

Accepted Manuscript

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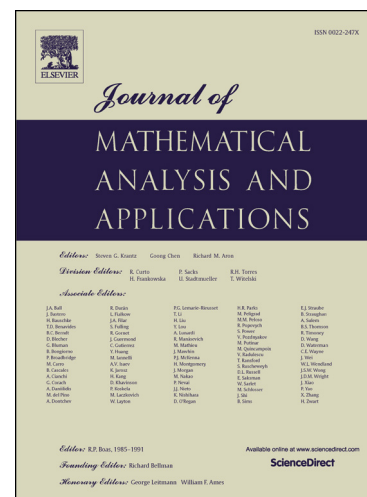
PII: S0022-247X(18)30407-4
DOI: <https://doi.org/10.1016/j.jmaa.2018.05.014>
Reference: YJMAA 22246

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 21 January 2018

Please cite this article in press as: K. Yagdjian, A. Balogh, The maximum principle and sign changing solutions of the hyperbolic equation with the Higgs potential, *J. Math. Anal. Appl.* (2018), <https://doi.org/10.1016/j.jmaa.2018.05.014>

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The maximum principle and sign changing solutions of the hyperbolic equation with the Higgs potential

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Abstract

In this article we discuss the maximum principle for the linear equation and the sign changing solutions of the semilinear equation with the Higgs potential. Numerical simulations indicate that the bubbles for the semilinear Klein-Gordon equation in the de Sitter space-time are created and apparently exist for all times.

Keywords: maximum principle; sign-changing solutions; semilinear Klein-Gordon equation; de Sitter space-time; global solutions; Higgs potential

1. Introduction

In this article we discuss the maximum principle for the linear equation and the sign changing solutions of the semilinear equation with the Higgs potential. The Klein-Gordon equation with the Higgs potential (the Higgs boson equation) in the de Sitter space-time is the equation

$$\psi_{tt} - e^{-2t}\Delta\psi + n\psi_t = \mu^2\psi - \lambda\psi^3, \quad (1)$$

where Δ is the Laplace operator in $x \in \mathbb{R}^n$, $n = 3$, $t > 0$, $\lambda > 0$, and $\mu > 0$. We assume that $\psi = \psi(x, t)$ is a real-valued function.

We focus on the zeros of the solutions to the linear and semilinear hyperbolic equation in the Minkowski and de Sitter space-times. One motivation for the study of the maximum principle, sign changing solutions and zeros of the solutions to the linear and semilinear hyperbolic equation comes from the cosmological contents and quantum field theory. It is of considerable

interest for particle physics and inflationary cosmology to study the so-called bubbles [3], [15], [26]. In [14] bubble is defined as a simply connected domain surrounded by a wall such that the field approaches one of the vacuums outside of a bubble. The creation and growth of bubbles is an interesting mathematical problem [3, Ch.7], [15]. In this paper, for the continuous solution $\psi = \psi(x, t)$ to the Klein-Gordon equation, for every given positive time t we define a bubble as a maximal connected set of points $x \in \mathbb{R}^n$ at which solution changes sign.

Another motivation to study all these closely related properties comes from the issue of the existence of a global in time solution to non-linear equation. Consider the Cauchy problem for the linear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad u_0, u_1 \in C^\infty(\mathbb{R}^n). \end{cases}$$

If one can prove that the solution $u = u(x, t)$ vanishes at some point (x_b, t_b) , then it opens the door to study the blowup phenomena for the equation

$$\begin{cases} \partial_t^2 v - \Delta v + (\partial_t v)^2 - |\nabla v|^2 = 0, \\ v(x, 0) = v_0(x), \quad v_t(x, 0) = v_1(x), \quad v_0, v_1 \in C^\infty(\mathbb{R}^n), \end{cases}$$

which is the Nirenberg's Example (see, e.g. [13]) of the quasilinear equation. Indeed, the transformation

$$u(x, t) = \exp(v(t, x))$$

shows that $u(x_b, t_b) = 0$ at some $t_b > 0$ and $x_b \in \mathbb{R}^n$ implies $v(x_b, t_b) = -\infty$. To avoid blowup phenomena one can restrict the initial data to be small in some norm. (For details, see, e.g., [28].) Therefore to guarantee existence of the global solution to quasilinear equation, the solution $u = u(x, t)$ of the related linear equation must keep sign for all $t > 0$ and all $x \in \mathbb{R}^n$. This link between sign preserving solutions and global in time solvability is especially easy to trace in the case of $n = 3$. In fact, the explicit representation formulas for the solutions to the linear equation play key role. On the other hand for the equations with the variable coefficients and, in particular, for the linear hyperbolic equations in the curved space-time, the new global in time explicit representation formulas were obtained very recently (see, [29, 34]). For the results on the sign changing solutions of the quasilinear equations one can consult [24].

The outline of the discussion in this paper is organized as follows. In Section 2 we describe the maximum principle for the wave equation in the Minkowski space-time when the initial data are subharmonic or superharmonic. In Section 3 we present the maximum principle for the linear Klein-Gordon equation in the de Sitter space-time. Theorem 3.1 of that section guarantees that the solution does not changes sign, that is, it provides with some necessary conditions to have a sign-changing solution. Section 4 is devoted to kernels of the integral transforms have been used in the proofs. Section 5 is a bridge between Section 6 and previous sections. It is aimed to give some theoretical background material about semilinear Klein-Gordon equation in the de Sitter space-time with the Higgs potential. Section 5 also prepares the reader to Section 6, which is about numerical simulations on the evolution of the bubbles in the de Sitter space-time.

2. The maximum principle in the Minkowski space-time

In [20] the following maximum principle is established for the wave operator

$$L := \partial_t^2 - \Delta,$$

where Δ is the Laplace operator in $x \in \mathbb{R}^n$. Denote

$$N = \begin{cases} \frac{n-2}{2} & \text{if } n \text{ even} \\ \frac{n-3}{2} & \text{if } n \text{ odd.} \end{cases}$$

Let u satisfy the differential inequality

$$\frac{\partial^N}{\partial t^N}(L[u]) \leq 0 \quad \text{for all } t \in [0, T],$$

and the initial conditions

$$\frac{\partial^k u}{\partial t^k}(x, 0) = 0, \quad k = 0, 1, \dots, N, \quad \frac{\partial^{N+1} u}{\partial t^{N+1}}(x, 0) \leq 0,$$

for all x in the domain $D_0 \subseteq \mathbb{R}^n$. Then

$$u(x, t) \leq 0$$

in the domain of dependence of D_0 , where $t \leq T$.

We recall definition of the forward light cone $D_+(x_0, t_0)$, and the backward light cone $D_-(x_0, t_0)$, in the Minkowski space-time for the point $(x_0, t_0) \in \mathbb{R}^{n+1}$:

$$D_{\pm}(x_0, t_0) := \left\{ (x, t) \in \mathbb{R}^{n+1}; |x - x_0| \leq \pm(t - t_0) \right\}.$$

For the domain $D_0 \subseteq \mathbb{R}^n$ define a dependence domain of D_0 as follows:

$$D(D_0) := \bigcup_{x_0 \in \mathbb{R}^n, t_0 \in [0, \infty)} \{D_-(x_0, t_0); D_-(x_0, t_0) \cap \{t = 0\} \subset D_0\}.$$

In particular, according to Theorem 1 [20], for $x \in \mathbb{R}^3$ if u satisfies the differential inequality

$$Lu \leq 0, \quad \text{for all } t \leq T, \quad (2)$$

and the initial conditions

$$u(x, 0) = 0, \quad u_t(x, 0) \leq 0, \quad \text{for all } x \in D_0 \subseteq \mathbb{R}^3,$$

then $u(x, t) \leq 0$ in the domain of dependence of D_0 , where $t \leq T$. The statement is a simple consequence of the Duhamel's principle and the well-known Kirchhoff formula

$$u(x_1, x_2, x_3, t) = \frac{1}{4\pi} \int_{S_t(x_1, x_2, x_3)} \frac{\varphi(\alpha_1, \alpha_2, \alpha_3)}{t} dS_t, \quad (3)$$

for the solution of the Cauchy problem for the wave equation (see, e.g., [21]), where

$$L[u] = 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = \varphi(x),$$

and $S_t(x_1, x_2, x_3)$ is a sphere of radius t centered at (x_1, x_2, x_3) . The kernel $1/t$ of the integral operator (3) is positive. The next statement also can be proved by the Kirchhoff formula.

