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Dynamics of a diffusive competition model in spatially heterogeneous environment

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ABSTRACT

We discuss the effects of dispersal rate and spatial heterogeneity on population dynamics via a diffusive Lotka–Volterra competition model under homogeneous Dirichlet boundary conditions. By the theory of monotone dynamical system, we show that in the weak competition case, for some ranges of dispersal rate of competing species 1, if dispersal rate of competing species 2 is sufficiently small, there exists a unique coexistence steady state of the model which is globally asymptotically stable. Moreover, for certain ranges of dispersal rate of competing species 1, we derive the limit behavior of the coexistence steady state as dispersal rate of competing species 2 tends to zero. On the other hand, we demonstrate that for some larger scopes of dispersal rate of competing species 1, if dispersal rate of competing species 2 is small enough, there is no coexistence steady state, whereas one of the semi-trivial steady states of the model is globally asymptotically stable.
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1. Introduction

The evolution of dispersal has been one of the central topics in recent theoretical studies of population dynamics [23]. In particular, spatially heterogeneous reaction–diffusion models for interacting biological species in a bounded isolated habitat patch are especially well-suited for capturing biogeographic features of species' interactions or movements. The joint action of dispersal and spatial heterogeneity often give rise to some interesting phenomenon. It is shown in [5] that for a spatially heterogeneous Lotka–Volterra competition model with the same resources, the slower diffuser always win. For another Lotka–Volterra competition system [10] with the total resource being fixed exactly at the same level, the environmental heterogeneity is usually superior to its homogeneous counterpart in the present of dispersal. It turns out that spatial

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varying environment does not only influence the dynamics of competition system, but also the predator–prey model. For instance, Lou and Wang [24] have successfully extended spatial heterogeneity to a diffusive predator–prey model with homogeneous Neumann boundary conditions and systematically investigated the stability of semi-trivial steady state of the model. Their results suggest that the dynamics of the predator–prey model with spatial heterogeneity is more complicated. There has been considerable interest, by both mathematicians and ecologists, in two-species Lotka–Volterra competition and predator–prey models with spatially heterogeneous interactions, see [3,4,6,16,17,23,25,28] and references therein.

In this paper, we study a diffusive Lotka–Volterra competition model in spatially heterogeneous environment with homogeneous Dirichlet boundary conditions

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mu\Delta u + u(m(x) - u - bv) & \text{in } \Omega\times(0,\infty),\\ \dfrac{\partial v}{\partial t} = \nu\Delta v + v(m(x) - cu - v) & \text{in } \Omega\times(0,\infty),\\ u = v = 0 & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $v(x,t)$ represent the population densities of competing species 1 and 2, and are therefore assumed to be nonnegative, with corresponding dispersal rates μ and ν. The function $m(x)$ stands for intrinsic growth rate of the two species, b and c are inter-specific competition coefficients. $\Delta := \sum_{i=1}^{N} \partial^2/\partial x_i^2$ denotes the Laplace operator in $\mathbb{R}^N$ which describes the random motion of the species, the habitat Ω is a bounded region in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. The homogeneous Dirichlet boundary conditions mean the exterior region of Ω is a completely hostile world for both species. We shall assume that μ, ν, b and c are nonnegative constants, $u_0(x)$ and $v_0(x)$ are nonnegative functions and not identically equal to zero.

There are many results concerning the Lotka–Volterra competition system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mu\Delta u + u(m(x) - u - bv) & \text{in } \Omega\times(0,\infty),\\ \dfrac{\partial v}{\partial t} = \nu\Delta v + v(m(x) - cu - v) & \text{in } \Omega\times(0,\infty),\\ \dfrac{\partial u}{\partial n} = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega\times(0,\infty),\\ u(x,0) = u_0(x), v(x,0) = v_0(x) & \text{in } \Omega \end{cases} \tag{1.2}$$

under the homogeneous Neumann boundary conditions. It is demonstrated in [18] that if μ and ν are sufficiently small, the function $m(x)$ is positive, and $0 < b, c < 1$, then (1.2) admits a unique, globally asymptotically stable positive steady state (u^*, v^*). Moreover, (u^*, v^*) converges to $(\frac{(1-b)m}{1-bc}, \frac{(1-c)m}{1-bc})$ in L^∞ as $\mu, \nu \to 0$. When μ and ν are sufficiently large, then (1.2) also has a unique, globally asymptotically stable positive steady state (u^{**}, v^{**}). Furthermore, (u^{**}, v^{**}) converges to $(\frac{1-b}{1-bc}\frac{1}{|\Omega|}\int_\Omega m, \frac{1-c}{1-bc}\frac{1}{|\Omega|}\int_\Omega m)$ as $\mu, \nu \to \infty$. Subsequently, Lam and Ni [21] filled the gap of the dynamics of (1.2) between the regions when μ and ν are both sufficiently small and large. Their results display that there exists $\delta > 0$ such that if $|\mu - \nu| < \delta$, (1.2) has a unique coexistence steady state (u^{***}, v^{***}) which is globally asymptotically stable. In addition, $(u^{***}, v^{***}) \to (\frac{(1-b)\theta_{\widehat{d}}}{1-bc}, \frac{(1-c)\theta_{\widehat{d}}}{1-bc})$ uniformly in $\overline{\Omega}$ as $\mu, \nu \to \widehat{d} > 0$, where $\theta_{\widehat{d}}$ is the unique positive solution of

$$\widehat{d}\Delta\theta_{\widehat{d}} + \theta_{\widehat{d}}(m(x) - \theta_{\widehat{d}}) = 0 \text{ in } \Omega, \quad \frac{\partial\theta_{\widehat{d}}}{\partial n} = 0 \text{ on } \partial\Omega.$$

However, if the competition system (1.2) is imposed on homogeneous Dirichlet boundary conditions, the dynamics of (1.1) is quite different from that of (1.2). Indeed, by the similar arguments to that of [18,20],

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we can show that if μ and ν are small enough, the function $m(x)$ is positive, and $0 < b, c < 1$, then (1.1) admits a unique, globally asymptotically stable positive steady state (u_*, v_*). Moreover, (u_*, v_*) converges to $(\frac{(1-b)m}{1-bc}, \frac{(1-c)m}{1-bc})$ uniformly on compact subset of Ω as $\mu, \nu \to 0$. Recently, the authors [7] have generalized similar conclusion to a Lotka–Volterra competition system with the different resources and spatially varying inter and intra-specific competition coefficients under the homogeneous Dirichlet boundary conditions. On the other hand, for larger μ and ν, it is known [1,8] that there exists some $d^* > 0$ such that if $\mu \geq d^*$ and $\nu \geq d^*$, (1.1) has no coexistence steady state. To our knowledge, with the help of the method in [21], we are able to demonstrate that there exists small $\eta > 0$ such that if $|\mu - \nu| < \eta$, (1.1) has a unique coexistence steady state (u_{**}, v_{**}) which is globally asymptotically stable. Furthermore, $(u_{**}, v_{**}) \to (\frac{(1-b)\theta_{\overline{d}}}{1-bc}, \frac{(1-c)\theta_{\overline{d}}}{1-bc})$ uniformly on compact subset of Ω as $\mu, \nu \to \overline{d} \in (0, d^*)$, where $\theta_{\overline{d}}$ is the unique positive solution of

$$\overline{d}\Delta\theta_{\overline{d}} + \theta_{\overline{d}}(m(x) - \theta_{\overline{d}}) = 0 \text{ in } \Omega, \quad \theta_{\overline{d}} = 0 \text{ on } \partial\Omega.$$

In this paper, we would like to investigate how dispersal rate and spatial heterogeneity affect the dynamics of (1.1). Thus we shall always assume that the function $m(x)$ is non-constant for reflecting the spatial heterogeneity of the environment. More precisely, we suppose that

$$m(x) \text{ is non-constant, Hölder continuous and } |\{x \in \Omega : m(x) > 0\}| \neq \varnothing. \tag{1.3}$$

It is well known [1,8] that under the assumption (1.3), the following logistic equation

$$\mu\Delta\theta + \theta(m(x) - \theta) = 0 \text{ in } \Omega, \quad \theta = 0 \text{ on } \partial\Omega \tag{1.4}$$

has a unique positive solution for every $\mu \in (0, \gamma)$, denoted by $\theta(x, \mu)$, where γ is uniquely determined by the eigenvalue problem (2.1) in Section 2 ($\gamma := 1/\sigma_1$). Moreover, for any $\mu \in (0, \gamma)$, $\theta(x, \mu) \in C^2(\overline{\Omega})$. For convenience, we frequently write $\theta(x, \mu)$ as θ_μ to denote its dependence on the dispersal rate μ. Clearly, (1.1) has two semi-trivial nonnegative steady states for every $\mu, \nu \in (0, \gamma)$, i.e., $(\theta_\mu, 0)$ and $(0, \theta_\nu)$ respectively.

