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Zero entropy for some birational maps of $\mathbb{C}^2$

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In this study, we consider a special case of the family of birational maps $f : \mathbb{C}^2 \to \mathbb{C}^2$, which were dynamically classified by [13]. We identify the zero entropy subfamilies of f and explicitly give the associated invariant fibrations. In particular, we highlight all of the integrable and periodic mappings.

1. Introduction

In this study, we consider the family of linear fractional maps $f : \mathbb{C}^2 \to \mathbb{C}^2$ of the form:

$$f(x, y) = \left(\alpha_0 + \alpha_1 x + \alpha_2 y, \frac{\beta_0 + \beta_1 x + \beta_2 y}{\gamma_0 + \gamma_1 x + \gamma_2 y}\right), (\gamma_1, \gamma_2) \neq (0, 0), \tag{1}$$

where the parameters $\alpha_i,\ \beta_i,\ \gamma_i,\ i \in \{0, 1, 2\}$ are complex numbers.

We require that the family of mappings $f(x, y)$ in (1) is birational. The values of the parameters $\alpha_i,\ \beta_i,\ \gamma_i,\ i \in \{0, 1, 2\}$ for which $f(x, y)$ is a birational mapping are discussed in Lemma 1. The dynamics generated by birational mappings in the plane and their classification have been discussed widely in recent years (see [1–4,7,8,10,11,14–16,19–25,28]). The family of mappings (1) was classified in a previous study by [13,27]. In this study, we consider f when it exhibits some degenerate behavior for general values of parameters, specifically when $\alpha_2\gamma_1 - \alpha_1\gamma_2 = 0$ or when $\beta_2\gamma_1 - \beta_1\gamma_2 = 0$.

For a birational map $f(x, y)$ the sequence of degrees d_n for the iterates of f satisfies a homogeneous linear recurrence (see [17]), which is governed by the characteristic polynomial $\mathcal{X}(x)$ of a certain matrix associated

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with F, where $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ is the extension of $f : \mathbb{C}^2 \to \mathbb{C}^2$ in the projective plane $P\mathbb{C}^2$. The sequence of degrees allows the introduction of a quantity called the *dynamical degree of* F, which is defined as:

$$\delta(F) := \lim_{n\to\infty} (\deg(F^n))^{\frac{1}{n}}, \tag{2}$$

where F^n represents the iterates of F. The logarithm of $\delta(F)$ is the *algebraic entropy of* F ([3–5,16,17,26]).

Considering the embedding $(x_1, x_2) \in \mathbb{C}^2 \mapsto [1 : x_1 : x_2] \in P\mathbb{C}^2$ into the projective space, the induced map $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ has three components $F_i[x_0 : x_1 : x_2]$, $i = 1, 2, 3$ which are homogeneous polynomials comprising $F[x_0 : x_1 : x_2] = [F_1[x_0 : x_1 : x_2] : F_2[x_0 : x_1 : x_2] : F_3[x_0 : x_1 : x_2]]$, where:

$$\begin{aligned} F_1[x_0 : x_1 : x_2] &= x_0(\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2), \\ F_2[x_0 : x_1 : x_2] &= (\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2)(\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2), \\ F_3[x_0 : x_1 : x_2] &= x_0(\beta_0 x_0 + \beta_1 x_1 + \beta_2 x_2). \end{aligned} \tag{3}$$

The map F has degree two because the components of F do not have common factors for general values of the parameters. Similarly, the degree of each iterate of F can generally be found by iterating F and removing the common homogeneous components of $F^n = F \circ \cdots \circ F$ for each $n \in \mathbb{N}$.

The birational mappings $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ have an indeterminacy set $\mathcal{I}(F)$ of points where F is ill defined as a continuous map. Hence, they also have a set of curves that are sent to a single point called the *exceptional locus* of F denoted as $\mathcal{E}(F)$. Generically, the mappings of the form (1) have three indeterminacy points. The exceptional locus is formed by three straight lines, where each two of them intersect on a single indeterminate point of F. However, in some cases, the exceptional locus is formed by only two straight lines. In this case, these mappings are identified as *degenerate mappings*. Lemma 1 in the following section states the conditions for the birationality and degeneracy of the family (1). We consider all the subfamilies of $f(x, y)$ for which f exhibits degenerate behavior for general values of the parameters. The cases where the exceptional locus is formed by three straight lines were discussed previously by [12] and [13]. These cases are recognized as *nondegenerate mappings*.

The first aim of this study is to search for the sequence of degrees d_n for iterates of $f(x, y)$ in (1), which is achieved by performing a series of blow-ups in order to find the characteristic polynomial that determines the behavior of d_n.

The second aim is to identify the values of the parameters for which $f(x, y)$ has zero algebraic entropy and extract dynamical consequences. In particular, we employ the results reported by Diller and Favre ([17]), which characterize the growth rate of d_n when invariant fibrations exist. We find all of the prescribed invariant fibrations for each of these cases. We focus on the elements of the family $f(x, y)$, which are integrable mappings. We also distinguish all of the periodic mappings $f(x, y)$ and give a pair of first integrals for them that are generically transverse.

The remainder of this article is organized as follows. In Section 2, we give some preliminary results for the family (1) and the birational maps (with descriptions of the blow-up process and the Picard group). In Section 3, we consider the subfamily $\alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$. In Section 4, we investigate the subfamily $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$.