Theorem 2.1. *Assume that the C^2 - functions $u = u(x, t)$, $\varphi_0 = \varphi_0(x)$, $\varphi_1 = \varphi_1(x)$ satisfy the differential inequality*

$$L[u] + \Delta\varphi_0 + t\Delta\varphi_1 \leq 0, \quad \text{for all } t \leq T, \quad (4)$$

and u takes the initial values

$$u(x, 0) = \varphi_0(x), \quad u_t(x, 0) = \varphi_1(x) \quad \text{for all } x \in D_0 \subseteq \mathbb{R}^3. \quad (5)$$

Then $u(x, t) \leq \varphi_0(x) + t\varphi_1(x)$ in the domain of dependence of D_0 , where $t \leq T$.

Proof. For $w = u - \varphi_0 - t\varphi_1(x)$ we have

$$L[w] = L[u] - L[\varphi_0] - L[t\varphi_1] = L[u] + \Delta\varphi_0 + t\Delta\varphi_1 \leq 0, \quad \text{for all } t \leq T,$$

and the initial conditions

$$w(x, 0) = 0, \quad w_t(x, 0) = 0 \quad \text{for all } x \in D_0 \subseteq \mathbb{R}^3.$$

Then we apply Theorem 1 [20]. □

Corollary 2.2. *Assume that the function $u \in C^2$ satisfies*

$$L[u] + \Delta\varphi_0 \leq 0, \quad \text{for all } t \leq T,$$

and u takes the initial values (5), where $\varphi_1(x) \leq 0$ in D_0 . Then $u(x, t) \leq \varphi_0(x)$ in the domain of dependence of D_0 , where $t \leq T$.

The definition of the superharmonic functions will be used in the next corollaries can be found in [27]. We are not going to prove the next statements for the less smooth superharmonic functions or for superharmonic function of higher order.

Corollary 2.3. *Assume that the function u satisfies*

$$L[u] \leq 0, \quad \text{for all } t \leq T,$$

and u takes the initial values (5), where $\varphi_1(x) \leq 0$ in D_0 . Suppose that $\varphi_0 \in C^2$ is superharmonic in $D_0 \subseteq \mathbb{R}^3$. Then $u(x, t) \leq \varphi_0(x)$ in the domain of dependence of D_0 , where $t \leq T$.

Corollary 2.4. *Assume that the function u satisfies*

$$L[u] \leq 0, \quad \text{for all } t \leq T,$$

and u takes the initial values (5). Suppose that $\varphi_0, \varphi_1 \in C^2$ are superharmonic in $D_0 \subseteq \mathbb{R}^3$. Then $u(x, t) \leq \varphi_0(x) + t\varphi_1(x)$ in the domain of dependence of D_0 , where $t \leq T$.

Remark 2.5. *The analogous statements are valid with the subharmonic functions $\varphi_0, \varphi_1 \in C^2$.*

We also note that the conditions on the first and second initial data of the solution to the partial differential inequalities (2) and (4) are asymmetric. The asymmetry exists also in the Cauchy problem but it reveals itself only in the loss of regularity in one derivative in the Sobolev spaces $H_{(s)}(\mathbb{R}^n)$.

Thus, Theorem 2.1, in particular, gives sufficient conditions for the solution of the linear equation to be sign-preserving. If we turn to the linear Klein-Gordon equation in the Minkowski space

$$u_{tt} - \Delta u + m^2 u = f,$$

with $m > 0$, then the functional $F(t) := \int_{\mathbb{R}^3} u(x, t) dx$ solves the differential equation $F'' + m^2 F = \int_{\mathbb{R}^3} f(x, t) dx$. The solution $u = u(x, t)$ cannot preserve the sign, for instance, if $u(x, 0) = 0$, $f = f(x)$, and

$$2 \left| \int_{\mathbb{R}^3} f(x, t) dx \right| < \left| \int_{\mathbb{R}^3} u_t(x, 0) dx \right|,$$

since

$$F(t) = \frac{1}{m} \sin(mt) \int_{\mathbb{R}^3} u_t(x, 0) dx + \int_0^t \frac{1}{m} \sin(m(t - \tau)) \int_{\mathbb{R}^3} f(x, \tau) dx d\tau.$$

On the other hand, for the linear Klein-Gordon operator with the imaginary mass $L_{KGM} := \partial_t^2 - \Delta - M^2$, if

$$L_{KGM}[u] = f, \tag{6}$$

then for the functional F with $F(0) = 0$, we have

$$F(t) = \frac{1}{M} \sinh(Mt) \int_{\mathbb{R}^3} u_t(x, 0) dx + \int_0^t \frac{1}{M} \sinh(M(t - \tau)) \int_{\mathbb{R}^3} f(x, \tau) dx d\tau.$$

Although the functional F for $f \leq 0$ and $\int_{\mathbb{R}^3} u_t(x, 0) dx \leq 0$ is non-positive if t is large, we cannot conclude that the solution u is sign preserving.

On the other hand, we can apply the integral transform approach (see [34] and references therein) and obtain the following result for the equation (6).

Theorem 2.6. *Assume that the function u satisfies*

$$L_{KGM}[u] \leq 0, \quad \text{for all } t \leq T,$$

and $L_{KGM}[u] \in C^2$ is a superharmonic in x function. Suppose that $u(x, 0)$ and $u_t(x, 0)$ are superharmonic non-positive functions in $D_0 \subseteq \mathbb{R}^3$. Then

$$\begin{aligned} u(x, t) \leq & \int_0^t L_{KGM}[u](x, b) \frac{1}{M} \sinh(M(t-b)) db \\ & + \cosh(Mt)u(x, 0) + \frac{1}{M} \sinh(Mt)u_t(x, 0) \quad \text{for all } t \leq T \end{aligned} \quad (7)$$

in the domain of dependence of D_0 . In particular,

$$u(x, t) \leq 0 \quad \text{for all } t \leq T$$

in the domain of dependence of D_0 .

