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Existence and nonexistence of solutions to elliptic equations involving the Hardy potential

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ABSTRACT

The purpose of this paper is to study the nonexistence of nonnegative super solutions to the problem

$$(-\Delta)^\alpha u + \frac{\mu}{|x|^{2\alpha}} u \geq Qu^p \quad \text{in } \mathbb{R}^N \setminus \mathcal{K}, \quad (0.1)$$

where $\alpha \in (0, 1]$, $\mu \in \mathbb{R}$, $p > 0$, \mathcal{K} is a compact set in \mathbb{R}^N with $N \geq 1$ and Q is a potential in $\mathbb{R}^N \setminus \mathcal{K}$ satisfying that $\liminf_{|x| \rightarrow +\infty} Q(x)|x|^\gamma > 0$ for some $\gamma < 2\alpha$. When $\alpha = 1$, $(-\Delta)^\alpha$ is the Laplacian operator, and when $\alpha \in (0, 1)$, it is the fractional Laplacian which is a typical nonlocal operator. In this paper, we find the critical exponent $p^* > 1$ depending on α , μ and γ such that problem (0.1) has no nontrivial nonnegative super solutions for $0 < p < p^*$. Furthermore, we also consider the existence and nonexistence of isolated singular solutions to the equation

$$\begin{cases} (-\Delta)^\alpha u + \frac{\mu}{|x|^{2\alpha}} u = Qu^p & \text{in } \mathbb{R}^N \setminus \{0\}, \\ \lim_{|x| \rightarrow +\infty} u(x) = 0, \end{cases}$$

where $\mu > 0$, $p > 0$ and $Q(x) = (1 + |x|)^{-\gamma}$ with $\gamma \in (0, 2\alpha)$.

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1. Introduction

We are concerned with the nonexistence of nontrivial nonnegative super solutions to the problem

$$(-\Delta)^\alpha u + \frac{\mu}{|x|^{2\alpha}} u \geq Qu^p \quad \text{in } \mathbb{R}^N \setminus \mathcal{K}, \quad (1.1)$$

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where $\alpha \in (0, 1]$, $\mu \in \mathbb{R}$, $p > 0$, \mathcal{K} is a compact set in \mathbb{R}^N with $N \geq 1$ and Q is a potential in $\mathbb{R}^N \setminus \mathcal{K}$ satisfying that $\liminf_{|x| \rightarrow +\infty} Q(x)|x|^\gamma > 0$ for some $\gamma < 2\alpha$. When $\alpha = 1$, the operator $(-\Delta)^\alpha$ is the Laplacian, and when $\alpha \in (0, 1)$, it is the fractional Laplacian defined in the principle value sense as

$$(-\Delta)^\alpha u(x) = c_{N,\alpha} \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(0)} \frac{u(x) - u(x+z)}{|z|^{N+2\alpha}} dz,$$

where $B_\epsilon(0)$ is the ball centered at the origin with radius ϵ , $c_{N,\alpha}$ is the normalized constant

$$c_{N,\alpha} = 2^{2\alpha} \alpha \pi^{-\frac{N}{2}} \frac{\Gamma(\frac{N+2\alpha}{2})}{\Gamma(1-\alpha)}$$

and Γ is the Gamma function. The fractional Laplacian is a nonlocal operator, so if Lebesgue measure $|\mathcal{K}| \neq 0$, we have to assume moreover that $u \geq 0$ a.e. in \mathcal{K} . The semilinear elliptic equations involving the fractional Laplacian and the related Sobolev spaces have been studied extensively, see [1,5,10–12,20,21] and the references therein.

It is known that the fundamental solution and Comparison Principle play an important role in the obtention of the nonexistence of solutions to semilinear elliptic equations. In the Laplacian case, the authors in [2,6] used the fundamental solution of Laplacian and Comparison Principle to obtain the nonexistence of positive solutions to the problem

$$-\Delta u = Qu^p \quad \text{in } \mathbb{R}^N \setminus \mathcal{K}.$$

In the fractional case, i.e. $\alpha \in (0, 1)$, [14] shows the nonexistence results of (1.1) when $\mu = 0$, $\mathcal{K} = \emptyset$, $Q = 1$ and $p \leq \frac{N}{N-2\alpha}$, by using the fundamental solution of the fractional Laplacian and Comparison Principle.

To study the nonexistence of nonnegative nontrivial super solutions of (1.1), we first clarify the fundamental solution of $(-\Delta)^\alpha + \frac{\mu}{|x|^{2\alpha}}$ as follows.

Proposition 1.1. Assume that $\alpha \in (0, 1]$ and $N \in \mathbb{N}$.

(i) When $N > 2\alpha$, let us denote

$$\bar{\tau} = -\frac{N-2\alpha}{2} \quad \text{and} \quad \mu_0 = \begin{cases} -\frac{(N-2)^2}{4} & \text{if } \alpha = 1, \\ -2^{2\alpha-1} c_{N,\alpha} \frac{\Gamma^2(\frac{N+2\alpha}{4})}{\Gamma^2(\frac{N-2\alpha}{4})} & \text{if } \alpha \in (0, 1), \end{cases} \quad (1.2)$$

then

$$(-\Delta)^\alpha |x|^{\bar{\tau}} + \mu_0 |x|^{\bar{\tau}-2\alpha} = 0, \quad \forall x \in \mathbb{R}^N \setminus \{0\}.$$

For $\mu > \mu_0$, there exists a unique $\tau_\alpha(\mu) \in (-N, \bar{\tau})$ such that

$$\phi_{\tau_\alpha(\mu)}(x) := |x|^{\tau_\alpha(\mu)} \quad (1.3)$$

is a fundamental solution of $(-\Delta)^\alpha + \frac{\mu}{|x|^{2\alpha}}$, i.e.

$$(-\Delta)^\alpha \phi_{\tau_\alpha(\mu)} + \frac{\mu}{|x|^{2\alpha}} \phi_{\tau_\alpha(\mu)} = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}. \quad (1.4)$$

(ii) When $N \leq 2\alpha$, then for $\mu > 0$, there exists a unique

$$\tau_\alpha(\mu) \in \begin{cases} (-\infty, 0) & \text{if } \alpha = 1, \\ (-N, 0) & \text{if } \alpha \in (0, 1) \end{cases}$$

such that $\phi_{\tau_\alpha(\mu)}$ defined in (1.3) is a fundamental solution of $(-\Delta)^\alpha + \frac{\mu}{|x|^{2\alpha}}$.

Furthermore, the mapping $\mu \mapsto \tau_\alpha(\mu)$ is strictly decreasing in $(\mu_0, +\infty)$ if $N > 2\alpha$ and in $(0, +\infty)$ if $N \leq 2\alpha$, and

$$\lim_{\mu \rightarrow +\infty} \tau_\alpha(\mu) = \begin{cases} -\infty & \text{if } \alpha = 1, \\ -N & \text{if } \alpha \in (0, 1). \end{cases}$$

Note that for $\alpha = 1$, $\tau_\alpha(\mu)$ has the explicit formula

$$\tau_\alpha(\mu) = -\frac{(N-2)_+ + \sqrt{[(N-2)_+]^2 + 4\mu}}{2}.$$

For $\alpha \in (0, 1)$ and $N > 2\alpha$, denote $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^N$,

$$c_\alpha(\tau) = -\frac{c_{N,\alpha}}{2} \int_{\mathbb{R}^N} \frac{|x - e_1|^\tau + |x + e_1|^\tau - 2}{|x|^{N+2\alpha}} dx, \quad (1.5)$$

then the function $c_\alpha(\cdot) + \mu$ has two zero points for $\mu \in (\mu_0, 0)$, one zero point when $\mu = \mu_0$, and $\tau_\alpha(\mu)$ is the smaller zero point. μ_0 is the best constant of the fractional Hardy inequalities, see the references [4,15]. Moreover, Lemma 3.1 in [13] provides an explicit expression of $c_\alpha(\tau)$,

$$c_\alpha(\tau) = 2^{2\alpha} \frac{\Gamma(\frac{N+\tau}{2})\Gamma(\frac{2\alpha-\tau}{2})}{\Gamma(\frac{-\tau}{2})\Gamma(\frac{N+\tau-2\alpha}{2})}. \quad (1.6)$$

Then the nonexistence of super solutions to (1.1) states as follows.

