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Critical points of orthogonal polynomials

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ABSTRACT

We study properties of the critical points of orthogonal polynomials with respect to a measure on the unit circle (OPUC). The main result states that, under some conditions, the asymptotic distribution of the critical points of OPUC coincides with the asymptotic distribution of its zeros and each Nevai–Totik point attracts the same number of critical points as zeros of the OPUC. Analogous results are also presented for paraorthogonal polynomials and for orthogonal polynomials with respect to a regular measure supported on a continuum set.

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1. Introduction

The critical points of a polynomial are the equilibrium points in a certain force field. The Gauss–Lucas theorem states that the critical points of a polynomial lie in the convex hull of their zeros (see [6, Sect. 2.1]). The Jentzsch–Szegő theorem tells us that for a power series with finite radius of convergence there is an infinite sequence of partial sums, the zeros of which are "equidistributed" with respect to the angular measure on the boundary circle of the disk of convergence. Since a series and its derivative have equal radii of convergence, the result of Jentzsch–Szegő also holds for the corresponding sequence of the derivative. Both theorems have been generalized in different directions (see [11, Sect. 2.1] and [2]). Another interesting open problem in this issue which deserves attention is Sendov's conjecture: "*If p is a polynomial of degree ≥ 2 having all its zeros in the closed unit disk $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ and if z_0 is any one of such zero, then at least one critical point of p lies on the disk $\{z \in \mathbb{C} : |z - z_0| \leq 1\}$*" (see [11, Sect. 7.3]). Sendov's conjecture has been proved for polynomials of large degree (see [3]). The asymptotic behavior of the critical

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points of a sequence of polynomials whose almost all zeros lie in a given convex bounded domain has been proved in [19].

In this paper we study properties of the critical points of orthogonal polynomials with respect to a measure on the unit circle (OPUC). A polynomial whose zeros lie in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ is a term of a sequence of polynomials orthogonal with respect to a Bernstein–Szegő measure in the unit circle. Thus, our results want to be in some sense a contribution for a better understanding of the kind of problems above mentioned.

Let μ be a probability measure on $[0, 2\pi)$ and $\{\Phi_n\}_{n\geq 0}$ be the associated sequence of monic OPUC. They satisfy the recurrence relations

$$\begin{cases} \Phi_{n+1}(z) = z\Phi_n(z) + \Phi_{n+1}(0)\Phi_n^*(z), \\ \Phi_{n+1}^*(z) = \Phi_n^*(z) + \overline{\Phi_{n+1}(0)} z\Phi_n^*(z), \end{cases} \tag{1}$$

for $n \geq 0$, where $\Phi_0(z) = 1$ and the reverse polynomials are $\Phi_n^*(z) := z^n \overline{\Phi_n(1/\overline{z})}$. As all the zeros of Φ_n are inside $\mathbb{D}$, we have $|\Phi_{n+1}(0)| < 1$ for all $n \geq 0$. According to Verblunsky theorem the map $\mu \mapsto \{\Phi_{n+1}(0)\}_{n\geq 0}$ is a one–one correspondence between non-trivial probability positive Borel measures and sequences in $\mathbb{D}$. Let $\int |\Phi_n(e^{i\theta})|^2 \, d\mu(\theta) =: \kappa_n^{-2}$; then, analogous formulae to (1) hold for orthonormal polynomials $\{\varphi_n := \kappa_n \Phi_n\}_{n\geq 0}$. All these results can be found in [13, Chap. 1].

Nevai and Totik [9] proved that if

$$\rho = \limsup_n |\Phi_n(0)|^{1/n} < 1, \tag{2}$$

then $S(z)^{-1}$ can be analytically continued to all the region $\{z \in \mathbb{C} : |z| < 1/\rho\}$, where $S(z)$ represents the Szegő function

$$S(z) = \exp\left(\frac{1}{4\pi}\int_0^{2\pi} \log \mu'(\theta) \frac{e^{i\theta} + z}{e^{i\theta} - z} \, d\theta\right), \quad z \in \mathbb{D}. \tag{3}$$

This extention $S_{\text{ext}}(z)^{-1}$ has no zeros in $\overline{\mathbb{D}}$. The Nevai–Totik points are the zeros of $S_{\text{ext}}(1/\overline{z})^{-1}$ in $\{z \in \mathbb{C} : \rho < |z| < 1\}$. Since

$$\lim_n \varphi_n^*(z) = S_{\text{ext}}(z)^{-1} \tag{4}$$

holds uniformly on compact subsets of $\{z \in \mathbb{C} : |z| < 1/\rho\}$, the Nevai–Totik points are precisely the limit points of zeros of $\{\varphi_n\}_{n\geq 0}$ in $\{z \in \mathbb{C} : \rho < |z| < 1\}$.

As usual, we say that μ satisfies the Szegő condition or μ belongs to the Szegő class if

$$\int_0^{2\pi} |\log \mu'(\theta)| \, d\theta < \infty.$$

According to Szegő theorem in this case the convergence of $\varphi_n^*(z)$ to $S(z)^{-1}$ is uniform on compact subsets of $\mathbb{D}$.

Let p be a polynomial of degree n and ν_p its normalized counting measure which gives weight k/n to each zero of p with multiplicity k. For $r > 0$, let m_r denote the arc-measure on the circle $C_r := \{z \in \mathbb{C} : |z| = r\}$; i.e. $m_r(\{r\, e^{i\theta} : \theta_1 \leq \theta \leq \theta_2\}) = \theta_2 - \theta_1$ where $0 \leq \theta_1 < \theta_2 < 2\pi$. When $r = 0$, we let m_0 be the delta distribution with mass 1 supported at $z = 0$. If either (2) or

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}\Phi_k(0)=0 \quad \text{if } \rho=1 \tag{5}$$

holds, then Mhaskar and Saff [8] proved that $\lim_{n\in\Lambda}\nu_{\Phi_n}=m_\rho$ weakly. Here and in what follows, Λ is a subsequence of index such that $\lim_{n\in\Lambda}|\Phi_n(0)|^{1/n}=\limsup_n|\Phi_n(0)|^{1/n}$. Detailed asymptotics of the zeros of OPUC can be found in [7,14,15], and [16].