The main conclusions of this paper are as follows:

Theorem 1. *Assume that (1.3) and (2.2) hold. For any given $b, c \in (0, 1)$, the following conclusions hold.*

(i) There exist small $\eta_1 > 0$ and some $\underline{\mu} > 0$ such that if $\nu \leq \eta_1$ and $\mu < \underline{\mu}$, (1.1) admits a unique coexistence steady state (u^*, v^*) which is globally asymptotically stable for non-negative, non-trivial initial data (u_0, v_0). Moreover, for any $\mu \in (0, \underline{\mu})$,

$$\lim_{\nu \to 0}(u^*, v^*) = (\widetilde{\theta}_\mu, m - c\widetilde{\theta}_\mu)$$

uniformly on compact subset of Ω, where $\widetilde{\theta}_\mu$ is the unique positive solution of

$$\mu\Delta u + u[(1-b)m(x) - (1-bc)u] = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

(ii) There exist small $\eta_2 > 0$ and some $\underline{\nu} > 0$ such that if $\mu \leq \eta_2$ and $\nu < \underline{\nu}$, (1.1) has a unique coexistence steady state (u^{**}, v^{**}) which is globally asymptotically stable for non-negative, non-trivial initial data (u_0, v_0). Furthermore, for every $\nu \in (0, \underline{\nu})$,

$$\lim_{\mu \to 0}(u^{**}, v^{**}) = (m - b\widetilde{\theta}_\nu, \widetilde{\theta}_\nu)$$

uniformly on compact subset of Ω, where $\widetilde{\theta}_\nu$ is the unique positive solution of

$$\nu\Delta v + v[(1-c)m(x) - (1-bc)v] = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega.$$

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Theorem 1 (i) implies that for some ranges of dispersal rate of competing species 1, if dispersal rate of competing species 2 is small enough, two competing species can coexist. In contrast to previous results [7,18], this conclusion extends the scopes of dispersal rates of two competing species which make two competing species coexist. Theorem 1 (ii) can be explained similarly.

Theorem 2. *Suppose that m satisfies (1.3) and (2.2), then the following conclusions hold.*

(i) For any given $b, c \in (0,1)$, there exists small $\delta_1 > 0$ and some $\mu^* > 0$ such that if $0 < \nu \leq \delta_1$ and $\mu^* < \mu < \gamma$, $(0, \theta_\nu)$ is globally asymptotically stable among non-negative, non-trivial initial data (u_0, v_0).

(ii) For any given $b, c \in (0,1)$, there exists small $\delta_2 > 0$ and some $\nu^* > 0$ such that if $0 < \mu \leq \delta_2$ and $\nu^* < \nu < \gamma$, $(\theta_\mu, 0)$ is globally asymptotically stable among non-negative, non-trivial initial data (u_0, v_0).

(iii) For any given $b \in (0,1)$, there exist small $\delta_3 > 0$ and some $\mu^*, \nu_* > 0$ such that if $0 < c \leq \delta_3$ and $\mu^* < \mu < \gamma$, $(0, \theta_\nu)$ is globally asymptotically stable among non-negative, non-trivial initial data (u_0, v_0) provided that $\nu \in (0, \nu_*)$.

(iv) For any given $c \in (0,1)$, there exist small $\delta_4 > 0$ and some $\mu_*, \nu^* > 0$ such that if $0 < b \leq \delta_4$ and $\nu^* < \nu < \gamma$, $(\theta_\mu, 0)$ is globally asymptotically stable among non-negative, non-trivial initial data (u_0, v_0) provided that $\mu \in (0, \mu_*)$.

Theorem 2 (i) indicates that for certain scopes of competing species 1, if dispersal rate of competing species 2 is sufficiently small, the semi-trivial steady state $(0, \theta_\nu)$ is the global attractor of (1.1). Consequently, the joint action of spatial heterogeneity and diffusion can drive one of the two competing species to extinction. Theorem 2 (iii) manifests that for some ranges of the dispersal rates of the two competing species, if one of the inter-specific competition coefficients is small enough, then one of semi-trivial steady states of (1.1) is also a global attractor of (1.1). Such "diffusion-driven extinction" phenomena have been discussed in [11–13,21,22]. By Theorems 1 and 2, we see that only for some special regions of dispersal rates and the inter-specific competition coefficients, the dynamics of (1.1) is well understood. For other ranges of dispersal rates and the inter-specific competition coefficients, a further understanding of the dynamics of (1.1) seems to be a very interesting and challenging problem, awaiting further investigation.

The rest of this paper is organized as follows: In Section 2 we present some preliminary results Lemmas 3–6. Section 3 is devoted to the proof of Lemmas 7–11 and Theorem 1. In Section 4, we establish Lemmas 12–13 and Theorem 2.

2. Preliminaries

In this section, we will introduce some lemmas which shall be utilized in subsequence analysis.

Lemma 3. *Assume that (1.3) holds. The following eigenvalue problem*

$$\Delta\phi + \sigma m(x)\phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega \tag{2.1}$$

admits a positive principal eigenvalue $\sigma_1(m)$ which is given by

$$\frac{1}{\sigma_1(m)} = \max_{\phi \in H_0^1(\Omega), \phi \neq 0} \frac{\int_\Omega m\phi^2}{\int_\Omega |\nabla\phi|^2}.$$

Furthermore, $\sigma_1(m)$ is strictly decreasing with respect to m in the sense that if $m_1 \leq m_2$ then $\sigma_1(m_1) \geq \sigma_1(m_2)$, and if $m_1 < m_2$ on a subset of positive measure then $\sigma_1(m_1) > \sigma_1(m_2)$.

Proof. The existence and uniqueness of $\sigma_1(m)$ is well known, we refer the proof to [1,15]. Now it remains to discuss the monotonicity of $\sigma_1(m)$. Let ϕ_1 be the eigenfunction associated with $\sigma_1(m_1)$, then

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$$\frac{1}{\sigma_1(m_1)} = \frac{\int_\Omega m_1\phi_1^2}{\int_\Omega |\nabla\phi_1|^2} \le \frac{\int_\Omega m_2\phi_1^2}{\int_\Omega |\nabla\phi_1|^2} \le \max_{\phi\in H_0^1(\Omega),\phi\neq 0} \frac{\int_\Omega m_2\phi^2}{\int_\Omega |\nabla\phi|^2} = \frac{1}{\sigma_1(m_2)}.$$

Since $\phi_1 > 0$ in Ω, the first inequality is strict if $m_1(x) < m_2(x)$ on a set of positive measure. □

Lemma 4. *Suppose that m satisfies (1.3), then the following conclusions hold.*

(i) $\mu \mapsto \theta_\mu$ is a continuous mapping from $(0,\gamma)$ to $C^2(\overline{\Omega})$. Moreover, θ_μ has the following property:

$$\lim_{\mu\to 0} \theta_\mu = m^+$$

uniformly on compact subset of Ω, where $m^+ = \max\{m(x), 0\}$.

(ii) If m also fulfills the following hypotheses:

$$-\Delta m \ge 0 \text{ in } \Omega \quad \text{and} \quad m|_{\partial\Omega} \ge 0, \tag{2.2}$$

for every $\mu \in (0,\gamma)$, then $\theta_\mu \le m$ in $\overline{\Omega}$. In addition, θ_μ is strictly decreasing in μ.

Proof. (i) The continuous dependence of θ_μ on μ can be established by an application of implicit function theorem [2]. The limiting behavior of θ_μ as μ goes to zero is well known, for instance, see [8]. We skip it here.

(ii) By the maximum principle, we see that $m(x) > 0$ in Ω. Moreover, it is not hard to verify that $m(x)$ is a supersolution of (1.3). Hence we have $\theta_\mu \le m(x)$ in $\overline{\Omega}$ for every $\mu > 0$. The monotonicity of θ_μ with respect to μ can be found in [8, Lemma 3.3]. □

Lemma 5. *Suppose that $k(x) \in L^\infty(\Omega)$. The principal eigenvalue $\lambda_1(\mu, k)$ of*

$$\mu\Delta\varphi + k(x)\varphi + \lambda\varphi = 0 \text{ in } \Omega, \quad \varphi = 0 \text{ on } \partial\Omega$$

is uniquely determined by

$$\lambda_1(\mu,k) = \min_{\varphi\in H_0^1(\Omega),\varphi\neq 0} \frac{\int_\Omega [\mu|\nabla\varphi|^2 - k(x)\varphi^2]}{\int_\Omega \varphi^2}. \tag{2.3}$$

Moreover,

(i) $\lambda_1(\mu,k)$ is strictly increasing in μ, and decreasing in k in the sense that if $k_1 \le k_2$ then $\lambda_1(\mu,k_1) \ge \lambda_1(\mu,k_2)$, and if $k_1 < k_2$ on a subset of positive measure then $\lambda_1(\mu,k_1) > \lambda_1(\mu,k_2)$.