Theorem 1.1. Suppose that $\alpha \in (0, 1]$, $N \in \mathbb{N}$ and Q is a nonnegative function satisfying

$$\liminf_{|x| \rightarrow +\infty} Q(x)|x|^\gamma > 0$$

for some $\gamma < 2\alpha$. Then problem (1.1) has no nontrivial nonnegative super solution for $0 < p < p_{\mu,\gamma}^*$, where

$$p_{\mu,\gamma}^* = \begin{cases} 1 + \frac{2\alpha-\gamma}{-\tau_\alpha(\mu)} & \text{if } \mu > 0, \\ 1 + \frac{2\alpha-\gamma}{N-2\alpha} & \text{if } \mu \leq 0 \text{ and } N > 2\alpha, \\ +\infty & \text{if } \mu \leq 0 \text{ and } N \leq 2\alpha. \end{cases} \quad (1.7)$$

We remark that if $\mu > 0$, the mapping $\mu \mapsto \frac{2\alpha-\gamma}{-\tau_\alpha(\mu)}$ is decreasing and

$$\lim_{\mu \rightarrow +\infty} \frac{2\alpha-\gamma}{-\tau_\alpha(\mu)} = \begin{cases} 0 & \text{if } \alpha = 1, \\ \frac{2\alpha-\gamma}{N} & \text{if } \alpha \in (0, 1). \end{cases}$$

When $N > 2\alpha$ and $\mu \in (\mu_0, 0]$, we observe that $1 + \frac{2\alpha-\gamma}{-\tau_\alpha(\mu)} > 1 + \frac{2\alpha-\gamma}{N-2\alpha}$, but we do not know the nonexistence results when $p \in \left[1 + \frac{2\alpha-\gamma}{N-2\alpha}, 1 + \frac{2\alpha-\gamma}{-\tau_\alpha(\mu)}\right)$ due to the lack of comparison principle for the operator $(-\Delta)^\alpha +$

$\frac{\mu}{|x|^{2\alpha}}$ in unbounded domains. To prove the nonexistence results, an essential tool is the Hadamard property derived by the fundamental solution and the Comparison Principle, see [2,6,14]. In this paper, we employ a new method to show the nonexistence. Our idea is to obtain an initial decay at infinity from the fundamental solution by applying the Comparison Principle, then an iterating technique is used to improve the power of the decay for $p < p_{\mu,\gamma}^*$ until it makes the solution blow up everywhere. It is worthy to point out that for $\mu \in (\mu_0, 0)$, the initial decay is not $|x|^{\tau_\alpha(\mu)}$, but $|x|^{2\alpha-N}$. This leads to the independence of the critical exponent $p_{\mu,\gamma}^*$ with the parameter μ in $(\mu_0, 0)$.

Our second aim of this paper is to show that $p_{\mu,\gamma}^*$ is sharp for the nonexistence when $\mu > 0$. To this end, we take $\mathcal{K} = \{0\}$, $0 < \gamma < 2\alpha$,

$$Q(x) = (1 + |x|)^{-\gamma}, \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \quad (1.8)$$

and to consider solutions of the problem

$$\begin{cases} (-\Delta)^\alpha u + \frac{\mu}{|x|^{2\alpha}} u = Qu^p & \text{in } \mathbb{R}^N \setminus \{0\}, \\ \lim_{|x| \rightarrow +\infty} u(x) = 0. \end{cases} \quad (1.9)$$

Theorem 1.2. *Let $\alpha \in (0, 1]$, $\mu > 0$ and the function Q satisfy (1.8) with $0 < \gamma < 2\alpha$.*

(i) *When*

$$0 < p < p_{\mu,\gamma}^*, \quad (1.10)$$

then problem (1.9) has no nontrivial nonnegative solution.

(ii) *When*

$$p_{\mu,\gamma}^* \leq p < p_{\mu,0}^*, \quad (1.11)$$

then there exists $k^ > 0$ such that for any $k \in (0, k^*)$, problem (1.9) has a positive solution u satisfying*

$$\lim_{|x| \rightarrow 0^+} u(x)|x|^{-\tau_\alpha(\mu)} = k. \quad (1.12)$$

Note that $p_{\mu,\gamma}^* < p_{\mu,0}^* = 1 + \frac{2\alpha}{-\tau_\alpha(\mu)}$. When $p < p_{\mu,\gamma}^*$, the nonexistence result in Theorem 1.2 (i) follows by Theorem 1.1 directly and for the existence, we have to mention Lions' work [18], where an equivalence is built between the elliptic problem

$$-\Delta u = u^p \quad \text{in } \Omega \setminus \{0\}, \quad u = 0 \quad \text{in } \partial\Omega \quad (1.13)$$

and the one involving the Dirac mass at the origin

$$-\Delta u = u^p + k\delta_0 \quad \text{in } \Omega, \quad (1.14)$$

where $k > 0$ and Ω is a bounded, smooth domain containing the origin. Then for $1 < p < \frac{N}{N-2\alpha}$, positive singular solutions of (1.13) could be obtained by iterating the sequence of

$$v_0 = \mathbb{G}_1[k\delta_0] \quad \text{and} \quad v_n = \mathbb{G}_1[v_{n-1}^p] \quad \text{for } n = 1, 2, \dots,$$

where $k > 0$, \mathbb{G}_1 is the Green operator of $-\Delta$. When $k > 0$ suitably small, a barrier function of (1.14) could be constructed and then the limit of $\{v_n\}_n$ is the desired solution. More general second order problems with

Hardy potentials in bounded domain considered in [3,16,17] and the singularities of semilinear fractional equations in bounded domain have been studied in [7,8].

However, for $\mu > 0$, the singularity of (1.9) could not be expressed by the Dirac mass due to the Hardy potential, i.e. the term $\frac{\mu}{|x|^{2\alpha}}u$ is no longer in $L^1_{loc}(\mathbb{R}^N)$. Another difficulty is the unbounded domain $\mathbb{R}^N \setminus \{0\}$. To overcome these difficulties, we consider the sequence $\{v_n\}_n$, the solution of equation

$$\begin{cases} (-\Delta)^\alpha v_n + \frac{\mu}{|x|^{2\alpha}} v_n = Qv_{n-1}^p & \text{in } \mathbb{R}^N \setminus \{0\}, \\ \lim_{|x| \rightarrow +\infty} v_n(x) = 0 & \text{and} \quad \lim_{|x| \rightarrow 0^+} v_n(x)|x|^{-\tau_\alpha(\mu)} = k, \end{cases}$$

where $v_0 = k|x|^{\tau_\alpha(\mu)}$, then we approximate the singular solutions of (1.9) by $\{v_n\}_n$ as $n \rightarrow \infty$. An upper bound has to be constructed to control the singularity at the origin and the decay at infinity.

The rest of the paper is organized as follows. In Section 2, we clarify the fundamental solutions of (1.4) and provide a version of Comparison Principle. Section 3 is devoted to prove the nonexistence of nontrivial nonnegative solutions to problem (1.1). Finally, we prove the existence of isolated singular solutions of (1.9) in Section 4.