The zeros of OPUC in the region $\{z\in\mathbb{C}:|z|<\rho\}$ can have a chaotic behavior (see [7] and [18]). A control of a such situation can be obtained with the assumption

$$\Phi_n(0)=\sum_{j=1}^{J}C_jb_j^n+O((\rho\Delta)^n) \tag{6}$$

where the b_j's are distinct, $C_j\neq 0$ for all j, $|b_j|=\rho$, $j=1,2,\dots,J$, and $0<\rho<1$, $0\leq\Delta<1$ (see [15]). Condition (6) extends the periodic asymptotic behavior for Verblunsky coefficients studied in [1].

For the critical points of a polynomial we have the additional problem that they can be swept up to the convex hull of the zeros as occur in the trivial example $p(z)=z^n-1$. Nevertheless, under assumption (6) this can not happen asymptotically for the corresponding OPUC as the following result states.

Theorem 1. *If (6) holds, then for each $k\geq 1$ we have*

$$\lim_n \nu_{\Phi_n^{(k)}}=m_\rho.$$

For n large enough the polynomials $\Phi_n^{(k)}$ and Φ_n have the same number of zeros, counted according to their multiplicities, inside every small open neighborhood of each Nevai–Totik point. Moreover, in each compact subset of $\{z\in\mathbb{C}:|z|<\rho\}$ the number of critical points is uniformly bounded in n. If $\lim_n|\Phi_n(0)|^{1/n}=0$, then $\lim_n\nu_{\Phi_n^{(k)}}=m_0$.

The paraorthogonal polynomials play an important role in the Szegő quadrature formula (see [5]). For a sequence $\{w_n\}_{n\geq 1}$ in the unit circle they are defined by

$$B_n(z,w_n):=\Phi_n^*(z)\overline{\Phi_n^*(w_n)}-\Phi_n(z)\overline{\Phi_n(w_n)},\quad n=1,\dots \tag{7}$$

Theorem 2. *If μ satisfies the Szegő condition and the Szegő function is not a constant in $\mathbb{D}$, then for each $k\geq 0$ and for any sequence $\{w_n\}_{n\geq 1}$ in the unit circle we have*

$$\lim_n \nu_{B_{n+1}^{(k)}}=m_1.$$

The structure of this paper is as follows. Theorems 1 and 2 will be proved in Section 3. In Section 2, we include some auxiliary results that we need in order to prove these two theorems. Section 4 contains a theorem on the distribution of critical points of orthogonal polynomials with respect to a regular measure and a study of differential properties of Akhiezer–Chebyshev's polynomials which is linked to the behavior of its critical points. We also include pictures in Section 4 which are consistent with our theorems.

2. Auxiliary results

Let us consider the set

$$\mathbb{A}:=\bigcap_{n=1}^{\infty}\overline{\bigcup_{k=n,\,k\in\Lambda}^{\infty}\{z\in\mathbb{C}:\Phi_k^*(z)=0\}}.$$

Lemma 1. *If either (2) or (5) holds and $z \notin \mathbb{A}$, we have*

$$\lim_{n\in\Lambda} \frac{(\Phi_n^*)'(z)}{n\Phi_n^*(z)} = \int \frac{dm_{1/\rho}(t)}{z-t} = \begin{cases} 0 & \text{if } |z| < \frac{1}{\rho}, \\ \frac{1}{z} & \text{if } |z| > \frac{1}{\rho}. \end{cases} \tag{8}$$

Moreover, for all $k \geq 1$, we have

$$\lim_{n\in\Lambda} \frac{(\Phi_n^*)^{(k)}(z)}{n(\Phi_n^*)^{(k-1)}(z)} = \frac{1}{z} \tag{9}$$

uniformly on compact subsets of $\{z \in \mathbb{C} : |z| > \frac{1}{\rho}\} \setminus \mathbb{A}$.

Proof. Let $\{z_{k,n}^*\}$ denote the zeros of Φ_n^*. So we can write

$$\frac{(\Phi_n^*)'(z)}{n\Phi_n^*(z)} = \frac{1}{n}\sum_k \frac{1}{z - z_{k,n}^*} = \int \frac{d\nu_{\Phi_n^*}(t)}{z-t}.$$

By the Mhaskar–Saff Theorem, we know that $\lim_{n\in\Lambda} \nu_{\Phi_n^*} = m_{1/\rho}$. Since

$$\int \frac{dm_{1/\rho}(t)}{z-t} = \begin{cases} 0 & \text{if } |z| < \frac{1}{\rho}, \\ \frac{1}{z} & \text{if } |z| > \frac{1}{\rho}, \end{cases}$$

the proof of (8) has been concluded.

From

$$\frac{1}{n}\left(\frac{(\Phi_n^*)^{(k)}(z)}{n(\Phi_n^*)^{(k-1)}(z)}\right)' = \frac{(\Phi_n^*)^{(k)}(z)}{n(\Phi_n^*)^{(k-1)}(z)}\left(\frac{(\Phi_n^*)^{(k+1)}(z)}{n(\Phi_n^*)^{(k)}(z)} - \frac{(\Phi_n^*)^{(k)}(z)}{n(\Phi_n^*)^{(k-1)}(z)}\right),$$

the relation (9) follows intermediately by induction. □

Lemma 2.

(i) If z_0 is a zero of $(\Phi_n')^*$, then

$$z_0(\Phi_n^*)'(z_0) = n\Phi_n^*(z_0). \tag{10}$$

(ii) If (2) holds the sequence of polynomials $\{(\Phi_n')^* : n \in \Lambda\}$ has a number of zeros which is uniformly bounded in n inside each closed set in the disk $\{z \in \mathbb{C} : |z| < 1/\rho\}$.

(iii) If $\rho < 1$ the polynomials $\Phi_n^{(k)}$ and Φ_n, for $n \in \Lambda$ large enough, have the same number of zeros, counted according to their multiplicities, inside small open neighborhood of each Nevai–Totik point.

Notice that the reverse polynomial $(\Phi_n')^*$ has degree at most $n-1$.

Proof. *(i)* If P is a polynomial, we have $z(P^*)'(z) = nP^*(z) - (P')^*(z)$. In particular,

$$z(\Phi_n^*)'(z) = n\Phi_n^*(z) - (\Phi_n')^*(z), \tag{11}$$

and we get (10).

(ii) Suppose, to get a contradiction, that there exist $z_0 \in \{z \in \mathbb{C} : 1 \leq |z_0| < \frac{1}{\rho}\}$, such that $\overline{z_0}^{-1}$ is not a Nevai–Totik point, and a sequence $\{z_{n_j} : n_j \in \Lambda\}$ verifying $(\Phi_{n_j}')^*(z_{n_j}) = 0$ and $\lim_j z_{n_j} = z_0$. Then,

$$\lim_j \frac{(\Phi^*_{n_j})'(z_{n_j})}{n_j \Phi^*_{n_j}(z_{n_j})} = \lim_j \frac{1}{z_{n_j}} = \frac{1}{z_0}.$$

This is impossible according to Lemma 1. As the number of Nevai–Totik points in each closed set in $\{z \in \mathbb{C} : |z| < 1/\rho\}$ is finite, the statement *(ii)* follows.

To prove *(iii)*, let C be a small circle with center at w_0 such that $\overline{w_0}^{-1}$ is a Nevai–Totik point. By Lemma 1 and (11), we have

$$\lim_n \frac{(\Phi'_n)^*(z)}{n\Phi^*_n(z)} = 1,$$

uniformly on C. Therefore, for $z \in C$ and $n \in \Lambda$ large enough, we obtain

$$|(\Phi'_n)^*(z) - n\Phi^*_n(z)| < |n\Phi^*_n(z)|;$$

now the conclusion *(iii)* follows from Rouché's Theorem. $\square$

Lemma 3. *If μ is in the Szegő class, then*

$$\lim_n \Phi'_n(z) = 0, \tag{12}$$

uniformly on compact subsets of $\mathbb{D}$.

Proof. Since μ is in the Szegő class, $\lim_n \frac{\Phi_n(z)}{\Phi^*_n(z)} = 0$ and

$$\lim_n \left(\frac{\Phi_n(z)}{\Phi^*_n(z)}\right)' = 0,$$

both limits hold uniformly on compact subsets of $\mathbb{D}$. Further, we have

$$\lim_n \Phi^*_n(z) = S(0)^{-1}S(z)^{-1}, \quad \lim_n (\Phi^*_n)'(z) = S(0)^{-1}(S(z)^{-1})'$$

and

$$\left(\frac{\Phi_n(z)}{\Phi^*_n(z)}\right)' = \frac{\Phi'_n(z)}{\Phi^*_n(z)} - \frac{\Phi_n(z)}{\Phi^*_n(z)}\frac{\Phi^{*\prime}_n(z)}{\Phi^*_n(z)}$$
$$\Leftrightarrow \Phi'_n(z) = \Phi^*_n(z)\left(\frac{\Phi_n(z)}{\Phi^*_n(z)}\right)' + \frac{\Phi_n(z)}{\Phi^*_n(z)}\Phi^{*\prime}_n(z),$$

and these relations yield (12). $\square$

3. Proofs of the main results

3.1. Proof of Theorem 1

Let us consider only $k = 1$ because it is easy to get, by induction, the general result.

When $\rho = 0$ Lemma 2 tell us that for each $R > 0$ the sequence of polynomials $\{(\varphi'_n)^* : n \in \Lambda\}$ has a number of zeros uniformly bounded in n inside the disk $\{z \in \mathbb{C} : |z| < R\}$; so, $\{\varphi'_n : n \in \Lambda\}$ has a number of zeros which is uniformly bounded in n inside $\{z \in \mathbb{C} : |z| > 1/R\}$ and this is equivalent to

$$\lim_{n\in\Lambda} \nu_{\Phi'_n} = m_0.$$

When $0 < \rho < 1$ we know by Lemma 2 that any partial limit ν of $\{\nu_{\Phi_n'}\}$ has its support included in $\{z \in \mathbb{C} : |z| \le \rho\}$. Let us see that the support of ν is $\{z \in \mathbb{C} : |z| = \rho\}$. From (6) it is proved in [15, Theorem 2.2] that

$$\lim_n \left(\frac{\varphi_n(z)}{\rho^n} - Q_n(z) \right) = 0 \tag{13}$$

uniformly on each compact subset of $\{z \in \mathbb{C} : |z| < \rho\}$, where

$$Q_n(z) := \sum_{j=1}^{L} \overline{C}_j \overline{\omega}_j^n (\overline{b}_j - z)^{-1} S(z)^{-1},$$

$\omega_j := \frac{b_j}{\rho}$ and so $|\omega_j| = 1$, $j = 1, \ldots, L$. For any sequence of natural numbers there is a subsequence $\{n_k\}$ such that the limit

$$\lim_k \sum_{j=1}^{L} \overline{C}_j \overline{\omega}_j^{n_k} \prod_{\ell \neq j} (z - \overline{b}_\ell) =: P_\infty(z)$$

exists being $P_\infty(z)$ a nonzero polynomial of degree at most $L - 1$.

Let $T(z) := \prod_{\ell=1}^{L} (z - \overline{b}_\ell)$. Multiplying (13) by $S(z)T(z)$ and taking derivative we obtain

$$\lim_k \left(\frac{\varphi_{n_k}'(z)}{\rho^{n_k}} S(z)T(z) + \frac{\varphi_{n_k}(z)}{\rho^{n_k}} S'(z)T(z) + \frac{\varphi_{n_k}(z)}{\rho^{n_k}} S(z)T'(z) \right) = P_\infty'(z)$$

which is equivalent to

$$\begin{aligned} \lim_k \frac{\varphi_{n_k}'(z)}{\rho^{n_k}} S(z)T(z) &= P_\infty'(z) - P_\infty(z)\frac{S'(z)}{S(z)} - P_\infty(z)\frac{T'(z)}{T(z)} \\ &= S(z)T(z)\left(P_\infty(z)S(z)^{-1}T(z)^{-1}\right)'. \end{aligned}$$

So, by Hurwitz's theorem $\{\varphi_{n_k}'(z)\}$ has a number of zeros which is uniformly bounded in n in each compact subset of $\{z \in \mathbb{C} : |z| < \rho\}$, and the support of ν is a subset of $\{z \in \mathbb{C} : |z| = \rho\}$. Since $\lim_k |\Phi_{n_k}(0)|^{1/n_k} = \rho$, by (9) in Lemma 1 the measures ν and m_ρ have the same moments and the same support $\{z \in \mathbb{C} : |z| = \rho\}$. Thus, $\nu = m_\rho$.

3.2. *Proof of Theorem 2*

According to the definition of B_n in (7), we get

$$\frac{\partial}{\partial z} B_n(z, w_n) = (\Phi_n^*)'(z)\overline{\Phi_n^*(w_n)} - \Phi_n'(z)\overline{\Phi_n(w_n)},$$

and

$$\frac{\frac{\partial}{\partial z} B_n(z, w_n)}{B_n(z, w_n)} = \frac{(\Phi_n^*)'(z)}{\Phi_n^*(z)} \frac{1 - \frac{\Phi_n'(z)}{(\Phi_n^*)'(z)} \overline{\left(\frac{\Phi_n(w_n)}{\Phi_n^*(w_n)}\right)}}{1 - \frac{\Phi_n(z)}{\Phi_n^*(z)} \overline{\left(\frac{\Phi_n(w_n)}{\Phi_n^*(w_n)}\right)}}.$$

Because of $\frac{\Phi_n(w_n)}{\Phi_n^*(w_n)}$ has modulus 1, by the Szegő theorem (see (4)) and Lemma 3, we obtain

$$\lim_n \frac{\frac{\partial}{\partial z} B_n(z,w_n)}{B_n(z,w_n)} = \frac{(S(z)^{-1})'}{S(z)^{-1}}, \tag{14}$$

uniformly on compact subsets of $\mathbb{D}$. Taking $(S(z)^{-1})' \not\equiv 0$ into account, we have that the number of critical points of B_n is uniformly bounded in n in each closed set of $\{z \in \mathbb{C} : |z| < 1\}$.

Thus, a partial limit ν of $\nu_{B'_n(z,w_n)}$ is a probability measure supported in $\{z \in \mathbb{C} : |z| = 1\}$. For $|z| > 1$, we have

$$\begin{aligned} 0 = \lim_n \frac{\partial}{\partial z} \frac{\frac{\partial}{\partial z} B_n(z,w_n)}{n^2 B_n(z,w_n)} &= \lim_n \left(\frac{\frac{\partial^2}{\partial z^2} B_n(z,w_n)}{n^2 B_n(z,w_n)} - \frac{(\frac{\partial}{\partial z} B_n(z,w_n))^2}{n^2 B_n^2(z,w_n)} \right) \\ &= \lim_n \left(\frac{\frac{\partial^2}{\partial z^2} B_n(z,w_n)}{n \frac{\partial}{\partial z} B_n(z,w_n)} \frac{\frac{\partial}{\partial z} B_n(z,w_n)}{n B_n(z,w_n)} - \left(\frac{(\frac{\partial}{\partial z} B_n(z,w_n))}{n B_n(z,w_n)} \right)^2 \right), \end{aligned}$$

which yields according to zero distribution of paraorthogonal polynomials (see [5])

$$\lim_n \frac{\frac{\partial^2}{\partial z^2} B_n(z,w_n)}{n \frac{\partial}{\partial z} B_n(z,w_n)} = \lim_n \frac{\frac{\partial}{\partial z} B_n(z,w_n)}{n B_n(z,w_n)} = \frac{1}{z} = \int \frac{d\nu(t)}{z-t}$$

in $\{z \in \mathbb{C} : |z| > 1\}$.

Therefore, any partial limit ν of $\{\nu_{B'_n}\}$ is the normalized Lebesgue measure m_1 in $\{z \in \mathbb{C} : |z| = 1\}$.

Remark. If the Szegő function is constant, then

$$\lim_n \nu_{B^{(k)}_{n+1}} = m_0, \quad k \geq 1.$$

4. Critical points for OP with respect to a regular measure

In this section we consider a measure μ regular in the sense of Stahl and Totik [17, p. 61], which has compact connected support, $\mathrm{supp}(\mu)$, with positive logarithm capacity, $\mathrm{cap}(\mathrm{supp}(\mu)) > 0$, so $\mathrm{supp}(\mu)$ is a continuum with more than a single point, and the interior of the polynomial convex hull of its support is the empty set, $\mathrm{int}(\mathrm{Pc}(\mathrm{supp}(\mu))) = \emptyset$. Let $\{p_n\}_{n\geq 0}$ denote the sequence of orthonormal polynomials with respect to μ. Then, if ω_μ is the equilibrium measure of $\mathrm{supp}(\mu)$ it is known that

$$\lim_n \nu_{p_n} = \omega_\mu \tag{15}$$

weakly (see [17, Theorem 3.1.4]). Thus, we have

$$\lim_n \frac{p_n'(z)}{n p_n(z)} = \int \frac{d\omega_\mu(\zeta)}{z-\zeta}, \tag{16}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{Co}(\mathrm{supp}(\mu))$, where $\mathrm{Co}(\mathrm{supp}(\mu))$ is the convex hull of $\mathrm{supp}(\mu)$ (see [17, Theorem 2.1.1]). Actually, the convergence in (16) is in $\overline{\mathbb{C}} \setminus (\mathrm{supp}(\mu) \cup \mathbb{B})$. The set $\mathbb{B}$ is defined by

$$\mathbb{B} := \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n,\, k\in\Lambda}^{\infty} \{z \in \mathbb{C} : p_n(z) = 0\}}.$$

Under above conditions on the measure this set only has possible accumulation points in $\mathrm{supp}(\mu)$.

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If ϕ_μ is the Riemann conformal transform of $\overline{\mathbb{C}} \setminus \operatorname{supp}(\mu)$ onto $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, with $\phi_\mu(\infty) = \infty$, and $\lim_{z\to\infty} \phi_\mu(z)/z > 0$, we know (see [12, p. 53 and p. 109])

$$\int \log|z-\zeta|\, d\omega_\mu(\zeta) = \log|\phi_\mu(z)| + \log \operatorname{cap}(\operatorname{supp}(\mu)), \quad z \in \overline{\mathbb{C}} \setminus \operatorname{supp}(\mu).$$

Hence, fixing any branch of logarithm functions in the corresponding regions, there exists a constant $C \in \mathbb{R}$ such that

$$\int \log(z-\zeta)\, d\omega_\mu(\zeta) = \log \phi_\mu(z) + \log \operatorname{cap}(\operatorname{supp}(\mu)) + iC, \quad z \in \overline{\mathbb{C}} \setminus \operatorname{supp}(\mu). \tag{17}$$

Combining (16) and (17), we obtain

$$\lim_n \frac{p_n'(z)}{np_n(z)} = \frac{\phi_\mu'(z)}{\phi_\mu(z)}, \tag{18}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus (\operatorname{supp}(\mu) \cup \mathbb{B})$. Since $\phi_\mu(z)$ is a conformal mapping, $\phi_\mu'(z) \neq 0$. This fact, (18) and the argument principle yield near to each point of $\mathbb{B}$ there is a critical point of p_n. Taking derivative in (18) we get

$$\lim_n \left(\frac{p_n''(z)}{np_n'(z)} - \frac{p_n'(z)}{np_n(z)} \right) = 0,$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus (\operatorname{supp}(\mu) \cup \mathbb{B})$. Therefore, from the former formula and (16), if we use induction in k, we get the following result.

Theorem 3. *Let μ be a regular measure in the sense of Stahl and Totik, its support is a continuum with more than a single point and* $\operatorname{int}(\operatorname{Pc}(\operatorname{supp}(\mu))) = \emptyset$. *Then for each $k \geq 0$ we have*

$$\lim_n \nu_{p_n^{(k)}} = \omega_\mu.$$

For n large enough near to each point of $\mathbb{B}$ there is a critical point of p_n.

4.1. An example: Akhiezer–Chebyshev's polynomials

Let Δ_α denote the arc $\Delta_\alpha := \{z \in \mathbb{C} : z = e^{i\theta},\ \alpha < \theta < 2\pi - \alpha\}$, with $0 < \alpha < \pi$, and define

$$W(z) := \frac{\sin(\alpha/2)}{2\sin(\theta/2)\sqrt{|z-e^{i\alpha}|\,|z-e^{-i\alpha}|}}, \quad z \in \Delta_\alpha.$$

Let φ_n be Akhiezer–Chebyshev's polynomials orthogonal with respect to W in Δ_α; i.e.

$$K\varphi_n(z) := \frac{w^n(v)}{1-\beta v} + \frac{v\, w^n(1/v)}{v-\beta}, \quad n \geq 1, \tag{19}$$

where $K = \sqrt{(1+\sin(\alpha/2))/(2\sin(\alpha/2))}$, $z = w(v)\, w(\frac{1}{v}) =: h(v)$, with $w(v) := i(1-\beta v)/(v+\beta)$, $\beta := i\tan((\pi-\alpha)/4)$ (see [4, Equation (22) there]).

According to Theorem 3 and since $\frac{1}{2\pi}\frac{\sin(\theta/2)\, d\theta}{\sqrt{\cos^2(\alpha/2)-\cos^2(\theta/2)}}$, $e^{i\theta} \in \Delta_\alpha$, is the equilibrium measure on the arc Δ_α (see [10]), we have

$$\lim_n \nu_{\varphi_n^{(k)}} = \frac{1}{2\pi}\frac{\sin(\theta/2)d\theta}{\sqrt{\cos^2(\alpha/2)-\cos^2(\theta/2)}}, \tag{20}$$

and

$$\begin{aligned}\lim_n \frac{\varphi_n'(z)}{n\varphi_n(z)} = \lim_n \frac{\varphi_n''(z)}{n\varphi_n'(z)} = \frac{1}{2\pi}\int_{\alpha}^{2\pi-\alpha} \frac{\sin(\theta/2)d\theta}{(z-e^{i\theta})\sqrt{\cos^2(\alpha/2)-\cos^2(\theta/2)}} \\ = \frac{1}{2z} + \frac{z-1}{2z\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}},\end{aligned} \tag{21}$$

uniformly on compact subset of $\mathbb{C}\setminus\Delta_\alpha$.

But the above relations can be also proved using differential properties of the polynomials φ_n as we see below.