The results are given in the following. The theorems provide the results for the dynamical degree and the growth of d_n, and the propositions give the results for the zero entropy and the existence of invariant fibrations. Thus, in Section 3, we present Theorem 4 and Proposition 5 concerning the subfamily $\alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$. In Section 4, we deal with the mappings that satisfy $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$ in three subsections. In the first subsection, we present Theorem 6 and Proposition 7, which correspond to the case where $\gamma_1\gamma_2 \neq 0$. In the next subsection, we analyze the case where $\gamma_1 = 0$, and we prove Theorem 8 and Proposition 9 (resp. Proposition 11) when $\alpha_2 \neq 0$ (resp. $\alpha_2 = 0$). In the last subsection, we present Theorem 12 and Proposition 13, which correspond to the case where $\gamma_2 = 0$.

2. Preliminary results

Consider the mapping $F[x_0 : x_1 : x_2] : P\mathbb{C}^2 \to P\mathbb{C}^2$ defined in (3). The exceptional locus of $F[x_0 : x_1 : x_2]$ is $\mathcal{E}(F) = \{S_0, S_1, S_2\}$, where:

$$S_0 = \{x_0 = 0\}, \quad S_1 = \{\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2 = 0\},$$
$$S_2 = \{(\alpha_1(\beta\gamma)_{02} - \alpha_2(\beta\gamma)_{01})\, x_0 + \alpha_1(\beta\gamma)_{12} x_1 + \alpha_2(\beta\gamma)_{12} x_2 = 0\}.$$

We employ the following notation: $(\delta\epsilon)_{ij} = \delta_i\epsilon_j - \delta_j\epsilon_i$. The exceptional locus of $F^{-1}[x_0 : x_1 : x_2]$ is $\mathcal{E}(F^{-1}) = \{T_0, T_1, T_2\}$, where:

$$\begin{aligned}
&T_0 = \{(\gamma_0(\alpha\beta)_{12} - \gamma_1(\alpha\beta)_{02} + \gamma_2(\alpha\beta)_{01})\, x_0 - (\beta\gamma)_{12} x_1 = 0\},\\
&T_1 = \{(\alpha\beta)_{12} x_0 - (\alpha\gamma)_{12} x_2 = 0\}, \qquad T_2 = \{x_0 = 0\}.
\end{aligned}$$

The birational map $F[x_0 : x_1 : x_2]$ has an indeterminacy set $\mathcal{I}(F)$ of points where F is ill defined as a continuous map. This set is given by:

$$\{[x_0 : x_1 : x_2] \in P\mathbb{C}^2 : F_1[x_0 : x_1 : x_2] = 0, F_2[x_0 : x_1 : x_2] = 0, F_3[x_0 : x_1 : x_2] = 0]\},$$

which gives:

$$\mathcal{I}(F) = \{O_0, O_1, O_2\},$$

where

$$\begin{aligned}
O_0 &= [(\beta\gamma)_{12} : (\beta\gamma)_{20} : (\beta\gamma)_{01}],\\
O_1 &= [0 : \alpha_2 : -\alpha_1],\\
O_2 &= [0 : \gamma_2 : -\gamma_1],
\end{aligned}$$

and $(\beta\gamma)_{ij} := \beta_i\gamma_j - \gamma_j\beta_i$ for $i, j = 0, 1, 2$.

By referring to $g(x, y)$ as the inverse of $f(x, y)$ given in (1) and considering $G[x_0 : x_1 : x_2]$ as its extension on $P\mathbb{C}^2$, then an indeterminacy set $\mathcal{I}(G)$ also exists, i.e., $\mathcal{I}(G) = \{A_1, A_2, A_3\}$, where:

$$\begin{aligned}
A_0 &= [0 : 1 : 0],\\
A_1 &= [0 : 0 : 1],\\
A_2 &= [(\beta\gamma)_{12}\,(\alpha\gamma)_{12}, (\alpha_0\,(\beta\gamma)_{12} - \alpha_1\,(\beta\gamma)_{02} + \alpha_2\,(\beta\gamma)_{01})\,(\alpha\gamma)_{12} : (\alpha\beta)_{12}\,(\beta\gamma)_{12}].
\end{aligned}$$

We are interested in the birational mappings (1) when the corresponding F only has two distinct exceptional curves. The next lemma describes the set of parameters considered in this study.

We recall that a birational map is a map $f : \mathbb{C}^2 \to \mathbb{C}^2$ with rational components such that an algebraic curve V exists and another rational map g exists such that $f \circ g = g \circ f = \mathrm{Id}$ in $\mathbb{C}^2 \setminus V$.

Lemma 1. *Consider the mappings*

$$f(x_1, x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right), \, (\gamma_1, \gamma_2) \neq (0, 0).$$

Then:

(a) The mapping f is birational if and only if the vectors $(\beta_0, \beta_1, \beta_2)$, $(\gamma_0, \gamma_1, \gamma_2)$ are linearly independent and $((\alpha\beta)_{12}, (\alpha\gamma)_{12}) \neq (0,0)$, $((\alpha\gamma)_{12}, (\beta\gamma)_{12}) \neq (0,0)$, and either $((\alpha\beta)_{12}, (\beta\gamma)_{12}) \neq (0,0)$ or $(\beta_1, \beta_2) = (0,0)$.
(b) The mapping f is degenerate if and only if $(\beta\gamma)_{12} = 0$ or $(\alpha\gamma)_{12} = 0$.

Proof. The conditions in *(a)* are necessary for f to be invertible because if the vectors $(\beta_0, \beta_1, \beta_2)$, $(\gamma_0, \gamma_1, \gamma_2)$ are linearly dependent, then the second component of f is a constant. In addition, if $((\alpha\beta)_{12}, (\alpha\gamma)_{12}) = (0,0)$ or $((\alpha\gamma)_{12}, (\beta\gamma)_{12}) = (0,0)$, then f only depends on $\alpha_1 x_1 + \alpha_2 x_2$ or on $\gamma_1 x_1 + \gamma_2 x_2$. If $((\alpha\beta)_{12}, (\beta\gamma)_{12}) = (0,0)$ and $(\beta_1, \beta_2) \neq (0,0)$, then f only depends on $\beta_1 x_1 + \beta_2 x_2$.