2. Preliminary

This section is devoted to clarify the fundamental solution of $(-\Delta)^\alpha + \frac{\mu}{|x|^{2\alpha}}$ and the Comparison Principle. For convenience, let us denote

$$\phi_\tau(x) = |x|^\tau, \quad x \in \mathbb{R}^N \setminus \{0\}, \quad (2.1)$$

where $\tau < 0$. By direct computation, we have that

$$-\Delta \phi_\tau(x) = -\tau(\tau + N - 2)|x|^{\tau-2}, \quad \forall x \in \mathbb{R}^N \setminus \{0\},$$

and the mapping $\tau \mapsto -\tau(\tau + N - 2)$ is strictly concave in $(-\infty, 0)$ and

$$\lim_{\tau \rightarrow -\infty} [-\tau(\tau + N - 2)] = -\infty.$$

When $\alpha \in (0, 1)$, by the definition of the fractional Laplacian, we have that

$$\begin{aligned} (-\Delta)^\alpha \phi_\tau(x) &= -\frac{c_{N,\alpha}}{2} \int_{\mathbb{R}^N} \frac{|x+y|^\tau + |x-y|^\tau - 2|x|^\tau}{|y|^{N+2\alpha}} dy \\ &= -\frac{c_{N,\alpha}}{2} |x|^{\tau-2\alpha} \int_{\mathbb{R}^N} \frac{|e_x+z|^\tau + |e_x-z|^\tau - 2}{|z|^{N+2\alpha}} dz, \end{aligned}$$

where $e_x = \frac{x}{|x|}$. We know that $\int_{\mathbb{R}^N} \frac{|e_x+z|^\tau + |e_x-z|^\tau - 2}{|z|^{N+2\alpha}} dz$ is independent of x and so we may replace e_x by $e_1 = (1, 0, \dots, 0)$, combining with the formula of $c_\alpha(\tau)$ in (1.5), we have that

$$(-\Delta)^\alpha \phi_\tau(x) = c_\alpha(\tau)|x|^{\tau-2\alpha}, \quad x \in \mathbb{R}^N \setminus \{0\}. \quad (2.2)$$

From (1.6), we have the following estimates.

Lemma 2.1. Suppose that $\alpha \in (0, 1)$.

(i) If $N \leq 2\alpha$, then $c_\alpha(\tau) < 0$ for $\tau \in (-N, 0)$.

(ii) If $N > 2\alpha$, then

$$c_\alpha(\tau) \begin{cases} < 0, & \tau \in (-N, 2\alpha - N), \\ = 0, & \tau = 2\alpha - N, \\ > 0, & \tau \in (2\alpha - N, 0). \end{cases} \quad (2.3)$$

Moreover, $c_\alpha(\cdot)$ is strictly concave in $(-N, 0)$,

$$\lim_{\tau \rightarrow (-N)^+} c_\alpha(\tau) = -\infty \quad \text{and} \quad \lim_{\tau \rightarrow 0^-} c_\alpha(\tau) = 0. \quad (2.4)$$

Proof. From (1.6), we have that

$$c_\alpha(\tau) = 2^{2\alpha} \frac{\Gamma(\frac{N+\tau}{2})\Gamma(\frac{2\alpha-\tau}{2})}{\Gamma(\frac{-\tau}{2})\Gamma(\frac{N+\tau-2\alpha}{2})},$$

where $\Gamma(\frac{N+\tau}{2})$, $\Gamma(\frac{2\alpha-\tau}{2})$, $\Gamma(\frac{-\tau}{2}) > 0$, the signs of $c_\alpha(\cdot)$ are decided by the ones of $\Gamma(\frac{N+\tau-2\alpha}{2})$. When $\tau \rightarrow 0^-$, we have that $\Gamma(\frac{-\tau}{2}) \rightarrow +\infty$, the others keep bounded and when $\tau \rightarrow (-N)^+$, $\Gamma(\frac{N+\tau}{2}) \rightarrow +\infty$, the others keep bounded.

For the convexity, by directly calculus, we have that

$$c'_\alpha(\tau) = -\frac{c_{N,\alpha}}{2} \int_{\mathbb{R}^N} \frac{|e_1 - x|^\tau \log |e_1 - x| + |e_1 + x|^\tau \log |e_1 + x|}{|x|^{N+2\alpha}} dx$$

and

$$c''_\alpha(\tau) = -\frac{c_{N,\alpha}}{2} \int_{\mathbb{R}^N} \frac{|e_1 - x|^\tau (\log |e_1 - x|)^2 + |e_1 + x|^\tau (\log |e_1 + x|)^2}{|x|^{N+2\alpha}} dx < 0.$$

The proof ends. \square

Proposition 2.1. Assume that $\alpha \in (0, 1)$, $N > 2\alpha$, ϕ_τ is given by (2.1) and

$$\bar{\tau} = -\frac{N - 2\alpha}{2}, \quad \mu_0 = -2^{2\alpha-1} c_{N,\alpha} \frac{\Gamma^2(\frac{N+2\alpha}{4})}{\Gamma^2(\frac{N-2\alpha}{4})}.$$

Then

$$(-\Delta)^\alpha \varphi_{\bar{\tau}}(x) + \mu_0 |x|^{\bar{\tau}-2\alpha} = 0, \quad x \in \mathbb{R}^N \setminus \{0\} \quad (2.5)$$

and

$$\mu_0 = - \sup_{\tau \in (-N, 0)} c_\alpha(\tau). \quad (2.6)$$

Proof. From Lemma 3.1 in [13], we have that (2.5) holds true. To obtain (2.6), on the one hand, for any $\tau \in (-N, 0)$ and $u \in C_c^\infty(\mathbb{R}^N)$, we have that

$$[u(x) - u(y)]^2 + u(x)^2 \frac{\phi_\tau(y) - \phi_\tau(x)}{\phi_\tau(x)} + u(y)^2 \frac{\phi_\tau(x) - \phi_\tau(y)}{\phi_\tau(y)} = \phi_\tau(x) \phi_\tau(y) \left[\frac{u(x)}{\phi_\tau(x)} - \frac{u(y)}{\phi_\tau(y)} \right]^2 \geq 0,$$

then

$$\begin{aligned}
 & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{[u(x) - u(y)]^2}{|x - y|^{N-2\alpha}} dx dy \\
 & \geq \int_{\mathbb{R}^N} \frac{u(x)^2}{\phi_\tau(x)} \left(\int_{\mathbb{R}^N} \frac{\phi_\tau(x) - \phi_\tau(y)}{|x - y|^{N+2\alpha}} dy \right) dx + \int_{\mathbb{R}^N} \frac{u(y)^2}{\phi_\tau(y)} \left(\int_{\mathbb{R}^N} \frac{\phi_\tau(y) - \phi_\tau(x)}{|x - y|^{N+2\alpha}} dx \right) dy \\
 & = \frac{2}{c_{N,\alpha}} \int_{\mathbb{R}^N} u(x)^2 \frac{(-\Delta)^\alpha \phi_\tau(x)}{\phi_\tau(x)} dx \\
 & = \frac{2c_\alpha(\tau)}{c_{N,\alpha}} \int_{\mathbb{R}^N} \frac{u(x)^2}{|x|^{2\alpha}} dx.
 \end{aligned}$$

On the other hand, Yafaev in [22] indicated that $-\frac{2}{c_{N,\alpha}}\mu_0$ is the sharp constant in the Hardy–Rellich inequality as

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}} dx dy \geq -\frac{2}{c_{N,\alpha}}\mu_0 \int_{\mathbb{R}^N} \frac{u(x)^2}{|x|^{2\alpha}} dx.$$

Then

$$\mu_0 \leq -\sup_{\tau \in (-N, 0)} c_\alpha(\tau),$$

which, combining with the fact that $c_\alpha(\bar{\tau}) = -\mu_0$, implies that $c_\alpha(\bar{\tau}) \geq \sup_{\tau \in (-N, 0)} c_\alpha(\tau)$. As a consequence, (2.6) holds true. \square

Proof of Proposition 1.1. When $\alpha = 1$, the statements in Proposition 1.1 are clear. For $\alpha \in (0, 1)$, by Proposition 2.1, we have that $c_\alpha(\bar{\tau}) = -\mu_0$ for $\bar{\tau} = -\frac{N-2\alpha}{2}$.