Proof. If we denote $f := (\partial_t^2 - \Delta - M^2)u$, $\varphi_0 := u(x, 0)$, and $\varphi_1 := u_t(x, 0)$, then according to the integral transform approach formulas [34] we can write

$$\begin{aligned} u(x, t) = & \int_0^t db \int_0^{t-b} I_0 \left(M \sqrt{(t-b)^2 - r^2} \right) v_f(x, r; b) dr \\ & + v_{\varphi_0}(x, t) + \int_0^t \frac{\partial}{\partial t} I_0 \left(M \sqrt{t^2 - r^2} \right) v_{\varphi_0}(x, r) dr \\ & + \int_0^t I_0 \left(M \sqrt{t^2 - r^2} \right) v_{\varphi_1}(x, r) dr, \quad x \in \mathbb{R}, \quad t > 0, \end{aligned}$$

where the function $v_f(x, t; b)$ is the solution to the Cauchy problem for the wave equation

$$v_{tt} - \Delta v = 0, \quad v(x, 0; b) = f(x, b), \quad v_t(x, 0) = 0,$$

while v_φ is the solution of the Cauchy problem

$$v_{tt} - \Delta v = 0, \quad v(x, 0) = \varphi(x), \quad v_t(x, 0) = 0.$$

Here $I_0(z)$ is the modified Bessel function of the first kind. Then the statement of this theorem follows from Theorem 2.1 and the properties of the function $I_0(z)$. Indeed, due to Corollary 2.3, we have

$$v_f(x, r; b) \leq f(x, b), \quad v_{\varphi_0}(x, r) \leq \varphi_0(x), \quad v_{\varphi_1}(x, r) \leq \varphi_1(x)$$

for all corresponding x, r , and b . The function $I_0(z)$ is positive while $I'_0(z)$ is non-negative for $z > 0$. Thus, the inequality

$$\begin{aligned} u(x, t) \leq & \int_0^t db f(x, b) \int_0^{t-b} I_0 \left(M \sqrt{(t-b)^2 - r^2} \right) dr \\ & + \varphi_0(x) + \varphi_0(x) \int_0^t \frac{\partial}{\partial t} I_0 \left(M \sqrt{t^2 - r^2} \right) dr \\ & + \varphi_1(x) \int_0^t I_0 \left(M \sqrt{t^2 - r^2} \right) dr, \quad x \in \mathbb{R}, \quad t > 0, \end{aligned}$$

and the result of integrations prove theorem. \square

On the other hand, in order to prove a sign changing property of the solutions to the semilinear equations for those no explicit formulas are available, the F -functional method can be applied. For details see [31].

3. The maximum principle in the de Sitter space-time

For the hyperbolic equation with variable coefficients such maximum principle is known only in the one dimensional case (see, e.g., [17]) and for Euler-Poisson-Darboux equation [27]. We consider the linear part of the equation

$$u_{tt} - e^{-2t} \Delta u - M^2 u = -e^{\frac{n}{2}t} V'(e^{-\frac{n}{2}t} u), \quad (8)$$

with $M \geq 0$ and the potential function $V = V(\psi)$. If we denote the non-covariant Klein-Gordon operator in the de Sitter space-time

$$L_{KGdS} := \partial_t^2 - e^{-2t} \Delta - M^2,$$

then (8) can be written as follows:

$$L_{KGdS}[u] = -e^{\frac{n}{2}t} V'(e^{-\frac{n}{2}t} u).$$

The equation (8) covers two important cases. The first one is the Higgs boson equation (1) that leads to (8) if $\psi = e^{-\frac{n}{2}t} u$. Here $V'(\psi) = \lambda \psi^3$ and $M^2 = \mu^2 + n^2/4$ with $\lambda > 0$ and $\mu > 0$, while $n = 3$. The second case is the case of the covariant Klein-Gordon equation

$$\psi_{tt} + n\psi_t - e^{-2t} \Delta \psi + m^2 \psi = -V'(\psi),$$

with small physical mass, that is $0 \leq m \leq n/2$. For the last case $M^2 = n^2/4 - m^2$. It is evident that the last equation is related to the equation (8) via transform $\psi = e^{-\frac{n}{2}t}u$.

It is known that the Klein-Gordon quantum fields whose squared physical masses are negative (imaginary mass) represent tachyons. (See, e.g., [2].) In [2] the Klein-Gordon equation with imaginary mass is considered. It is shown that localized disturbances spread with at most the speed of light, but grow exponentially. The conclusion is made that free tachyons have to be rejected on stability grounds.

The Klein-Gordon quantum fields on the de Sitter manifold with imaginary mass present scalar tachyonic quantum fields. Epstein and Moschella [5] give an exhaustive study of scalar tachyonic quantum fields which are linear Klein-Gordon quantum fields on the de Sitter manifold whose masses take an infinite set of discrete values $m^2 = -k(k+n)$, $k = 0, 1, 2, \dots$. The corresponding linear equation is

$$\psi_{tt} + n\psi_t - e^{-2t}\Delta\psi + m^2\psi = 0.$$

If n is an odd number, then m takes value at the knot points set [33].

The nonexistence of a global in time solution of the semilinear Klein-Gordon massive tachyonic (self-interacting quantum fields) equation in the de Sitter space-time is proved in [30]. More precisely, consider the semilinear equation

$$\psi_{tt} + n\psi_t - e^{-2t}\Delta\psi - m^2\psi = c|\psi|^{1+\alpha},$$

which is commonly used model for general nonlinear problems. Then, according to Theorem 1.1 [30], if $c \neq 0$, $\alpha > 0$, and $m \neq 0$, then for every positive numbers ε and s there exist functions $\psi_0, \psi_1 \in C_0^\infty(\mathbb{R}^n)$ such that their norms in the Sobolev space $H_{(s)}(\mathbb{R}^n)$ are small, $\|\psi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\psi_1\|_{H_{(s)}(\mathbb{R}^n)} \leq \varepsilon$, but the solution $\psi = \psi(x, t)$ with the initial values

$$\psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x),$$

blows up in finite time. This implies also blowup for the sign-preserving solutions of the equation

$$\psi_{tt} + n\psi_t - e^{-2t}\Delta\psi - m^2\psi = c|\psi|^\alpha\psi.$$

The next theorem gives certain kind of maximum principle for the non-covariant Klein-Gordon equation in the de Sitter space-time. Define the “forward light cone” $D_+^{dS}(x_0, t_0)$ and the “backward light cone” $D_-^{dS}(x_0, t_0)$, in the de Sitter space-time for the point $(x_0, t_0) \in \mathbb{R}^{n+1}$, as follows

$$D_{\pm}^{dS}(x_0, t_0) := \left\{ (x, t) \in \mathbb{R}^{n+1} ; |x - x_0| \leq \pm(e^{-t_0} - e^{-t}) \right\}.$$

For the domain $D_0 \subseteq \mathbb{R}^n$ define dependence domain of D_0 as follows:

$$D^{dS}(D_0) := \bigcup_{x_0 \in \mathbb{R}^n, t_0 \in [0, \infty)} \left\{ D_-^{dS}(x_0, t_0) ; D_-^{dS}(x_0, t_0) \cap \{t = 0\} \subset D_0 \right\}.$$

Theorem 3.1. *Assume that $M > 1$ and the function u satisfies*

$$L_{KGdS}[u] \leq 0, \quad \text{for all } t \leq T,$$

and $L_{KGdS}[u] \in C^2$ is a superharmonic in x function. Suppose that $u(x, 0)$ and $u_t(x, 0)$ are superharmonic non-positive functions in $D_0 \subseteq \mathbb{R}^3$. Then

$$\begin{aligned} u(x, t) \leq & \int_0^t L_{KGdS}[u](x, b) \frac{1}{M} \sinh(M(t - b)) db \\ & + \cosh(Mt)u(x, 0) + \frac{1}{M} \sinh(Mt)u_t(x, 0) \end{aligned} \quad (9)$$

for all $t \in [\ln(M/(M - 1)), T]$ in the domain of dependence of D_0 . In particular,

$$u(x, t) \leq 0 \quad \text{for all } t \in [\ln(M/(M - 1)), T] \quad (10)$$

in the domain of dependence of D_0 .

If $u(x, 0) \equiv 0$, then the statements (9), (10) hold also for all $t \in [0, T]$ and each $M \geq 0$.