Now, we assume that the conditions in (a) are satisfied. Then, the inverse of f is formally defined as:

$$f^{-1}(x,y) = \left(\frac{-(\alpha\beta)_{02} + \beta_2 x + (\alpha\gamma)_{02} y - \gamma_2 xy}{(\alpha\beta)_{12} - (\alpha\gamma)_{12} y}, \frac{(\alpha\beta)_{01} - \beta_1 x + (\alpha\gamma)_{10} y + \gamma_1 xy}{(\alpha\beta)_{12} - (\alpha\gamma)_{12} y} \right),$$

and it is well defined. Furthermore, the numerators of the determinants of the Jacobian of f and f^{-1} are:

$$\alpha_1(\beta\gamma)_{02} - \alpha_2(\beta\gamma)_{01} + \alpha_1(\beta\gamma)_{12} x + \alpha_2(\beta\gamma)_{12} y \tag{4}$$

and

$$\alpha_0(\beta\gamma)_{12} - \alpha_1(\beta\gamma)_{02} + \alpha_2(\beta\gamma)_{01} - (\beta\gamma)_{12} y, \tag{5}$$

respectively. Clearly, the conditions in (a) imply that both (4) and (5) are not identically zero. Hence, $f \circ f^{-1} = f^{-1} \circ f = id$ in $\mathbb{C}^2 \setminus V$, where V is the algebraic curve determined by the common zeros of (4) and (5).

For *(b)*, we know that S_i is mapped to A_i, which implies that the points A_0, A_1, A_2 are not all distinct. $A_0 \neq A_1$ so we have two possibilities: $A_0 = A_2$ or $A_1 = A_2$. The condition that $A_0 = A_2$ is written as $(\beta\gamma)_{12}\,(\alpha\gamma)_{12} = 0$ and $(\alpha\beta)_{12}(\beta\gamma)_{12} = 0$. From *(a)*, the vector $((\alpha\beta)_{12}, (\alpha\gamma)_{12}) \neq (0,0)$. Hence, $(\beta\gamma)_{12}$ must be zero. In a similar manner, we can see that $A_1 = A_2$ if and only if $(\alpha\gamma)_{12} = 0$. □

We observe that F maps each S_i to A_i and that the inverse of F maps T_i to O_i for $i \in \{0,1,2\}$. To specify this behavior, we write $F : S_i \twoheadrightarrow A_i$ (also $F^{-1} : T_i \twoheadrightarrow O_i$). It is known that the dynamical degree depends on the orbits of A_0, A_1, A_2 under the action of F (see the proposition in Section 2). Indeed, the key point is whether the iterates of A_0, A_1, A_2 coincide with any of the indeterminacy points of F. After finding one orbit of F that ends at some indeterminacy point of F, we perform a series of blow-ups in order to remove the indeterminacy of F in a new extended space.

For $X = \{((x,y),[u:v]) \in \mathbb{C}^2 \times P\mathbb{C}^1 \,:\, xv = yu\}$ and $p \in \mathbb{C}^2$, let (X, π) be the blowing up of $\mathbb{C}^2$ at the point p. By translating p at the origin, $\pi^{-1}p = \pi^{-1}(0,0) = \{((0,0),[u:v])\} := E_p \simeq P\mathbb{C}^1$ and $\pi^{-1}q = \pi^{-1}(x,y) = ((x,y),[x:y]) \in X$ for $q = (x,y) \neq (0,0)$. Every blow-up gives a new expanded space X and a new induced map $\tilde{F} : X \to X$ is defined based on it. The indeterminacy sets and exceptional locus can also be defined by considering the meromorphic functions on the complex manifolds X that we obtain after a series of blow-ups. Consider the Picard group of X denoted by $\mathcal{P}ic(X)$, where X is the complex manifold. For a generic line $L \in P\mathbb{C}^2$, $\mathcal{P}ic(P\mathbb{C}^2)$ is generated based on the class of L. If the base points of the blow-ups are $\{p_1, p_2, \ldots, p_k\} \subset P\mathbb{C}^2$ and $E_i := \pi^{-1}\{p_i\}$, then it is known that $\mathcal{P}ic(X)$ is generated by $\{\hat{L}, E_1, E_2, \ldots, E_k\}$, ([3,4]). The curve $\hat{L}$ is the *strict transform* of $L \subset \mathbb{C}^2$, which is the adherence of $\pi^{-1}(C \setminus \{p\})$, in the Zariski topology. Furthermore, $\pi : X \longrightarrow P\mathbb{C}^2$ induces a morphism of groups $\pi^* : \mathcal{P}ic(P\mathbb{C}^2) \longrightarrow \mathcal{P}ic(X)$, which have the property that for any complex curve $C \subset P\mathbb{C}^2$:

$$\pi^*(C) = \hat{C} + \sum m_i\, E_i, \tag{6}$$

where m_i is the algebraic multiplicity of C at p_i. For $F \in P\mathbb{C}^2$, we denote $\tilde{F}$ as the natural extension of F on X and it induces a morphism of groups, $\tilde{F}^* : \mathcal{P}ic(X) \to \mathcal{P}ic(X)$, by considering the classes of preimages such that $\tilde{F}^*(\hat{L}) = d\,\hat{L} + \sum_{i=1}^{k} c_i\, E_i, c_i \in \mathbb{Z}$, where d is the degree of F. By iterating F, we obtain the corresponding formula by changing F by F^n and d by d_n. To determine the behavior of the sequence of degrees d_n, we consider the maps $\tilde{F}$ such that:

$$(\tilde{F}^n)^* = (\tilde{F}^*)^n. \tag{7}$$

The maps $\tilde{F}$ that satisfy condition (7) are called *algebraically stable maps* (AS maps) ([17]). In order to obtain the AS maps, we use the following useful result presented by Fornaess and Sibony ([18]) (also see Theorem 1.14) and by [17]:

$$\text{The map } \tilde{F} \text{ is AS if and only if for every exceptional curve } C \text{ and all } n \geq 0\,,\ \tilde{F}^n(C) \notin \mathcal{I}(\tilde{F}). \tag{8}$$