Since $c_\alpha(\cdot)$ is concave and strictly increasing in $(-N, \bar{\tau})$. Then for any $\mu \in [\mu_0, +\infty)$, there exists a unique $\tau_\alpha(\mu) \in (-N, \bar{\tau})$ such that

$$\mu = -c_\alpha(\tau_\alpha(\mu)).$$

The proof ends. \square

Remark 2.1. For $\mu \in (\mu_0, 0)$, there exists a unique $\varsigma_\alpha(\mu) \in (\bar{\tau}, 0)$ such that

$$c_\alpha(\varsigma_\alpha(\mu)) + \mu = 0.$$

It is obvious that $\varsigma_\alpha(\mu_0) = \tau_\alpha(\mu_0)$ and $\mu \mapsto \varsigma_\alpha(\mu)$ is increasing for $\mu \in (\mu_0, 0)$.

The following Comparison Principle plays an important role in the obtention of nonexistence results for (1.1).

Lemma 2.2. Let $\alpha \in (0, 1)$, $\mu \geq 0$, \mathcal{D} be a C^2 domain such that $0 \notin \mathcal{D}$, functions $f_1, f_2 \in C(\mathcal{D})$ satisfy that $f_2 \geq f_1$ in \mathcal{D} and $g_1, g_2 \in L^1(\mathbb{R}^N \setminus \mathcal{D}, \frac{dx}{1+|x|^{N+2\alpha}})$ satisfy that $g_2 \geq g_1$ a.e. in $\mathbb{R}^N \setminus \mathcal{D}$. Assume more that u_1 is a super solution of

$$\begin{aligned} (-\Delta)^\alpha u + \frac{\mu}{|x|^{2\alpha}} u &= f_i \quad \text{in } \mathcal{D}, \\ u &= g_i \quad \text{in } \mathbb{R}^N \setminus \mathcal{D} \end{aligned} \quad (2.7)$$

with $i = 1$ and u_2 is a sub solution of (2.7) with $i = 2$. Suppose that

$$\liminf_{x \rightarrow \partial \mathcal{D}} (u_2 - u_1)(x) \geq 0, \quad (2.8)$$

in addition, if \mathcal{D} is an unbounded domain, u_1 and u_2 satisfy that

$$\liminf_{|x| \rightarrow +\infty} (u_2 - u_1)(x) \geq 0,$$

then

$$u_2 \geq u_1 \quad \text{in } \mathcal{D}.$$

Proof. If $\inf_{x \in \mathcal{D}} (u_2 - u_1)(x) < 0$, then there exists a point $x_0 \in \mathcal{D}$ such that

$$(u_2 - u_1)(x_0) = \inf_{x \in \mathcal{D}} (u_2 - u_1)(x) = \operatorname{ess\,inf}_{x \in \mathbb{R}^N} (u_2 - u_1)(x) < 0,$$

which implies that $\frac{\mu}{|x_0|^{2\alpha}} (u_2 - u_1)(x_0) < 0$ and

$$(-\Delta)^\alpha (u_2 - u_1)(x_0) = \int_{\mathbb{R}^N} \frac{(u_2 - u_1)(x_0) - (u_2 - u_1)(z)}{|x_0 - z|^{N+2\alpha}} dz < 0.$$

Thus,

$$(-\Delta)^\alpha (u_2 - u_1)(x_0) + \frac{\mu}{|x|^{2\alpha}} (u_2 - u_1)(x_0) < 0,$$

combining with (2.7), this contradicts $f_2(x_0) - f_1(x_0) \geq 0$. \square

Remark 2.2. Lemma 2.2 holds for $\alpha = 1$ by replacing the boundary type condition in (2.7) by

$$u = g_i \quad \text{on } \partial \mathcal{D}.$$

Corollary 2.1. Let $\alpha \in (0, 1]$, $\mu \geq 0$, functions $f_1, f_2 \in C(\mathbb{R}^N \setminus \{0\})$ satisfy that $f_2 \geq f_1$ in $\mathbb{R}^N \setminus \{0\}$. Assume that u_1 is a positive super solution of

$$(-\Delta)^\alpha u + \frac{\mu}{|x|^{2\alpha}} u = f_i \quad \text{in } \mathbb{R}^N \setminus \{0\} \quad (2.9)$$

with $i = 1$ and u_2 is a positive sub solution of (2.9) with $i = 2$. Suppose that

$$\liminf_{x \neq 0, x \rightarrow 0} \frac{u_2(x)}{u_1(x)} > 1,$$

and

$$\liminf_{|x| \rightarrow +\infty} u_2(x) \geq \limsup_{|x| \rightarrow +\infty} u_1(x).$$

Then

$$u_2 \geq u_1 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Remark 2.3. If $\alpha \in (0, 1]$ and $\mu \in (\mu_0, 0)$, the above Comparison Principle fails. We give a counterexample as follows. Let $\epsilon \in (0, |\mu|)$, $u_1(x) = |x|^{\varsigma_\alpha(\mu+\epsilon)}$ and $u_2(x) = |x|^{\tau_\alpha(\mu)}$, where $\varsigma_\alpha(\mu)$ is given by Remark 2.1, we observe that

$$\liminf_{x \neq 0, x \rightarrow 0} \frac{u_2(x)}{u_1(x)} > 1,$$

$$\liminf_{|x| \rightarrow +\infty} u_2(x) \geq \limsup_{|x| \rightarrow +\infty} u_1(x),$$

and

$$(-\Delta)^\alpha u_2 + \frac{\mu}{|x|^{2\alpha}} u_2 = 0 > \frac{-\epsilon}{|x|^{2\alpha}} u_1(x) = (-\Delta)^\alpha u_1 + \frac{\mu}{|x|^{2\alpha}} u_1 \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

But it is not true that $u_2 \geq u_1$ in $\mathbb{R}^N \setminus \{0\}$.

3. Nonexistence

We prove the nonexistence of nontrivial nonnegative solutions of (1.1) by contradiction. Let u be a nonnegative nontrivial solution of problem (1.1), then we will obtain a contradiction from the decay of u at infinity. Without loss generality, we may assume that

$$\mathcal{K} \subset B_1(0) \quad \text{and} \quad Q(x) \geq q_0 |x|^{-\gamma} \quad \text{for } |x| > 4, \quad (3.1)$$

where $q_0 > 0$.

