J. Math. Anal. Appl. 463 (2018) 895–909

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Weighted Hardy inequalities and Ornstein–Uhlenbeck type operators perturbed by multipolar inverse square potentials [☆]

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ARTICLE INFO

Article history:
Received 9 February 2018
Available online 28 March 2018
Submitted by A. Cianchi

Keywords:
Kolmogorov operators
Multipolar potentials
Weighted Hardy inequalities
Optimal constant

ABSTRACT

In this paper our main results are the multipolar weighted Hardy inequality

$$c\sum_{i=1}^{n}\int_{\mathbb{R}^N}\frac{\varphi^2}{|x-a_i|^2}\,d\mu \le \int_{\mathbb{R}^N}|\nabla\varphi|^2 d\mu + K\int_{\mathbb{R}^N}\varphi^2 d\mu, \qquad c\le c_o,$$

where the functions φ belong to a weighted Sobolev space H^1_μ, and the proof of the optimality of the constant $c_o = c_o(N) := \left(\frac{N-2}{2}\right)^2$. The Gaussian probability measure $d\mu$ is the unique invariant measure for Ornstein–Uhlenbeck type operators. This estimate allows us to get necessary and sufficient conditions for the existence of positive solutions to a parabolic problem corresponding to the Kolmogorov operators defined on smooth functions and perturbed by a multipolar inverse square potential

$$Lu + Vu = \left(\Delta u + \frac{\nabla\mu}{\mu}\cdot\nabla u\right) + \sum_{i=1}^{n}\frac{c}{|x-a_i|^2}u, \qquad x\in\mathbb{R}^N,$$

$c>0$, $a_1,\dots,a_n\in\mathbb{R}^N$.

1. Introduction

The paper deals with a class of Kolmogorov operators defined on smooth functions

$$Lu = \Delta u + \frac{\nabla\mu}{\mu}\cdot\nabla u, \tag{1.1}$$

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https://doi.org/10.1016/j.jmaa.2018.03.059
0022-247X/

where μ is a probability density on $\mathbb{R}^N$, perturbed by a multipolar inverse square potential

$$V(x) = \sum_{i=1}^{n} \frac{c}{|x-a_i|^2}, \quad x \in \mathbb{R}^N, \quad c > 0, \quad a_1, \dots, a_n \in \mathbb{R}^N. \tag{1.2}$$

From the mathematical point of view, the interest in inverse square potentials of type $V \sim \frac{c}{|x|^2}$ relies in the criticality: they have the same homogeneity as the Laplacian and do not belong to the Kato's class, then they cannot be regarded as a lower order perturbation term. Furthermore the study of such singular potentials is motived by applications to many fields, for example in many physical contexts as molecular physics [13], quantum cosmology (see e.g. [3]), quantum mechanics [2] and combustion models [11].

Multipolar potentials are associated with the interaction of a finite number of electric dipoles as, for example, in molecular systems consisting of n nuclei of unit charge located in a finite number of points $a_1, \dots, a_n$ and of n electrons. The Hartree–Fock model describes these systems (see [7]).

It is well known that if $L = \Delta$ and $V \le \frac{c}{|x|^{2-\varepsilon}}$, $c > 0$, $\varepsilon > 0$, then the corresponding initial value problem is well-posed. But for $\varepsilon = 0$ the problem may not have positive solution. In [2] Baras and Goldstein showed that the evolution problem associated to $\Delta + V$ admits a unique positive solution if $c \le c_o = c_o(N) := \left(\frac{N-2}{2}\right)^2$ and no positive solutions exist if $c > c_o$ (see also [5] for a different approach involving the Hardy inequality). When it exists, the solution is exponentially bounded, on the contrary, if $c > c_o$, there is the so called instantaneous blowup phenomena.

A similar behaviour was obtained in [12] with the potential $V = \frac{c}{|x|^2}$ replacing the Laplacian by the Kolmogorov operator L. See also [6] where the hypotheses on μ allow the drift term to be of the type $\frac{\nabla \mu}{\mu} = -|x|^{m-2}\, x$, $m > 0$.

In this paper we consider the generalized Ornstein–Uhlenbeck operator

$$Lu = \Delta u - \sum_{i=1}^{n} A(x-a_i) \cdot \nabla u, \tag{1.3}$$

where A is a positive definite real Hermitian $N \times N$-matrix, and the associated evolution problem

$$(P) \quad \begin{cases} \partial_t u(x,t) = Lu(x,t) + V(x)u(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(\cdot, 0) = u_0 \ge 0 \in L^2_\mu, \end{cases}$$

with the multipolar singular potential V defined in (1.2) and L^2_μ a suitable weighted space.

We state existence and nonexistence results using the relationship between the weak solution of (P) and the *bottom of the spectrum* of the operator $-(L+V)$

$$\lambda_1(L+V) := \inf_{\varphi \in H^1_\mu \setminus \{0\}} \left(\frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2 \, d\mu - \int_{\mathbb{R}^N} V \varphi^2 \, d\mu}{\int_{\mathbb{R}^N} \varphi^2 \, d\mu} \right).$$

Here H^1_μ denotes a suitable weighted Sobolev space defined in the next Section.

When $\mu = 1$ Cabré and Martel in [5] showed that the boundedness of $\lambda_1(\Delta + V)$, $0 \le V \in L^1_{loc}(\mathbb{R}^N)$, is a necessary and sufficient condition for the existence of positive exponentially bounded in time solutions to the associated initial value problem. Later in [12] the authors extended such a result to the case of Kolmogorov operators.

The estimate of the bottom of the spectrum $\lambda_1(L+V)$ is equivalent to the weighted Hardy inequality with $V(x) = \sum_{i=1}^{n} \frac{c}{|x-a_i|^2}$, $c \le c_o$,

$$\int_{\mathbb{R}^N} V\,\varphi^2\,d\mu \le \int_{\mathbb{R}^N} |\nabla\varphi|^2 d\mu + K\int_{\mathbb{R}^N} \varphi^2 d\mu, \quad \varphi\in H^1_\mu, \qquad K>0, \tag{1.4}$$

and to the sharpness of the best possible constant.

Then the existence of positive solutions to (P) is related to the Hardy inequality (1.4) and the nonexistence is due to the optimality of the constant c_o.