(ii) The limit $\lim_{\mu\to 0}\lambda_1(\mu,k)$ is given by

$$\lim_{\mu\to 0} \lambda_1(\mu,k) = \min_{\overline{\Omega}}[-k(x)].$$

Proof. The monotonicity of $\lambda_1(\mu,k)$ with respect to μ can be directly obtained by the variational characterization (2.3), while the monotonicity of $\lambda_1(\mu,k)$ with regard to k can be proved by the similar argument to that of Lemma 3. The asymptotic behavior of $\lambda_1(\mu,k)$ as μ tends to zero is well known, we refer the proof to [8,16]. □

Lemma 6. *The semi-trivial steady state $(0,\theta_\nu)$ is stable/unstable if and only if the following eigenvalue problem, for $(\lambda_1,\psi) \in \mathbb{R}\times C^2(\overline{\Omega})$, has a positive/negative principle eigenvalue (denoted as λ_1):*

$$\begin{cases} \mu\Delta\psi + [m(x) - b\theta_\nu]\psi + \lambda\psi = 0 \text{ in } \Omega, \\ \psi = 0 \text{ on } \partial\Omega, \quad \psi > 0 \text{ in } \Omega. \end{cases} \tag{2.4}$$

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Proof. Set $X = \{(u,v) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega) : u = v = 0 \text{ on } \partial\Omega\}$ and $Y = L^p(\Omega) \times L^p(\Omega)$ with $p > N$. Define the operator $F(u,v) : X \to Y$ by

$$F(u,v) = \begin{pmatrix} -\mu\Delta u - u(m(x) - u - bv) \\ -\nu\Delta v - v(m(x) - cu - v) \end{pmatrix},$$

then we have

$$D_{(u,v)}F|_{(0,\theta_\nu)} = \begin{pmatrix} -\mu\Delta - (m(x) - b\theta_\nu) & 0 \\ c\theta_\nu & -\nu\Delta - (m(x) - 2\theta_\nu) \end{pmatrix}.$$

By (1.4) and the positivity of θ_ν, we see that zero is the smallest eigenvalue of the operator $-\nu\Delta - (m - \theta_\nu)$ with homogeneous Dirichlet boundary conditions. By the comparison principle for eigenvalues and the positivity of θ_ν, the least eigenvalue of the operator $-\nu\Delta - (m - 2\theta_\nu)$ with homogeneous Dirichlet boundary conditions is strictly positive. Hence, to determine the stability of semi-trivial steady state $(0, \theta_\nu)$, it suffices to inquire the sign of the least eigenvalue of (2.4). □

3. Global stability of coexistence steady state

The following lemma derived from the theory of monotone dynamical system [14, Proposition 9.1] will be frequently used to establish Theorems 1 and 2. For convenience, we display it here.

Lemma 7. *For any $\mu, \nu > 0$, assume that every coexistence steady state of (1.1), if exists, is asymptotically stable, then one of the following alternatives holds:*

(i) There exists a unique coexistence steady state of (1.1), if the unique positive steady state solution is stable, then it is globally asymptotically stable.

(ii) System (1.1) has no coexistence steady state and either one of semi-trivial steady states $(\theta_\mu, 0)$ and $(0, \theta_\nu)$ is globally asymptotically stable, while the other one is unstable.

Before applying Lemma 7 to prove Theorem 1, we need to make some preparations. Firstly, we investigate the stability of semi-trivial steady states of (1.1).

Lemma 8. *Assume that m satisfies (1.3) and $b, c \in (0,1)$, then the following conclusions hold.*

(i) For every $\nu \in (0, \gamma)$, there exist some $\mu_* > 0$ and $\mu^* > 0$ such that if $\mu < \mu_*$, the semi-trivial steady state $(0, \theta_\nu)$ is linearly unstable, while stable if $\mu > \mu^*$.

(ii) For any $\mu \in (0, \gamma)$, there exist some $\nu_* > 0$ and $\nu^* > 0$ such that if $\nu < \nu_*$, the semi-trivial steady state $(\theta_\mu, 0)$ is linearly unstable, while stable if $\nu > \nu^*$.

Proof. By Lemma 6, we know that the stability of semi-trivial steady state $(0, \theta_\nu)$ is determined by the sign of the least eigenvalue (denoted by λ_1) of the following eigenvalue problem:

$$\begin{cases} \mu\Delta\psi + [m(x) - b\theta_\nu]\psi + \lambda\psi = 0 & \text{in } \Omega, \\ \psi = 0 \quad \text{on } \partial\Omega, \quad \psi > 0 & \text{in } \Omega. \end{cases} \tag{3.1}$$

Obviously, λ_1 is a function of both μ and ν. We occasionally write λ_1 as $\lambda_1(\mu, \nu)$ to denote its dependence on μ and ν.

Applying the maximum principle to the equation of θ_μ, we have $\max_{\overline{\Omega}} \theta_\mu \le \max_{\overline{\Omega}} m(x)$. Set $m(x_0) = \max_{\overline{\Omega}} m(x)$, then

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$$m(x_0) = \max_{\overline{\Omega}} m(x) \geq \max_{\overline{\Omega}} \theta_\mu \geq \theta_\mu(x_0) > b\theta_\mu(x_0).$$

In other words, there exists some point $x_0 \in \Omega$ such that $m(x_0) - b\theta_\mu(x_0) > 0$. By Lemma 3, we see that the eigenvalue problem

$$\Delta\varphi + \sigma[m(x) - b\theta_\nu]\varphi = 0 \text{ in } \Omega, \quad \varphi = 0 \text{ on } \partial\Omega$$

has a positive principal eigenvalue $\sigma_1(\nu)$ which is characterized by

$$\frac{1}{\sigma_1(\nu)} = \max_{\varphi \in H_0^1(\Omega), \varphi \neq 0} \frac{\int_\Omega (m - b\theta_\nu)\varphi^2}{\int_\Omega |\nabla\varphi|^2}. \tag{3.2}$$

Let $\widetilde{\mu}(\nu) = 1/\sigma_1(\nu)$. By (3.2), there exists some $\psi > 0$ such that

$$\widetilde{\mu}(\nu)\Delta\psi + [m(x) - b\theta_\nu]\psi = 0 \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega.$$

By (3.1) and (3.2), we see that $\lambda_1 = 0$ when $\mu = \widetilde{\mu}(\nu)$. Since λ_1 is strictly increasing with respect to μ, we have $\lambda_1 > 0$ if $\mu > \widetilde{\mu}(\nu)$ and $\lambda_1 < 0$ if $\mu < \widetilde{\mu}(\nu)$.

Similarly, the stability of semi-trivial steady state $(\theta_\mu, 0)$ is determined by the sign of the least eigenvalue of the following eigenvalue problem:

$$\nu\Delta\phi + [m(x) - c\theta_\mu]\phi + \lambda\phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega.$$

By the similar arguments as before, we can show that there exists a positive principal eigenvalue $\sigma_1(\mu)$ determined by

$$\frac{1}{\sigma_1(\mu)} = \max_{\phi \in H_0^1(\Omega), \phi \neq 0} \frac{\int_\Omega (m - c\theta_\mu)\phi^2}{\int_\Omega |\nabla\phi|^2}$$

such that

$$\Delta\phi + \sigma_1(\mu)[m(x) - c\theta_\mu]\phi = 0 \text{ in } \Omega, \quad \phi = 0 \text{ on } \partial\Omega.$$

Set $\widetilde{\nu}(\mu) = 1/\sigma_1(\mu)$. Since λ_1 is strictly increasing in ν, we have $\lambda_1 > 0$ if $\nu > \widetilde{\nu}(\mu)$, $\lambda_1 = 0$ if $\nu = \widetilde{\mu}(\nu)$ and $\lambda_1 < 0$ if $\nu < \widetilde{\nu}(\mu)$.