Proof. We are going to apply the integral transform and the kernel functions $E(x, t; x_0, t_0; M)$, $K_0(z, t; M)$, and $K_1(z, t; M)$ from [32]. First we introduce the function

$$\begin{aligned} E(x, t; x_0, t_0; M) = & 4^{-M} e^{M(t_0+t)} \left((e^{-t} + e^{-t_0})^2 - (x - x_0)^2 \right)^{-\frac{1}{2}+M} \\ & \times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-t_0} - e^{-t})^2 - (x - x_0)^2}{(e^{-t_0} + e^{-t})^2 - (x - x_0)^2}\right). \end{aligned}$$

Here $F(a, b; c; \zeta)$ is the hypergeometric function. (See, e.g., [1].) Next we define the kernels $K_0(z, t; M)$ and $K_1(z, t; M)$ by

$$\begin{aligned} K_0(z, t; M) &:= - \left[\frac{\partial}{\partial b} E(z, t; 0, b; M) \right]_{b=0} \\ &= 4^{-M} e^{tM} ((1 + e^{-t})^2 - z^2)^{-\frac{1}{2}+M} \frac{1}{(1 - e^{-t})^2 - z^2} \\ &\quad \times \left[(e^{-t} - 1 + M(e^{-2t} - 1 - z^2)) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2}\right) \right. \\ &\quad \left. + (1 - e^{-2t} + z^2) \left(\frac{1}{2} + M\right) F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2}\right) \right] \end{aligned} \quad (11)$$

and $K_1(z, t; M) := E(z, t; 0, 0; M)$, that is,

$$\begin{aligned} K_1(z, t; M) &= 4^{-M} e^{Mt} ((1 + e^{-t})^2 - z^2)^{-\frac{1}{2}+M} \\ &\quad \times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - z^2}{(1 + e^{-t})^2 - z^2}\right), \quad 0 \leq z \leq 1 - e^{-t}, \end{aligned}$$

respectively. These kernels have been introduced and used in [29, 30] in the representation of the solutions of the Cauchy problem. The positivity of the kernels E , K_0 , and K_1 is proved in the next section.

The solution $u = u(x, t)$ to the Cauchy problem

$$u_{tt} - e^{-2t} \Delta u - M^2 u = f, \quad u(x, 0) = 0, \quad u_t(x, 0) = 0,$$

with $f \in C^\infty(\mathbb{R}^{n+1})$ and with vanishing initial data is given [32] by the next expression

$$u(x, t) = 2 \int_0^t db \int_0^{e^{-b}-e^{-t}} dr v_f(x, r; b) E(r, t; 0, b; M), \quad (12)$$

where the function $v_f(x, t; b)$ is a solution to the Cauchy problem for the wave equation

$$v_{tt} - \Delta v = 0, \quad v(x, 0; b) = f(x, b), \quad v_t(x, 0) = 0. \quad (13)$$

If the superharmonic function f is also non-positive, $f(x, t) \leq 0$, then due to Corollary 2.4 we conclude

$$v_f(x, r; b) \leq f(x, b) \leq 0,$$

in the domain of dependence of D_0 . Due to the relation

$$\int_0^{e^{-b}-e^{-t}} E(r, t; 0, b; M) dr = \frac{1}{2M} \sinh(M(t-b)) db \quad (14)$$

of Proposition 1 [30], it follows

$$\begin{aligned} & \int_0^t db \int_0^{e^{-b}-e^{-t}} dr v_f(x, r; b) E(r, t; 0, b; M) \\ & \leq \int_0^t f(x, b) db \int_0^{e^{-b}-e^{-t}} E(r, t; 0, b; M) dr \\ & \leq \int_0^t f(x, b) \frac{1}{2M} \sinh(M(t-b)) db \leq 0, \end{aligned}$$

provided that $E(r, t; 0, b; M) \geq 0$.

The solution $u = u(x, t)$ to the Cauchy problem

$$u_{tt} - e^{-2t} \Delta u - M^2 u = 0, \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad (15)$$

with $u_0, u_1 \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$, can be represented [32] as follows:

$$\begin{aligned} u(x, t) &= e^{\frac{t}{2}} v_{u_0}(x, \phi(t)) + 2 \int_0^1 v_{u_0}(x, \phi(t)s) K_0(\phi(t)s, t; M) \phi(t) ds \\ &+ 2 \int_0^1 v_{u_1}(x, \phi(t)s) K_1(\phi(t)s, t; M) \phi(t) ds, \quad x \in \mathbb{R}^n, \quad t > 0, \end{aligned}$$

where $\phi(t) := 1 - e^{-t}$. Here, for $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, the function $v_\varphi(x, \phi(t)s)$ coincides with the value $v(x, \phi(t)s)$ of the solution $v(x, t)$ of the Cauchy problem

$$v_{tt} - \Delta v = 0, \quad v(x, 0) = \varphi(x), \quad v_t(x, 0) = 0.$$

For the function u_1 , which is superharmonic, from Corollary 2.3 we conclude

$$v_{u_1}(x, r) \leq u_1(x).$$

From the definition of $K_1(r, t; M)$ and relation (14) with $b = 0$ it follows

$$\begin{aligned} 2 \int_0^{\phi(t)} v_{u_1}(x, r) K_1(r, t; M) dr &\leq 2u_1(x) \int_0^{\phi(t)} K_1(r, t; M) dr \\ &= \frac{1}{M} \sinh(Mt) u_1(x). \end{aligned}$$

since $K_1(r, t; M) \geq 0$. In particular, if $u_1(x) \leq 0$, then

$$2 \int_0^{\phi(t)} v_{u_1}(x, r) K_1(r, t; M) dr \leq 0.$$

Further, if $u_0 \in C^2$ is superharmonic, that is $\Delta u_0 \leq 0$, then, according to Corollary 2.3, $(\partial_t^2 - \Delta)v_{u_0} = 0$ implies $v_{u_0}(x, t) \leq u_0(x)$. Consequently, if $M > 1$, then $K_0(r, t; M) \geq 0$ for all $t \in [\ln(M/(M-1)), T]$, and

$$\begin{aligned} & e^{\frac{t}{2}} v_{u_0}(x, \phi(t)) + 2 \int_0^{\phi(t)} v_{\varphi_0}(x, r) K_0(r, t; M) dr \\ & \leq u_0(x) \left[e^{\frac{t}{2}} + 2 \int_0^{\phi(t)} K_0(r, t; M) dr \right] = \cosh(Mt) u_0(x). \end{aligned}$$

Here the following relation has been used

$$e^{\frac{t}{2}} + 2 \int_0^{\phi(t)} K_0(r, t; M) dr = \cosh(Mt).$$

To verify the last relation we note that according to Theorem 1.1 [34] the left-hand side solves the problem (15) for the hyperbolic equation with $u_0(x, t) = 1$, $u_1(x, t) = 0$. Meanwhile the right-hand side solves the same problem. The cone of dependence in the problem and the uniqueness complete a verification of the relation. Theorem 3.1 is proved. \square

We do not know if the condition of superharmonicity can be relaxed.

4. The positivity of the kernel functions E , K_0 and K_1

Proposition 4.1. *Assume that $M \geq 0$. Then*

$$\begin{aligned} & E(r, t; 0, b; M) > 0, \quad \text{for all } 0 \leq b \leq t, \quad r \leq e^{-b} - e^{-t}, \quad t \in [0, \infty), \\ & K_1(r, t; M) > 0 \quad \text{for all } r \leq 1 - e^{-t}, \quad t \in [0, \infty). \end{aligned}$$

If we assume that $M > 1$, then

$$K_0(r, t; M) > 0 \quad \text{for all } r \leq 1 - e^{-t} \quad \text{and for all } t > \ln \frac{M}{M-1}.$$

Proof. Indeed, for $0 \leq b \leq t$ and $r \leq e^{-b} - e^{-t}$ we have

$$\begin{aligned} E(r, t; 0, b; M) &= 4^{-M} e^{M(b+t)} \left((e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2}+M} \\ &\quad \times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2}\right). \end{aligned}$$

For $M \geq 0$ the parameters $a = b = 1/2 - M$ and $c = 1$ of the function $F(a, b; c; z)$ satisfy the relation $a + b \leq c$. Then, we denote

$$z := \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2} \leq 1, \quad 0 \leq b \leq t, \quad r \leq e^{-b} - e^{-t}.$$

Hence, it remains to check the sign of the function $F(a, a; 1; z)$ with parameter $a \leq 1/2$ and $z \in (0, 1)$. If a is not a non-positive integer then the series

$$F(a, a; 1; x) = \sum_{n=0}^{\infty} \frac{[(a)_n]^2}{[n!]^2} x^n, \quad (a)_n := a(a+1) \cdots (a+n-1),$$

is a convergent series for all $x \in [0, 1)$. If a is negative integer, $a = -k$, then $F(a, a; 1; x)$ is polynomial with the positive coefficients:

$$F(a, a; 1; x) = \sum_{n=0}^k \frac{[(a)_n]^2}{[n!]^2} x^n.$$

Since $K_1(r, t; M) := E(r, t; 0, 0; M)$, the first two statements of the proposition are proved.