Our results about Hardy-type inequalities (1.4) (see Theorem 3.1 and Theorem 3.2 in Section 3) fit into the context of the so-called *multipolar Hardy inequalities*.

When $\mu=1$ the behaviour of the operator with a multipolar inverse square potential has been investigated in literature. In particular if $\mathcal{L}$ is the Schrödinger operator

$$\mathcal{L} = -\Delta - \sum_{i=1}^{n} \frac{c_i^+}{|x-a_i|^2},$$

$n\ge 2$, $c_i\in\mathbb{R}$, $c_i^+=\max\{c_i,0\}$, for any $i\in\{1,\dots,n\}$, Felli, Marchini and Terracini in [10] proved that the associated quadratic form

$$Q(\varphi) := \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx - \sum_{i=1}^{n} c_i \int_{\mathbb{R}^N} \frac{\varphi^2}{|x-a_i|^2}\,dx$$

is positive if $\sum_{i=1}^n c_i^+ < \frac{(N-2)^2}{4}$, conversely if $\sum_{i=1}^n c_i^+ > \frac{(N-2)^2}{4}$ there exists a configuration of poles such that Q is not positive. Later Bosi, Dolbeault and Esteban in [4] proved that for any $c\in\left(0,\frac{(N-2)^2}{4}\right]$ there exists a positive constant K such that (1.4) holds. Recently Cazacu and Zuazua in [9], improving a result stated in [4], obtained the inequality (1.4) with $K=0$ and $V=c\sum_{1\le i<j\le n}\frac{|a_i-a_j|^2}{|x-a_i|^2|x-a_j|^2}$ (see also Cazacu [8] for estimates for the Hardy constants in bounded domains).

As far as we know there are no results in the literature about the weighted multipolar Hardy inequalities.

In this paper we are motivated to consider the Gaussian measure $d\mu(x) = \mu(x)dx = Ce^{-\frac{1}{2}\sum_{i=1}^n\langle A(x-a_i),(x-a_i)\rangle}dx$, with C normalization constant, which is the unique invariant measure for the Ornstein–Uhlenbeck type operator (1.3) whose drift term is unbounded at infinity.

In Section 3 we will prove the inequality (1.4) which is the main result. Our technique to get the inequality, unlike the vector field method used in literature in the case $n=1$ (see, e.g., [12] for the weighted case), allow us to overcome the difficulties due to the mutual interaction among the poles and to achieve the constant c_o in the left-hand side in (1.4).

We obtain the estimate in a direct way starting from the result obtained in [4] with the Lebesgue measure and exploiting a suitable bound which the function μ we consider satisfies.

The optimality of the constant c_o is less immediate to obtain. The crucial points to estimate the bottom of the spectrum are the choice of a suitable function φ which involves only one pole and the connection we state between the weight functions in the case of one pole and in the case of multiple poles.

Afterwards, in Section 4, we will get in another way the proof of the weighted inequality through the so called *IMS* (Ismaligov, Morgan, Morgan-Simon, Sigal) method and reasoning as in [4]. To this aim we need to use a Hardy inequality in the case $n=1$ which we need to prove. Indeed in the IMS method a fundamental tool is an estimate with a single pole which allows us to achieve the optimal constant c_o in the inequality.

In Section 5 we will state an existence and nonexistence result, Theorem 5.1, putting together weighted Hardy inequality and Theorem 2.2 in Section 2. Furthermore, using the bilinear form associated to the operator $-(L+V)$, we will state the generation of an analytic C_0-semigroup and the positivity of the solution arguing as in [1].

2. Notation and preliminary results

Let us consider Kolmogorov operators L defined in (1.1) and the functions $\mu \in C^{1,\alpha}_{loc}\left(\mathbb{R}^N\right)$ for some $\alpha \in (0,1)$, $\mu(x) > 0$ for all $x \in \mathbb{R}^N$.

It is known that the operator L with domain

$$D_{max}(L) = \{u \in C_b(\mathbb{R}^N) \cap W^{2,p}_{loc}(\mathbb{R}^N) \text{ for all } 1 < p < \infty, Lu \in C_b(\mathbb{R}^N)\}$$

is the weak generator of a not necessarily C_0-semigroup $\{T(t)\}_{t\geq 0}$ in $C_b(\mathbb{R}^N)$. Since $\int_{\mathbb{R}^N} Lu\, d\mu = 0$ for any $u \in C_c^\infty(\mathbb{R}^N)$, where $d\mu = \mu(x)dx$, then $d\mu$ is the invariant measure for $\{T(t)\}_{t\geq 0}$ in $C_b(\mathbb{R}^N)$. So we can extend it to a positive preserving and analytic C_0-semigroup on $L^2_\mu := L^2(\mathbb{R}^N, d\mu)$, whose generator is still denoted by L.

Furthermore we denote by H^1_μ be the set of all the functions $f \in L^2_\mu$ having distributional derivative ∇f in $(L^2_\mu)^N$.

We recall the following proposition (see [14, Chapter 8] for more details).

Proposition 2.1. *The following assertions hold:*

i) *$C_c^\infty(\mathbb{R}^N)$ is a core for L in L^2_μ;*
ii) *$D(L)$ is continuously and densely embedded in H^1_μ;*
iii) *$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v\, d\mu = -\int_{\mathbb{R}^N} (Lu)v\, d\mu, \quad u \in D(L),\ v \in H^1_\mu$;*
iv) *for any $t > 0$, $T(t)L^2_\mu \subset H^1_\mu$.*

From *i)* and *ii)* it follows that $C_c^\infty(\mathbb{R}^N)$ is densely embedded in H^1_μ. Then we can regard H^1_μ also as the completion of $C_c^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|^2_{H^1_\mu} := \|u\|^2_{L^2_\mu} + \|\nabla u\|^2_{L^2_\mu}.$$

The operator L can also be defined via the bilinear form

$$a_\mu(u,v) = \int_{\mathbb{R}^N} \nabla u \cdot \nabla v\, d\mu \tag{2.1}$$

on H^1_μ. This is immediately clear by integrating by parts in (2.1). Indeed

$$a_\mu(u,v) = -\int_{\mathbb{R}^N} Luv\, d\mu, \qquad u, v \in C_c^\infty(\mathbb{R}^N).$$