It is known (see Theorem 0.1 given by [17]) that we can always make a birational map AS by performing a finite number of blow-ups. If this is the case, we refer to $\mathcal{X}(x) = x^k + \sum_{i=0}^{k-1} c_i\, x^i$ as the characteristic polynomial of $A := (\tilde{F}^*)$. Then, $\mathcal{X}(A) = 0$ and d_i is the $(1,1)$ term of A^i, so we find that $d_k = -(c_0 + c_1 d_1 + c_2 d_2 + \cdots + c_{k-1} d_{k-1})$, i.e., the sequence d_n satisfies a homogeneous linear recurrence with constant coefficients. Thus, the dynamical degree is the largest real root of $\mathcal{X}(x)$. In the following, we state Theorem 2, which is useful for our analysis and it is a direct consequence of Theorem 0.2 given by [17] and Theorem A given by [6].

Given a birational map F of $P\mathbb{C}^2$, let $\tilde{F}$ be its regularized map such that the induced map $\tilde{F}^* : \mathcal{P}ic(X) \to \mathcal{P}ic(X)$ satisfies $(\tilde{F}^n)^* = (\tilde{F}^*)^n$. Then:

Theorem 2. *(See [6,17]) Let $F : P\mathbb{C}^2 \to P\mathbb{C}^2$ be a birational map and let $d_n = deg(F^n)$. Then, up to bimeromorphic conjugacy, exactly one of the following holds.*

- *The sequence d_n grows quadratically and $\tilde{F}$ is an automorphism that preserves an elliptic fibration.*
- *The sequence d_n grows linearly and $\tilde{F}$ preserves a rational fibration. In this case, $\tilde{F}$ cannot be conjugated to an automorphism.*
- *The sequence d_n is bounded and $\tilde{F}$ is an automorphism that preserves two generically transverse rational fibrations.*
- *The sequence d_n grows exponentially.*

In the first three cases, $\delta(F) = 1$, whereas $\delta(F) > 1$ in the last case. Furthermore, in the first and second cases, the invariant fibrations are unique.

We are only interested in degenerate maps $f(x, y)$ of type (1), so we have to consider two subfamilies: $(\beta\gamma)_{12} = 0$ and $(\alpha\gamma)_{12} = 0$. First, we consider the simpler case where $(\alpha\gamma)_{12} = 0$.

3. Subfamily $(\alpha\gamma)_{12} = 0$

Lemma 3. *Consider the birational mappings*

$$f(x_1, x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right), (\gamma_1, \gamma_2) \neq (0, 0)$$

with the condition that $(\alpha\gamma)_{12} = \alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$. Then, either:

(i) *The four numbers* $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ *are different from zero.*
(ii) $\alpha_1 = 0, \gamma_1 = 0$ *and* $\alpha_2 \neq 0 \neq \gamma_2$.
(iii) $\alpha_2 = 0, \gamma_2 = 0$ *and* $\alpha_1 \neq 0 \neq \gamma_1$.

Proof. From Lemma 1, we know that $(\alpha_1, \alpha_2) \neq (0, 0)$. Therefore, if α_1 (resp. γ_1) is zero, then α_2 (resp. γ_2) is not, and from $\alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$, we find that γ_1 (resp. α_1) must be zero. $\square$

Theorem 4. *Consider the birational mappings*

$$f(x_1, x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right), (\gamma_1, \gamma_2) \neq (0, 0)$$

with the condition that $(\alpha\gamma)_{12} = \alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$. *The following hold:*

(i) *If* $\alpha_1 \neq 0, \alpha_2 \neq 0, \gamma_1 \neq 0$ *and* $\gamma_2 \neq 0$, *then* $\delta(F) = 2$.
(ii) *If* $\alpha_1 = \gamma_1 = 0$, *then* $\delta(F) = \frac{1+\sqrt{5}}{2}$ *and* $d_{n+2} = d_{n+1} + d_n$.
(iii) *If* $\alpha_2 = \gamma_2 = 0$, *then* $\delta(F) = 1$ *and* $d_n = 1 + n$.

Proof. From the hypothesis, we have: $\mathcal{E}(F) = \{S_0, S_1\}, \mathcal{I}(F) = \{O_0, O_1\}$, $\mathcal{E}(F^{-1}) = \{T_0, T_1\}$ and $\mathcal{I}(F^{-1}) = \{A_0, A_1\}$ with

$$S_0 = \{x_0 = 0\}, \, S_1 = \{\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2 = 0\},$$
$$O_0 = [(\beta\gamma)_{12} : (\beta\gamma)_{20} : (\beta\gamma)_{01})], \, O_1 = [0 : \alpha_2 : -\alpha_1],$$
$$T_0 = \{(\beta_2(\alpha\gamma)_{01} - \beta_1(\alpha\gamma)_{12})x_0 - (\beta\gamma)_{12}x_1 = 0\}, \, T_1 = \{x_0 = 0\},$$
$$A_0 = [0 : 1 : 0], \, A_1 = [0 : 0 : 1].$$

When $\alpha_1, \alpha_2, \gamma_1$ and γ_2 are non-zero, and we observe that $A_0 \neq O_0$ and $A_0 \neq O_1$. Hence, $F(A_0) = [0 : \alpha_1\,\gamma_1 : 0] = A_0$ and $F(A_1) = [0 : \alpha_2\,\gamma_2 : 0] = A_0$, so we find that F is AS, which implies that $d_n = 2^n$ and thus $\delta(F) = 2$.