In order to verify the last statement it suffices to verify the inequality $K_0(r, t; M) > 0$, where $r \in (0, 1)$. Denote $M = (2k+1)/2$. Then $1/2 - M = -k < 0$ and we can write (11) in the equivalent form as follows

$$\begin{aligned} K_0(r, t; M) &= - \left[\frac{\partial}{\partial b} \left\{ 4^{-M} e^{M(b+t)} \left((e^{-t} + e^{-b})^2 - r^2 \right)^{-\frac{1}{2}+M} \right\} \right]_{b=0} \\ &\quad \times F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1 - e^{-t})^2 - r^2}{(1 + e^{-t})^2 - r^2}\right) \\ &\quad - 4^{-M} e^{Mt} \left((e^{-t} + 1)^2 - r^2 \right)^{-\frac{1}{2}+M} \\ &\quad \times \left[\frac{\partial}{\partial b} \left\{ F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^{-b} - e^{-t})^2 - r^2}{(e^{-b} + e^{-t})^2 - r^2}\right) \right\} \right]_{b=0}. \end{aligned}$$

Then we use the relation (20) [1, Sec.2.8]:

$$\begin{aligned}
 & K_0(r, t; M) \\
 = & 4^{-k-1} ((e^{-t} + 1)^2 - r^2)^k e^{(k+\frac{1}{2})t} ((e^t + 1)^2 - r^2 e^{2t})^{-1} \\
 & \times \left\{ (e^{2t}(2k(r^2 + 1) + r^2 - 1) - 2k - 2e^t - 1) \right. \\
 & \quad \times F\left(-k, -k; 1; \frac{(1 - e^{-t})^2 - r^2}{(1 + e^{-t})^2 - r^2}\right) \\
 & \quad \left. + \frac{8k^2 e^t ((r^2 + 1)e^{2t} - 1)}{(e^t + 1)^2 - r^2 e^{2t}} F\left(1 - k, 1 - k; 2; \frac{(1 - e^{-t})^2 - r^2}{(1 + e^{-t})^2 - r^2}\right) \right\}.
 \end{aligned} \tag{16}$$

The functions $F(-k, -k; 1; z)$ and $F(1 - k, 1 - k; 2; z)$ are defined as follows

$$\begin{aligned}
 F(-k, -k; 1; z) &= \sum_{n=0}^{\infty} \frac{[(-k)(-k+1) \cdots (-k+n-1)]^2}{[n!]^2} z^n, \\
 F(1 - k, 1 - k; 2; z) &= \sum_{n=0}^{\infty} \frac{[(1 - k)_n]^2}{[n!]^2 (n+1)} z^n.
 \end{aligned}$$

Here we have denoted

$$z := \frac{(1 - e^{-t})^2 - r^2}{(1 + e^{-t})^2 - r^2} \in [0, 1] \quad \text{for all } t \in [0, \infty), \quad r \in (0, 1 - e^{-t}).$$

Thus,

$$\begin{aligned}
 F(-k, -k; 1; z) &\geq 1 \quad \text{for all } z \in [0, 1), \\
 F(1 - k, 1 - k; 2; z) &\geq 1 \quad \text{for all } z \in [0, 1).
 \end{aligned}$$

On the other hand,

$$e^{2t} (2k(r^2 + 1) + r^2 - 1) - 2k - 2e^t - 1 > 0 \quad \text{for all } r \geq 0 \text{ and } t > \ln \frac{M}{M-1}.$$

Since all terms of (16) are positive, the proposition is proved. \square

Remark 4.2. The graph of the $K_0(r, t; \frac{3}{4})$ shows that the K_0 changes a sign.

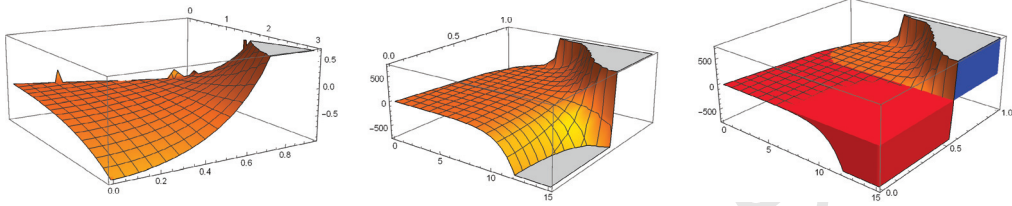


Figure 1: The graph of $K_0(z, t, \frac{3}{4})$, $t \in (0, 3)$ and $t \in (0, 15)$, $z \in (0, 1 - \exp(-t))$

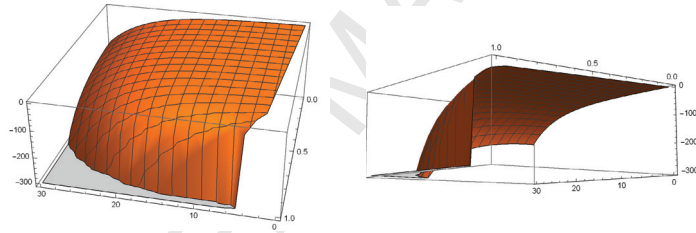


Figure 2: The graph of $K_0(z, t, \frac{1}{6})$, $t \in (0, 30)$, $z \in (0, 1 - \exp(-t))$

Remark 4.3. The graph of the $K_0(r, t; \frac{1}{6})$ shows that the K_0 does not change a sign.

For $M = 1/2$ the kernels are (see [33])

$$E\left(r, t; 0, b; \frac{1}{2}\right) = \frac{1}{2}e^{\frac{1}{2}(b+t)}, \quad K_0\left(r, t; \frac{1}{2}\right) = -\frac{1}{4}e^{\frac{1}{2}t}, \quad K_1\left(r, t; \frac{1}{2}\right) = \frac{1}{2}e^{\frac{1}{2}t}$$

and the solution can be written as follows

$$\begin{aligned} u(x, t) = & \int_0^t db \int_0^{e^{-b}-e^{-t}} dr e^{\frac{1}{2}(b+t)} v_f(x, r; b) + e^{\frac{1}{2}t} v_{u_0}(x, \phi(t)) \\ & - \frac{1}{2}e^{\frac{1}{2}t} \int_0^{\phi(t)} v_{u_0}(x, r) dr + e^{\frac{1}{2}t} \int_0^{\phi(t)} v_{u_1}(x, r) dr, \end{aligned}$$

where $x \in \mathbb{R}^n$, $t > 0$. In particular, if $f = 0$ and $u_1 = 0$, then

$$u(x, t) = e^{\frac{1}{2}t} v_{u_0}(x, \phi(t)) - \frac{1}{2} e^{\frac{1}{2}t} \int_0^{\phi(t)} v_{u_0}(x, r) dr, \quad x \in \mathbb{R}^n, \quad t > 0,$$

solves (15). The second term of the last expression is the so-called tail. The tail is of considerable interest in many aspects in physics, and, in particular, in the General Relativity [22].

Remark 4.4. *If we assume that $u(x, t) \leq 0$, then*

$$v_{u_0}(x, s) - \frac{1}{2} \int_0^s v_{u_0}(x, r) dr \leq 0 \quad \text{for all } s \in [0, 1],$$

and the Gronwall's lemma implies $v_{u_0}(x, s) \leq 0$.

The converse statement is not true in general. Indeed, according to the Figure 3, for $v(s) = -e^{-\frac{s^2}{1.2-s^3}} < 0$, the function

$$v(s) - \frac{1}{2} \int_0^s v(r) dr,$$

is positive when $s \geq 0.9$:

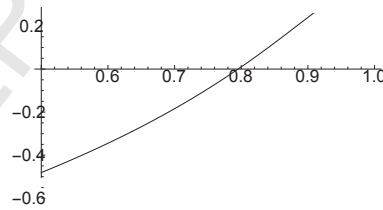


Figure 3: The graph of $v(s) - \frac{1}{2} \int_0^s v(r) dr$ with $v(s) = -e^{-\frac{s^2}{1.2-s^3}}$

If $u_0 = u_0(x)$ is harmonic function in \mathbb{R}^n , then $v_{u_0}(x, r) = u_0(x)$ and

$$u(x, t) = \cosh\left(\frac{1}{2}t\right) u_0(x), \quad x \in \mathbb{R}^n, \quad t > 0.$$

Remark 4.5. *We do not know if the value $M = 1$, that is $m = \sqrt{5}/2$, has some physical significance similar to one when $M = 1/2$, that is $m = \sqrt{2}$, which is the end point of the Higuchi bound [32].*

Conjecture 4.6. *Assume that $M \in [0, 1/2]$. Then*

$$K_0(r, t; M) \leq 0 \quad \text{for all } r \leq 1 - e^{-t} \quad \text{and for all } t > 0.$$

5. The sign-changing solutions for the semilinear Klein-Gordon equation in the de Sitter space with Higgs potential

We are interested in sign-changing solutions of the equation for the Higgs real-valued scalar field in the de Sitter space-time

$$\psi_{tt} + 3\psi_t - e^{-2t}\Delta\psi = \mu^2\psi - \lambda\psi^3. \quad (17)$$

The constants $\psi = \pm\mu/\sqrt{\lambda}$ are non-trivial real-valued solutions of the equation (17). The x -independent solution of (17) solves the Duffing's-type equation

$$\ddot{\psi} + 3\dot{\psi} = \mu^2\psi - \lambda\psi^3,$$

which describes the motion of a mechanical system in a twin-well potential field. Unlike the equation in the Minkowski space-time, that is, the equation

$$\psi_{tt} - \Delta\psi = \mu^2\psi - \lambda\psi^3, \quad (18)$$

the equation (17) has no other time-independent solution. For the equation (18) the existence of a weak global solution in the energy space is known (see, e.g., [7, 8]). The equation (18) for the Higgs scalar field in the Minkowski space-time has the time-independent flat solution

$$\psi_M(x) = \frac{\mu}{\sqrt{\lambda}} \tanh\left(\frac{\mu^2}{2}N \cdot (x - x_0)\right), \quad N, x_0, x \in \mathbb{R}^3, \quad (19)$$

where N is the unit vector. The solution (19), after Lorentz transformation, gives rise to a traveling solitary wave

$$\psi_M(x, t) = \frac{\mu}{\sqrt{\lambda}} \tanh\left(\frac{\mu^2}{2}[N \cdot (x - x_0) \pm v(t - t_0)]\frac{1}{\sqrt{1 - v^2}}\right),$$

where $N, x_0, x \in \mathbb{R}^3$, $t \geq t_0$, and $0 < v < 1$. The set of zeros of the solitary wave $\psi = \psi_M(x, t)$ is the moving boundary of the *wall*.

A global in time solvability of the Cauchy problem for equation (17) is not known, and the only estimate for the lifespan is given by Theorem 0.1 [35]. The local solution exists for every smooth initial data. (See, e.g., [19].) The C^2 solution of the equation (17) is unique and obeys the finite speed of the propagation property. (See, e.g., [11].)

In order to make our discussion more transparent we appeal to the function $u = e^{\frac{3}{2}t}\psi$. For this new unknown function $u = u(x, t)$, the equation (17) takes the form of the semilinear Klein-Gordon equation

$$u_{tt} - e^{-2t} \Delta u - M^2 u = -\lambda e^{-3t} u^3, \quad (20)$$

where a positive number M is defined as follows:

$$M^2 := \frac{9}{4} + \mu^2 > 0.$$

The equation (20) is the equation with imaginary mass. Next, we use the fundamental solution of the corresponding linear operator in order to reduce the Cauchy problem for the semilinear equation to the integral equation and to define a weak solution. We denote by G the resolving operator of the problem

$$u_{tt} - e^{-2t} \Delta u - M^2 u = f, \quad u(x, 0) = 0, \quad \partial_t u(x, 0) = 0.$$

Thus, $u = G[f]$. The operator G is explicitly written in [29] for the case of the real mass. The analytic continuation with respect to the parameter M of this operator allows us also to use G in the case of imaginary mass. More precisely, for $M \geq 0$ we define the operator G acting on $f(x, t) \in C^\infty(\mathbb{R}^3 \times [0, \infty))$ by (12),

$$G[f](x, t) = 2 \int_0^t db \int_0^{e^{-b}-e^{-t}} dr v(x, r; b) E(r, t; 0, b; M),$$

where the function $v(x, t; b)$ is a solution to the Cauchy problem for the wave equation (13).

Let $u_0 = u_0(x, t)$ be a solution of the Cauchy problem

$$\partial_t^2 u_0 - e^{-2t} \Delta u_0 - M^2 u_0 = 0, \quad u_0(x, 0) = \varphi_0(x), \quad \partial_t u_0(x, 0) = \varphi_1(x). \quad (21)$$

Then any solution $u = u(x, t)$ of the equation (20), which takes initial value $u(x, 0) = \varphi_0(x)$, $\partial_t u(x, 0) = \varphi_1(x)$, solves the integral equation

$$u(x, t) = u_0(x, t) - G[\lambda e^{-3\cdot} u^3](x, t). \quad (22)$$

We use the last equation to define a weak solution of the problem for the partial differential equation.

Definition 5.1. *If u_0 is a solution of the Cauchy problem (21), then the solution $u = u(x, t)$ of (22) is said to be a weak solution of the Cauchy problem for the equation (20) with the initial conditions $u(x, 0) = \varphi_0(x)$, $\partial_t u(x, 0) = \varphi_1(x)$.*

It is suggested in [31] to measure a variation of the sign of the function ψ by the deviation from the Hölder inequality

$$\left| \int_{\mathbb{R}^n} u(x) dx \right|^3 \leq C_{supp u} \int_{\mathbb{R}^n} |u(x)|^3 dx$$

of the inequality between the integral of the function and the self-interaction functional:

$$\left| \int_{\mathbb{R}^n} u(x) dx \right|^3 \leq \nu(u) \left| \int_{\mathbb{R}^n} u^3(x) dx \right|$$

provided that $\int_{\mathbb{R}^n} u^3(x) dx \neq 0$. For the solutions with the initial data with supports in some bounded ball of radius R due to finite speed of propagation the constant $C_{supp u}$ depends of R alone and is the same for all solutions. The constant $\nu(u)$ depends on function, but for the solution $u = u(x, t)$ of the equation (22) it is regulated by the equation, that is, $\nu(u) = \nu(t)$ is a function of time universal for all functions. For the sign preserving global in time solutions the rate of growth of the function $\nu(t)$ is restricted from below.

The next definition is a particular case of Definition 1.2 [31]. Time t is regarded as a parameter.

Definition 5.2. *The real valued-function $\psi \in C([0, \infty); L^1(\mathbb{R}^3) \cap L^3(\mathbb{R}^3))$ is said to be asymptotically time-weighted L^3 -non-positive (non-negative), if there exist number $C_\psi > 0$ and positive non-decreasing function $\nu_\psi \in C([0, \infty))$ such that with $\sigma = 1$ ($\sigma = -1$) one has*

$$\left| \int_{\mathbb{R}^3} \psi(x, t) dx \right|^3 \leq -\sigma C_\psi \nu_\psi(t) \int_{\mathbb{R}^3} \psi^3(x, t) dx \quad \text{for all sufficiently large } t.$$

It is evident that any sign preserving function $\psi \in L^3(\mathbb{R}^3)$ with a compact support satisfies the last inequality with $\nu_\psi(t) \equiv 1$ and either $\sigma = 1$ or $\sigma = -1$, while $C_\psi^{1/2}$ is a measure of the support.

As a result of the finite speed of propagation property of the equation (18), any smooth global non-positive (non-negative) solution $\psi = \psi(x, t)$ of (18) with compactly supported initial data is also asymptotically time-weighted L^3 -non-positive (non-negative) with the weight $\nu_\psi(t) = (1+t)^6$.

The following statement follows from Theorem 1.3 [31]. Let $u = u(x, t) \in C([0, \infty); L^q(\mathbb{R}^3))$, $2 \leq q < \infty$, be a global solution of the equation

$$u(x, t) = u_0(x, t) - G[\lambda u^3(y, \cdot)](x, t). \quad (23)$$

where $u_0(x, t)$ solves initial value problem (21) with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ such that

$$\sigma \left(M \int_{\mathbb{R}^3} \varphi_0(x) dx + \int_{\mathbb{R}^3} \varphi_1(x) dx \right) > 0. \quad (24)$$

Assume also that the self-interaction functional satisfies

$$\sigma \int_{\mathbb{R}^3} u^3(z, t) dz \leq 0$$

for all t outside of the sufficiently small neighborhood of zero. Then, the global solution $u = u(x, t)$ cannot be an asymptotically time-weighted L^3 -non-positive (-non-negative) with the weight $\nu_u = \text{const} > 0$.

Thus, the the last statement shows that the continuous global solution of the equation (23) cannot be negative sign preserving provided that it is generated by the function $u_0 = u_0(x, t)$, which obeys (24). Thus, *it takes positive value at some point, that is, it changes a sign.*

An application of the last theorem to the Higgs real-valued scalar field equation (17) with $\mu > 0$ results in the following statement (see also Corollary 1.4 [31]). Let $\psi = \psi(x, t) \in C([0, \infty); L^q(\mathbb{R}^3))$, $2 \leq q < \infty$, be a global weak solution of the equation (17). Assume also that the initial data of $\psi = \psi(x, t)$ satisfy

$$\sigma \left(\left(\sqrt{9 + 4\mu^2} + 3 \right) \int_{\mathbb{R}^3} \psi(x, 0) dx + 2 \int_{\mathbb{R}^3} \partial_t \psi(x, 0) dx \right) > 0 \quad (25)$$

with $\sigma = 1$ ($\sigma = -1$), while

$$\sigma \int_{\mathbb{R}^3} \psi^3(x, t) dx \leq 0$$

is fulfilled for all t outside of the sufficiently small neighborhood of zero.

Then, the global solution $\psi = \psi(x, t)$ cannot be an asymptotically time-weighted L^3 -non-positive (-non-negative) solution with the weight $\nu_\psi(t) = e^{a_\psi t} t^{b_\psi}$, where $a_\psi < \sqrt{9 + 4\mu^2} - 3$, $b_\psi \in \mathbb{R}$.

For the solution $\psi = \psi(x, t) \in C^2(\mathbb{R}^3 \times [0, \infty))$ with the compactly supported smooth initial data $\psi(x, 0), \psi_t(x, 0) \in C_0^\infty(\mathbb{R}^3)$, the finite propagation speed property for (17) with $\mu > 0$ implies that the solution has a support in some cylinder $B_R \times [0, \infty)$, and consequently, if it is sign preserving, it is also asymptotically time-weighted L^3 -non-positive (-non-negative) solution with the weight $\nu_\psi(t) \equiv 1$. This contradicts to the previous statement. Hence, *the global solution with data satisfying (25) and $\psi(x, 0) \leq 0$ must take positive value at some point and, consequently, must take zero value inside of some section $t = \text{const} > 0$. It gives rise to the creation of a bubble.*

6. Evolution of bubbles

Since an issue of the global solution for equation (17) is not resolved, we present some simulation that shows evolution of the bubbles in time.

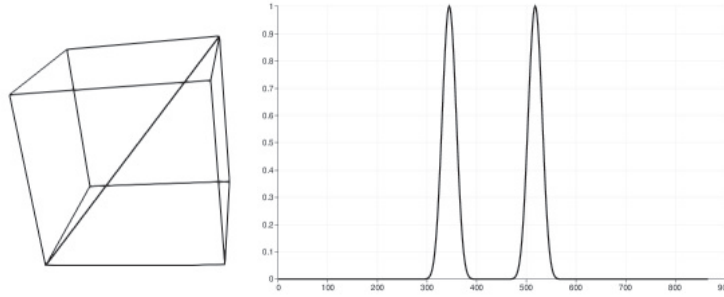
Our numerical approach uses a fourth order finite difference method in space [12] and an explicit fourth order Runge-Kutta method in time [4] for the discretization of the Higgs boson equation. The numerical code has been programmed using the Community Edition of PGI CUDA Fortran [18] on NVIDIA Tesla K40c GPU Accelerators. The grid size in space was $n \times n \times n = 501 \times 501 \times 501$, resulting in a uniform spatial grid spacing of $\delta x_1 = \delta x_2 = \delta x_3 = 2 \times 10^{-3}$. The time step $\delta t = 10^{-4}$ ensured that the Courant–Friedrichs–Lewy (CFL) condition [23] for stability $\left(|\psi| < \frac{\delta x}{\sqrt{3}\delta t} \approx 11.54\right)$ was satisfied for all time. As first initial data ψ_0 we choose the combination of two bell-shaped, infinitely smooth exponential functions

$$\psi_0(x) = B_1(x) + B_2(x) \quad \forall x = (x_1, x_2, x_3) \in \Omega, ,$$

where

$$B_i(x) = \begin{cases} \exp\left(\frac{1}{R_i^2} - \frac{1}{R_i^2 - |x - C_i|^2}\right) & \text{if } |x - C_i| < R_i, \\ 0 & \text{if } |x - C_i| \geq R_i \end{cases}$$

Figure 4: Computational domain and first initial data
Computational domain Initial data along a diagonal line segment



for $i = 1, 2$ with the center of the bell-shapes at $C_1 = (0.4, 0.4, 0.4)$, $C_2 = (0.6, 0.6, 0.6)$, and the radii of the bell-shapes $R_1 = R_2 = 0.2$. Figure 4 shows the computational domain with a diagonal line segment and the line plot of the first initial data $\psi_0(x)$ along that line segment. Note that the initial data is nonnegative with a compact support. The finite cone of influence [11] enables us to use zero boundary conditions on the unit box $\Omega = (0, 1) \times (0, 1) \times (0, 1)$ as computational domain, since the solution's domain of support stayed inside the unit box. As second initial data ψ_1 we choose a constant multiple of the first initial data

$$\psi_1(x) = -5\phi_0(x) \quad \forall x = (x_1, x_2, x_3) \in \Omega.$$

The parameter values are $\lambda = \mu^2 = 0.1$. Initially there is no bubble present. Figure 5 shows the formation and interactions of bubbles. After the two bubbles form around time $t = 0.08$, their size grows continuously. Around time $t = 0.69$ the two bubbles touch, and from that time on they are attached to each other. At time $t = 0.8$ (shown on part (d) of Figure 5) an additional tiny bubble forms inside each of the now merged bubbles. These additional bubbles grow (part (e) of Figure 5 at time $t = 1$); then they flatten and become concave (part (f) of Figure 5 at time $t = 2$). Later hole forms in them and they become toroidal (part (g) of Figure 5 at time $t = 2.15$), and finally they disappear (part (h) of Figure 5 at time $t = 3$). The growth of the larger outer bubble exponentially slows down and it does not seem to change shape after time $t = 3$.

Acknowledgments

The authors are indebted to the anonymous referee for the remarks and suggestions which improved the readability of the text. The authors acknowledge the Texas Advanced Computing Center at the University of Texas at Austin for providing high performance computing and visualization resources that have contributed to the research results reported within this paper. We also gratefully acknowledge the support of NVIDIA Corporation with the donation of the Tesla K40 GPU used for this research. K. Yagdjian was supported by the 2016-17 Research Enhancement Seed Grant from College of Sciences, the University of Texas Rio Grande Valley.

References

- [1] H. Bateman, A. Erdelyi, Higher Transcendental Functions. vol. 1,2, New York: McGraw-Hill, 1953.
- [2] A. Bers, R. Fox, C. G Kuper, S. G. Lipson, The impossibility of free tachyons, in Relativity and Gravitation, eds. C. G. Kuper and Asher Peres, New York, Gordon and Breach Science Publishers, 1971, 41–46.
- [3] S. Coleman, Aspects of Symmetry: Selected Erice Lectures. Cambridge University Press, 1985.
- [4] K. Dekker, J.G. Verwer, Stability of Runge-Kutta methods for Stiff Nonlinear Differential Equations., North-Holland, Amsterdam: Elsevier Science Ltd., 1984.
- [5] H. Epstein, U. Moschella, de Sitter tachyons and related topics. Comm. Math. Phys. 336 (1) (2015) 381–430.
- [6] A. Galstian, K. Yagdjian, Global in time existence of self-interacting scalar field in de Sitter space-times. Nonlinear Analysis: Real World Applications 34 (2017) 110–139.
- [7] J. Ginibre, G. Velo, The global Cauchy problem for the nonlinear Klein-Gordon equation. Math. Z. 189, no. 4: (1985) 487–505.
- [8] J. Ginibre, G. Velo, The global Cauchy problem for the nonlinear Klein-Gordon equation. II. *Ann. Inst. H. Poincaré Anal. Non Linéaire*. 6, no. 1 (1989) 15–35.

- [9] S. W. Hawking, G. F. R. Ellis, The large scale structure of space-time. Cambridge Monographs on Mathematical Physics, No. 1. London-New York: Cambridge University Press, 1973.
- [10] P.W. Higgs, Broken symmetries and the masses of gauge bosons. Phys. Rev. Lett. 13, no. 16 (1964) 508–509.
- [11] L. Hörmander, Lectures on nonlinear hyperbolic differential equations. Berlin: Springer-Verlag, 1997.
- [12] H.B. Keller, V. Pereyra, Symbolic generation of finite difference formulas. Math. Comp. 32 (144) (1978) 955–971.
- [13] S. Klainerman, Global existence for nonlinear wave equations. Comm. Pure Appl. Math. 33 , no. 1 (1980) 43–101.
- [14] T.D. Lee, G.C. Wick, Vacuum stability and Vacuum Excitation in Spin-0 Field. Phys. Rev. D 9 (8) (1974) 2291–2316.
- [15] A. Linde, Particle Physics and Inflationary Cosmology. Harwood, Chur, Switzerland, 1990.
- [16] C. Møller, The theory of relativity. Oxford: Clarendon Press, 1952.
- [17] M.H. Protter, H.F. Weinberger, Maximum Principles in Differential Equations. by Springer-Verlag New York Inc., 1984
- [18] NVIDIA Corporation: PGI CUDA Fortran Compiler. (2017)
- [19] A. Rendall, Partial differential equations in general relativity. Oxford Graduate Texts in Mathematics, 16, Oxford: Oxford University Press, 2008.
- [20] D. Sather, A maximum property of Cauchy’s problem for the wave operator. Arch. Rational Mech. Anal. 21 (1966) 303–309.
- [21] J. Shatah, M. Struwe, Geometric wave equations. Courant Lect. Notes Math., 2. New York Univ., New York: Courant Inst. Math. Sci., 1998.
- [22] S. Sonogo, V. Faraoni, Huygens’ principle and characteristic propagation property for waves in curved space-times. J. Math. Phys. 33 , no. 2 (1992) 625–632.

- [23] G. Strang, Computational science and engineering. Wellesley, MA: Wellesley-Cambridge Press, 2007.
- [24] J. Speck, Finite-time degeneration of hyperbolicity without blowup for quasilinear wave equations, *Analysis&PDE*, 10, no. 8 (2017) 2001–2030.
- [25] A. Vasy, The wave equation on asymptotically de Sitter-like spaces. *Adv. Math.* 223, no. 1 (2010) 49–97.
- [26] N.A. Voronov, A.L. Dyshko, , N.B. Konyukhova, On the Stability of a Self-Similar Spherical Bubble of a Scalar Higgs Field in de Sitter Space. *Physics of Atomic Nuclei* 68, no. 7: (2005) 1218–1226.
- [27] A. Weinstein, On a Cauchy problem with subharmonic initial values. *Ann. Mat. Pura Appl.* (4) 43 (1957) 325–340.
- [28] K. Yagdjian, Global existence in the Cauchy problem for nonlinear wave equations with variable speed of propagation, *New trends in the theory of hyperbolic equations*, *Oper. Theory Adv. Appl.*, 159, Birkhäuser, Basel, (2005) 301–385.
- [29] K. Yagdjian, A. Galstian, Fundamental Solutions for the Klein-Gordon Equation in de Sitter space-time, *Comm. Math. Phys.* 285 (2009) 293–344.
- [30] K. Yagdjian, The semilinear Klein-Gordon equation in de Sitter space-time, *Discrete Contin. Dyn. Syst. Ser. S* 2 (3) (2009) 679–696.
- [31] K. Yagdjian, On the global solutions of the Higgs boson equation, *Comm. Partial Differential Equations* **37** (3) (2012) 447–478.
- [32] K. Yagdjian, Global existence of the scalar field in de Sitter space-time, *J. Math. Anal. Appl.* 396 (1) (2012) 323–344.
- [33] K. Yagdjian, Huygens’ Principle for the Klein-Gordon equation in the de Sitter space-time, *J. Math. Phys.* 54, no. 9 (2013) 091503.
- [34] K. Yagdjian, Integral transform approach to solving Klein-Gordon equation with variable coefficients, *Mathematische Nachrichten*, **288** (17-18) (2015) 2129-2152.
- [35] K. Yagdjian, Global existence of the self-interacting scalar field in the de Sitter universe, *arXiv:1706.07703*.

Figure 5: Formation and interaction of two bubbles

(a) 3D bubbles at $t = 0.08$ (b) 3D bubbles at $t = 0.2$



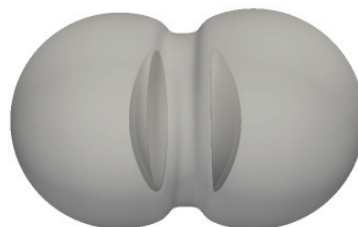
(c) 3D bubbles at $t = 0.69$

(d) 3D bubbles at $t = 0.8$



(e) 3D bubbles at $t = 1$

(f) 3D bubbles at $t = 2$



(g) 3D bubbles at $t = 2.15$

(h) 3D bubbles at $t = 3$