Let us recall the problem

$$(P) \quad \begin{cases} \partial_t u(x,t) = Lu(x,t) + V(x)u(x,t), & t > 0, x \in \mathbb{R}^N, N \geq 3, \\ u(\cdot,t) = u_0 \in L^2_\mu, \end{cases}$$

where L is as in (1.1). We say that u is a weak solution to (P) if, for each $T, R > 0$, we have

$$u \in C([0,T], L^2_\mu), \quad Vu \in L^1(B_R \times (0,T), d\mu dt)$$

and

$$\int_0^T \int_{\mathbb{R}^N} u(-\partial_t \phi - L\phi)\, d\mu dt - \int_{\mathbb{R}^N} u_0 \phi(\cdot, 0)\, d\mu = \int_0^T \int_{\mathbb{R}^N} V u \phi \, d\mu dt$$

for all $\phi \in W_2^{2,1}(\mathbb{R}^N \times [0,T])$ having compact support with $\phi(\cdot, T) = 0$, where B_R denotes the open ball of $\mathbb{R}^N$ of radius R centered at 0.

For any $\Omega \subset \mathbb{R}^N$, $W_2^{2,1}(\Omega \times (0,T))$ is the parabolic Sobolev space of the functions $u \in L^2(\Omega \times (0,T))$ having weak space derivatives $D_x^\alpha u \in L^2(\Omega \times (0,T))$ for $|\alpha| \leq 2$ and weak time derivative $\partial_t u \in L^2(\Omega \times (0,T))$ equipped with the norm

$$\|u\|_{W_2^{2,1}(\Omega\times(0,T))} := \Bigg(\|u\|^2_{L^2(\Omega\times(0,T))} + \|\partial_t u\|^2_{L^2(\Omega\times(0,T))} + \sum_{1 \leq |\alpha| \leq 2} \|D^\alpha u\|^2_{L^2(\Omega\times(0,T))} \Bigg)^{\frac{1}{2}}.$$

We will use the following result in Section 5.

Theorem 2.2. *Assume $0 < \mu \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ is a probability density on $\mathbb{R}^N$ and $0 \leq V \in L^1_{loc}(\mathbb{R}^N)$. Then the following assertions hold:*

(i) *If $\lambda_1(L+V) > -\infty$, then there exists a positive weak solution $u \in C([0,\infty), L^2_\mu)$ of (P) satisfying*

$$\|u(t)\|_{L^2_\mu} \leq M e^{\omega t} \|u_0\|_{L^2_\mu}, \quad t \geq 0 \tag{2.2}$$

for some constants $M \geq 1$ and $\omega \in \mathbb{R}$.

(ii) *If $\lambda_1(L+V) = -\infty$, then for any $0 \leq u_0 \in L^2_\mu \setminus \{0\}$, there is no positive weak solution of (P) satisfying (2.2).*

The proof of the Theorem is based on Cabré-Martel's idea in [5] and it was proved in [12] for functions μ belonging to $C^{1,\alpha}_{loc}(\mathbb{R}^N)$. The proof relies on certain properties of the operator L and its corresponding semigroup $\{T(t)\}_{t\geq 0}$ in L^2_μ. Furthermore the strict positivity on compact sets of $T(t)u_0$, $t > 0$, if $0 \leq u_0 \in L^2_\mu \setminus \{0\}$ is required.

3. Weighted Hardy inequality and optimality of the constant

Let us consider the following Gaussian measure

$$d\mu = \mu(x)dx = C\, e^{-\frac{1}{2}\sum_{i=1}^n \langle A(x-a_i), x-a_i\rangle}\, dx \tag{3.1}$$

with

$$C = \Bigg(\int_{\mathbb{R}^N} e^{-\frac{1}{2}\sum_{i=1}^n \langle A(x-a_i), x-a_i\rangle}\, dx \Bigg)^{-1} \tag{3.2}$$

and A positive definite real Hermitian $N \times N$-matrix, which is the unique invariant probability measure for Ornstein–Uhlenbeck type operators

$$Lu = \Delta u - \sum_{i=1}^{n} A(x - a_i) \cdot \nabla u.$$

So the operator L, with domain $H^2_\mu := \{u \in H^1_\mu : D_k u \in H^1_\mu\}$, generates an analytic semigroup $\{T(t)\}_{t\geq 0}$ on L^2_μ (cf. [15]).

The operators we consider are perturbed by the multipolar inverse square potential

$$V(x) = \sum_{i=1}^{n} \frac{c}{|x - a_i|^2} = c\, V_n, \tag{3.3}$$

where $x \in \mathbb{R}^N$, $c > 0$, $a_i \in \mathbb{R}^N$, $i = 1, \dots, n$.

We state the following weighted Hardy inequality.

Theorem 3.1. *Assume $N \geq 3$, $n \geq 2$, A a positive definite real Hermitian $N \times N$-matrix and let $r_0 = \min_{i \neq j} |a_i - a_j|/2$, $i, j = 1, \dots, n$. Then there exists a constant $k \in [0, \pi^2)$ such that*

$$\begin{aligned} c \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\varphi^2}{|x - a_i|^2}\, d\mu \leq & \int_{\mathbb{R}^N} |\nabla \varphi|^2\, d\mu \\ & + \left[\frac{k + (n+1)c}{r_0^2} + \frac{n}{2} \operatorname{Tr} A \right] \int_{\mathbb{R}^N} \varphi^2\, d\mu \end{aligned} \tag{3.4}$$

for all $\varphi \in H^1_\mu$, where $c \in (0, c_o]$ with $c_o = c_o(N) := \left(\frac{N-2}{2}\right)^2$ optimal constant.

Proof.
Step 1 (Inequality)

By density we can consider functions $\varphi \in C_c^\infty(\mathbb{R}^N)$.

The starting point is the following inequality, stated by Bosi, Dolbeault and Esteban in [4, Theorem 1]:

$$c \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\varphi^2}{|x - a_i|^2}\, dx \leq \int_{\mathbb{R}^N} |\nabla \varphi|^2\, dx + \left[\frac{k + (n+1)c}{r_0^2} \right] \int_{\mathbb{R}^N} \varphi^2\, dx \tag{3.5}$$

for all $\varphi \in H^1(\mathbb{R}^N)$, with $n \geq 2$, $k \in \left[0, \pi^2\right)$ and $c \in (0, c_o]$. The proof of (3.5) is based on IMS truncation method. In the Section 4 we will prove the weighted version of the inequality (3.5) reasoning as in [4, Theorem 1].