To prove (ii), we observe that $\alpha_1 = \gamma_1 = 0$ implies that $(\alpha_2, \gamma_2) \neq (0, 0)$ but also that $\beta_1 \neq 0$ (if this is not the case, then f would only depend on y and it would not be birational). Now, $A_0 = O_1 \in \mathcal{I}(F)$ and we have to blow-up this point. Let E_0 be the principal divisor at this point and consider a point $[u : v]_{E_0} \in E_0$. In order to extend F on E_0, we consider $[u : v]_{E_0}$ as $\lim_{t\to 0}[tu : 1 : tv]$ and we evaluate $F[tu : 1 : tv]$:

$$F[tu : 1 : tv] = [u(\gamma_0 u + \gamma_2 v)t : (\alpha_0 u + \alpha_2 v)(\gamma_0 u + \gamma_2 v)t : \beta_1 u + (\beta_0 u + \beta_2 v)ut].$$

By taking the limit when t tends to zero, we find that when $u \neq 0$, $\tilde{F}[u : v]_{E_0} = [0 : 0 : 1]$, while $[0 : 1]_{E_0}$ becomes an indeterminacy point for $\tilde{F}$.

To understand the action of $\tilde{F}$ on S_0, we consider the point $[0 : x_1 : x_2]$ as $\lim_{t\to 0}[t : x_1 : x_2]$. Then, for $t \to 0$ (and $x_2 \neq 0$):

$$\lim_{t\to 0} F[t : x_1 : x_2] = \lim_{t\to 0}[\gamma_2 x_2 t : \alpha_2\gamma_2 x_2^2 : (\beta_1 x_1 + \beta_2 x_2)t] = [\gamma_2 x_2 : \beta_1 x_1 + \beta_2 x_2]_{E_0}.$$

The above considerations imply that $\mathcal{I}(\tilde{F}) = \{O_0, [0 : 1]_{E_0}\}, \mathcal{E}(\tilde{F}) = \{\hat{S}_1, E_0\}$ with $\hat{S}_1 \twoheadrightarrow A_1$ and $E_0 \twoheadrightarrow A_1$. To follow the orbit of A_1 under $\tilde{F}$, we observe that $A_1 = [0 : 0 : 1] \in S_0$ and thus $\tilde{F}[0 : 0 : 1] = [\gamma_2 : \beta_2]_{E_0} \neq [0 : 1]_{E_0}$, which is sent to A_1 again to give a two-periodic orbit. This implies that $\tilde{F} : X \longrightarrow X$ is AS. The Picard group of X is $Pic(X) =< \hat{L}, E_0 >$, where L is a generic line of $P\mathbb{C}^2$. Let $\tilde{F}^*$ denote the corresponding map on $Pic(X)$, which acts simply by taking preimages. Hence, $\tilde{F}^*(E_0) = \hat{S}_0$. In order to write $\hat{S}_0$ as a linear combination of $\hat{L}, E_0$, we use (6). We have $\pi^*(S_0) = \hat{S}_0 + E_0 = \hat{L}$, which implies that $\tilde{F}^*(E_0) = \hat{L} - E_0$. In addition, $\pi^*(F^{-1}(L)) = \hat{F^{-1}}(L) + E_0 = 2\hat{L}$, which implies that $\tilde{F}^*(\hat{L}) = 2\hat{L} - E_0$. Hence, the matrix of $\tilde{F}^*$ on $Pic(X) =< \hat{L}, E_0 >$ is: $\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$ with the characteristic polynomial $z^2 - z - 1$. Hence, $\delta(F) = \frac{1+\sqrt{5}}{2}$.

To prove (iii), we again observe that $\alpha_2 = \gamma_2 = 0$ implies that $(\alpha_1, \gamma_1) \neq (0, 0)$ but also that $\beta_2 \neq 0$ (if this not the case, then f would only depend on x and it would not be birational). Now, $A_1 = O_1 = [0 : 0 : 1] \in \mathcal{I}(F)$ and we have to blow-up this point. To understand the action of $\tilde{F}$ on S_1, we consider the point $[\gamma_1 x_0 : -\gamma_0 x_0 : \gamma_1 x_2]$ as $\lim_{t\to 0}[\gamma_1 x_0 : t - \gamma_0 x_0 : \gamma_1 x_2]$. Similar computations to those described above show that each point in $S_1 \setminus \{O_0, [0 : 0 : 1]\}$ is sent to the point $[\gamma_1 : (\alpha\gamma)_{01}]_{E_1}$, and thus $\hat{S}_1$ is still exceptional for $\tilde{F}$.

Now, we consider a point $[u : v]_{E_1} \in E_1$, which we observe as $\lim_{t\to 0}[tu : tv : 1]$. Then, for $t \to 0$,

$$\lim_{t\to 0} F[tu : tv : 1] = \lim_{t\to 0}[tu(\gamma_0 u + \gamma_1 v) : t(\gamma_0 u + \gamma_1 v)(\alpha_0 u + \alpha_1 v) : \beta_2 u].$$

If $\gamma_0 u + \gamma_1 v \neq 0$ and $u \neq 0$, then $\tilde{F}[u : v]_{E_1} = [u : \alpha_0 u + \alpha_1 v]_{E_1}$.

If $\gamma_0 u + \gamma_1 v = 0$, then in the computation given above with $[u : v]_{E_1} = [\gamma_1 : -\gamma_0]_{E_1}$, we consider the point $[\gamma_1 t : -\gamma_0 t : 1] \in S_1$ and we must apply $\tilde{F}$ to obtain $\tilde{F}[\gamma_1 t : -\gamma_0 t : 1] = [\gamma_1 : (\alpha\gamma)_{01}]_{E_1}$. We observe that $\lim_{u\to\gamma_1, v\to -\gamma_0} \tilde{F}[u : v]_{E_1} = [\gamma_1 : (\alpha\gamma)_{01}]_{E_1}$, i.e., $\tilde{F}$ is well defined.

In the case that $u = 0$, $F[0 : t : 1] = [0 : 1 : 0]$, which implies that $[0 : 1]_{E_1} \in \mathcal{I}(\tilde{F})$.

After this blow-up, we claim that the map $\tilde{F}$ is AS because $S_0 \twoheadrightarrow A_0$ and A_0 is a fixed point of F and $\hat{S}_1 \twoheadrightarrow [\gamma_1 : (\alpha\gamma)_{01}]_{E_1}$, and the iterates of this point never coincide with $[0 : 1]_{E_1}$. The Picard group of X is now $Pic(X) =< \hat{L}, E_1 >$ where L is a generic line of $P\mathbb{C}^2$, $\tilde{F}^*(E_1) = \hat{S}_1 + E_1$, and similar computations to those for (ii) give the matrix:

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

The characteristic polynomial is given by $(z - 1)^2$. Hence, $\delta(F) = 1$. Furthermore, $d_1 = 2$ and $d_2 = 3$, so we obtain $d_n = 1 + n$. □

Proposition 5. *Assume that:*

$$f(x_1, x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right), (\gamma_1, \gamma_2) \neq (0, 0)$$

under the condition that $(\alpha\gamma)_{12} = \alpha_1\gamma_2 - \alpha_2\gamma_1 = 0$ has zero entropy. Then, after an affine change of the coordinates, this can be written as:

$$f(x, y) = \left(\alpha_0 + \alpha_1 x, \frac{\beta_0 + y}{x}\right), \quad \alpha_1 \neq 0.$$

This map preserves the fibration $V(x, y) = x$ and this fibration is unique. If $m(x) := \alpha_0 + \alpha_1 x$ is periodic of period p, i.e., if $\alpha_1^p = 1$ for some $p > 1$, $\alpha_1 \neq 1$, then

$$W(x, y) = x \cdot m(x) \cdot m(m(x)) \cdots m^{p-1}(x)$$

is a first integral of $f(x, y)$. In addition, when $\alpha_1 = 1$ and $\alpha_0 = 0$, f is integrable.

Proof. From Theorem 4, we know that the only zero entropy maps in the family are those where $\alpha_2 = \gamma_2 = 0$, and we also know that in this case, β_2, α_1, and γ_1 are different from zero. Hence, we can conjugate $f(x, y)$ with $h(x, y) = \left(\frac{\beta_2}{\gamma_1}x - \frac{\gamma_0}{\gamma_1}, \frac{1}{\beta_2}y + \frac{\beta_1}{\gamma_1}\right)$. By renaming, the parameters we can see that the conjugate map is of the form:

$$f(x, y) = \left(\alpha_0 + \alpha_1 x, \frac{\beta_0 + y}{x}\right), \quad \alpha_1 \neq 0.$$

Clearly, this map preserves the fibration $V(x, y) = x$ and this fibration is unique according to Theorem 2. If $\alpha_1^p = 1$ for some $p > 1$, $\alpha_1 \neq 1$, then $W(f(x, y)) = W(x, y)$ and the result follows. When $\alpha_1 = 1$, then we can see that $f(x, y)$ is integrable if and only if $\beta_0 = 0$. □

Next, we consider the second subfamily.

4. Subfamily $(\beta\gamma)_{12} = 0$

We consider three different cases that depend on $\gamma_1\gamma_2 \neq 0$, $\gamma_1 = 0$, and $\gamma_2 = 0$. When $(\beta\gamma)_{12} = \beta_1\gamma_2 - \beta_2\gamma_1 = 0$, we have $\mathcal{E}(F) = \{S_0, S_1\}$, $\mathcal{I}(F) = \{O_0, O_1\}$, $\mathcal{E}(F^{-1}) = \{T_0, T_1\}$, and $\mathcal{I}(F^{-1}) = \{A_0, A_1\}$ with:

$$S_0 = \{x_0 = 0\}, \, S_1 = \{\gamma_0 x_0 + \gamma_1 x_1 + \gamma_2 x_2 = 0\}$$
$$O_0 = [0 : \gamma_2 : -\gamma_1], \, O_1 = [0 : \alpha_2 : -\alpha_1]$$
$$T_0 = \{x_0 = 0\}, \, T_1 = \{(\alpha\beta)_{12} x_0 - (\alpha\gamma)_{12} x_2 = 0\}$$
$$A_0 = [0 : 1 : 0], \, A_1 = [0 : 0 : 1].$$

4.1. Case where $(\beta\gamma)_{12} = 0$ with $\gamma_1\gamma_2 \neq 0$

Theorem 6. *We consider the birational mappings*

$$f(x_1, x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right), \, (\gamma_1, \gamma_2) \neq (0, 0)$$

under the conditions that $(\beta\gamma)_{12} = 0$ and $\gamma_1\gamma_2 \neq 0$. Then, either:

(i) $\alpha_1 \neq 0 \neq \alpha_2$ *and* $\delta(F) = 2$ *with* $d_n = 2^n$ *for all* $n \in \mathbb{N}$.
(ii) $\alpha_1 = 0$ *and the dynamical degree is* $\delta(F) = \frac{1+\sqrt{5}}{2}$ *with* $d_{n+2} = d_{n+1} + d_n$ *for all* $n \in \mathbb{N}$.
(iii) $\alpha_2 = 0$ *and the dynamical degree is* $\delta(F) = 1$ *with* $d_n = 1 + n$ *for all* $n \in \mathbb{N}$.

Proof. To prove (i), we observe that $S_0 \twoheadrightarrow A_0$ and $S_1 \twoheadrightarrow A_1$ with $F(A_0) = [0 : \alpha_1\gamma_1 : 0] = A_0 \notin \mathcal{I}(F)$ and $F(A_1) = [0 : \alpha_2\gamma_2 : 0] = A_0 \notin \mathcal{I}(F)$. Thus, by using (8), we see that F is AS, which implies that $d_n = 2^n$ and thus $\delta(F) = 2$.

Now, we consider that $\alpha_1 = 0$, which implies that $\alpha_2 \neq 0$. In this case, $S_0 \twoheadrightarrow A_0 = O_1 \in \mathcal{I}(F)$. Hence, we blow-up A_0 to obtain E_0. Similar computations to those described above show that $\tilde{F}$ sends $\hat{S}_0 \to E_0 \to \hat{T}_1$ and no new indeterminacy points are created.

Now, we must follow the orbit of A_1 under the action of $\tilde{F}$. As $A_1 \in S_0$, we find that $\tilde{F}(A_1) = [\gamma_2 : \beta_2]_{E_0}$ and $\tilde{F}[\gamma_2 : \beta_2]_{E_0} = [\gamma_1\gamma_2 : \alpha_0\gamma_2 + \alpha_2\beta_2 : \beta_1\gamma_2] \in T_1$. We observe that $\mathcal{I}(\tilde{F}) = \{O_0\}$ and $O_0 \in S_0 = T_0$. We know that the only points on T_0 that have preimages are A_0 and A_1, which implies that if the iterates of A_1 reach O_0 for some iterate of F, then O_0 should be equal to either A_0 or A_1. However, the conditions imposed on the parameters imply that $A_0 \neq O_0 \neq A_1$, which implies that O_0 has no preimages and thus the iterates of A_1 cannot reach O_0. Hence, we can see that $\tilde{F}$ is AS.

In this case, $\tilde{F}^*(\hat{L}) = 2\hat{L} - E_0$ and $\tilde{F}^*(E_0) = \hat{L} - E_0$. Hence, the characteristic polynomial of the corresponding matrix is $z^2 - z - 1$, which implies that the dynamical degree is $\delta(F) = \frac{1+\sqrt{5}}{2}$ and $d_{n+2} = d_{n+1} + d_n$ for all $n \in \mathbb{N}$.

Finally, for (iii), $\alpha_2 = 0$, so we find that $\alpha_1 \neq 0$. Now, we observe that S_0 collapses to $A_0 = [0 : 1 : 0] \in S_0$ and that $F[0 : 1 : 0] = [0 : \alpha_1\gamma_1 : 0] = [0 : 1 : 0]$. Hence, A_0 is a fixed point.

The other exceptional curve $S_1 \twoheadrightarrow A_1 = O_1 = [0 : 0 : \alpha_1] = [0 : 0 : 1] \in \mathcal{I}(F)$. Hence, we have to blow-up A_1 to obtain E_1. Similar computations to those described above show that $\tilde{F}$ sends $\hat{S}_1 \to E_1 \to \hat{T}_1$ and no new indeterminacy points are created. After this blow-up, the mapping $\tilde{F}$ is AS. We can see that $\tilde{F}^*(\hat{L}) = 2\hat{L} - E_0$ and $\tilde{F}^*(E_1) = \hat{L}$. Hence, the matrix of $\widetilde{F}^*$ is:

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}. \tag{9}$$

The characteristic polynomial is $(z-1)^2$, and thus the dynamical degree is 1. $d_1 = 2$, $d_2 = 3$, so we find that the sequence of degrees is $d_n = 1 + n$ for all $n \in \mathbb{N}$. $\square$

For the zero entropy mappings, we see that the only possible case is the third when $\alpha_2 = 0$. The result (and the proof) obtained is very similar to that stated in Proposition 5.

Proposition 7. *Let:*

$$f(x_1, x_2) = \left(\alpha_0 + \alpha_1 x_1 + \alpha_2 x_2, \frac{\beta_0 + \beta_1 x_1 + \beta_2 x_2}{\gamma_0 + \gamma_1 x_1 + \gamma_2 x_2}\right), \ (\gamma_1, \gamma_2) \neq (0, 0)$$

with the conditions $(\beta\gamma)_{12} = 0$ and $\gamma_1\gamma_2 \neq 0$, and assume that $f(x, y)$ has zero entropy. Then, after an affine change of the coordinates, this can be written as:

$$f(x, y) = \left(\alpha_0 + \alpha_1 x, \frac{\beta_0}{x + y}\right), \quad \alpha_1 \neq 0.$$

This map preserves the fibration $V(x, y) = x$ and this fibration is unique. If $m(x) := \alpha_0 + \alpha_1 x$ is periodic of period p, i.e., if $\alpha_1^p = 1$ for some $p > 1$, $\alpha_1 \neq 1$, then

$$W(x, y) = x \cdot m(x) \cdot m(m(x)) \cdots m^{p-1}(x)$$

is a first integral of $f(x, y)$. In addition, when $\alpha_1 = 1$ and $\alpha_0 = 0$, f is integrable.

4.2. Case where $(\beta\gamma)_{12} = 0$ with $\gamma_1 = 0$

In the next theorem, we discuss the behavior of d_n within this family. As shown in the following, after an affine change of the coordinates, these mappings can be studied easily and the sequence of degrees d_n can be deduced using elementary methods. We employed this approach in the proof of item (ii). However, in the first part, we prefer the blow-up approach. In fact, multiple blow-ups are implemented and it is interesting to see how this method detects the different behaviors of d_n.

Theorem 8. *Consider the birational mappings*

$$f(x_1,x_2)=\left(\alpha_0+\alpha_1x_1+\alpha_2x_2,\frac{\beta_0+\beta_1x_1+\beta_2x_2}{\gamma_0+\gamma_1x_1+\gamma_2x_2}\right),\ (\gamma_1,\gamma_2)\neq(0,0)$$

under the conditions that $(\beta\gamma)_{12}=0$ *and* $\gamma_1=0$.

(i) Assume that $\alpha_2\neq 0$. *Then, after an affine change of the coordinates,* $f(x,y)$ *can be written as:*

$$f(x,y)=\left(\alpha_0+\alpha_1x+y,\frac{\beta_0}{\gamma_0+y}\right),\ \alpha_1\neq 0\neq\beta_0 \tag{10}$$

and the following hold.

(a) If the one-dimensional mapping $h(y):=\frac{\beta_0}{\gamma_0+y}$ *is not a periodic map, then the sequence of degrees is* $d_n=1+n$.

(b) If $h(y)$ *is a* k*-periodic map and* $1+\alpha_1^k+\alpha_1^{2k}+\cdots+\alpha_1^{nk}\neq 0$ *for all* $n\in\mathbb{N}$, *then* $d_n=1+n$ *for all* $n\leq k-1$ *and* $d_n=k$ *for all* $n\geq k$.

(c) If $h(y)$ *is a* k*-periodic map and* $1+\alpha_1^k+\alpha_1^{2k}+\cdots+\alpha_1^{nk}=0$ *for some* $n\in\mathbb{N}$, *then* d_n *is an* $(n+1)k$*-periodic sequence.*

(ii) Assume that $\alpha_2=0$. *Then, after an affine change of the coordinates* $f(x,y)$ *can be written as:*

$$f(x,y)=\left(\alpha_0+\alpha_1x,\frac{\beta_0}{\gamma_0+y}\right),\ \alpha_1\neq 0\neq\beta_0 \tag{11}$$

and the following hold.

(a) If the one-dimensional mapping $h(y):=\frac{\beta_0}{\gamma_0+y}$ *is not a periodic map, then* $d_n=2$ *for all* $n\in\mathbb{N}$.

(b) If $h(y)$ *is a* k*-periodic map, then* d_n *is a* k*-periodic sequence.*

Proof. We note that since $\gamma_1=0$, $\gamma_2\neq 0$, then we can conjugate $f(x,y)$ with:

$$\psi(x,y)=\left(\frac{\alpha_2}{\gamma_2}x,\frac{1}{\gamma_2}y+\frac{\beta_2}{\gamma_2}\right).$$

By renaming the coefficients if necessary, we obtain the desired map (10). Now, we have:

$$S_0=\{x_0=0\},\ S_1=\{\gamma_0x_0+x_2=0\},\ A_0=[0:1:0],\ A_1=[0:0:1],$$

and

$$T_0=\{x_0=0\},\ T_1=\{x_2=0\},\ O_0=[0:1:0],\ O_1=[0:1:-\alpha_1].$$

$A_0=O_0$, so we have to blow-up this point to obtain E_0. Then:

$$\tilde{F}[u:v]_{E_0}=[\gamma_0u+v:\beta_0u]_{E_0}\ ,\ [u:v]_{E_0}\neq[1:-\gamma_0]_{E_0}$$

and

$$\hat{S}_0\twoheadrightarrow[1:0]_{E_0}.$$

The point $[1:-\gamma_0]_{E_0}$ is now an indeterminacy point of $\tilde{F}$. Hence, if $\tilde{F}^p[1:0]_{E_0}\neq[1:-\gamma_0]_{E_0}$ for all $p\in\mathbb{N}$, then since $\hat{S}_1\twoheadrightarrow A_1\in S_0$, $\tilde{F}(A_1)=[1:0]_{E_0}$ and we find that $\tilde{F}$ is AS. We can see that the matrix of $\tilde{F}^*:Pic(X)\to Pic(X)=<\hat{L},E_0>$ is:

$$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}. \tag{12}$$

The characteristic polynomial is $(z-1)^2$, and thus the dynamical degree is 1. $d_1 = 2$, $d_2 = 3$ so we find that the sequence of degrees is $d_n = 1 + n$ for all $n \in \mathbb{N}$.

Now, we assume that some $p \in \mathbb{N}$ exists such that $\tilde{F}^p[1:0]_{E_0} = [1:-\gamma_0]_{E_0}$. In this case, we claim that $\tilde{F}: E_0 \to E_0$ is a $(p+2)$-periodic map. To prove the claim, we distinguish between the case where $\gamma_0 = 0$ (which gives a 2-periodic map and corresponds to $p = 0$) and the case where $\gamma_0 \neq 0$. We have:

$$[1:0]_{E_0} \longrightarrow^{\tilde{F}^p} [1:-\gamma_0]_{E_0} \longrightarrow^{\tilde{F}} [0:1]_{E_0} \longrightarrow^{\tilde{F}} [1:0]_{E_0}.$$

Hence, $\tilde{F}^{p+2}$, which is in fact a Moebius map, and at least three different points are fixed. