 + \frac{1}{2} u_x^2 \right) - \frac{1 - 2\Omega A}{2} G * 1 - (1 - 2\Omega A) G * (\rho - 1) \\
&\quad - \frac{1 - 2\Omega A}{2} G * (\rho - 1)^2 + \Omega G * ((\rho - 1)^2 u) + 2\Omega G * ((\rho - 1)u) + \Omega G * u + \Omega \partial_x G * (\rho(\rho - 1)u_x) \\
&\quad + \Omega \partial_x G * ((\rho - 1)u_x) + \Omega \partial_x G * u_x \\
&\leq \frac{1}{2} u^2 + \Omega |u \rho^2| + |A| |\partial_x^2 G * u| - \frac{1 - 2\Omega A}{2} + (1 - 2\Omega A) |G * (\rho - 1)| + \Omega |G * ((\rho - 1)^2 u)| \\
&\quad + 2\Omega |G * ((\rho - 1)u)| + \Omega |G * u| + \Omega |\partial_x G * (\rho(\rho - 1)u_x)| + \Omega |\partial_x G * ((\rho - 1)u_x)| + \Omega |\partial_x G * u_x|,
\end{aligned} \tag{3.8}$$

where we use the fact $G * (u^2 + \frac{1}{2} u_x^2) \geq \frac{1}{2} u^2$ (cf. [6]). Applying Young's inequality, Lemma 3, and $G = \frac{\cosh(x - [x] - \frac{1}{2})}{2 \sinh(\frac{1}{2})}$ leads to

$$\frac{1}{2} u^2 \leq \frac{e+1}{4(e-1)} \|u\|_{H^1}^2 = \frac{e+1}{4(e-1)} (\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2), \tag{3.9}$$

$$|A| |\partial_x^2 G * u| = |A| |\partial_x G * u_x| \leq |A| \|\partial_x G\|_{L^2} \|u_x\|_{L^2} \leq \frac{e+1}{2(e-1)} + \frac{A^2}{4} \|u_x\|_{L^2}^2, \tag{3.10}$$

$$(1 - 2\Omega A) |G * (\rho - 1)| \leq (1 - 2\Omega A) \|G\|_{L^2} \|\rho - 1\|_{L^2} \leq \frac{(1 - 2\Omega A)(e+1)}{2(e-1)} + \frac{1 - 2\Omega A}{4} \|\rho - 1\|_{L^2}^2, \tag{3.11}$$

$$\Omega |G * u| \leq \Omega \|G\|_{L^2} \|u\|_{L^2} \leq \frac{e+1}{2(e-1)} + \frac{\Omega^2}{4} \|u\|_{L^2}^2, \tag{3.12}$$

$$\Omega |\partial_x G * u_x| \leq \Omega \|\partial_x G\|_{L^2} \|u_x\|_{L^2} \leq \frac{e+1}{2(e-1)} + \frac{\Omega^2}{4} \|u_x\|_{L^2}^2. \tag{3.13}$$

On the other hand, we obtain

$$\Omega |u \rho^2| \leq \Omega \|\rho\|_{L^\infty}^2 \|u\|_{L^\infty} \leq \frac{\Omega^2}{4} \|\rho\|_{L^\infty}^4 + \frac{e+1}{2(e-1)} (\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2), \tag{3.14}$$

$$\Omega |G * ((\rho - 1)^2 u)| \leq \frac{\Omega(e+1)}{2(e-1)} \|u\|_{L^\infty} \|\rho - 1\|_{L^2}^2 \leq \frac{\Omega^2}{4} \|\rho - 1\|_{L^2}^4 + \frac{1}{8} \left(\frac{e+1}{e-1} \right)^3 (\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2), \tag{3.15}$$

$$2\Omega |G * ((\rho - 1)u)| \leq 2\Omega \|G\|_{L^\infty} \|(\rho - 1)u\|_{L^1} \leq \frac{\Omega(e+1)}{e-1} \|(\rho - 1)u\|_{L^1} \leq \frac{\Omega(e+1)}{2(e-1)} (\|u\|_{L^2}^2 + \|\rho - 1\|_{L^2}^2), \tag{3.16}$$

$$\begin{aligned}
\Omega |\partial_x G * (\rho(\rho - 1)u_x)| &\leq \frac{\Omega(e+1)}{2(e-1)} \|\rho\|_{L^\infty} \|(\rho - 1)u_x\|_{L^1} \leq \frac{\Omega^2}{4} \|\rho\|_{L^\infty}^2 + \frac{1}{4} \left(\frac{e+1}{e-1} \right)^2 \|\rho - 1\|_{L^2}^2 \|u_x\|_{L^2}^2 \\
&\leq \frac{\Omega^2}{4} \|\rho\|_{L^\infty}^2 + \frac{1}{8} \left(\frac{e+1}{e-1} \right)^2 (\|u_x\|_{L^2}^4 + \|\rho - 1\|_{L^2}^4),
\end{aligned} \tag{3.17}$$

$$\Omega |\partial_x G * ((\rho - 1)u_x)| \leq \frac{\Omega(e+1)}{2(e-1)} \|(\rho - 1)u_x\|_{L^1} \leq \frac{\Omega(e+1)}{4(e-1)} (\|u_x\|_{L^2}^2 + \|\rho - 1\|_{L^2}^2). \tag{3.18}$$

Combining (3.9)–(3.18) and (3.7) together gives

$$\begin{aligned}
f &\leq \left[\frac{2\Omega + 3}{4} \left(\frac{e+1}{e-1} \right) + \frac{1}{8} \left(\frac{e+1}{e-1} \right)^3 + \frac{\Omega^2}{4} \right] \|u\|_{L^2}^2 \\
&\quad + \left[\frac{\Omega + 3}{4} \left(\frac{e+1}{e-1} \right) + \frac{1}{8} \left(\frac{e+1}{e-1} \right)^3 + \frac{\Omega^2 + A^2}{4} \right] \|u_x\|_{L^2}^2
\end{aligned} \tag{3.19}$$

$$\begin{aligned}
& + \frac{1}{8} \left(\frac{e+1}{e-1} \right)^2 \|u_x\|_{L^2}^4 + \left[\frac{1-2\Omega A}{4} + \frac{3\Omega}{4} \left(\frac{e+1}{e-1} \right) \right] \|\rho - 1\|_{L^2}^2 + \left[\frac{3}{2} \left(\frac{e+1}{e-1} \right) + \frac{1-2\Omega A}{e-1} \right] \\
& + \left[\frac{\Omega^2}{4} + \frac{1}{8} \left(\frac{e+1}{e-1} \right)^2 \right] \|\rho - 1\|_{L^2}^4 + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^4 e^{4C_1 t} + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^2 e^{2C_1 t} \\
& \leq \left[\frac{2\Omega+3}{2} \left(\frac{e+1}{e-1} \right) + \frac{1}{4} \left(\frac{e+1}{e-1} \right)^3 + \frac{\Omega^2}{2} \right] E(u_0, \rho_0 - 1) \\
& + \left[\frac{\Omega+3}{2} \left(\frac{e+1}{e-1} \right) + \frac{1}{4} \left(\frac{e+1}{e-1} \right)^3 + \frac{\Omega^2 + A^2}{2} \right] E(u_0, \rho_0 - 1) \\
& + \frac{1}{2} \left(\frac{e+1}{e-1} \right)^2 E^2(u_0, \rho_0 - 1) + \left[\frac{1}{2} + \frac{3\Omega}{2(1-2\Omega A)} \left(\frac{e+1}{e-1} \right) \right] E(u_0, \rho_0 - 1) + \left[\frac{3}{2} \left(\frac{e+1}{e-1} \right) + \frac{1-2\Omega A}{e-1} \right] \\
& + \left[\frac{\Omega^2}{(1-2\Omega A)^2} + \frac{1}{2(1-2\Omega A)^2} \left(\frac{e+1}{e-1} \right)^2 \right] E^2(u_0, \rho_0 - 1) + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^4 e^{4C_1 t} + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^2 e^{2C_1 t} \\
& \leq \left[\frac{3}{2} \left(\frac{e+1}{e-1} \right) + \frac{1-2\Omega A}{e-1} \right] \\
& + \left[\frac{6(\Omega+2)(1-2\Omega A) + 3\Omega}{2(1-2\Omega A)} \left(\frac{e+1}{e-1} \right) + \frac{1}{2} \left(\frac{e+1}{e-1} \right)^3 + \frac{2\Omega^2 + A^2}{2} + \frac{1}{2} \right] E(u_0, \rho_0 - 1) \\
& + \left[\frac{\Omega^2}{(1-2\Omega A)^2} + \frac{(1-2\Omega A)^2 + 1}{2(1-2\Omega A)^2} \left(\frac{e+1}{e-1} \right)^2 \right] E^2(u_0, \rho_0 - 1) + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^4 e^{4C_1 t} + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^2 e^{2C_1 t} \\
& := \kappa^2 + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^4 e^{4C_1 t} + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^2 e^{2C_1 t},
\end{aligned}$$

where κ denotes a constant that depends only on $E(u_0, \rho_0 - 1)$.

For any given $x \in \mathbb{S}$, define

$$Q(t) = m(t) - \|u_{0,x}\|_{L^\infty} - 2\kappa - \Omega \|\rho_0\|_{L^\infty}^2 e^{2C_1 t} - (\Omega + \sqrt{1-2\Omega A}) \|\rho_0\|_{L^\infty} e^{C_1 t}.$$

Observing that $Q(t)$ is a C^1 -differentiable function on $[0, t)$ and satisfies

$$Q(0) = m(0) - \|u_{0,x}\|_{L^\infty} - 2\kappa - \Omega \|\rho_0\|_{L^\infty}^2 - (\Omega + \sqrt{1-2\Omega A}) \|\rho_0\|_{L^\infty} \leq m(0) - \|u_{0,x}\|_{L^\infty} \leq 0,$$

we now claim

$$Q(t) \leq 0 \quad \forall t \in [0, T). \quad (3.20)$$

Assume the contrary that there is $t_0 \in [0, T)$ such that $Q(t_0) > 0$. Let

$$t_1 = \max\{t < t_0 : Q(t) = 0\}.$$

Then $Q(t_1) = 0$ and $Q'(t_1) \geq 0$, or equivalently,

$$m(t_1) = \|u_{0,x}\|_{L^\infty} + 2\kappa + \Omega \|\rho_0\|_{L^\infty}^2 e^{2C_1 t_1} + (\Omega + \sqrt{1-2\Omega A}) \|\rho_0\|_{L^\infty} e^{C_1 t_1} \quad (3.21)$$

and

$$m'(t_1) = 2C_1 \Omega \|\rho_0\|_{L^\infty}^2 e^{2C_1 t_1} + C_1 (\Omega + \sqrt{1-2\Omega A}) \|\rho_0\|_{L^\infty} e^{C_1 t_1} \geq 0 \quad \text{a.e. } t \in [0, T). \quad (3.22)$$

In view of (3.7), (3.19), (3.21) and the first equation in (3.5), it then follows that

$$\begin{aligned} m'(t_1) &= -\frac{1}{2}m^2(t_1) + \frac{1-2\Omega A}{2}\gamma^2(t_1) + f(t_1, \varphi(t_1, x)) \quad \text{a.e.} \quad t \in [0, T] \\ &\leq -\frac{1}{2} \left[\|u_{0,x}\|_{L^\infty} + 2\kappa + \Omega\|\rho_0\|_{L^\infty}^2 e^{2C_1 t_1} + (\Omega + \sqrt{1-2\Omega A})\|\rho_0\|_{L^\infty} e^{C_1 t_1} \right]^2 + \frac{1-2\Omega A}{2} \|\rho_0\|_{L^\infty}^2 e^{2C_1 t_1} \\ &\quad + \kappa^2 + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^4 e^{4C_1 t_1} + \frac{\Omega^2}{4} \|\rho_0\|_{L^\infty}^2 e^{2C_1 t_1} \\ &\leq 0, \end{aligned}$$

which is a contradiction to (3.22). This shows that the estimate in (3.20). Therefore, the arbitrary chosen of x implies

$$\sup_{x \in \mathbb{S}} u_x(t, x) \leq \|u_{0,x}\|_{L^\infty} - 2\kappa - \Omega\|\rho_0\|_{L^\infty}^2 e^{2C_1 t} - (\Omega + \sqrt{1-2\Omega A})\|\rho_0\|_{L^\infty} e^{C_1 t}.$$

Recalling

$$\inf_{x \in \mathbb{S}} u_x(t, x) \geq -C_1, \quad \forall t \in [0, T],$$

we obtain $|u_x(t, x)| \leq \infty$. Therefore, Theorem 2 implies that the maximal existence time $T = \infty$, which contradicts our assumption $T < \infty$. This completes the proof of Theorem 3. \square

Now, we will give an initial condition, which guarantees that wave-breaking phenomena occur in finite time.

Theorem 4. Suppose that $1 - 2\Omega A > 0$, $\sigma = 1$, and $\mu = 0$. Let $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ be the solution to (1.1) with initial data $X_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$ with $s > 3/2$ and T be the maximal existence time. If there is some $\bar{x} \in \mathbb{S}$ such that

$$\rho_0(\bar{x}) = 0 \tag{3.23}$$

and

$$u_{0,x}(\bar{x}) < -\kappa_1 - 2\Omega\kappa_2, \tag{3.24}$$

then the corresponding solution $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.1) blows up in finite time T_1 with

$$0 < T_1 \leq \left(\frac{u_{0,x}(\bar{x}) + \Omega\partial_{0,x}G * \rho^2(\bar{x}) + \Omega\kappa_1 - \kappa_2}{u_{0,x}(\bar{x}) + \Omega\partial_{0,x}G * \rho^2(\bar{x}) + \Omega\kappa_1 + \kappa_2} \right) \tag{3.25}$$

such that $\liminf_{t \uparrow T_1^-} \left\{ \inf_{x \in \mathbb{S}} u_x(t, x) \right\} = -\infty$, where

$$\kappa_1 = \frac{2e}{(e-1)(1-2\Omega A)} E(u_0, \rho - 1) + \frac{3e-1}{2(e-1)}$$

and

$$\kappa_2 = \sqrt{2\Omega\kappa_1 \sqrt{\frac{1-2\Omega A}{e} \left(\kappa_1 - \frac{3e-1}{2(e-1)} \right)} + \frac{1-2\Omega A}{2e} (A^2(e-1) + 7(e+1)) \left(\kappa_1 - \frac{3e-1}{2(e-1)} \right)}.$$

To deal with this wave-breaking phenomena, the following crucial lemma will be useful.

Lemma 5. Let $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ be the solution to (1.1) with initial data $X_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$ with $s > 3/2$ and T be the maximal existence time. Then $\Gamma = u + \Omega G * \rho^2$ satisfy

$$\Gamma_t + (\sigma u - \mu)\Gamma_x = \Omega(A - \mu)\partial_x G * \rho^2 + \Omega\sigma u\partial_x G * \rho^2 - \partial_x G * \left((\mu - A)u + \frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 \right) \quad (3.26)$$

and

$$\begin{aligned} \Gamma_{xt} + (\sigma u - \mu)\Gamma_{xx} = & -\frac{\sigma}{2}(\Gamma_x - \Omega\partial_x G * \rho^2)^2 + \frac{1+2\Omega(\mu-A)-2\Omega\sigma u}{2}\rho^2 + \Omega\sigma uG * \rho^2 \\ & + (A - \mu)\partial_x^2 G * u + \frac{3-\sigma}{2}u^2 - G * \left(\frac{3-\sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1-2\Omega(A-\mu)}{2}\rho^2 \right) \end{aligned} \quad (3.27)$$

Proof. Recall the first equation in (1.1)

$$u_t - u_{txx} - Au_x + 3uu_x = \sigma(2u_x u_{xx} + uu_{xxx}) - \mu u_{xxx} - (1 - 2\Omega A)\rho\rho_x + 2\Omega\rho(\rho u)_x \quad (3.28)$$

Using the second equation in (1.1), we can rewrite the high order nonlinear term $\rho(\rho u)_x$ as $-\rho\rho_t$. After rearranging (3.28) it is found that

$$u_t - u_{txx} + 2\Omega\rho\rho_t = A(u - u_{xx} + \Omega\rho^2)_x + (A - \mu)u_{xxx} - 3uu_x + \sigma(2u_x u_{xx} + uu_{xxx}) - \rho\rho_x. \quad (3.29)$$

In view of $uu_{xxx} = -(1 - \partial_x^2)uu_x + uu_x - 3u_x u_{xx}$, it then follows from (3.29) that

$$\begin{aligned} ((1 - \partial_x^2)u + \Omega\rho^2)_t = & A((1 - \partial_x^2)u + \Omega\rho^2)_x + (\mu - A)(1 - \partial_x^2)u_x + (A - \mu)u_x \\ & - \frac{\sigma}{2}(u_x^2)_x + \frac{\sigma-3}{2}(u^2)_x - \sigma(1 - \partial_x^2)uu_x - \frac{1}{2}(\rho^2)_x. \end{aligned} \quad (3.30)$$

Applying the operator $(1 - \partial_x^2)^{-1}$ to both sides of (3.30), we get

$$\begin{aligned} (u + \Omega(1 - \partial_x^2)^{-1}\rho^2)_t - A(u + \Omega(1 - \partial_x^2)^{-1}\rho^2)_x \\ = & (\mu - A)u_x - \partial_x(1 - \partial_x^2)^{-1} \left[(\mu - A)u + \frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 + \frac{1}{2}\rho^2 \right] - \sigma uu_x \\ = & (\mu - A)(u + \Omega(1 - \partial_x^2)^{-1}\rho^2)_x - (\mu - A)(\Omega(1 - \partial_x^2)^{-1}\rho^2)_x \\ & - \partial_x(1 - \partial_x^2)^{-1} \left[(\mu - A)u + \frac{\sigma}{2}u_x^2 + \frac{3-\sigma}{2}u^2 + \frac{1}{2}\rho^2 \right] \\ & - \sigma u(u + \Omega(1 - \partial_x^2)^{-1}\rho^2)_x + \sigma u(\Omega(1 - \partial_x^2)^{-1}\rho^2)_x. \end{aligned} \quad (3.31)$$

We conclude that (3.26) holds by taking $\Gamma = u + \Omega G * \rho^2$ and using the fact $(1 - \partial_x^2)^{-1}f = G * f$ in (3.31). In order to get (3.27), differentiating (3.26) with respect to x leads to

$$\begin{aligned}\Gamma_{xt} + (\sigma u - \mu)\Gamma_{xx} &= -\frac{\sigma}{2}u_x^2 + \Omega(A - \mu)(G * \rho^2 - \rho^2) + \Omega\sigma u(G * \rho^2 - \rho^2) + (A - \mu)\partial_x^2 G * u \\ &\quad + \frac{3 - \sigma}{2}u^2 + \frac{1}{2}\rho^2 - G * \left(\frac{3 - \sigma}{2}u^2 + \frac{\sigma}{2}u_x^2 + \frac{1}{2}\rho^2 \right),\end{aligned}\quad (3.32)$$

where we use $-\partial_x^2 G * f = f - G * f$. A rearrangement of (3.32) then gives the desired result (3.27). This completes the proof of Lemma 5. \square

Now we are in the position to give the proof of Theorem 4.

Proof of Theorem 4. Let $\sigma = 1$ and $\mu = 0$. We use a similar argument in the beginning of the proof of Theorem 3. So we take $s \geq 3$. Since $\sigma = 1$ and $\mu = 0$, we obtain from (3.27)

$$\begin{aligned}\Gamma_{xt} + u\Gamma_{xx} &= -\frac{1}{2}(\Gamma_x - \Omega\partial_x G * \rho^2)^2 + \frac{1 - 2\Omega A - 2\Omega u}{2}\rho^2 + \Omega u G * \rho^2 \\ &\quad + A\partial_x^2 G * u + u^2 - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1 - 2\Omega A}{2}\rho^2 \right)\end{aligned}\quad (3.33)$$

Given $x \in \mathbb{S}$, let

$$\bar{m}(t) = \Gamma_x(t, \varphi(t, x)), \quad \gamma(t) = \rho(t, \varphi(t, x)), \quad t \in [0, T], \quad (3.34)$$

where $\varphi(t, x)$ is defined by (2.2). Along the trajectory $\varphi(t, x)$, the second equation of (2.1) becomes

$$\gamma'(t) = -\gamma u_x, \quad t \in [0, T] \quad (3.35)$$

Thus,

$$\gamma(t) = \gamma(0)e^{-\int_0^t u_x(\tau) d\tau}. \quad (3.36)$$

By taking $x = \bar{x}$ and the assumption (3.23), we have $\gamma(0) = \rho_0(\bar{x}) = 0$. Hence, from (3.36), we see that

$$\gamma(t) \equiv 0, \quad t \in [0, T]. \quad (3.37)$$

Using the function $\bar{m}(t)$ as defined in (3.34), equation (3.33) becomes

$$\bar{m}'(t) = -\frac{1}{2}(\bar{m} - \Omega\partial_x G * \rho^2)^2 + f(t, \varphi(t, \bar{x})), \quad (3.38)$$

where $'$ denotes the derivative with respect to t and $f(t, \varphi(t, \bar{x}))$ is given by

$$f(t, \varphi(t, \bar{x})) = \Omega u G * \rho^2 + A\partial_x^2 G * u + u^2 - G * \left(u^2 + \frac{1}{2}u_x^2 + \frac{1 - 2\Omega A}{2}\rho^2 \right). \quad (3.39)$$

Since

$$\begin{aligned}|\partial_x G * \rho^2| &\leq |G * \rho^2| = |G * (\rho - 1)^2 + 2G * (\rho - 1) + G * 1| \\ &\leq \frac{e}{e-1} \|\rho - 1\|_{L^2}^2 + \frac{3e-1}{2(e-1)} \\ &\leq \frac{2e}{(e-1)(1-2\Omega A)} E(u_0, \rho - 1) + \frac{3e-1}{2(e-1)} := \kappa_1,\end{aligned}\quad (3.40)$$

$$\begin{aligned}
\Omega|uG * \rho^2| &= \Omega\|u\|_{L^\infty}\|G * \rho^2\|_{L^\infty} \\
&\leq \Omega\sqrt{\frac{2(e+1)}{e-1}}E(u_0, \rho-1) \left[\frac{2e}{(e-1)(1-2\Omega A)}E(u_0, \rho-1) + \frac{3e-1}{2(e-1)} \right] \\
&\leq \Omega\kappa_1\sqrt{\frac{1-2\Omega A}{e} \left(\kappa_1 - \frac{3e-1}{2(e-1)} \right)},
\end{aligned} \tag{3.41}$$

with (3.9), (3.10), and the fact $|G * (u^2 + \frac{1}{2}u_x^2)| \leq \frac{e+1}{2(e-1)}\|u\|_{L^2}^2 + \frac{e+1}{4(e-1)}\|u_x\|_{L^2}^2$, we have

$$\begin{aligned}
f &\leq \Omega|uG * \rho^2| + |A|\|\partial_x^2 G * u\| + u^2 + \left| G * \left(u^2 + \frac{1}{2}u_x^2 \right) \right| \\
&\leq \Omega|uG * \rho^2| + \frac{e+1}{2(e-1)} + \frac{A^2}{4}\|u_x\|_{L^2}^2 + \frac{e+1}{2(e-1)}(\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2) + \frac{e+1}{2(e-1)}\|u\|_{L^2}^2 + \frac{e+1}{4(e-1)}\|u_x\|_{L^2}^2 \\
&\leq \Omega\sqrt{\frac{2(e+1)}{e-1}}E(u_0, \rho-1) \left[\frac{2e}{(e-1)(1-2\Omega A)}E(u_0, \rho-1) + \frac{3e-1}{2(e-1)} \right] + \frac{e+1}{2(e-1)} \\
&\quad + \left[\frac{A^2}{2} + \frac{7(e+1)}{2(e-1)} \right] E(u_0, \rho_0-1) \\
&\leq \Omega\kappa_1\sqrt{\frac{1-2\Omega A}{e} \left(\kappa_1 - \frac{3e-1}{2(e-1)} \right)} + \frac{1-2\Omega A}{4e}(A^2(e-1) + 7(e+1)) \left(\kappa_1 - \frac{3e-1}{2(e-1)} \right) \\
&=: \frac{1}{2}\kappa_2^2,
\end{aligned} \tag{3.42}$$

where κ_1 and κ_2 denotes a constant that depends only on $E(u_0, \rho_0-1)$. From (3.38) and (3.42), we see that

$$\bar{m}'(t) \leq -\frac{1}{2}(\bar{m} - \Omega\partial_x G * \rho^2)^2 + \frac{1}{2}\kappa_2^2, \quad t \in [0, T]. \tag{3.43}$$

If \bar{x} satisfies the assumption (3.24), then we have

$$\bar{m}(0) = u_{0,x}(\bar{x}) + \Omega\partial_x G * \rho_0^2(\bar{x}) < u_{0,x}(\bar{x}) + \Omega\kappa_1 < -\kappa_2 - \Omega\kappa_1.$$

We now claim that

$$\bar{m}(t) < -\kappa_2 - \Omega\kappa_1, \quad \forall t \in [0, T]. \tag{3.44}$$

Indeed, $\bar{m}(0) < -\kappa_2 - \Omega\kappa_1$ and $\bar{m}(t)$ is continuous. If (3.44) does not holds, then suppose there is a $t_0 \in (0, T)$ such that $\bar{m}(t_0) < -\kappa_2 - \Omega\kappa_1$ on $[0, t_0)$, while $\bar{m}(t_0) = -\kappa_2 - \Omega\kappa_1$. But then we get by (3.43)

$$\frac{d\bar{m}(t)}{dt} < 0, \quad \text{a.e. } t \in [0, t_0]. \tag{3.45}$$

Being locally Lipschitz, the function $\bar{m}(t)$ is a absolutely continuous on $[0, t_0]$, and therefore an integration of the inequality (3.45) would lead to

$$\bar{m}(t_0) \leq \bar{m}(0) < -\kappa_2 - \Omega\kappa_1,$$

which contradicts our assumption $\bar{m}(t_0) = -\kappa_2 - \Omega\kappa_1$. Hence (3.44) holds, implying that $\bar{m}(t)$ is strictly decrease on $[0, T)$. Thus

$$\bar{m}'(t) \leq -\frac{1}{2}(\bar{m} + \Omega\kappa_1)^2 + \frac{1}{2}\kappa_2^2, \quad t \in [0, T]. \quad (3.46)$$

Solving the above inequality (3.46) gives

$$\frac{\bar{m}(0) + \Omega\kappa_1 + \kappa_2}{\bar{m}(0) + \Omega\kappa_1 - \kappa_2} e^{\kappa_2 t} - 1 \leq \frac{2\kappa_2}{\bar{m}(t) + \Omega\kappa_1 - \kappa_2} \leq 0.$$

Due to the fact that $0 < \frac{\bar{m}(0) + \Omega\kappa_1 - \kappa_2}{\bar{m}(0) + \Omega\kappa_1 + \kappa_2} < 1$ and the boundedness of $\partial_x G * \rho^2$ in (3.40), we conclude that there exists T_1 satisfying

$$0 < T_1 < \frac{1}{\kappa_2} \ln \left(\frac{\bar{m}(0) + \Omega\kappa_1 - \kappa_2}{\bar{m}(0) + \Omega\kappa_1 + \kappa_2} \right) = \frac{1}{\kappa_2} \ln \left(\frac{u_{0,x}(\bar{x}) + \Omega\partial_{0,x}G * \rho^2(\bar{x}) + \Omega\kappa_1 - \kappa_2}{u_{0,x}(\bar{x}) + \Omega\partial_{0,x}G * \rho^2(\bar{x}) + \Omega\kappa_1 + \kappa_2} \right)$$

such that $\lim_{t \uparrow T_1} \bar{m}(t) = -\infty$, i.e. $\lim_{t \uparrow T_1} u_x(t, x) = -\infty$. This completes the proof of Theorem 4. \square

4. Global existence

In this section, we provide a sufficient condition for the global solution of system (1.1) in the case $\sigma = 1$ and $\mu = 0$.

Theorem 5. Suppose that $1 - 2\Omega A > 0$, $\sigma = 1$, and $\mu = 0$. Let $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ be the solution to (1.1) with initial data $X_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$ with $s > 3/2$ and T be the maximal existence time. If

$$\rho_0(x) \neq 0, \quad \forall x \in \mathbb{S} \quad (4.1)$$

and

$$2\Omega \sqrt{\frac{2(e+1)}{e-1}} E(u_0, \rho_0 - 1) < 1 - 2\Omega A, \quad (4.2)$$

then $T = +\infty$ and the solution $X = \begin{pmatrix} u \\ \rho \end{pmatrix}$ is global.

Proof. Observe that system (1.1) becomes the following ordinary differential equations

$$\begin{cases} \bar{m}'(t) = -\frac{1}{2}(\bar{m} - \Omega\partial_x G * \rho^2)^2(t) + \frac{1-2\Omega A-2\Omega u}{2}\gamma^2 + f(t, \varphi(t, x)), \\ \gamma'(t) = -\gamma u_x = -\gamma\bar{m} + \Omega\gamma\partial_x G * \rho^2, \quad \text{a.e. } t \in [0, T], \end{cases} \quad (4.3)$$

where $\bar{m}(t) = u_x(t, \varphi(t, x)) + \Omega\partial_x G * \rho^2(t, \varphi(t, x))$, $\gamma(t) = \rho(t, \varphi(t, x))$, and f is defined in (3.34) and (3.39). The assumption (4.1) and the second equation (4.4) implies that $\gamma(t)$ and $\gamma(0)$ are of the same sign.

Inspired by [10] (see also [14]), we want to construct a Lyapunov function for our system. Because of the assumption (4.2) and $\gamma(0) = \rho_0(x) \neq 0$ for every $x \in \mathbb{S}$, we define the following strictly positive Lyapunov function

$$\bar{w}(t) = \left(1 - 2\Omega A - 2\Omega \sqrt{\frac{2(e+1)}{e-1}} E(u_0, \rho_0 - 1) \right) \gamma(0)\gamma(t) + \frac{\gamma(0)}{\gamma(t)} (1 + \bar{m}^2(t)).$$

Computing the evolution of \bar{w} and using (4.3), we get

$$\begin{aligned}
\bar{w}'(t) &= \left(1 - 2\Omega A - 2\Omega \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)}\right) \gamma(0) \gamma'(t) \\
&\quad - \frac{\gamma(0)}{\gamma(t)} \gamma'(t) (1 + \bar{m}^2(t)) + 2 \frac{\gamma(0)}{\gamma(t)} \bar{m}(t) \bar{m}'(t) \\
&= \frac{2\gamma(0) \bar{m}(t)}{\gamma(t)} \left(-\frac{\Omega^2}{2} (\partial_x G * \rho^2)^2 + \frac{1}{2} + f(t, \varphi(t, x)) \right) + \Omega \frac{\gamma(0)(\bar{m}^2(t) - 1)}{\gamma(t)} \partial_x G * \rho^2 \\
&\quad + \Omega \left(1 - 2\Omega A - 2\Omega \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)}\right) \gamma(0) \gamma(t) \partial_x G * \rho^2 \\
&\quad - 2\Omega \gamma(0) \gamma(t) \bar{m}(t) \left(u - \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)} \right) \\
&\leq \frac{\gamma(0)}{\gamma(t)} (1 + \bar{m}^2(t)) \left(\frac{\Omega^2}{2} |\partial_x G * \rho^2|^2 + |f(t, \varphi(t, x))| + \frac{1}{2} + \Omega |\partial_x G * \rho^2| \right) \\
&\quad + \Omega \left(1 - 2\Omega A - 2\Omega \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)}\right) \gamma(0) \gamma(t) |\partial_x G * \rho^2| \\
&\quad - 2\Omega \gamma(0) \gamma(t) \bar{m}(t) \left(u - \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)} \right) \\
&\leq \frac{1}{2} [\Omega^2 \kappa_1^2 + 1 + \kappa_2^2 + 4\Omega \kappa_1] \bar{w}(t) - 2\Omega \gamma(0) \gamma(t) \bar{m}(t) \left(u - \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)} \right) \\
&:= \kappa_3 \bar{w}(t) - 2\Omega \gamma(0) \gamma(t) \bar{m}(t) \left(u - \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)} \right),
\end{aligned} \tag{4.4}$$

where we have used the boundedness of $\partial_x G * \rho^2$ and $|f|$ in (3.40) and (3.42), respectively.

We will show that no wave breaking can occur in this case. Suppose the corresponding solution exists in finite time, then for some fixed $\xi \in \mathbb{S}$ we have

$$u_x(t, \varphi(t, \xi)) \rightarrow -\infty \quad \text{as } t \rightarrow T_a, \tag{4.5}$$

where T_a denotes the first time when wave breaking occurs. Hence there exists a time $0 \leq \bar{t} < T_a$ such that

$$u_x(\bar{t}, \varphi(\bar{t}, \xi)) \leq -\Omega \kappa_1 \quad \text{for } \bar{t} \leq t < T_a,$$

where κ_1 is defined in (3.40). From (3.34) and (3.40) we see that $\bar{m}(t, \varphi(t, \xi)) \leq 0$ for $\bar{t} \leq t < T_a$. As a consequence of

$$\bar{m}(t) \left(u - \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)} \right) \geq 0 \quad \text{for } \bar{t} \leq t < T_a,$$

we obtain from (4.4)

$$\bar{w}(t) \leq \bar{w}(\bar{t}) e^{\kappa_3(t-\bar{t})} \leq \bar{w}(\bar{t}) e^{\kappa_3 T_a}, \quad \text{for } \bar{t} \leq t < T_a. \tag{4.6}$$

As $\bar{t} \rightarrow T_a$, there exists a $C_2 > 0$ such that

$$u_x \geq -C_2, \quad \text{for } 0 \leq t \leq \bar{t}.$$

Thus

$$|\gamma(t)| \leq |\gamma(0)|e^{C_2 t} \leq \|\gamma(0)\|_{L^\infty} e^{C_2 T_a}, \quad \text{for } 0 \leq t \leq \bar{t}.$$

It then follows from (4.4) that

$$\begin{aligned} \bar{w}'(t) &\leq \kappa_3 \bar{w}(t) - 2\Omega \frac{\gamma(0)}{\gamma(t)} \bar{m}(t) \left(u - \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)} \right) \gamma^2(t), \\ &\leq \kappa_3 \bar{w}(t) + \Omega \frac{\gamma(0)}{\gamma(t)} (\bar{m}^2(t) + 1) \left| \left(u - \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)} \right) \gamma^2(t) \right| \\ &\leq \kappa_3 \bar{w}(t) + 2\Omega \frac{\gamma(0)}{\gamma(t)} (\bar{m}^2(t) + 1) \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)} \|\gamma(0)\|_{L^\infty}^2 e^{2C_2 T_a} \\ &\leq (\kappa_3 + \kappa_4) \bar{w}(t) \end{aligned} \quad (4.7)$$

which yields

$$\bar{w}(\bar{t}) \leq \bar{w}(0) e^{(\kappa_3 + \kappa_4) \bar{t}} \leq \bar{w}(0) e^{(\kappa_3 + \kappa_4) T_a}, \quad (4.8)$$

where

$$\begin{aligned} \bar{w}(0) &= \left(1 - 2\Omega A - 2\Omega \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)} \right) \gamma^2(0) + 1 + \bar{m}^2(0) \\ &\leq \left(1 - 2\Omega A - 2\Omega \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)} \right) \|\rho_0\|_{L^\infty}^2 + 1 + 2\|u_{0,x}\|_{L^\infty}^2 + 2\Omega^2 \kappa_1^2 \\ &:= \kappa_5. \end{aligned} \quad (4.9)$$

From (4.8), (4.9), and (4.6), it in turn implies that

$$\bar{w}'(t) \leq \kappa_5 e^{(2\kappa_3 + \kappa_4) T_a}. \quad (4.10)$$

Recalling that $\gamma(t)$ and $\gamma(0)$ are of the same sign, the definition of Lyapunov function $\bar{w}(t)$ in (4.2) implies

$$\sqrt{\left(1 - 2\Omega A - 2\Omega \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)} \right)} |\gamma(0)| |\bar{m}(t)| \leq \bar{w}(t). \quad (4.11)$$

On the other hand, we know that ρ_0 is a continuous function and $\rho_0 \rightarrow 1$ as $|x| \rightarrow \infty$ due to the assumption on $\rho_0 - 1 \in H^{s-1}(\mathbb{S})$ with $s < 3/2$. Therefore, the addition assumption (4.1) leads to

$$\inf_{x \in \mathbb{S}} \rho_0(x) > 0.$$

From (4.10) and (4.11), we obtain

$$\begin{aligned} |\bar{m}(t)| &= |u_x(t, \varphi(t, \xi)) + \Omega \partial_x G * \rho^2| \\ &\leq \frac{\bar{w}(t)}{\sqrt{\left(1 - 2\Omega A - 2\Omega \sqrt{\frac{2(e+1)}{e-1} E(u_0, \rho_0 - 1)} \right)} |\gamma(0)|} \end{aligned} \quad (4.12)$$

$$\begin{aligned}
&\leq \frac{\kappa_5 e^{2\kappa_3 + \kappa_4} T_a}{\sqrt{\left(1 - 2\Omega A - 2\Omega \sqrt{\frac{2(e+1)}{e-1}} E(u_0, \rho_0 - 1)\right)} |\rho_0(\xi)|} \\
&\leq \frac{\kappa_5 e^{2\kappa_3 + \kappa_4} T_a}{\sqrt{\left(1 - 2\Omega A - 2\Omega \sqrt{\frac{2(e+1)}{e-1}} E(u_0, \rho_0 - 1)\right)} \inf_{x \in \mathbb{S}} \rho_0(x)} < \infty,
\end{aligned}$$

implying $|u_x(t, \varphi(t, \xi))| < \infty$ for $t \in [\bar{t}, T_a)$, which is a contradiction to our assumption (4.5). Thus the proof of Theorem 5 is complete. \square

References

- [1] R. Camassa, D.D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.* 71 (1993) 1661–1664.
- [2] R. Camassa, D.D. Holm, J. Hyman, A new integrable shallow water equation, *Adv. Appl. Mech.* 31 (1994) 1–33.
- [3] C.S. Cao, D.D. Holm, E.S. Titi, Traveling wave solutions for a class of one-dimensional nonlinear shallow water wave models, *J. Dynam. Differential Equations* 16 (2004) 167–178.
- [4] R.M. Chen, Y. Liu, Wave breaking and global existence for a generalized two-component Camassa–Holm system, *Int. Math. Res. Not. IMRN* 6 (2011) 1381–1416.
- [5] C. Chen, Y. Yan, On the wave breaking phenomena for the generalized periodic two-component Dullin–Gottwald–Holm system, *J. Math. Phys.* 53 (2012) 103709.
- [6] A. Constantin, Global existence of solutions and breaking waves for a shallow water equation: a geometric approach, *Ann. Inst. Fourier (Grenoble)* 50 (2000) 321–362.
- [7] A. Constantin, J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.* 181 (1998) 229–243.
- [8] A. Constantin, J. Escher, Global existence and blow-up for a shallow water equation, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* 26 (1998) 303–328.
- [9] A. Constantin, J. Escher, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, *Math. Z.* 233 (2000) 75–91.
- [10] A. Constantin, R.I. Ivanov, On an integrable two-component Camassa–Holm shallow water system, *Phys. Lett. A* 372 (2008) 7129–7132.
- [11] R. Dullin, G. Gottwald, D. Holm, An integrable shallow water equation with linear and nonlinear dispersion, *Phys. Rev. Lett.* 87 (2001) 4501.
- [12] J. Escher, D. Henry, B. Kolev, T. Lyons, Two-component equations modelling water waves with constant vorticity, *Ann. Mat. Pura Appl.* 195 (2016) 249–271.
- [13] J. Escher, O. Lechtenfeld, Z. Yin, Well-posedness and blow-up phenomena for the 2-component Camassa–Holm equation, *Discrete Contin. Dyn. Syst.* 19 (3) (2007) 493–513.
- [14] L. Fan, H. Gao, Y. Liu, On the rotation-two-component Camassa–Holm system modelling the equatorial water waves, *Adv. Math.* 291 (2016) 59–89.
- [15] F. Genoud, D. Henry, Instability of equatorial water waves with an underlying current, *J. Math. Fluid Mech.* 16 (4) (2014) 661–667.
- [16] G. Gui, Y. Liu, On the global existence and wave-breaking criteria for the two-component Camassa–Holm system, *J. Funct. Anal.* 258 (2010) 4251–4278.
- [17] F. Guo, H.J. Gao, Y. Liu, On the wave-breaking phenomena for the two-component Dullin–Gottwald–Holm system, *J. Lond. Math. Soc.* 86 (2012) 810–834.
- [18] Y. Han, F. Guo, H.J. Gao, On solitary waves and wave-breaking phenomena for a generalized two-component integrable Dullin–Gottwald–Holm system, *J. Nonlinear Sci.* 23 (2013) 617–656.
- [19] R. Ivanov, Two-component integrable systems modelling shallow water waves: the constant vorticity case, *Wave Motion* 46 (2009) 389–396.
- [20] T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, in: *Spectral Theory and Differential Equations*, in: *Lecture Notes in Math.*, vol. 448, Springer Verlag, Berlin, 1975, pp. 25–70.
- [21] P. Olver, P. Rosenau, Tri-Hamiltonian duality between solitons and solitary wave solutions having compact support, *Phys. Rev. E* 53 (1996) 1900–1906.
- [22] Z. Yin, On the blow-up of solutions of the periodic Camassa–Holm equation, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* 12 (2005) 375–381.
- [23] P.Z. Zhang, Y. Liu, Stability of solitary waves and wave-breaking phenomena for the two-component Camassa–Holm system, *Int. Math. Res. Not. IMRN* 211 (2010) 1981–2021.