Now we state the weighted version of this result in a direct way.

Indeed, applying (3.5) to the function $\varphi\sqrt{\mu}$, we have

$$c \int_{\mathbb{R}^N} \sum_{i=1}^{n} \frac{\varphi^2}{|x - a_i|^2}\, d\mu \leq \int_{\mathbb{R}^N} |\nabla(\varphi\sqrt{\mu})|^2\, dx + \left[\frac{k + (n+1)c}{r_0^2} \right] \int_{\mathbb{R}^N} \varphi^2\, d\mu.$$

By means the easy calculation

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla(\varphi\sqrt{\mu})|^2\, dx &= \int_{\mathbb{R}^N} \left| (\nabla\varphi)\sqrt{\mu} + \varphi \frac{\nabla\mu}{2\sqrt{\mu}} \right|^2 dx \\ &= \int_{\mathbb{R}^N} |\nabla\varphi|^2\, d\mu + \int_{\mathbb{R}^N} \left(\frac{1}{4} \left| \frac{\nabla\mu}{\mu} \right|^2 - \frac{1}{2} \frac{\Delta\mu}{\mu} \right) \varphi^2\, d\mu, \end{aligned}$$

and observing that we can estimate the last integral above taking into account that

$$\begin{aligned}\frac{1}{4}\left|\frac{\nabla\mu}{\mu}\right|^2-\frac{1}{2}\frac{\Delta\mu}{\mu}&=\frac{1}{4}\left|\sum_{j=1}^{n}A(x-a_j)\right|^2\\&\quad-\frac{1}{2}\left[-n\operatorname{Tr}A+\left|\sum_{j=1}^{n}A(x-a_j)\right|^2\right]\le\frac{n}{2}\operatorname{Tr}A\end{aligned}\tag{3.6}$$

we get the result.

Step 2 (*Optimality*)

To state the optimality of the constant c_o we suppose that $c>c_o$.

Let us fix i and consider the function $\varphi=|x-a_i|^\gamma$, $\gamma\in(1-\frac{N}{2},0)$. The function φ belongs to H^1_μ and

$$\int_{\mathbb{R}^N}\left(|\nabla\varphi|^2-c\,\frac{\varphi^2}{|x-a_i|^2}\right)d\mu=(\gamma^2-c)\int_{\mathbb{R}^N}|x-a_i|^{2(\gamma-1)}\,d\mu.$$

Hence the bottom of the spectrum λ_1 of the operator $-(L+V)$ satisfies

$$\lambda_1\le(\gamma^2-c)\frac{\int_{\mathbb{R}^N}|x-a_i|^{2(\gamma-1)}\,d\mu}{\int_{\mathbb{R}^N}|x-a_i|^{2\gamma}\,d\mu}\tag{3.7}$$

since

$$\int_{\mathbb{R}^N}\left(|\nabla\varphi|^2-V\varphi^2\right)d\mu\le\int_{\mathbb{R}^N}\left(|\nabla\varphi|^2-c\,\frac{\varphi^2}{|x-a_i|^2}\right)d\mu.$$

We are able to state that for any $i\in\{1,\dots,n\}$ it holds

$$C_1\,e^{-\alpha_2(2n-1)\frac{|x-a_i|^2}{2}}\le e^{-\sum_{i=1}^{n}\frac{|A^{\frac{1}{2}}(x-a_i)|^2}{2}}\le C_2\,e^{-\alpha_1\frac{n+1}{2}\frac{|x-a_i|^2}{2}}\tag{3.8}$$

with $C_1=e^{-\alpha_2\sum_{i\neq j}|a_i-a_j|^2}$ and $C_2=e^{\frac{\alpha_1}{2}\sum_{i\neq j}|a_i-a_j|^2}$ which is a consequence of the inequalities

$$\alpha_1\sum_{i=1}^{n}|x-a_i|^2\le\sum_{i=1}^{n}|A^{\frac{1}{2}}(x-a_i)|^2\le\alpha_2\sum_{i=1}^{n}|x-a_i|^2,\qquad\alpha_1,\alpha_2>0,$$

and

$$\begin{aligned}-\sum_{j\neq i}|a_i-a_j|^2+\frac{n+1}{2}|x-a_i|^2&\le\sum_{i=1}^{n}|x-a_i|^2\\&\le(2n-1)|x-a_i|^2+2\sum_{j\neq i}|a_i-a_j|^2.\end{aligned}\tag{3.9}$$

The inequality (3.9) is proved in Appendix.

For simplicity in the following we place $\tilde\alpha_1=\alpha_1\frac{n+1}{2}$ and $\tilde\alpha_2=\alpha_2(2n-1)$.

The equivalence between the weight functions in the case of one pole and in the case of multiple poles allows us to calculate integrals in (3.7). Indeed, by a change of variables and by (3.8)

$$\begin{aligned}\int_{\mathbb{R}^N} |x-a_i|^{2\beta} e^{-\sum_{i=1}^n \frac{|A^{\frac{1}{2}}(x-a_i)|^2}{2}}\,dx &\le C_2 \int_{\mathbb{R}^N} |x-a_i|^{2\beta} e^{-\tilde{\alpha}_1 \frac{|x-a_i|^2}{2}}\,dx \\ &= C_2\, 2^{\beta+\frac{N}{2}} \tilde{\alpha}_1^{-\beta-\frac{N}{2}} \int_{\mathbb{R}^N} |x-a_i|^{2\beta} e^{-\frac{|x-a_i|^2}{2}}\,dx.\end{aligned} \tag{3.10}$$

Taking in mind the definition of Gamma integral function

$$\int_{\mathbb{R}^N} |x|^{2\beta} e^{-\frac{|x|^2}{2}}\,dx = \sigma_N\, 2^{\beta+\frac{N}{2}-1}\Gamma\left(\beta+\frac{N}{2}\right), \quad \beta+\frac{N}{2}>0,$$

we get from (3.10)

$$\begin{aligned}\int_{\mathbb{R}^N} |x-a_i|^{2\beta} e^{-\sum_{i=1}^n \frac{|A^{\frac{1}{2}}(x-a_i)|^2}{2}}\,dx &\le \\ &\le C_2\, 2^{2\beta+N-1} \tilde{\alpha}_1^{-\beta-\frac{N}{2}} \sigma_N \Gamma\left(\beta+\frac{N}{2}\right).\end{aligned} \tag{3.11}$$

Reasoning as above we obtain an estimate from below

$$\begin{aligned}\int_{\mathbb{R}^N} |x-a_i|^{2\beta} e^{-\sum_{i=1}^n \frac{|A^{\frac{1}{2}}(x-a_i)|^2}{2}}\,dx &\ge C_1 \int_{\mathbb{R}^N} |x-a_i|^{2\beta} e^{-\tilde{\alpha}_2 \frac{|x-a_i|^2}{2}}\,dx \\ &= C_1 \tilde{\alpha}_2^{-\beta-\frac{N}{2}} \int_{\mathbb{R}^N} |x-a_i|^{2\beta} e^{-\frac{|x-a_i|^2}{2}}\,dx \\ &= C_1\, 2^{\beta+\frac{N}{2}-1} \tilde{\alpha}_2^{-\beta-\frac{N}{2}} \sigma_N \Gamma\left(\beta+\frac{N}{2}\right).\end{aligned} \tag{3.12}$$

Therefore, using (3.11) and (3.12), we get

$$\begin{aligned}\frac{\int_{\mathbb{R}^N} |x-a_i|^{2(\gamma-1)}\,d\mu}{\int_{\mathbb{R}^N} |x-a_i|^{2\gamma}\,d\mu} &\ge \frac{C_1\, 2^{\gamma+\frac{N}{2}-2} \tilde{\alpha}_2^{-\gamma-\frac{N}{2}+1} \sigma_N \Gamma(\gamma+\frac{N}{2}-1)}{C_2\, 2^{2\gamma+N-1} \tilde{\alpha}_1^{-\gamma-\frac{N}{2}} \sigma_N \Gamma(\gamma+\frac{N}{2})} \\ &= \frac{C_1\, 2^{\gamma+\frac{N}{2}-2} \tilde{\alpha}_2^{-\gamma-\frac{N}{2}+1}}{C_2\, 2^{2\gamma+N-1} \tilde{\alpha}_1^{-\gamma-\frac{N}{2}} (\gamma+\frac{N}{2}-1)}.\end{aligned}$$

Then

$$\lambda_1 \le \lim_{\gamma\to(1-\frac{N}{2})^+} (\gamma^2-c) \frac{C_1\, 2^{\gamma+\frac{N}{2}-2} \tilde{\alpha}_2^{-\gamma-\frac{N}{2}+1}}{C_2\, 2^{2\gamma+N-1} \tilde{\alpha}_1^{-\gamma-\frac{N}{2}} (\gamma+\frac{N}{2}-1)} = -\infty.$$

Thus, for any $M>0$, there is $\varphi \in H^1_\mu$ such that

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2\,d\mu - c\int_{\mathbb{R}^N} \frac{\varphi^2}{|x-a_i|^2}\,d\mu < -M \int_{\mathbb{R}^N} \varphi^2\,d\mu.$$

By taking $M := \frac{k+(n+1)c}{r_0^2} + \frac{n}{2}\operatorname{Tr} A$ we find $\varphi \in H^1_\mu$ such that

$$c\int_{\mathbb{R}^N}\frac{\varphi^2}{|x-a_i|^2}\,d\mu > \int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu + \left[\frac{k+(n+1)c}{r_0^2}+\frac{n}{2}\operatorname{Tr}A\right]\int_{\mathbb{R}^N}\varphi^2\,d\mu$$

which leads to a contradiction with respect the weighted Hardy inequality (3.4) because, of course,

$$c\int_{\mathbb{R}^N}\frac{\varphi^2}{|x-a_i|^2}\,d\mu \le c\int_{\mathbb{R}^N}\sum_{i=1}^n\frac{\varphi^2}{|x-a_i|^2}\,d\mu.$$

This proves the optimality of c_o.

We remark that when $c\in(0,\frac{c_o}{n}]$ the constant on the right-hand side of (3.4) can be improved using a different proof based on the multipolar Hardy inequality in the case of Lebesgue measure.

Moreover the inequality (3.13) below holds also in the case $n=1$.

Theorem 3.2. *Assume $N\ge 3$ and $n\ge 1$. Then we get*

$$\frac{c_o}{n}\int_{\mathbb{R}^N}\sum_{i=1}^n\frac{\varphi^2}{|x-a_i|^2}\,d\mu \le \int_{\mathbb{R}^N}|\nabla\varphi|^2\,d\mu + \frac{n}{2}\operatorname{Tr}A\int_{\mathbb{R}^N}\varphi^2\,d\mu \tag{3.13}$$

for any $\varphi\in H^1_\mu$, where $c_o=c_o(N):=\left(\frac{N-2}{2}\right)^2$.

Proof. We start from the known inequality

$$\frac{c_o}{n}\int_{\mathbb{R}^N}\sum_{i=1}^n\frac{\varphi^2}{|x-a_i|^2}\,dx \le \int_{\mathbb{R}^N}|\nabla\varphi|^2\,dx \tag{3.14}$$

for all $\varphi\in H^1(\mathbb{R}^N)$, where $c_o=c_o(N):=\left(\frac{N-2}{2}\right)^2$, which we can get immediately by using the Hardy inequality with one pole.

Then we apply the inequality (3.14) to the function $\varphi\sqrt{\mu}$ and reason as in the proof of Theorem 3.1.

4. Proof of the weighted Hardy inequality via the IMS method

We can prove the inequality in Theorem 3.1 using the so-called IMS method, which consists in localizing the wave functions around the singularities by using a partition of unity.

We say that a finite family $\{J_i\}_{i=1}^{n+1}$ of real valued functions $J_i\in W^{1,\infty}(\mathbb{R}^N)$ is a *partition of unity* in $\mathbb{R}^N$ if $\sum_{i=1}^{n+1}J_i^2=1$.

Any family of this type has the following properties:

(a) $\sum_{i=1}^{n+1}J_i\partial_\alpha J_i=0$ for any $\alpha=1,\dots,N$;
(b) $J_{n+1}=\sqrt{1-\sum_{i=1}^n J_i^2}$;
(c) $\sum_{i=1}^{n+1}|\nabla J_i|^2\in L^\infty(\mathbb{R}^N)$.