Define

$$\mu_* = \min_{\nu \in [0,\gamma]} \widetilde{\mu}(\nu) = \min_{\nu \in [0,\gamma]} \max_{\varphi \in H_0^1(\Omega), \varphi \neq 0} \frac{\int_\Omega (m - b\theta_\nu)\varphi^2}{\int_\Omega |\nabla\varphi|^2},$$

$$\nu_* = \min_{\mu \in [0,\gamma]} \widetilde{\nu}(\mu) = \min_{\mu \in [0,\gamma]} \max_{\phi \in H_0^1(\Omega), \phi \neq 0} \frac{\int_\Omega (m - c\theta_\mu)\phi^2}{\int_\Omega |\nabla\phi|^2},$$

$$\mu^* = \max_{\nu \in [0,\nu_*]} \widetilde{\mu}(\nu) = \max_{\nu \in [0,\nu_*]} \max_{\varphi \in H_0^1(\Omega), \varphi \neq 0} \frac{\int_\Omega (m - b\theta_\nu)\varphi^2}{\int_\Omega |\nabla\varphi|^2},$$

$$\nu^* = \max_{\mu \in [0,\mu_*]} \widetilde{\nu}(\mu) = \max_{\mu \in [0,\mu_*]} \max_{\phi \in H_0^1(\Omega), \phi \neq 0} \frac{\int_\Omega (m - c\theta_\mu)\phi^2}{\int_\Omega |\nabla\phi|^2}.$$

Claim. $0 < \mu_* \leq \mu^* < \gamma$ and $0 < \nu_* \leq \nu^* < \gamma$.

We first prove that $0 < \mu_* < \gamma$. By (3.2), we see that for every $\nu \in (0, \gamma)$, $\widetilde{\mu}(\nu)$ is uniformly bounded from below by some positive constant. Moreover, by Lemma 4 we have

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$$\widetilde{\mu}(\nu) \to \max_{\varphi\in H_0^1(\Omega),\varphi\neq 0} \frac{(1-b)\int_\Omega m\varphi^2}{\int_\Omega |\nabla\varphi|^2} = (1-b)\gamma \quad \text{as } \nu \to 0,$$

$$\widetilde{\mu}(\nu) \to \max_{\varphi\in H_0^1(\Omega),\varphi\neq 0} \frac{\int_\Omega m\varphi^2}{\int_\Omega |\nabla\varphi|^2} = \gamma \quad \text{as } \nu \to \gamma,$$

$$\mu_* \leq \max_{\varphi\in H_0^1(\Omega),\varphi\neq 0} \frac{(1-b)\int_\Omega m\varphi^2}{\int_\Omega |\nabla\varphi|^2} = (1-b)\gamma < \gamma.$$

Hence $0 < \mu_* < \gamma$. In the same spirit, we can show that $0 < \nu_* < \gamma$. By the definition of μ_* and μ^*, we find that $\mu_* \leq \mu^*$. Now it suffices to prove that $\mu^* < \gamma$. For every $\nu \in [0, \nu_*]$, we have $m - b\theta_\nu < m$ in Ω. By Lemma 3, we obtain

$$\mu^* = \max_{\nu\in[0,\nu_*]} \max_{\varphi\in H_0^1(\Omega),\varphi\neq 0} \frac{\int_\Omega (m-b\theta_\nu)\varphi^2}{\int_\Omega |\nabla\varphi|^2} < \max_{\varphi\in H_0^1(\Omega),\varphi\neq 0} \frac{\int_\Omega m\varphi^2}{\int_\Omega |\nabla\varphi|^2} = \gamma.$$

It follows from the similar argument that $0 < \nu_* \leq \nu^* < \gamma$. If $\mu > \mu^*$, we have $\mu > \widetilde{\mu}(\nu)$, then $(0, \theta_\nu)$ is linearly stable. Similarly, we can illustrate that $(0, \theta_\nu)$ is unstable if $\mu < \mu_*$. By symmetry, we see that $(\theta_\mu, 0)$ is linearly stable if $\nu > \nu^*$ and unstable if $\nu < \nu_*$. The proof is complete. □

Since (1.1) is a monotone system, the following conclusion follows from Lemma 7.

Theorem 9. *Suppose that m satisfies (1.3). If $\mu < \mu_*$ and $\nu < \nu_*$, (1.1) admits at least one coexistence steady state.*

The following result characterizes the limit behavior of coexistence steady states of (1.1) for some ranges of μ as $\nu \to 0$.

Lemma 10. *Assume that (1.3) and (2.2) hold. Let (u, v) be any coexistence steady state of (1.1). For any given $b, c \in (0, 1)$ and $\mu \in (0, \mu_*)$, $\lim_{\nu\to 0}(u, v) = (\widetilde{\theta}_\mu, m - c\widetilde{\theta}_\mu)$ uniformly on compact subset of Ω, where $\widetilde{\theta}_\mu$ is uniquely determined by*

$$\mu\Delta w + w[(1-b)m(x) - (1-bc)w] = 0 \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega.$$

Proof. By Lemma 4 (ii), we have $m(x) > 0$ in Ω. Furthermore $\theta_\nu \leq m(x)$ for every $\nu \in (0, \nu_*)$.

We firstly employ a standard iteration scheme to derive an ordered sequence as $\nu \to 0$. To this end, set $\overline{v}_0 = \theta_\nu$, where θ_ν satisfies

$$-\nu\Delta w = w(m(x) - w) \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega. \tag{3.3}$$

From the equation of v, it is easy to testify that v and $\max_{\overline{\Omega}} m$ are subsolution and supersolution of (3.3) respectively. By the upper and lower solution method and the uniqueness of positive steady states of (3.3), we have $v \leq \overline{v}_0$. Now from the equation of u, we see that

$$-\mu\Delta u \geq u(m - u - b\overline{v}_0) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

That is to say, u is a supersolution of the following equation:

$$-\mu\Delta w = w(m - w - b\overline{v}_0) \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega. \tag{3.4}$$

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Since $\overline{v}_0 \to m$ uniformly as $\nu \to 0$, we have $m - b\overline{v}_0 \to (1-b)m > 0$ uniformly on compact subset of Ω. For every $\mu \in (0, \mu_*)$, it is not hard to verify that $\epsilon\psi_1$ is a subsolution of (3.4) for some small $\epsilon > 0$, where ψ_1 is the corresponding eigenfunction of the principal eigenvalue of

$$\mu\Delta\psi + (1-b)m(x)\psi + \lambda\psi = 0 \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega.$$

By a sub/super solution argument [26], we see that if $\mu < \mu_*$, (3.4) has a unique positive solution for sufficiently small ν, denoted as $\underline{u}_1$. Moreover, $\underline{u}_1 \le u$ in $\overline{\Omega}$. By the equation of v, we see that

$$-\nu\Delta v \le v(m - c\underline{u}_1 - v) \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega,$$

i.e., v is a subsolution of

$$-\nu\Delta w = w(m - c\underline{u}_1 - w) \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega. \tag{3.5}$$

Clearly, m is a supersolution of (3.5). Hence, by a sub/super solution argument again, (3.5) has a unique positive solution, denoted by $\overline{v}_1$. Furthermore, $v \le \overline{v}_1$ in $\overline{\Omega}$.

For every $k \ge 1$, we can construct $\underline{u}_k > 0$ and $\overline{v}_k > 0$ inductively by

$$\begin{cases} \mu\Delta\underline{u}_k + \underline{u}_k(m - \underline{u}_k - b\overline{v}_{k-1}) = 0 \text{ in } \Omega, \quad \underline{u}_k = 0 \text{ on } \partial\Omega, \\ \nu\Delta\overline{v}_k + \overline{v}_k(m - c\underline{u}_k - \overline{v}_k) = 0 \text{ in } \Omega, \quad \overline{v}_k = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.6}$$

The existence and uniqueness of $\{\underline{u}_k\}_{k=1}^{\infty}$ and $\{\overline{v}_k\}_{k=1}^{\infty}$ follows from a sub/super solution argument as above. Moreover, the following inequalities hold:

$$\begin{cases} \underline{u}_1 \le \cdots \le \underline{u}_{k-1} \le \underline{u}_k \le \cdots \le u \quad \text{in } \overline{\Omega}, \\ v \le \cdots \le \overline{v}_k \le \overline{v}_{k-1} \le \cdots \le \overline{v}_1 \le \overline{v}_0 \quad \text{in } \overline{\Omega}. \end{cases} \tag{3.7}$$

We argue by induction: note that $\underline{u}_1 \le u$ and $v \le \overline{v}_1$ in $\overline{\Omega}$. Suppose that $\underline{u}_{k-1}$ and $\overline{v}_{k-1}$ exist, $\underline{u}_{k-1} \le u$ and $v \le \overline{v}_{k-1}$ in $\overline{\Omega}$. Then by $v \le \overline{v}_{k-1}$, we see that u satisfies

$$-\mu\Delta u \ge u(m - u - b\overline{v}_{k-1}) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

That is, u is a supersolution of

$$-\mu\Delta w = w(m - w - b\overline{v}_{k-1}) \text{ in } \Omega, \quad w = 0 \text{ on } \partial\Omega. \tag{3.8}$$

Since $\overline{v}_{k-1} \le \overline{v}_{k-2}$, by (3.6) we see that $\underline{u}_{k-1}$ satisfies

$$-\mu\Delta\underline{u}_{k-1} \le \underline{u}_{k-1}(m - \underline{u}_{k-1} - b\overline{v}_{k-1}) \text{ in } \Omega, \quad \underline{u}_{k-1} = 0 \text{ on } \partial\Omega,$$

i.e., $\underline{u}_{k-1}$ is a subsolution of (3.8). Hence, by a sub/super solution argument we see that (3.8) has a unique positive solution $\underline{u}_k$ and $\underline{u}_{k-1} \le \underline{u}_k \le u$ in $\overline{\Omega}$. The existence and uniqueness of $\overline{v}_k$ can be established by a similar argument. Moreover, $v \le \overline{v}_k \le \overline{v}_{k-1}$ in $\overline{\Omega}$.

In the same manner, set $\overline{u}_0 = \theta_\mu$ and define $\{\overline{u}_k, \underline{v}_k\}_{k=1}^{\infty}$ successively by

$$\begin{cases} \nu\Delta\underline{v}_k + \underline{v}_k(m - c\overline{u}_{k-1} - \underline{v}_k) = 0 \text{ in } \Omega, \quad \underline{v}_k = 0 \text{ on } \partial\Omega, \\ \mu\Delta\overline{u}_k + \overline{u}_k(m - \overline{u}_k - b\underline{v}_k) = 0 \text{ in } \Omega, \quad \overline{u}_k = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.9}$$

Furthermore, the following inequalities hold:

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$$\begin{cases} u \le \cdots \le \overline{u}_k \le \overline{u}_{k-1} \le \cdots \le \overline{u}_1 \le \overline{u}_0 & \text{in } \overline{\Omega}, \\ \underline{v}_1 \le \cdots \le \underline{v}_{k-1} \le \underline{v}_k \le \cdots \le v & \text{in } \overline{\Omega}. \end{cases} \tag{3.10}$$

For every $k \ge 1$, we investigate the limits of $(\underline{u}_k, \overline{v}_k)$ and $(\overline{u}_k, \underline{v}_k)$ as $\nu \to 0$. To this end, let

$$\begin{cases} f(\mu, U, V) = \mu \Delta U + U(m - U - bV) & \text{with } \mu < \mu_*, \\ g(U, V) = m - cU - V. \end{cases}$$

Define $\overline{V}_0(x) = m(x)$ for every $x \in \overline{\Omega}$. By a sub/super solution argument, we see that if $\mu < \mu_*$, the following problem

$$-\mu \Delta U = U(m - U - b\overline{V}_0) \text{ in } \Omega, \quad U = 0 \text{ on } \partial\Omega$$

admits a unique positive solution, denoted by $\underline{U}_1(x)$. The maximum principle implies that $\underline{U}_1(x) \le m(x)$ in $\overline{\Omega}$. It is easy to see that $g(\underline{U}_1(x), V) = 0$ has a unique root $\overline{V}_1(x) > 0$. Moreover, $\overline{V}_1(x) \le \overline{V}_0(x)$ in $\overline{\Omega}$.

For every $k \ge 1$, we can construct $\underline{U}_k > 0$ and $\overline{V}_k > 0$ successively by

$$\begin{cases} f(\mu, \underline{U}_k, \overline{V}_{k-1}) = 0 \text{ in } \Omega, \quad \underline{U}_k = 0 \text{ on } \partial\Omega, \\ g(\underline{U}_k, \overline{V}_k) = 0 \quad \text{with } \overline{V}_0(x) = m(x). \end{cases} \tag{3.11}$$

In addition, the following inequalities hold:

$$\begin{cases} \underline{U}_1 \le \cdots \le \underline{U}_{k-1} \le \underline{U}_k \le \cdots \le U & \text{in } \overline{\Omega}, \\ V \le \cdots \le \overline{V}_k \le \overline{V}_{k-1} \le \cdots \le \overline{V}_1 \le \overline{V}_0 & \text{in } \overline{\Omega}. \end{cases} \tag{3.12}$$

In the same way, we can define $\{\overline{U}_k\}_{k=1}^{\infty}$ and $\{\underline{V}_k\}_{k=1}^{\infty}$ successively by

$$\begin{cases} g(\overline{U}_{k-1}, \underline{V}_k) = 0 \quad \text{with } \overline{U}_0(x) = m(x). \\ f(\mu, \overline{U}_k, \underline{V}_k) = 0 \text{ in } \Omega, \quad \overline{U}_k = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.13}$$

Moreover, the following inequalities hold:

$$\begin{cases} U \le \cdots \le \overline{U}_k \le \overline{U}_{k-1} \le \cdots \le \overline{U}_1 \le \overline{U}_0 & \text{in } \overline{\Omega}, \\ \underline{V}_1 \le \cdots \le \underline{V}_{k-1} \le \underline{V}_k \le \cdots \le V & \text{in } \overline{\Omega}. \end{cases} \tag{3.14}$$

Claim. $(\underline{u}_k, \overline{v}_k) \to (\underline{U}_k, \overline{V}_k)$ and $(\overline{u}_k, \underline{v}_k) \to (\overline{U}_k, \underline{V}_k)$ uniformly on compact subset of Ω as $\nu \to 0$. We argue again by induction. By (3.6) and Lemma 4 (i), we see that $\lim_{\nu\to 0} \underline{u}_1 = \underline{U}_1$ uniformly on compact subset of Ω as $\nu \to 0$. A similar argument yields that $\overline{v}_1 \to \overline{V}_1$ uniformly on compact subset of Ω as $\nu \to 0$. Suppose that $\lim_{\nu\to 0} \underline{u}_{k-1} = \underline{U}_{k-1}$ and $\lim_{\nu\to 0} \overline{v}_{k-1} = \overline{V}_{k-1}$. Since $\lim_{\nu\to 0} \overline{v}_{k-1} = \overline{V}_{k-1}$, by (3.6) and (3.11) we easily see that $\lim_{\nu\to 0} \underline{u}_k = \underline{U}_k$ uniformly on compact subset of Ω. That is, for any $\varepsilon > 0$, if ν is sufficiently small, then $\underline{U}_k - \varepsilon \le \underline{u}_k \le \underline{U}_k + \varepsilon$. Thus by (3.6), $\underline{v}_k$ satisfies

$$\overline{v}_k(m - c(\underline{U}_k + \varepsilon) - \overline{v}_k) \le \overline{v}_k(m - c\underline{u}_k - \overline{v}_k) = -\nu\Delta\overline{v}_k \le \overline{v}_k(m - c(\underline{U}_k - \varepsilon) - \overline{v}_k).$$

Hence, $\underline{v}_k$ is a supersolution of $-\nu\Delta w = w(m - c(\underline{U}_k + \varepsilon) - w)$ and a subsolution of $-\nu\Delta w = w(m - c(\underline{U}_k - \varepsilon) - w)$. Now by Lemma 4 (i) again, we get

$$m - c(\underline{U}_k + \varepsilon) \le \lim_{\nu\to 0} \overline{v}_k \le m - c(\underline{U}_k - \varepsilon)$$

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uniformly on compact subset of Ω. Therefore, $\lim_{\nu\to 0}\overline{v}_k = m - c\underline{U}_k = \overline{V}_k$ uniformly on compact subset of Ω. In the same manner, we can deduce that the other case holds.

It follows from (3.12) and (3.14) that

$$\begin{cases} \underline{U}_1 \le \cdots \le \underline{U}_k \le \cdots \le \underline{\lim}_{\nu\to 0} u \le \overline{\lim}_{\nu\to 0} u \le \cdots \le \overline{U}_k \le \cdots \le \overline{U}_1 \le \overline{U}_0 & \text{in } \overline{\Omega}, \\ \underline{V}_1 \le \cdots \le \underline{V}_k \le \cdots \le \underline{\lim}_{\nu\to 0} v \le \overline{\lim}_{\nu\to 0} v \le \cdots \le \overline{V}_k \le \cdots \le \overline{V}_1 \le \overline{V}_0 & \text{in } \overline{\Omega}. \end{cases}$$

Clearly, $\{\underline{U}_k, \overline{V}_k, \overline{U}_k, \underline{V}_k\}_{k=1}^{\infty}$ are all monotone sequences, hence we may assume that

$$\begin{cases} \underline{U}(x) = \lim\limits_{k\to\infty} \underline{U}_k(x), & \overline{U}(x) = \lim\limits_{k\to\infty} \overline{U}_k(x), \\ \underline{V}(x) = \lim\limits_{k\to\infty} \underline{V}_k(x), & \overline{V}(x) = \lim\limits_{k\to\infty} \overline{V}_k(x). \end{cases}$$

Letting $k \to \infty$ in (3.11) and (3.13), we have

$$\begin{cases} f(\mu, \underline{U}(x), \overline{V}(x)) = g(\underline{U}(x), \overline{V}(x)) = 0, \\ f(\mu, \overline{U}(x), \underline{V}(x)) = g(\overline{U}(x), \underline{V}(x)) = 0. \end{cases}$$

It is not difficult to see that $\underline{U}(x), \overline{V}(x), \overline{U}(x), \underline{V}(x)$ are all non-negative. Then $\underline{U}(x) = \overline{U}(x) = \widetilde{\theta}_\mu$ and $\overline{V}(x) = \underline{V}(x) = m - c\widetilde{\theta}_\mu$ in Ω. By a well known calculus lemma [27], we see that $\underline{U}_k(x), \overline{U}_k(x) \to \widetilde{\theta}_\mu$ and $\underline{V}_k(x), \overline{V}_k(x) \to m - c\widetilde{\theta}_\mu$ uniformly on compact subset of Ω as $k \to \infty$. The proof is complete. □

Combining Lemmas 4 and 10, we have the following limit behavior of coexistence steady state of (1.1) as $\mu, \nu \to 0$.

Corollary 1. *Assume that (1.3) and (2.2) hold. Let (u, v) be any coexistence steady state of (1.1), then $\lim_{\mu,\nu\to 0}(u, v) = (\frac{(1-b)m}{1-bc}, \frac{(1-c)m}{1-bc})$ uniformly on compact subset of Ω.*

After some preparations, we now begin to inquire the stability of coexistence steady state of (1.1) derived from Theorem 9.

Lemma 11. *Assume that (1.3) and (2.2) hold, there exist some $\underline{\mu} = \underline{\mu}(m, \Omega) > 0$ and small $\delta_1 > 0$ such that for any compact subset $K \subset (0, \underline{\mu})$, if $0 < \nu \le \delta_1$, any coexistence steady state derived from Theorem 9 is linearly stable for every $\mu \in K$.*

Proof. Linearizing system (1.1) at a coexistence steady state (u, v), we have

$$\begin{cases} \mu\Delta\varphi + (m - 2u - bv)\varphi - bu\psi + \lambda\varphi = 0 & \text{in } \Omega, \\ \nu\Delta\psi - cv\varphi + (m - cu - 2v)\psi + \lambda\psi = 0 & \text{in } \Omega, \\ \varphi = \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.15}$$

Since (1.1) is a monotone system, by the Krein–Rutman Theorem [19], (3.15) has a principal eigenvalue, denoted as λ_1, and its corresponding eigenfunction (φ, ψ) can be chosen such that $\varphi > 0 > \psi$ in Ω and $\varphi = \psi = 0$ on $\partial\Omega$. To show that coexistence steady state (u, v) of (1.1) is linearly stable for sufficiently small ν, it suffices to prove that $\lambda_1 > 0$. We argue by contradiction: passing to a subsequence if necessary, assume that there exist some sequence $\{\nu_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \nu_k = 0$ and $\varphi_k > 0 > \psi_k$ in Ω and $\varphi_k = \psi_k = 0$ on $\partial\Omega$ such that $\lambda_1 \le 0$. For the sake of simplicity, we drop the subscript k in the rest of the proof.

Let $\varphi(x_1) = \max_{\overline{\Omega}} \varphi$ and normalize it such that $\max_{\overline{\Omega}} \varphi = 1$. Passing to a subsequence if necessary, we may assume that $x_1 \to \overline{x} \in \Omega$ as $\nu \to 0$. If not, suppose that $x_1 \to \overline{x} \in \partial\Omega$ as $\nu \to 0$, this is impossible. Denote $\psi(x_2) = \min_{\overline{\Omega}} \psi$. A similar argument deduces that $x_2 \to \widetilde{x} \in \Omega$ as $\nu \to 0$.

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We claim that λ_1 is bounded from below for sufficiently small ν. If this is true, together with the assumption $\lambda_1 \leq 0$, passing to a necessary if necessary, we may suppose that $\lambda_1 \to \overline{\lambda}_1 \leq 0$ as $\nu \to 0$. We argue by contradiction again: passing to a subsequence if necessary, assume that $\lambda_1 \to -\infty$ as $\nu \to 0$. Applying the maximum principle to the equation of φ and ψ in (3.15) respectively, we see that

$$\max_{\overline{\Omega}} \varphi \cdot [2u(x_1) + bv(x_1) - m(x_1) - \lambda_1] \leq b \max_{\overline{\Omega}}(-\psi) \cdot u(x_1) \tag{3.16}$$

and

$$\max_{\overline{\Omega}}(-\psi) \cdot [cu(x_2) + 2v(x_2) - m(x_2) - \lambda_1] \leq cv(x_2) \cdot \varphi(x_2). \tag{3.17}$$

Since $\lim_{\nu\to 0}(u, v) = (\widetilde{\theta}_\mu, m - c\widetilde{\theta}_\mu)$ uniformly on compact subset of Ω, by (3.17), we obtain $\max_{\overline{\Omega}}(-\psi) \leq C_1/(C_1 - \lambda_1) \to 0$ as $\nu \to 0$, where C_1 is some positive constant independent of μ and ν. Similarly, by (3.16) we have $\max_{\overline{\Omega}} \varphi \leq C_2 \cdot \max_{\overline{\Omega}}(-\psi)/(-C_2 - \lambda_1) \to 0$ for some positive constant C_2 which is also independent of μ and ν. Obviously, this is a contradiction.

Let $\widetilde{\varphi}(y) = \varphi(x_1 + \sqrt{\nu}y)$ and $\widetilde{\psi}(y) = \psi(x_1 + \sqrt{\nu}y)$ for any y satisfying $x_1 + \sqrt{\nu}y \in \Omega$. Define $\widetilde{u}(y) = u(x_1 + \sqrt{\nu}y)$ and $\widetilde{v}(y) = v(x_1 + \sqrt{\nu}y)$, then we have

$$\begin{cases} \dfrac{\mu}{\nu}\Delta_y\widetilde{\varphi} + [m(x_1 + \sqrt{\nu}y) - 2\widetilde{u} - b\widetilde{v}]\widetilde{\varphi} - b\widetilde{u}\widetilde{\psi} + \lambda_1\widetilde{\varphi} = 0, \\ \Delta_y\widetilde{\psi} - c\widetilde{v}\widetilde{\varphi} + [m(x_1 + \sqrt{\nu}y) - c\widetilde{u} - 2\widetilde{v}]\widetilde{\psi} + \lambda_1\widetilde{\psi} = 0. \end{cases}$$

It is not hard to see that $0 \leq \widetilde{\varphi}(y) \leq 1$ with $\widetilde{\varphi}(0) = 1$. Since $\lim_{\nu\to 0}(u, v) = (\widetilde{\theta}_\mu, m - c\widetilde{\theta}_\mu)$ uniformly on compact subset of Ω, by (3.17) we see that $\max_{\overline{\Omega}}(-\psi) \leq C_3/(C_3 - \lambda_1) < +\infty$, where $C_3 > 0$ independent of μ and ν. A similar argument yields that $-\psi(x_1) \geq C_4 > 0$ for some positive constant C_4 independent of μ and ν. That is, $-C_3 \leq \widetilde{\psi}(y) \leq 0$ and $\widetilde{\psi}(0) = \psi(x_1) \leq -C_4$ for every $y \in \mathbb{R}^N$. By elliptic regularity theory and Sobolev embedding theorem [9], passing to a subsequence if necessary, we may suppose that $(\widetilde{\varphi}, \widetilde{\psi}) \to (\overline{\varphi}, \overline{\psi})$ in $C^1(K)$ and weakly in $W^{2,p}(K)$ for every $p > 1$, where K is any compact subset of $\mathbb{R}^N$. Moreover, $\overline{\varphi}$ and $\overline{\psi}$ fulfill

$$\begin{cases} \Delta_y\overline{\varphi} = 0, & y \in \mathbb{R}^N, \\ \Delta_y\overline{\psi} - c(m(\overline{x}) - c\widetilde{\theta}_\mu)\overline{\varphi} - (m(\overline{x}) - c\widetilde{\theta}_\mu)\overline{\psi} + \overline{\lambda}_1\overline{\psi} = 0, & y \in \mathbb{R}^N, \\ 0 \leq \overline{\varphi}(y) \leq 1, \quad \overline{\varphi}(0) = 1 \text{ and } -C_3 \leq \overline{\psi}(y) \leq 0, \quad \overline{\psi}(0) \leq -C_4, & y \in \mathbb{R}^N. \end{cases} \tag{3.18}$$

By the boundedness of $\overline{\varphi}$ and $\overline{\varphi}(0) = 1$, we have $\overline{\varphi}(y) \equiv 1$ in $\mathbb{R}^N$. Since $\overline{\psi}$ is bounded, from (3.18) we conclude

$$\overline{\psi}(y) \equiv \frac{c(m(\overline{x}) - c\widetilde{\theta}_\mu(\overline{x}))}{\overline{\lambda}_1 - (m(\overline{x}) - c\widetilde{\theta}_\mu(\overline{x}))}, \quad y \in \mathbb{R}^N.$$

Since $x_1 \to \overline{x}$ and $(u, v) \to (\widetilde{\theta}_\mu, m - c\widetilde{\theta}_\mu)$ uniformly as $\nu \to 0$, by passing to the limit in (3.16), we get

$$\overline{\lambda}_1^2 - [bm(\overline{x}) + (2 - bc - c)\widetilde{\theta}_\mu(\overline{x})]\overline{\lambda}_1 + [2(1 - bc)\widetilde{\theta}_\mu(\overline{x}) - (1 - b)m(\overline{x})](m(\overline{x}) - c\widetilde{\theta}_\mu(\overline{x})) \leq 0. \tag{3.19}$$

Recall that $\widetilde{\theta}_\mu$ satisfies

$$\mu\Delta\widetilde{\theta}_\mu + \widetilde{\theta}_\mu[(1 - b)m(x) - (1 - bc)\widetilde{\theta}_\mu] = 0 \text{ in } \Omega, \quad \widetilde{\theta}_\mu = 0 \text{ on } \partial\Omega.$$

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By virtue of Lemma 4 (i), we see that $\widetilde{\theta}_\mu \to \frac{(1-b)m}{1-bc}$ uniformly on compact subset of Ω as $\mu \to 0$, and $\widetilde{\theta}_\mu \to 0$ uniformly as $\mu \to \mu_*$. Since $\widetilde{\theta}_\mu$ is strictly decreasing in μ by Lemma 4 (ii), there exists a unique $\underline{\mu} = \underline{\mu}(m, \Omega) > 0$ such that $\widetilde{\theta}_\mu > \frac{(1-b)m}{2(1-bc)}$ if $\mu < \underline{\mu}$ and $\widetilde{\theta}_\mu < \frac{(1-b)m}{2(1-bc)}$ if $\mu > \underline{\mu}$.

Since $\overline{\lambda}_1 \le 0$ and $2(1-bc)\widetilde{\theta}_\mu > (1-b)m$ on compact subset of Ω for any $\mu \in (0, \underline{\mu})$, we see that inequality (3.19) can not hold, contradiction. This finishes the proof. □

In particular, from Theorem 9 and Lemma 11, we have the following

Corollary 2. *There exists small $\tau > 0$ such that if $0 < \mu, \nu \le \tau$, any coexistence steady state derived from Theorem 9 is globally asymptotically stable among non-negative, non-trivial initial data (u_0, v_0).*

The proof of Theorem 1. Here we only prove (i), since (ii) can be established by similar arguments. From Lemma 8, we see that for every $\mu \in (0, \gamma)$, if $\nu < \nu_*$, semi-trivial steady state $(\theta_\mu, 0)$ is unstable, while $(0, \theta_\nu)$ is also unstable for any $\nu \in (0, \gamma)$ provided that $\mu < \mu_*$. Hence there exist at least one coexistence steady state of (1.1). By virtue of Lemma 11, we find that for any $\mu \in (0, \underline{\mu})$ and sufficiently small ν, the coexistence steady state of (1.1) is linearly stable. It follows from Lemma 7 (i) that the first conclusion of Theorem 1 holds. The proof of Theorem 1 is complete. □

4. Global stability of semi-trivial steady states

To prove one of the semi-trivial steady states $(\theta_\mu, 0)$ and $(0, \theta_\nu)$ is globally asymptotically stable, by Lemma 7 (ii), it suffices to exhibit that (1.1) has no coexistence steady state.

Lemma 12. *Fix $b, c \in (0, 1)$. For any compact subset $K \subset (\mu^*, \gamma)$, there exists $\delta_3 > 0$ such that if $0 < \nu \le \delta_3$, (1.1) has no coexistence steady state for every $\mu \in K$.*

Proof. We argue by contradiction: passing to a subsequence if necessary, assume that there exist some sequence $\{\mu_k, \nu_k\}_{k=1}^\infty$ with $\lim_{k\to\infty} \mu_k = \widehat{\mu} \in K$ and $\lim_{k\to\infty} \nu_k = 0$ such that (1.1) has a coexistence steady state (u_k, v_k) for every $k \ge 1$, i.e.,

$$\begin{cases} \mu_k \Delta u_k + u_k(m(x) - u_k - bv_k) = 0 & \text{in } \Omega, \\ \nu_k \Delta v_k + v_k(m(x) - cu_k - v_k) = 0 & \text{in } \Omega, \\ u_k = v_k = 0 & \text{on } \partial\Omega. \end{cases}$$

Consider the following single equation

$$\nu_k \Delta v + v(m(x) - v) = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega. \tag{4.1}$$

It is easy to see that v_k is a subsolution of (4.1) and $m(x)$ is a supersolution of (4.1). By a sub/super solution argument, (4.1) has a unique positive solution, denoted by $v_{k,1}$, and $v_k \le v_{k,1}$, $v_{k,1} \to m(x)$ uniformly on compact subset of Ω as $k \to \infty$. Since $v_k \le v_{k,1}$, we see that u_k satisfies

$$-\mu_k \Delta u_k \ge u_k(m(x) - u_k - bv_{k,1}) \text{ in } \Omega, \quad u_k = 0 \text{ on } \partial\Omega.$$

That is, u_k is a supersolution of

$$-\mu_k \Delta u = u(m(x) - u - bv_{k,1}) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{4.2}$$

Obviously, 0 is a subsolution of (4.2). By a sub/super solution argument again, we see that (4.2) admits a unique non-negative solution, denoted by $u_{k,1}$, and $u_{k,1} \le u_k$ in $\overline{\Omega}$.

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For every $k \geq 1$, we can construct $\{u_{k,i}\}_{i=1}^{\infty}$ and $\{v_{k,i}\}_{i=1}^{\infty}$ as follows: $u_{k,i}$ is the unique non-negative solution of

$$-\mu_k \Delta u_{k,i} = u_{k,i}(m(x) - u_{k,i} - bv_{k,i}) \text{ in } \Omega, \quad u_{k,i} = 0 \text{ on } \partial\Omega, \tag{4.3}$$

and $v_{k,i+1}$ is the unique positive solution of

$$-\nu_k \Delta v_{k,i+1} = v_{k,i+1}(m(x) - cu_{k,i} - v_{k,i+1}) \text{ in } \Omega, \quad v_{k,i+1} = 0 \text{ on } \partial\Omega. \tag{4.4}$$

Furthermore, the following inequalities hold:

$$\begin{cases} u_{k,1} \leq \cdots \leq u_{k,i} \leq u_{k,i+1} \leq \cdots \leq u_k & \text{in } \overline{\Omega}, \\ v_k \leq \cdots \leq v_{k,i+1} \leq v_{k,i} \leq \cdots \leq v_{k,1} & \text{in } \overline{\Omega}. \end{cases} \tag{4.5}$$

The existence and uniqueness of $u_{k,i}$ and $v_{k,i}$ can be established by a sub/super solution argument as before, and the above inequalities can be obtained at the same time.