Proposition 3.1. Assume that $\alpha \in (0, 1]$, $\gamma < 2\alpha$, $p > 0$ and u is a nonnegative nontrivial solution of (1.1). Then for some $b_0 > 0$,

(i) if $\mu > 0$, we have that

$$u(x) \geq b_0 |x|^{\tau_\alpha(\mu)}, \quad \forall |x| > 4; \quad (3.2)$$

(ii) if $N > 2\alpha$ and $\mu \leq 0$, we have that

$$u(x) \geq b_0 |x|^{2\alpha-N}, \quad \forall |x| > 4; \quad (3.3)$$

(ii) if $N \leq 2\alpha$ and $\mu \leq 0$, for any $\tau < 0$, we have that

$$u(x) \geq b_0 |x|^\tau, \quad \forall |x| > 4. \quad (3.4)$$

Proof. For $\alpha = 1$, the proof follows the procedure of the case of $\alpha \in (0, 1)$. So we concentrate on the case $\alpha \in (0, 1)$. We first deal with part (i). To this end, let us denote

$$w_{t,\nu}(x) = \begin{cases} t|x|^{\tau_\alpha(\mu)} & \text{for } |x| \geq 3, \\ t(|x|^{\tau_\alpha(\mu)} + \nu) & \text{for } 2 < |x| < 3, \\ 0 & \text{for } |x| \leq 2, \end{cases} \quad (3.5)$$

where $t, \nu > 0$ will be chosen later. We observe that

$$(-\Delta)^\alpha w_{t,\nu}(x) = t(-\Delta)^\alpha w_{1,\nu}(x) \quad \text{for } |x| > 4$$

and it follows by [Proposition 1.1](#) that for $|x| > 4$,

$$\begin{aligned} (-\Delta)^\alpha w_{1,\nu}(x) + \frac{\mu}{|x|^{2\alpha}} w_{1,\nu}(x) &= c_{N,\alpha} \int_{B_2(0)} \frac{|y|^{\tau_\alpha(\mu)}}{|x-y|^{N+2\alpha}} dy - c_{N,\alpha} \nu \int_{B_3(0) \setminus \overline{B_2(0)}} \frac{1}{|x-y|^{N+2\alpha}} dy \\ &\leq 2^{-N-2\alpha} c_{N,\alpha} \left[\int_{B_2(0)} |y|^{\tau_\alpha(\mu)} dy - \nu |B_3(0) \setminus B_2(0)| \right] \\ &\leq 0, \end{aligned}$$

if we choose that $\nu = \nu_0 = \frac{\int_{B_2(0)} |y|^{\tau_\alpha(\mu)} dy}{|B_3(0) \setminus B_2(0)|}$. Since u is continuous in $\mathbb{R}^N \setminus \overline{B_1(0)}$, there exists $t_0 > 0$ such that

$$u \geq t_0 w_{1,\nu_0} = w_{t_0,\nu_0} \quad \text{in } \overline{B_4(0) \setminus B_2(0)}.$$

Observe that

$$(-\Delta)^\alpha u(x) + \frac{\mu}{|x|^{2\alpha}} u(x) \geq Q(x) u(x)^p \geq 0 \quad \text{and} \quad \limsup_{|x| \rightarrow +\infty} w_{t,\nu}(x) = 0,$$

then it follows by [Lemma 2.2](#) that $u \geq t_0 w_{1,\nu_0}$, which implies [\(3.2\)](#).

We next prove part (ii). Re-define the barrier function $w_{t,\nu}$ replacing $\tau_\alpha(\mu)$ by $2\alpha - N$. By direct computation, we have that

$$(-\Delta)^\alpha w_{1,\nu_0}(x) \leq 0 \leq \frac{-\mu}{|x|^{2\alpha}} u(x) + Q(x) u(x)^p = (-\Delta)^\alpha u(x), \quad \forall x \in \mathbb{R}^N \setminus \overline{B_4(0)},$$

where $\nu_0 = \frac{\int_{B_2(0)} |x|^{2\alpha-N} dx}{|B_3(0) \setminus B_2(0)|}$. Taking $t = t_0 > 0$ small, we have that

$$u \geq w_{t_0,\nu_0} \quad \text{in } \overline{B_4(0) \setminus B_2(0)},$$

so [\(3.3\)](#) holds true for $\mu \leq 0$.

Finally, we deal with part (iii). In this case, the power $\tau_\alpha(\mu)$ of the barrier function $w_{t,\nu}$ could be replaced by any $\tau \in (-N, 0)$ and we may have that $(-\Delta)^\alpha w_{1,\nu_0}(x) \leq 0$. We omit the rest of the proof. \square

The next step is to improve the decay of u at infinity. To this end, we introduce some notations. Let

$$\tau_0 = \begin{cases} \tau_\alpha(\mu) & \text{if } \mu > 0, \\ 2\alpha - N & \text{if } \mu \leq 0 \quad \text{and } N > 2\alpha \end{cases} \quad (3.6)$$

and $\{\tau_j\}_j$ be the sequence generated by

$$\tau_j = 2\alpha - \gamma + p\tau_{j-1} \quad \text{for } j = 1, 2, 3, \dots \quad (3.7)$$

Lemma 3.1. Assume that $\mu > 0$ with $N \geq 1$ or $\mu \leq 0$ with $N > 2\alpha$,

$$p \in \left(0, 1 + \frac{\gamma - 2\alpha}{\tau_0}\right), \quad (3.8)$$

then $\{\tau_j\}_j$ is a strictly increasing sequence and there exists $j_0 \in \mathbb{N}$ such that

$$\tau_{j_0} \geq 0 \quad \text{and} \quad \tau_{j_0-1} < 0. \quad (3.9)$$

Proof. For $p \in (0, 1 + \frac{\gamma-2\alpha}{\tau_0})$, we have that

$$\tau_1 - \tau_0 = 2\alpha - \gamma + (p-1)\tau_0 > 0$$

and

$$\tau_j - \tau_{j-1} = p(\tau_{j-1} - \tau_{j-2}) = p^{j-1}(\tau_1 - \tau_0), \quad (3.10)$$

which imply that the sequence $\{\tau_j\}_j$ is increasing. If $p \geq 1$, our conclusions are obvious. If $p \in (0, 1)$, we have that in the case that $\tau_1 \geq 0$, we are done, and in the case that $\tau_1 < 0$, we deduce from (3.10) that

$$\begin{aligned} \tau_j &= \frac{1-p^j}{1-p}(\tau_1 - \tau_0) + \tau_0 \\ &\rightarrow \frac{1}{1-p}(\tau_1 - \tau_0) + \tau_0 = \frac{2\alpha - \gamma}{1-p} > 0 \quad \text{as } j \rightarrow +\infty, \end{aligned}$$

then there exists $j_0 > 0$ satisfying (3.9). \square

Remark 3.1. We note that

$$p_{\mu,\gamma}^* = 1 + \frac{\gamma - 2\alpha}{\tau_0}.$$

From the strictly increasing monotonicity of the sequence $\{\tau_j\}_j$, we have that

$$\tau_{j-1} < \tau_j = 2\alpha - \gamma + p\tau_{j-1},$$

that is,

$$\tau_{j-1}p - \gamma > \tau_{j-1} - 2\alpha. \quad (3.11)$$

Proposition 3.2. Let τ_0 and $\{\tau_j\}_j$ be defined by (3.6) and (3.7) respectively, and u be a nonnegative solution of (1.1) satisfying

$$u(x) \geq c_j |x|^{\tau_j}, \quad \forall |x| > 4 \quad (3.12)$$

for some $c_j > 0$ and $j \leq j_0 - 2$, where j_0 is from Lemma 3.1. Then for $p \in (0, 1 + \frac{\gamma-2\alpha}{\tau_0})$, there exists $c_{j+1} > 0$ such that

$$u(x) \geq c_{j+1} |x|^{\tau_{j+1}}, \quad \forall |x| > 4. \quad (3.13)$$

Proof. Case 1. $\mu > 0$. Let

$$v_{t,\nu}(x) = \begin{cases} t|x|^{\tau_{j+1}} & \text{for } |x| \geq 3, \\ t(|x|^{\tau_{j+1}} + \nu) & \text{for } 2 < |x| < 3, \\ 0 & \text{for } |x| \leq 2, \end{cases}$$

where $t, \nu > 0$ will be chosen later. We observe that

$$(-\Delta)^\alpha v_{t,\nu}(x) = t(-\Delta)^\alpha v_{1,\nu}(x) \quad \text{for } |x| > 4$$

and it follows by [Proposition 1.1](#) that for $|x| > 4$,

$$\begin{aligned} & (-\Delta)^\alpha v_{1,\nu}(x) + \frac{\mu}{|x|^{2\alpha}} v_{1,\nu}(x) \\ &= (-\Delta)^\alpha |x|^{\tau_{j+1}} + \frac{\mu}{|x|^{2\alpha}} |x|^{\tau_{j+1}} + c_{N,\alpha} \int_{B_2(0)} \frac{|y|^{\tau_{j+1}}}{|x-y|^{N+2\alpha}} dy - c_{N,\alpha} \nu \int_{B_3(0) \setminus \overline{B_2(0)}} \frac{1}{|x-y|^{N+2\alpha}} dy \\ &\leq (c_\alpha \tau_{j+1} + \mu) |x|^{\tau_{j+1}-2\alpha} + 2^{-N-2\alpha} c_{N,\alpha} \left[\int_{B_2(0)} |x|^{\tau_{j+1}} dx - \nu |B_3(0) \setminus B_2(0)| \right] \\ &\leq (c_\alpha \tau_{j+1} + \mu) |x|^{\tau_{j+1}-2\alpha}, \end{aligned}$$

if we choose that $\nu = \nu_j = \frac{\int_{B_2(0)} |x|^{\tau_{j+1}} dx}{|B_3(0) \setminus B_2(0)|}$. From [\(3.12\)](#), we have that

$$Q(x)u(x)^p \geq q_0 c_j^p |x|^{p\tau_j - \gamma},$$

and then there exists $t_{j_1} > 0$ such that for $t \in (0, t_{j_1}]$,

$$(-\Delta)^\alpha v_{t,\nu}(x) + \frac{\mu}{|x|^{2\alpha}} v_{t,\nu}(x) \leq (-\Delta)^\alpha u(x) + \frac{\mu}{|x|^{2\alpha}} u(x) \quad \text{for } |x| > 4.$$

Furthermore, since u is continuous in $\mathbb{R}^N \setminus \overline{B_1(0)}$, there exists $t_j \leq t_{j_1}$ such that

$$u \geq v_{t_j, \nu_j} = t_j v_{1, \nu_j} \quad \text{in } \overline{B_4(0) \setminus B_2(0)}.$$

Applying [Lemma 2.2](#), we have that $u \geq t_j v_{1, \nu_j}$, which implies [\(3.13\)](#) with $c_{j+1} = t_j$.

Case 2. $\mu < 0$ with $N > 2\alpha$. Since $\tau_j > \tau_0 = 2\alpha - N$, we have that

$$(-\Delta)^\alpha v_{1, \nu_j}(x) \leq c_\alpha (\tau_{j+1}) |x|^{\tau_{j+1}-2\alpha}.$$

By [\(3.11\)](#), we have that

$$(-\Delta)^\alpha u(x) = Q(x)u(x)^p + \frac{-\mu}{|x|^{2\alpha}} u(x) \geq c_j^p |x|^{p\tau_j - \gamma}$$

and choosing $t = t_j > 0$ small, we have that

$$(-\Delta)^\alpha u \geq (-\Delta)^\alpha v_{t_j, \nu_j} \quad \text{in } \mathbb{R}^N \setminus \overline{B_4(0)}, \quad u \geq v_{t_j, \nu_j} \quad \text{in } \overline{B_4(0) \setminus B_2(0)},$$

by [Lemma 2.2](#), we have that $u \geq t_j v_{1, \nu_j}$, which implies [\(3.13\)](#) with $c_{j+1} = t_j$. \square

Proof of Theorem 1.1. By contradiction, we assume that problem [\(1.1\)](#) has a nonnegative nontrivial super solution $u \geq 0$. We first claim that $u > 0$ in $\mathbb{R}^N \setminus \overline{B_1(0)}$. For $\alpha = 1$, the positivity of u follows by the strong maximum principle. For $\alpha \in (0, 1)$, since u is nonnegative nontrivial, then

$$\left\{x \in \mathbb{R}^N \setminus \overline{B_1(0)} : u(x) > 0\right\} \neq \emptyset.$$

If there exists $x_0 \in \mathbb{R}^N \setminus \overline{B_1(0)}$ such that $u(x_0) = 0$, then

$$(-\Delta)^\alpha u(x_0) = - \int_{\mathbb{R}^N} \frac{u(z)}{|x_0 - z|^{N-2\alpha}} dz < 0,$$

which implies that

$$(-\Delta)^\alpha u(x_0) + \frac{\mu}{|x_0|^{2\alpha}} u(x_0) < 0 = Q(x_0)u(x_0)^p,$$

which is impossible from (1.1). Thus, $u > 0$ in $\mathbb{R}^N \setminus \overline{B_1(0)}$.

In the case that $\mu > 0$ or $\mu \leq 0$ with $N > 2\alpha$, from Proposition 3.1, there exists $c_0 > 0$ such that

$$u(x) \geq c_0|x|^{\tau_0}, \quad \forall |x| > 4, \quad (3.14)$$

where τ_0 is defined by (3.6). For $p \in (0, p_{\mu,\gamma}^*)$, we use Proposition 3.2 to iterate, then we obtain that for any $j \leq j_0 - 1$,

$$u(x) \geq c_j|x|^{\tau_j}, \quad \forall |x| > 4, \quad (3.15)$$

where $\{\tau_j\}_j$ is defined by (3.7) and $c_j > 0$.

In case that $\mu \leq 0$ with $N \leq 2\alpha$, (3.15) also holds when we take $\tau = \tau_{j_0-1} = -\frac{2\alpha-\gamma}{2p}$ by Proposition 3.1 (iii).

Next we use the decay of (3.15) with $j = j_0 - 1$ to derive that u blows up everywhere. From (3.15) and (3.11) with $j = j_0 - 1$, there exists $r_0 > 4$ such that for $|x| > r_0$,

$$\begin{aligned} (-\Delta)^\alpha u(x) &\geq Q(x)u(x)^p - \frac{\mu}{|x|^{2\alpha}}u(x) \\ &\geq q_0 c_{j_0-1}^p |x|^{\tau_{j_0-1}p-\gamma} - c_{j_0-1}\mu |x|^{\tau_{j_0-1}-2\alpha} \geq \frac{q_0}{2} c_{j_0-1}^p |x|^{\tau_{j_0-1}p-\gamma}. \end{aligned}$$

We note that if $\mu \leq 0$, the above inequality holds directly.

In order to obtain a contradiction, we introduce some auxiliary functions v_r with $r > 8r_0$, which is the solution of problem

$$\begin{cases} (-\Delta)^\alpha v_r(x) = f_r(x), & \forall x \in \mathbb{R}^N \setminus \overline{B_{r_0}(0)}, \\ v_r(x) = 0, & \forall x \in \overline{B_{r_0}(0)}, \\ \lim_{|x| \rightarrow +\infty} v_r(x) = 0, \end{cases} \quad (3.16)$$

where

$$f_r(x) = \frac{q_0}{2} c_{j_0-1}^p |x|^{\tau_{j_0-1}p-\gamma} \chi_{B_r(0) \setminus B_{4r_0}(0)}(x)$$

and $\chi_{B_r(0) \setminus B_{4r_0}(0)}$ is the characteristic function of the set $B_r(0) \setminus B_{4r_0}(0)$. By Lemma 2.2, for all $r > 8r_0$, we have that

$$u(x) \geq v_r(x), \quad \forall x \in \mathbb{R}^N.$$

Then for any $x \in B_{6r_0}(0) \setminus B_{4r_0}(0)$, $z \in B_r(0) \setminus B_{8r_0}(0)$, we have that $|x - z| \leq 2|z|$ and

$$\begin{aligned} u(x) &\geq v_r(x) \geq \bar{c} c_{j_0-1}^p \int_{B_r(0) \setminus B_{8r_0}(0)} \frac{|z|^{\tau_{j_0-1}p-\gamma}}{|x-z|^{N-2\alpha}} dz \\ &\geq \bar{c} \int_{B_r(0) \setminus B_{8r_0}(0)} |z|^{2\alpha-N+\tau_{j_0-1}p-\gamma} dz \\ &\geq \begin{cases} c[r^{\tau_{j_0}} - (8r_0)^{\tau_{j_0}}], & \text{if } \tau_{j_0} > 0 \\ c[\log r - \log(8r_0)] & \text{if } \tau_{j_0} = 0 \end{cases} \\ &\rightarrow +\infty \quad \text{as } r \rightarrow +\infty, \end{aligned}$$

which contradicts that u satisfies (1.1). The proof ends. \square

Proof of Theorem 1.2 (i). If problem (1.9) has a nontrivial nonnegative solution u , then u satisfies (1.1), which contradicts Theorem 1.1 with $0 < p < p_{\mu,\gamma}^*$. \square

4. Existence

In this section, we are concerned with the existence of positive singular solutions of problem (1.9) when $p_{\mu,\gamma}^* \leq p < p_{\mu,0}^*$.

Lemma 4.1. Assume that $\alpha \in (0, 1]$, $\mu > 0$, Q satisfies (1.8) with $0 < \gamma < 2\alpha$, f is a nonnegative function in $C^1(\mathbb{R}^N \setminus \{0\})$ satisfying

$$\sup_{x \in \mathbb{R}^N \setminus \{0\}} f(x)|x|^{-\tau+2\alpha} < +\infty \quad (4.1)$$

for

$$\tau \in (\tau_\alpha(\mu), \tau_\alpha(\mu) + \gamma].$$

Then problem

$$\begin{cases} (-\Delta)^\alpha u + \frac{\mu}{|x|^{2\alpha}} u = Qf & \text{in } \mathbb{R}^N \setminus \{0\}, \\ \lim_{|x| \rightarrow +\infty} u(x) = 0 \end{cases} \quad (4.2)$$

has a minimal positive solution u_f such that

$$\limsup_{|x| \rightarrow 0^+} u_f(x)|x|^{-\tau} < +\infty \quad \text{and} \quad u_f(x) \leq c|x|^{\tau_\alpha(\mu)}, \quad x \in \mathbb{R}^N \setminus \{0\}, \quad (4.3)$$

for some $c > 0$. Moreover, the mapping $f \mapsto u_f$ is increasing.

Proof. Let $\eta_0 : [0, +\infty) \rightarrow [0, 1]$ be a C^∞ nondecreasing function such that

$$\eta_0 = 1 \quad \text{in } [2, +\infty) \quad \text{and} \quad \eta_0 = 0 \quad \text{in } [0, 1],$$

denote

$$\eta_n(t) = \eta_0(nt)[1 - \eta_0(n^{-1}t)] \quad \text{for } n \in \mathbb{N}. \quad (4.4)$$

We observe that the problem

$$\begin{cases} (-\Delta)^\alpha u + \frac{\mu}{|x|^{2\alpha}} u = Qf\eta_n & \text{in } \mathbb{R}^N \setminus \{0\}, \\ \lim_{|x| \rightarrow +\infty} u(x) = 0 \end{cases} \quad (4.5)$$

has a unique solution $w_n \geq 0$ satisfying

$$\limsup_{|x| \rightarrow 0^+} w_n(x)|x|^{-\tau_\alpha(\mu)} = 0.$$

We first claim that $\{w_n\}_n$ is increasing. By contradiction, we may assume that there exist n and $x_0 \neq 0$ such that $w_{n-1}(x_0) > w_n(x_0)$. In fact, $\{f\eta_n\}_n$ is an increasing sequence, then $w := w_{n-1} - w_n$ satisfying

$$(-\Delta)^\alpha w + \frac{\mu}{|x|^{2\alpha}} w \leq 0$$

and

$$\limsup_{|x| \rightarrow 0^+} w(x)|x|^{-\tau_\alpha(\mu)} = 0,$$

then, for any $\varepsilon > 0$, by [Corollary 2.1](#), we have that

$$w(x) \leq \varepsilon |x|^{\tau_\alpha(\mu)}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

thus,

$$\varepsilon \geq \frac{w_{n-1}(x_0) - w_n(x_0)}{|x_0|^{\tau_\alpha(\mu)}},$$

which contradicts the arbitrary of $\varepsilon > 0$. Then $\{w_n\}_n$ is an increasing sequence.

Next we construct a uniform bound for $\{w_n\}_n$. By [\(4.1\)](#), there exists $c > 0$ such that

$$Q(x)f(x) \leq c|x|^{\tau-2\alpha}(1+|x|)^{-\gamma} \leq c|x|^{\tau-2\alpha}, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

We observe that

$$(-\Delta)^\alpha |x|^\tau + \frac{\mu}{|x|^{2\alpha}} |x|^\tau = (c_\alpha(\tau) + \mu)|x|^{\tau-2\alpha},$$

where $c_\alpha(\tau) + \mu > 0$ by the fact that $\tau > \tau_\alpha(\mu)$. Therefore, $\frac{c}{c_\alpha(\tau) + \mu} |x|^\tau$ is a super solution of [\(4.2\)](#). By [Corollary 2.1](#), we have that for any n ,

$$w_n(x) \leq \frac{c}{c_\alpha(\tau) + \mu} |x|^\tau, \quad x \in \mathbb{R}^N \setminus \{0\}.$$

Taking $u_f = \lim_{n \rightarrow +\infty} w_n$, for any $x_0 \in \mathbb{R}^N \setminus \{0\}$, [\[9, Lemma 3.1\]](#) implies that for some $\beta \in (0, \alpha)$

$$\|w_n\|_{C^\beta(B_{\frac{|x_0|}{4}}(x_0))} \leq c,$$

where $c > 0$ is independent of n , thus, from [19, Corollary 2.4], we have that

$$\|w_n\|_{C^{2\alpha+\beta}(B_{\frac{|x_0|}{8}}(x_0))} \leq c \left(\|w_n\|_{C^\beta(B_{\frac{|x_0|}{4}}(x_0))} + \|Qf\|_{L^\infty(B_{\frac{|x_0|}{2}}(x_0))} \right) \leq c,$$

where $C^{2\alpha+\beta}$ is the space of $C^{[2\alpha+\beta], (2\alpha+\beta)-[2\alpha+\beta]}$ and $[2\alpha+\beta]$ is integer part.

Therefore, by stability theorem [9, Theorem 2.2] (also see Theorem 2.2 in [7]), u_f is a classical solution of (4.2) and

$$u_f(x) \leq \frac{c}{c_\alpha(\tau) + \mu} |x|^\tau, \quad x \in \mathbb{R}^N \setminus \{0\}. \quad (4.6)$$

In order to get a better decay estimate of u_f at infinity, we construct a new upper barrier function. Since $\tau - \gamma \in (\tau_\alpha(\mu) - \gamma, \tau_\alpha(\mu))$, let us fix

$$\tau_\infty \in (\tau - \gamma, \tau_\alpha(\mu))$$

and denote

$$v_{t,s}(x) = t|x|^{\tau_\alpha(\mu)} - s|x|^{\tau_\infty} \chi_{\mathbb{R}^N \setminus B_1(0)}(x), \quad x \in \mathbb{R}^N \setminus \{0\},$$

where $t > s > 0$ will be chosen later. We observe that for $|x| > 2$,

$$\begin{aligned} (-\Delta)^\alpha v_{t,s}(x) + \frac{\mu}{|x|^{2\alpha}} v_{t,s}(x) &\geq -sc_{\tau_\infty} |x|^{\tau_\infty-2\alpha} + c_{N,\alpha} s(|x|+1)^{-N-2\alpha} \\ &\geq -sc_{\tau_\infty} |x|^{\tau_\infty-2\alpha}, \end{aligned}$$

where $-c_{\tau_\infty} > 0$ by the fact that $\tau_\infty < \tau_\alpha(\mu)$.

Since $\tau - \gamma - 2\alpha < \tau_\infty - 2\alpha$ and

$$Q(x)f(x) \leq c|x|^{\tau-\gamma-2\alpha},$$

so we may fix

$$s = \frac{1}{-c_{\tau_\infty}} \sup_{|x|>1} (Q(x)f(x)|x|^{-\tau_\infty+2\alpha}) < +\infty$$

and choose $t > s$ such that

$$v_{t,s}(x) \geq \frac{c}{c_\alpha(\tau) + \mu} |x|^\tau \geq u_f(x) \quad \text{for } 0 < |x| \leq 2,$$

then by Corollary 2.1, we have that

$$u_f \leq v_{t,s} \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

which implies that

$$u_f(x) \leq c|x|^{\tau_\alpha(\mu)} \quad \text{for } |x| > 2. \quad (4.7)$$

The estimates (4.6) and (4.7) imply (4.3). \square

Corollary 4.1. Assume that $\mu > 0$, p satisfies (1.11), Q satisfies (1.8) with $0 < \gamma < 2\alpha$. Then

$$\begin{cases} (-\Delta)^\alpha u + \frac{\mu}{|x|^{2\alpha}} u = Q|x|^{\tau_\alpha(\mu)p} & \text{in } \mathbb{R}^N \setminus \{0\}, \\ \lim_{|x| \rightarrow +\infty} u(x) = 0 \end{cases} \quad (4.8)$$

has a minimal positive solution v such that

$$\limsup_{|x| \rightarrow 0^+} v(x)|x|^{-\tau_\alpha(\mu)p-2\alpha} < +\infty.$$

Furthermore, we have that

$$v(x) \leq c|x|^{\tau_\alpha(\mu)} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\} \quad (4.9)$$

for some $c > 0$.

Proof. We apply Lemma 4.1 with $f(x) = |x|^{\tau_\alpha(\mu)p}$ and $\tau = \tau_\alpha(\mu)p + 2\alpha$. From (1.11), we have that

$$\tau \in (\tau_\alpha(\mu), \tau_\alpha(\mu) + \gamma]$$

and then (4.9) follows by (4.3). \square

Proof of Theorem 1.2 (ii). Here we only provide the proof when $\alpha \in (0, 1)$. For $\alpha = 1$, the proof is very similar, so we omit it.

For $k > 0$, we define

$$v_0(x) = k|x|^{\tau_\alpha(\mu)} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}$$

and

$$v_n = v_0 + w_n,$$

where w_n is the minimal positive solution of

$$\begin{cases} (-\Delta)^\alpha w_n + \frac{\mu}{|x|^{2\alpha}} w_n = Qv_{n-1}^p & \text{in } \mathbb{R}^N \setminus \{0\}, \\ \lim_{|x| \rightarrow +\infty} w_n(x) = 0. \end{cases} \quad (4.10)$$

Note that $w_n(x) \leq c|x|^{\tau_\alpha(\mu)p+2\alpha}$, where $\tau_\alpha(\mu)p + 2\alpha > \tau_\alpha(\mu)$ for $p \in [p_{\mu,\gamma}^*, p_{\mu,0}^*)$.

By Lemma 4.1, we have that $v_1 \geq v_0$. By iterative argument, we assume that $v_{N-1} \geq v_{N-2}$, then $Qv_{N-1}^p \geq Qv_{N-2}^p$ and Lemma 4.1 implies that $w_n \geq w_{n-1}$, so $v_n \geq v_{n-1}$ in $\mathbb{R}^N \setminus \{0\}$, that is, the sequence $\{v_n\}_n$ is an increasing sequence with respect to n .

We next build an upper bound for the sequence $\{v_n\}_n$. For $t > 0$, denote

$$\bar{w}_t(x) = tk^p w_1(x) + k|x|^{\tau_\alpha(\mu)} \leq (ctk^p + k)|x|^{\tau_\alpha(\mu)},$$

where $c > 0$ is from Corollary 4.1, then

$$\begin{aligned} Q(x)\bar{w}_t(x)^p &\leq Q(x)(ctk^p + k)^p|x|^{\tau_\alpha(\mu)p} \\ &\leq tk^p Q(x)|x|^{\tau_\alpha(\mu)p} = (-\Delta)^\alpha \bar{w}_t(x) + \frac{\mu}{|x|^{2\alpha}} \bar{w}_t(x), \end{aligned}$$

if

$$(ctk^p + k)^p \leq tk^p,$$

that is,

$$(ctk^{p-1} + 1)^p \leq t. \quad (4.11)$$

Note that the convex function $h_k(t) = (ctk^{p-1} + 1)^p$ can intersect the line $g(t) = t$, if

$$ck^{p-1} \leq \frac{1}{p} \left(\frac{p-1}{p} \right)^{p-1}. \quad (4.12)$$

Let $k^* = \left(\frac{1}{cp} \right)^{\frac{1}{p-1}} \frac{p-1}{p}$, then if $k \leq k^*$, it always hold that $h_k(t_p) \leq t_p$ for $t_p = \left(\frac{p}{p-1} \right)^p$. Hence, by the definition of \bar{w}_{t_p} , we have that $\bar{w}_{t_p} \geq v_0$ and $Q\bar{w}_{t_p}^p \geq Qv_0^p$, by [Corollary 2.1](#), we have that $v_1 \leq \bar{w}_{t_p}$, which implies $Q\bar{w}_{t_p}^p \geq Qv_1^p$. Inductively, we obtain that

$$v_n \leq \bar{w}_{t_p} \quad (4.13)$$

for all $n \in \mathbb{N}$. Therefore, the sequence $\{v_n\}_n$ converges. Let us denote $u_k = \lim_{n \rightarrow \infty} v_n$.

Since $\{v_n\}_n$ is uniformly locally bounded and $\bar{w}_{t_p} \in L^1(\mathbb{R}^N, \frac{1}{1+|x|^{N+2\alpha}} dx)$, then for any $x_0 \in \mathbb{R}^N \setminus \{0\}$, [\[9, Lemma 3.1\]](#) implies that

$$\|w_n\|_{C^\beta(B_{\frac{|x_0|}{4}}(x_0))} \leq c \left(\|\bar{w}_{t_p}\|_{L^\infty(B_{\frac{|x_0|}{2}}(x_0))} + \|Q\bar{w}_{t_p}^p\|_{L^\infty(B_{\frac{|x_0|}{2}}(x_0))} + \|\bar{w}_{t_p}\|_{L^1(\mathbb{R}^N, \frac{1}{1+|x|^{N+2\alpha}} dx)} \right),$$

thus, from [\[19, Corollary 2.4\]](#),

$$\begin{aligned} \|w_n\|_{C^{2\alpha+\beta}(B_{\frac{|x_0|}{8}}(x_0))} &\leq c \left(\|w_n\|_{C^\beta(B_{\frac{|x_0|}{4}}(x_0))} + \|Q\bar{w}_{t_p}^p\|_{L^\infty(B_{\frac{|x_0|}{2}}(x_0))} \right) \\ &\leq c \left(\|\bar{w}_{t_p}\|_{L^\infty(B_{\frac{|x_0|}{2}}(x_0))} + \|Q\bar{w}_{t_p}^p\|_{L^\infty(B_{\frac{|x_0|}{2}}(x_0))} + \|\bar{w}_{t_p}\|_{L^1(\mathbb{R}^N, \frac{1}{1+|x|^{N+2\alpha}} dx)} \right). \end{aligned}$$

By stability theorem [\[9, Theorem 2.2\]](#) (also see Theorem 2.2 in [\[7\]](#)), we have that u_k is a classical solution of [\(1.9\)](#). From [\(4.13\)](#) and the fact that $u_k \geq v_0$, we have that

$$v_0 \leq u_k \leq \bar{w}_{t_p},$$

which implies [\(1.12\)](#). \square

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