Proposition 1. *The Akhiezer–Chebyshev polynomials have the following differential properties:*

1.

$$(z-1)(z-e^{i\alpha})(z-e^{-i\alpha})\varphi_n'(z) = C_n(z)\varphi_n(z) - D_n(z)\varphi_{n-1}(z), \quad n \geq 2, \tag{22}$$

where

$$\begin{aligned}C_n(z) &= nz^2 - ((1+\cos\alpha)n + \sin(\alpha/2) - 1)z + (n-1)\cos\alpha - \sin(\alpha/2),\\ D_n(z) &= \cos(\alpha/2)(nz^2 + (1-2\sin(\alpha/2)-2n)z + n - 1).\end{aligned}$$

2.

$$\alpha(z,n)\varphi_n'' - \beta(z,n)\varphi_n' + \sin(\alpha/2)\gamma(z,n)\varphi_n = 0, \quad n \geq 2, \tag{23}$$

where

$$\begin{aligned}\alpha(z,n) &= z(z-e^{i\alpha})(z-e^{-i\alpha})B_n(z),\\ \beta(z,n) &= \beta_0(z) + \beta_1(z)n + \beta_2(z)n^2,\\ \gamma(z,n) &= \gamma_0(z) + \gamma_1(z)n + \gamma_2(z)n^2 + \gamma_3(z)n^3,\end{aligned}$$

with

$$\begin{aligned}B_n(z) &= (z-1)^2 n + (1-2\sin(\alpha/2))z - 1,\\ \beta_0(z) &= 2 + z(-1+(3-2z)z) - 5z\cos\alpha\\ &\quad + 3z^2\cos\alpha + 2z\sin(\alpha/2)(1-3z\cos\alpha+2z^2),\\ \beta_1(z) &= (1-z)(-3+(z-4)z^2-(z-7)z\cos\alpha) - 2z\sin(\alpha/2)(1-2z\cos\alpha+z^2),\\ \beta_2(z) &= (z-1)^2(z-e^{i\alpha})(z-e^{-i\alpha}),\\ \gamma_0(z) &= 2(1-\sin(\alpha/2)), \quad \gamma_3(z) = (z-1)^2\sin(\alpha/2),\\ \gamma_1(z) &= -3 + z(3+z) - z\cos\alpha - (z-5)\sin(\alpha/2),\\ \gamma_2(z) &= 1 + (z-3)z + z\cos\alpha + \sin(\alpha/2)(z-1)(z+4).\end{aligned}$$

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Proof.

1. To prove (22) we take the derivative of $\varphi_n(z)$ in (19) and use the above expressions of $w(v)$ and $w(1/v)$; we obtain

$$K\frac{d\varphi_n}{dz}(z) = i\frac{(1+\beta v)^2(\beta(v+\beta)-n(1+\beta^2))}{2\beta(1+\beta^2)(1-\beta v)(v^2-1)}w^{n-1}(v) \\ + i\frac{(v+\beta)^2(-\beta(1+\beta v)+n(1+\beta^2)v)}{2\beta(1+\beta^2)(v-\beta)(v^2-1)}w^{n-1}(1/v).$$

Since $\varphi_n(z)$ and $\varphi_{n-1}(z)$ can be also written in terms of $w^{n-1}(v)$ and $w^{n-1}(1/v)$ (see (19)), to get $C_n(z)$ and $D_n(z)$ in (22), we have to solve the system of linear equations

$$\begin{pmatrix} i\frac{\beta(v+\beta)-n(1+\beta^2)}{(v+\beta)^2} \\ i\frac{-\beta(1+\beta v)+n(1+\beta^2)v}{(1+\beta v)^2} \end{pmatrix} = \begin{pmatrix} w(v) & 1 \\ v\,w(1/v) & v \end{pmatrix}\begin{pmatrix} a_n(v) \\ b_n(v) \end{pmatrix},$$

and

$$\frac{C_n(z)}{(z-1)(z-e^{i\alpha})(z-e^{-i\alpha})} = \frac{(v+\beta)^2(\beta v+1)^2}{2\beta(\beta^2+1)(v^2-1)}a_n(v), \\ \frac{-D_n(z)}{(z-1)(z-e^{i\alpha})(z-e^{-i\alpha})} = \frac{(v+\beta)^2(\beta v+1)^2}{2\beta(\beta^2+1)(v^2-1)}b_n(v). \tag{24}$$

It yields

$$a_n(v) = \frac{1}{v^2-1}\left(-\frac{\beta}{1+\beta^2}(\beta(v+\frac{1}{v})+2) + n\frac{(1+v\beta)^2+(v+\beta)^2}{(v+\beta)(1+\beta v)}\right), \tag{25}$$

and

$$b_n(v) = \frac{i}{v^2-1}\left(\frac{\beta}{1+\beta^2}(v+\frac{1}{v}-2\beta) + 2n(\beta^2-1)\frac{v}{(v+\beta)(1+\beta v)}\right). \tag{26}$$

From the expression $z = h(v)$ we get

$$1-z = \frac{2(1+\beta^2)v}{(v+\beta)(1+\beta v)}, \quad v+\frac{1}{v} = \frac{1+\beta^2}{\beta}\frac{1+z}{1-z},$$

and

$$\left(v-\frac{1}{v}\right)^2 = \frac{(1-\beta^2)^2(z-e^{i\alpha})(z-e^{-i\alpha})}{\beta^2(z-1)^2}.$$

In the end, taking these last expressions in (24), (25), (26), and doing some more calculations, we get the expressions for $C_n(z)$ and $D_n(z)$ aforementioned in (22).

2. To prove the Lamé-type differential equation in (23), we take derivative in (22), then it follows that

$$f(z)\varphi_n''(z) + (f'(z)-C_n(z))\varphi_n'(z) \\ - C_n'(z)\varphi_n(z) + D_n(z)\varphi_{n-1}'(z) + D_n'(z)\varphi_{n-1}(z) = 0,$$

where $f(z) := (z-1)(z^2-2\cos\alpha\, z+1) = (z-1)(z-e^{i\alpha})(z-e^{-i\alpha})$.

Now $\varphi'_{n-1}(z)$ can be removed from the above equation. For this end we multiply the above equation by $f(z)$ and use again (22) for $\varphi'_{n-1}(z)$. Then we have

$$\begin{aligned}&f(z)^2\varphi''_n(z)+f(z)(f'(z)-C_n(z))\varphi'_n(z)-f(z)C'_n(z)\varphi_n(z)\\&+(D_n(z)C_{n-1}(z)+f(z)D'_n(z))\varphi_{n-1}(z)-D_{n-1}(z)D_n(z)\varphi_{n-2}(z)=0.\end{aligned}$$

The next step is to remove $\varphi_{n-2}(z)$ from this equation. By the recurrence relation

$$z\varphi_{n-2}(z)=\frac{z+1}{\cos(\alpha/2)}\varphi_{n-1}(z)-\varphi_n(z),$$

we obtain

$$\begin{aligned}&zf(z)^2D_n\varphi''_n(z)+zf(z)(f'(z)-C_n(z))D_n(z)\varphi'_n(z)\\&\quad+(-zf(z)\,C'_n(z)+D_{n-1}(z)D_n(z))D_n(z)\varphi_n(z)\\&\quad+(z\,D_n(z)C_{n-1}(z)-D_{n-1}(z)D_n(z)\frac{z+1}{\cos(\alpha/2)}\\&\qquad+zf(z)D'_n(z))D_n(z)\varphi_{n-1}(z)=0.\end{aligned}$$

Finally $\varphi_{n-1}(z)$ can be removed from the above equation using again (22). It turns out

$$\begin{aligned}&zf(z)^2D_n(z)\varphi''_n(z)\\&\quad+(zf(z)(f'(z)-C_n(z))D_n(z)-f(z)(z\,D_n(z)C_{n-1}(z)\\&\qquad-D_{n-1}(z)D_n(z)\frac{z+1}{\cos(\alpha/2)}+zf(z)D'_n(z))\varphi'_n(z)\\&\quad+((-zf(z)\,C'_n(z)+D_{n-1}(z)D_n(z))D_n(z)+C_n(z)(z\,D_n(z)C_{n-1}(z)\\&\qquad-D_{n-1}(z)D_n(z)\frac{z+1}{\cos(\alpha/2)}+zf(z)D'_n(z)))\varphi_n(z)=0.\end{aligned}$$

The final expression for the coefficients in (23) have been obtained analytically with quite lengthy calculations. Furthermore, we have also checked these equations with numerical and symbolical calculations using Mathematica. □

Now we go back to prove the results in (21). Dividing (22) by $n\varphi_n$ and taking limit as n tends to infinity, we get (21) for $\lim_{n\to\infty}\frac{\varphi'_n}{n\varphi_n}$. Here we use the relation

$$\lim_{n\to\infty}\frac{\varphi_{n-1}(z)}{\varphi_n(z)}=\frac{1}{w(v)}=\frac{2\cos(\alpha/2)}{z+1+\sqrt{(z+1)^2-4z\cos^2(\alpha/2)}},\quad z=h(v),\ |v|<1,$$

which follows immediately from (19).

On the other hand, dividing (23) by $n^2\varphi'_n$, and taking limit as $n\to\infty$ we obtain (21) for $\lim_{n\to\infty}\frac{\varphi''_n}{n\varphi'_n}$.

4.2. Pictures

In Fig. 1 we show a picture of zeros of OPUC for $\Phi_n(0)=\left(\frac{1}{2}\right)^n$ and $n=50$. Observe that there exists a Nevai–Totik point. Around it we find a zero and a critical point of the OPUC.

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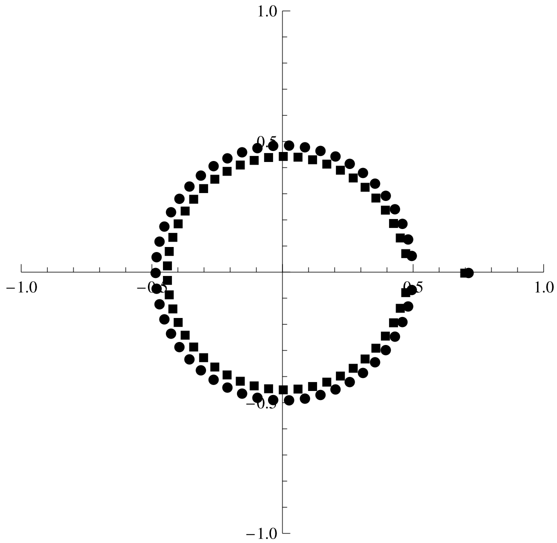

Fig. 1. Zeros of OPUC with disks and their critical points with squares for $\Phi_n(0) = \left(\frac{1}{2}\right)^n$, $n = 50$. Observe that there exists a Nevai–Totik point. Around it there is a zero and a critical point of the OPUC.

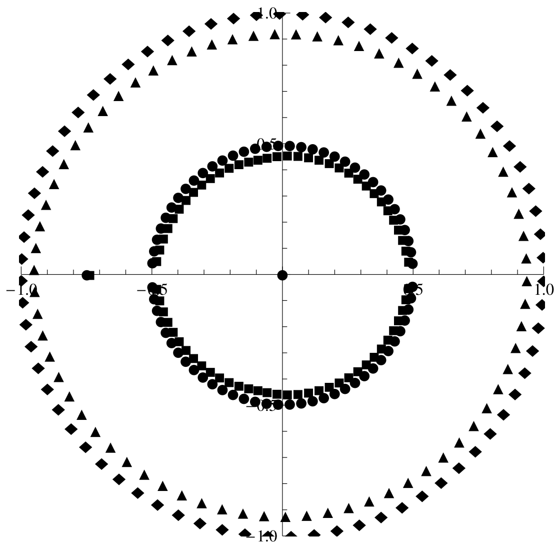

Fig. 2. Zeros of paraorthogonal polynomials B_n are plotted with diamonds and those of OPUC φ_n with disks; with squares, the zeros of φ_n' and with triangles those of B_n'. We use Verblusnky's coefficients $(-1/2)^{n+1} + (1/2)^{n+1}$, $n = 70$, and parameters $w_n = i$.

In Fig. 2, the zeros of paraorthogonal polynomials B_n are plotted with diamonds and those of OPUC φ_n with disks; with squares, the zeros of φ_n' and with triangles those of B_n'. We use Verblusnky's coefficients $(-1/2)^{n+1} + (1/2)^{n+1}$, $n = 70$, and parameters $w_n = i$ (see (2)).

In Fig. 3, the zeros of Akhiezer–Chebyshev's polynomial φ_n are plotted with disks; with squares those of φ_n'; with diamonds the zeros of φ_n'' for $\alpha = \pi/4$, $n = 50$.

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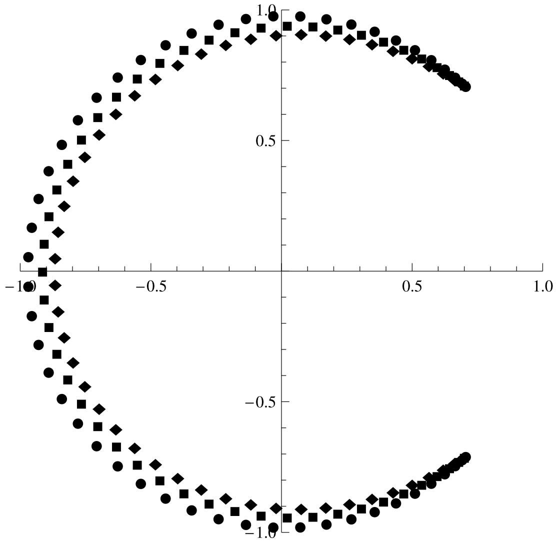

Fig. 3. Zeros of Akhiezer–Chebyshev's polynomials φ_n with disks; with squares, zeros of φ'_n; with diamond, zeros of φ''_n for $\alpha = \pi/4$, $n = 50$.

References

[1] D. Barrios Rolanía, G. López Lagomasino, E.B. Saff, Asymptotics of orthogonal polynomials inside the unit circle and Szegő–Padé approximants, J. Comput. Appl. Math. 133 (1–2) (2001) 171–181.
[2] B. de la Calle Ysern, The Jentzsch–Szegő theorem and balayage measures, Constr. Approx. 40 (2014) 307–327.
[3] J. Dégot, Sendov conjecture for high degree polynomials, Proc. Amer. Math. Soc. 142 (4) (2014) 1337–1349.
[4] L. Golinskii, Akhiezer's orthogonal polynomials and Bernstein–Szegő method for a circular arc, J. Approx. Theory 95 (2) (1998) 229–263.
[5] L. Golinskii, Quadrature formula and zeros of para-orthogonal polynomials on the unit circle, Acta Math. Hungar. 96 (3) (2002) 169–186.
[6] M. Marden, Geometry of Polynomials, second edition, Math. Surveys, vol. 3, AMS, Providence, R.I., 1966.
[7] A. Martínez Finkelshtein, K.T.-R. McLaughlin, E.B. Saff, Szegő orthogonal polynomials with respect to an analytic weight: canonical representation and strong asymptotics, Constr. Approx. 24 (3) (2006) 319–363.
[8] H.N. Mhaskar, E.B. Saff, On the distribution of zeros of polynomials orthogonal on the unit circle, J. Approx. Theory 63 (1) (1990) 30–38.
[9] P. Nevai, V. Totik, Orthogonal polynomials and their zeros, Acta Sci. Math. (Szeged) 53 (1–2) (1989) 99–104.
[10] F. Peherstorfer, R. Steinbauer, Asymptotic behaviour of orthogonal polynomials on the unit circle with asymptotically periodic reflection coefficients. II. Weak asymptotics, J. Approx. Theory 105 (1) (2000) 102–128.
[11] Q.I. Rahman, G. Schmeisser, Analytic Theory of Polynomials, Oxford University Press, 2002.
[12] E.B. Saff, V. Totik, Logarithmic Potential with External Fields, Springer-Verlag, Berlin, 1997.
[13] B. Simon, Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, Amer. Math. Soc. Colloq. Publ., vol. 54, AMS, Providence, RI, 2005.
[14] B. Simon, Fine structure of the zeros of orthogonal polynomials. I. A tale of two pictures, Electron. Trans. Numer. Anal. 25 (2006) 328–368.
[15] B. Simon, Fine structure of the zeros of orthogonal polynomials. II. OPUC with competing exponential decay, J. Approx. Theory 135 (1) (2005) 125–139.
[16] B. Simon, Fine structure of the zeros of orthogonal polynomials. III. Periodic recursion coefficients, Comm. Pure Appl. Math. 59 (7) (2006) 1042–1062.
[17] H. Stahl, V. Totik, General Orthogonal Polynomials, Encyclopedia Math. Appl., vol. 43, Cambridge University Press, Cambridge, 1992.
[18] J. Szabados, On some problems connected with polynomials orthogonal on the complex unit circle, Acta Math. Acad. Sci. Hung. 33 (1–2) (1979) 197–210.
[19] V. Totik, The Gauss–Lucas theorem in an asymptotic sense, Bull. Lond. Math. Soc. 48 (5) (2016) 848–854.

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