Claim. For every $i \geq 1$, $u_{k,i} \to U_i$ and $v_{k,i} \to V_i$ uniformly on compact subset of Ω as $k \to \infty$. Moreover, U_i and V_i fulfill

$$\begin{cases} \widehat{\mu}\Delta U_i + U_i(m(x) - U_i - bV_i) = 0 \text{ in } \Omega, \quad U_i = 0 \text{ on } \partial\Omega. \\ m(x) - cU_i - V_i = 0 \text{ in } \Omega. \end{cases} \tag{4.6}$$

We argue by induction on $i \geq 1$: note that $v_{k,1} \to m(x) = V_1$ uniformly on compact subset of Ω as $k \to \infty$. By elliptic regularity theory and Sobolev embedding theorem, (4.3) and (4.6), we see that $u_{k,1} \to U_1$ in $C^2(\Omega')$ uniformly as $k \to \infty$, where $\Omega' \subset\subset \Omega$. Suppose that $u_{k,i} \to U_i$ and $v_{k,i} \to V_i$ uniformly on compact subset of Ω as $k \to \infty$, then for any $\epsilon > 0$, there exists some $N > 0$ such that if $k > N$, $U_i - \epsilon \leq u_{k,i} \leq U_i + \epsilon$. Thus by (4.4), $v_{k,i+1}$ satisfies

$$\begin{aligned} v_{k,i+1}[m - c(U_i + \epsilon) - v_{k,i+1}] &\leq v_{k,i+1}(m - cu_{k,i} - v_{k,i+1}) \\ &= -\nu_k \Delta v_{k,i+1} \leq v_{k,i+1}[m - c(U_i - \epsilon) - v_{k,i+1}], \end{aligned}$$

i.e., $v_{k,i+1}$ is a supersolution of $-\nu_k \Delta w = w[m - c(U_i + \epsilon) - w]$ and a subsolution of $-\nu_k \Delta w = w[m - c(U_i - \epsilon) - w]$. By Lemma 4 (i), we obtain

$$m - c(U_i + \epsilon) \leq \lim_{k\to\infty} v_{k,i+1} \leq m - c(U_i - \epsilon)$$

uniformly on compact subset of Ω. In other words, $\lim_{k\to\infty} v_{k,i+1} = m - cU_i = V_{i+1}$ uniformly on compact subset of Ω. By elliptic regularity theory and Sobolev embedding theorem, (4.3) and (4.6) again, we have $u_{k,i+1} \to U_{i+1}$ in $C^2(\Omega')$ uniformly as $k \to \infty$. This we complete the proof of our assertion.

Letting $k \to \infty$ in (4.5), we find

$$\begin{cases} 0 \leq U_1 \leq \cdots \leq U_i \leq U_{i+1} \leq \cdots \leq \underline{\lim}_{k\to\infty} u_k & \text{in } \overline{\Omega}, \\ 0 \leq \overline{\lim}_{k\to\infty} v_k \leq \cdots \leq V_{i+1} \leq V_i \leq \cdots \leq V_1 & \text{in } \overline{\Omega}. \end{cases} \tag{4.7}$$

Set

$$U(x) = \lim_{i\to\infty} U_i(x), \quad V(x) = \lim_{i\to\infty} V_i(x). \tag{4.8}$$

Since both $\{U_i\}_{i=1}^{\infty}$ and $\{V_i\}_{i=1}^{\infty}$ are monotone sequences of functions and both limits $U(x)$ and $V(x)$ are continuous functions, it follows from [27, Theorem 7.13] that $U_i(x) \to U(x)$ and $V_i(x) \to V(x)$ uniformly on compact subset of Ω as $i \to \infty$. Combining (4.6) with (4.8), we have

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$$\begin{cases} \widehat{\mu}\Delta U + U(m(x) - U - bV) = 0 \text{ in } \Omega, \quad U = 0 \text{ on } \partial\Omega, \\ m(x) - cU - V = 0 \text{ in } \Omega. \end{cases} \tag{4.9}$$

Substituting $V = m - cU$ into the equation of U in (4.9), then

$$\widehat{\mu}\Delta U + U[(1-b)m(x) - (1-bc)U] = 0 \text{ in } \Omega, \quad U = 0 \text{ on } \partial\Omega.$$

Hence $U(x) = 0$ or $U(x) = \widetilde{\theta}_{\widehat{\mu}}$.

Case 1. $U(x) = 0$. By the first inequality of (4.7), we see that $u_k \geq 0$ for large k. On the other hand, by (4.9), we have $V(x) = m(x)$ in Ω, which implies $v_k \leq m(x)$ for large k. Integrating the equation of u_k and applying the boundary condition, we obtain

$$0 > \mu_k \int\limits_{\partial\Omega} \frac{\partial u_k}{\partial n} = \int\limits_{\Omega} u_k(m(x) - u_k - bv_k) > 0 \text{ for large } k,$$

where the first strict inequality follows from the Hopf lemma. Clearly, this is a contradiction.

Case 2. $U(x) = \widetilde{\theta}_{\widehat{\mu}}$. This means that $\widehat{\mu} < \mu^*$. Recall the definition of μ^*, this is impossible. Thus, we complete the proof. □

Not only the dispersal rate of competing species affect the dynamics of (1.1), but also the inter-specific competition coefficients. We are going to show that for some ranges of dispersal rates of two competing species, if one of the inter-specific competition coefficients is sufficiently small, (1.1) has no coexistence steady state.

Lemma 13. *Fix $b \in (0,1)$. For any compact subset $K \subset (\mu^*, \gamma) \times (0, \nu_*)$, there exists $\delta_2 > 0$ such that if $0 < c \leq \delta_2$, (1.1) has no coexistence steady state for $(\mu, \nu) \in K$.*

Proof. We argue by contradiction: passing to a subsequence if necessary, suppose that there exist some sequence $(\mu_k, \nu_k) \in K, c_k \to 0$ such that (1.1) has a coexistence steady state (u_k, v_k) for every $k \geq 1$, where $u_k, v_k > 0$ in Ω and $u_k = v_k = 0$ on $\partial\Omega$. By elliptic regularity theory and Sobolev embedding theorem, passing to a subsequence if necessary, we may assume that $(\mu_k, \nu_k) \to (\widehat{\mu}, \widehat{\nu}) \in K, c_k \to 0$, and $(u_k, v_k) \to (U, V)$ in $C^2(\overline{\Omega})$ with

$$\begin{cases} \widehat{\mu}\Delta U + U(m(x) - U - bV) = 0 & \text{in } \Omega, \\ \widehat{\nu}\Delta V + V(m(x) - V) = 0 & \text{in } \Omega, \\ U = V = 0 & \text{on } \partial\Omega. \end{cases}$$

By the equation of V, we see that $V(x) = 0$ or $V(x) = \theta_{\widehat{\nu}}$.

Case 1. $V(x) = 0$. That is to say, $v_k \to 0$ in $C^2(\Omega')$ with $\Omega' \subset\subset \Omega$ as $k \to \infty$. From the equation of u_k, we have $u_k \to \theta_{\widehat{\mu}}$ in $C^2(\Omega')$ as $k \to \infty$. Integrating the equation of v_k and applying the boundary condition, we have

$$0 > \nu_k \int\limits_{\partial\Omega} \frac{\partial v_k}{\partial n} = \int\limits_{\Omega} v_k(m(x) - cu_k - v_k) \to 0 \quad \text{as } k \to \infty,$$

where the strict inequality follows from Hopf boundary point lemma. Obviously, we arrive at a contradiction.

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Case 2. $V(x) = \theta_{\widehat{\nu}}$. Then U satisfies

$$\widehat{\mu}\Delta U + U(m(x) - b\theta_{\widehat{\nu}} - U) = 0 \text{ in } \Omega, \quad U = 0 \text{ on } \partial\Omega.$$

That fact $(\mu, \nu) \in K$ implies that $U \equiv 0$, i.e., $u_k \to 0$ in $C^2(\Omega')$. Setting $\widetilde{u}_k = u_k/\|u_k\|_{L^\infty(\Omega)}$ and letting $k \to \infty$ (passing to a subsequence if necessary), we find

$$\widehat{\mu}\Delta\widehat{U} + \widehat{U}(m(x) - b\theta_{\widehat{\nu}}) = 0 \text{ in } \Omega, \quad \widehat{U} = 0 \text{ on } \partial\Omega,$$

where $\widehat{U} > 0$ in Ω because $\|\widehat{U}\|_{L^\infty(\Omega)} = 1$. This indicates that

$$\widehat{\mu} = \max_{\varphi \in H_0^1(\Omega), \varphi \neq 0} \frac{\int_\Omega (m - b\theta_{\widehat{\nu}})\varphi^2}{\int_\Omega |\nabla\varphi|^2} \leq \mu^*,$$

which contradicts our hypotheses. The proof is complete. □

The proof of Theorem 2. In view of the ranges of μ and ν, we see that $(\theta_\mu, 0)$ is unstable, while $(0, \theta_\nu)$ is stable. On the other hand, by Lemma 13, (1.1) has no coexistence steady state, it follows from Lemma 7 (ii) that the first conclusion of Theorem 2 holds. It is not difficult to see that other conclusions of Theorem 2 hold via the similar arguments. This completes the proof of Theorem 2. □

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