Furthermore we require that

$$\Omega_i\cap\Omega_j=\emptyset\quad\text{for any } i,j=1,\dots,n,\ i\ne j, \tag{4.1}$$

where $\overline{\Omega}_i=\operatorname{supp}(J_i)$, $i=1,\dots,n$. By the property (a) we get

$$\sum_{\alpha=1}^{N} |J_{n+1}\partial_\alpha J_{n+1}|^2 = \sum_{\alpha=1}^{N} \left|\sum_{j=1}^{n} J_j \partial_\alpha J_j\right|^2 = \sum_{\alpha=1}^{N}\sum_{j=1}^{n} |J_j\partial_\alpha J_j|^2,$$

from which

$$|\nabla J_{n+1}|^2 = \sum_{i=1}^{n} \frac{J_i^2}{1-J_i^2}|\nabla J_i|^2.$$

As a consequence we obtain an explicit formula for the sum of the gradients:

(d) $\sum_{i=1}^{n+1} |\nabla J_i|^2 = \sum_{i=1}^{n} |\nabla J_i|^2 + \sum_{i=1}^{n} \frac{J_i^2}{1-J_i^2}|\nabla J_i|^2 = \sum_{i=1}^{n} \frac{|\nabla J_i|^2}{1-J_i^2}$,

Note that to avoid a singularity for the gradient of J_{n+1} at the points where $1 - J_i^2 = 0$, from (d) we shall assume the additional constraint $|\nabla J_i|^2 = F(x)(1 - J_i^2)$, for $i = 1, \ldots, n$ and for some $F \in L^\infty(\mathbb{R}^N)$.

By proceeding as in [4, Lemma 2], we are able to state the following result.

Lemma 4.1. *Let $\{J_i\}_{i=1}^{n+1}$ be a partition of unity satisfying (4.1), and $d\mu$ the Gaussian measure defined in (3.1). For any $u \in H_\mu^1$ and any $V \in L^1_{loc}(\mathbb{R}^N)$ we get*

$$\int_{\mathbb{R}^N} \left(|\nabla\varphi|^2 - V\varphi^2\right) d\mu = \sum_{i=1}^{n+1} \int_{\mathbb{R}^N} (|\nabla(J_i\varphi)|^2 - V(J_i\varphi)^2) d\mu - \int_{\mathbb{R}^N} \sum_{i=1}^{n+1} |\nabla J_i|^2 \varphi^2 \, d\mu.$$

Proof. We can immediately observe that

$$\int_{\mathbb{R}^N} V \left(\sum_{i=1}^{n+1} (J_i\varphi)^2\right) d\mu = \int_{\mathbb{R}^N} V \left(\sum_{i=1}^{n+1} J_i^2\right) \varphi^2 \, d\mu = \int_{\mathbb{R}^N} V\varphi^2 \, d\mu. \tag{4.2}$$

On the other hand,

$$\begin{aligned} \sum_{i=1}^{n+1} |\nabla (J_i\varphi)|^2 &= \sum_{i=1}^{n+1} |(\nabla J_i)\varphi + (\nabla\varphi)J_i|^2 \\ &= \sum_{i=1}^{n+1} |\nabla J_i|^2\varphi^2 + \sum_{i=1}^{n+1} |\nabla\varphi|^2 J_i^2 + 2\sum_{i=1}^{n+1} (J_i\nabla J_i)(\varphi\nabla\varphi) \qquad (4.3) \\ &= \sum_{i=1}^{n+1} |\nabla J_i|^2\varphi^2 + |\nabla\varphi|^2 + \left(\sum_{i=1}^{n+1} J_i\nabla J_i\right)\nabla\varphi^2. \end{aligned}$$

By property (a) it follows that $\left(\sum_{i=1}^{n+1} J_i\nabla J_i\right)\nabla\varphi^2 = 0$, then by integrating (4.3) on $\mathbb{R}^N$ we obtain

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 \, d\mu = \int_{\mathbb{R}^N} \sum_{i=1}^{n+1} |\nabla (J_i\varphi)|^2 \, d\mu - \int_{\mathbb{R}^N} \sum_{i=1}^{n+1} |\nabla J_i|^2\varphi^2 \, d\mu. \tag{4.4}$$

From (4.2) and (4.4) we get the result.

Taking in mind that

$$V_n(x) = \sum_{i=1}^{n} \frac{1}{|x-a_i|^2},$$

as defined in (3.3), we recall a preliminary lemma, stated by Bosi, Dolbeault and Esteban in [4], about the case $n=2$, with $a_1 = a$, $a_2 = -a$ and $0 < r_0 \le |a|$.

Lemma 4.2. *There is a partition of the unity $\{J_i\}_{i=1}^3$ satisfying (4.1) with $J_1 \equiv 1$ on $B(a, \frac{r_0}{2})$, $J_1 \equiv 0$ on $B(a, r_0)^c$, $J_2(x) = J_1(-x)$ for any $x \in \mathbb{R}^N$, $0 < r_0 \le |a|$, such that, for any $c > 0$, there exists a constant $k \in \left[0, \pi^2\right)$ for which, almost everywhere for all $x \in \Omega := \mathrm{supp}(J_1) \cup \mathrm{supp}(J_2)$, we have*

$$\sum_{i=1}^{3} |\nabla J_i|^2 + c\, J_3^2\, V_2(x) = \sum_{i=1,2} \frac{|\nabla J_i|^2}{1-J_i^2} + c\, J_3^2\, V_2(x) \le \frac{k+2c}{r_0^2}. \tag{4.5}$$

Now we are able to proceed with the proof.

Proof of Theorem 3.1. Let us define the following quadratic form

$$Q[\varphi] := \int_{\mathbb{R}^N} \left(|\nabla\varphi|^2 - cV_n(x)\varphi^2\right) d\mu, \qquad \varphi \in H^1_\mu. \tag{4.6}$$

By virtue of Lemma 4.1 we are able to write (4.6) as follows

$$Q[\varphi] = \sum_{i=1}^{n} Q[J_i\varphi] + R_n, \qquad \varphi \in H^1_\mu \tag{4.7}$$

where

$$R_n = \int_{\mathbb{R}^N} |\nabla(J_{n+1}\varphi)|^2\, d\mu - c\int_{\mathbb{R}^N} V_n |J_{n+1}\varphi|^2\, d\mu - \sum_{i=1}^{n+1} \int_{\mathbb{R}^N} |\nabla J_i|^2 \varphi^2\, d\mu.$$

Thanks to the property (d) we have

$$\begin{aligned} R_n &= \int_{\mathbb{R}^N} |\nabla(J_{n+1}\varphi)|^2\, d\mu - c\int_{\mathbb{R}^N} V_n\left(1 - \sum_{i=1}^{n} J_i^2\right)\varphi^2\, d\mu \\ &\quad - \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{|\nabla J_i|^2}{1-J_i^2}\varphi^2\, d\mu \\ &\ge -c\int_{\mathbb{R}^N} V_n(x)\left(1 - \sum_{i=1}^{n} J_i^2\right)\varphi^2\, d\mu - \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{|\nabla J_i|^2}{1-J_i^2}\varphi^2\, d\mu. \end{aligned}$$

Let us consider a partition of unity $\{J_i\}_{i=1}^{n+1}$ satisfying (4.1), and the sets $\Omega_i = B(a_i, r_0)$ such that $\overline{\Omega}_i = \mathrm{supp}(J_i)$, $i = 1, \dots, n$. If we set $\Omega = \cup_{i=1}^{n} \overline{\Omega}_i$ and $\Gamma = \mathbb{R}^N \setminus \Omega$, then $|x - a_i| \ge r_0$ in $\overline{\Omega}_j$ for $i \ne j$, and $V_n(x) \le \frac{n}{r_0^2}$ on Γ. Moreover, using the condition (4.1) we get

$$R_n \geq -\sum_{i=1}^{n} \int\limits_{\Omega_i} \left[\frac{|\nabla J_i|^2}{1-J_i^2} + c\left(1-J_i^2\right) V_n(x) \right] \varphi^2 \, d\mu - \frac{c\, n}{r_0^2} \int\limits_{\Gamma} \varphi^2 \, d\mu.$$

Taking into account that $J_j = 0$ on Ω_i for any $j \neq i$, we have for $j \neq i$

$$\begin{aligned} R_n \geq &-\sum_{i=1}^{n} \int\limits_{\Omega_i} \Bigg[\frac{|\nabla J_i|^2}{1-J_i^2} + \frac{|\nabla J_j|^2}{1-J_j^2} + c\left(1-J_i^2-J_j^2\right) \left(\frac{1}{|x-a_i|^2} + \frac{1}{|x-a_j|^2} \right) \\ &+ c\left(1-J_i^2\right) \left(\sum_{k\neq i,j} \frac{1}{|x-a_k|^2} \right) \Bigg] \varphi^2 \, d\mu - \frac{c\, n}{r_0^2} \int\limits_{\Gamma} \varphi^2 \, d\mu. \end{aligned}$$

Now, taking $\left\{ J_i, J_j, \sqrt{1-J_i^2-J_j^2} \right\}$ as the partition of unity, we can apply Lemma 4.2 on Ω_i with $(a_i, a_j) = (-a, a)$ up to a change of coordinates. In this way we get

$$\begin{aligned} R_n \geq &-\sum_{i=1}^{n} \int\limits_{\Omega_i} \left[\frac{k+2c}{r_0^2} + c(1-J_i^2) \left(\sum_{k\neq i,j} \frac{1}{|x-a_k|^2} \right) \right] \varphi^2 \, d\mu \\ &- \frac{c\, n}{r_0^2} \int\limits_{\Gamma} \varphi^2 \, d\mu \\ \geq &-\sum_{i=1}^{n} \int\limits_{\Omega_i} \left[\frac{k+2c}{r_0^2} + \frac{(n-2)c}{r_0^2}(1-J_i^2) \right] \varphi^2 \, d\mu - \frac{c\, n}{r_0^2} \int\limits_{\Gamma} \varphi^2 \, d\mu, \end{aligned} \tag{4.8}$$

since we can estimate $\frac{1}{|x-a_k|^2}$ by $\frac{1}{r_0^2}$ for all $k \neq i, j$. Taking into account (4.6) and using the weighted Hardy inequality (3.13) with $n = 1$ we get

$$\begin{aligned} Q[J_i\varphi] &= \int\limits_{\mathbb{R}^N} |\nabla J_i \varphi|^2 \, d\mu - c \int\limits_{\mathbb{R}^N} \left(\frac{1}{|x-a_i|^2} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{|x-a_j|^2} \right) |J_i\varphi|^2 \, d\mu \\ &\geq -\left[\frac{1}{2}\mathrm{Tr}A + \frac{(n-1)c}{r_0^2} \right] \int\limits_{\Omega_i} |J_i\varphi|^2 \, d\mu, \end{aligned}$$

from which

$$\sum_{i=1}^{n} Q[J_i\varphi] \geq -\frac{1}{2}\mathrm{Tr}A \sum_{i=1}^{n} \int\limits_{\Omega_i} \varphi^2 \, d\mu - \frac{(n-1)c}{r_0^2} \sum_{i=1}^{n} \int\limits_{\Omega_i} J_i^2 \varphi^2 \, d\mu \tag{4.9}$$

From (4.7), (4.8) and (4.9) we deduce

$$\begin{aligned} Q[\varphi] \geq &-\sum_{i=1}^{n} \int\limits_{\Omega_i} \left[\frac{k+2c}{r_0^2} + \frac{(n-2)c}{r_0^2}(1-J_i^2) + \frac{1}{2}\mathrm{Tr}A + \frac{(n-1)c}{r_0^2} J_i^2 \right] \varphi^2 \, d\mu \\ &- \frac{c\, n}{r_0^2} \int\limits_{\Gamma} \varphi^2 \, d\mu. \end{aligned}$$

Since

$$k + 2c + c(n-2)(1-J_i^2) + c(n-1)J_i^2 = k + cn + cJ_i^2 \leq k + c(n+1),$$

we finally obtain

$$\begin{aligned} Q[\varphi] &\geq -\left[\frac{k+(n+1)c}{r_0^2} + \frac{1}{2}\mathrm{Tr}A\right]\int\limits_{\Omega} \varphi^2\, d\mu - \frac{c\,n}{r_0^2}\int\limits_{\Gamma} \varphi^2\, d\mu \\ &\geq -\left[\frac{k+(n+1)c}{r_0^2} + \frac{1}{2}\mathrm{Tr}A\right]\int\limits_{\mathbb{R}^N} \varphi^2\, d\mu, \end{aligned}$$

from which we get inequality (3.4).

5. Existence of solutions via weighted Hardy inequality

The potential $V(x) = \sum_{i=1}^{n} \frac{c}{|x-a_i|^2}$ and the Gaussian density $\mu(x)$ satisfy the hypotheses of the Theorem 2.2. We can therefore state the following existence and nonexistence result as a consequence of the weighted Hardy inequality (3.4) and of the Theorem 2.2.

Theorem 5.1. *Assume that $N \geq 3$, A a positive definite real Hermitian $N \times N$-matrix and $0 \leq V(x) \leq \sum_{i=1}^{n} \frac{c}{|x-a_i|^2}$, with $c > 0$, $x, a_i \in \mathbb{R}^N$, $i \in \{1, \ldots, n\}$. Let L the Ornstein–Uhlenbeck type operator (1.3). Then the following assertions hold:*

i) *If $c \leq c_o$ there exists a positive weak solution $u \in C\left([0,\infty), L^2_\mu\right)$ of*

$$\begin{cases} \partial_t u(x,t) = L + V(x)u(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(\cdot,t) = u_0 \in L^2_\mu, \end{cases} \tag{5.1}$$

satisfying

$$\|u(t)\|_{L^2_\mu} \leq M e^{\omega t}\|u_0\|_{L^2_\mu}, \qquad t \geq 0 \tag{5.2}$$

for some constants $M \geq 1$, $\omega \in \mathbb{R}$, and any $u_0 \in L^2_\mu$.

ii) *If $c > c_o$ there exists no positive weak solution of (5.1) with $V(x) = \sum_{i=1}^{n} \frac{c}{|x-a_i|^2}$ satisfying (5.2) for any $0 \leq u_0 \in L^2_\mu$, $u_0 \neq 0$.*

Following a different approach based on bilinear forms associated to the operator $-(L+V)$, we obtain an existence result. We state the generation of an analytic C_0-semigroup.

Let us define the bilinear form

$$a_c(u,v) := \int\limits_{\mathbb{R}^N} \nabla u \cdot \nabla v\, d\mu - c\sum_{i=1}^{n}\int\limits_{\mathbb{R}^N} \frac{uv}{|x-a_i|^2}\, d\mu \tag{5.3}$$

for $u, v \in D(a_c) = H^1_\mu$, $N \geq 3$ and $c > 0$.

Arguing as in [1, Propositions 2.2 and 2.3], we can get the next result.

Proposition 5.2. *The following statements hold:*

i) *a_c is closed if $c < c_o$;*

ii) *a_{c_o} is closable.*

Furthermore a_c is quasi-accretive for all $c \in (0, c_o]$. In fact by the weighted Hardy inequality (3.4) we immediately get

$$a_c(u,u) \geq -K\,(u,u)_{H^1_\mu}$$

for all $u \in H^1_\mu$, with K the constant on the right-hand side in the inequality.

For $c < c_o$, if $\mathcal{A}$ is the associated operator defined by

$$D(\mathcal{A}) = \left\{ u \in D(a_c) : \exists\, v \in L^2_\mu \quad s.\ t. \quad a_c(u,\phi) = \int\limits_{\mathbb{R}^N} v\phi\, d\mu \quad \forall \phi \in D(a_c) \right\},$$

$$\mathcal{A}u = v,$$

then $-\mathcal{A} = L + V$ generates an analytic C_0-semigroup $\{S(t)\}_{t\geq 0}$ on L^2_μ satisfying

$$\|S(t)\| \leq e^{Kt}, \quad t \geq 0.$$

For the case $c = c_o$ the same conclusion holds taking the closure $\overline{a_{c_o}}$ instead of a_{c_o} in the definition of $\mathcal{A}$.

The positivity of the solution u can be obtained as in [1, Section 2]. Indeed, we can regard $S(t)$ as the limit of positive preserving semigroups described by cut-off potentials.

Let $\mathcal{A}_k = L + \min(V, ck)$, $k \in \mathbb{N}$. Since L is the generator of a positive preserving semigroup on L^2_μ and $\min(V, ck)$ is bounded and non-negative, $\mathcal{A}_k$ generates a positive preserving semigroup, denoted by $S_k(t)$. Moreover

$$0 \leq S_k(t) \leq S_{k+1}(t).$$

If $c \leq c_o$ it follows from the monotone convergence theorem for forms (cf. [16, Theorem S.14]) that

$$\lim_{k\to\infty} S_k(t) = S(t)$$

strongly in L^2_μ. Then $u(t) = S(t)u_0$ is positive.

Finally, as in [1, Proposition 2.5], we can observe that if $c > c_o$ then

$$\lim_{k\to\infty} \|S_k(t)\| = \infty, \quad t > 0.$$

Appendix A

Let us state the following estimates

$$\begin{aligned} -\sum_{j\neq i} |a_i - a_j|^2 + \frac{n+1}{2}|x-a_i|^2 &\leq \sum_{i=1}^{n} |x-a_i|^2 \\ &\leq (2n-1)|x-a_i|^2 + 2\sum_{j\neq i} |a_i - a_j|^2 \end{aligned} \tag{A.1}$$

for any $i, j \in \{1, \ldots, n\}$.

In fact, starting from the inequalities

$$|x-a_j|^2 = |x - a_i + a_i - a_j|^2 \leq 2|x-a_i|^2 + 2|a_i - a_j|^2$$

$$|x-a_j|^2 \geq \frac{|x-a_i|^2}{2} - |a_i-a_j|^2,$$

as a consequence we obtain

$$\sum_{i=1}^{n} |x-a_i|^2 = |x-a_i|^2 + \sum_{j\neq i} |x-a_j|^2 \leq |x-a_i|^2 + 2(n-1)|x-a_i|^2 + 2\sum_{i\neq j}^{n} |a_i-a_j|^2$$

and

$$\sum_{i=1}^{n} |x-a_i|^2 \geq |x-a_i|^2 + \frac{n-1}{2}|x-a_i|^2 - \sum_{i\neq j}^{n} |a_i-a_j|^2.$$

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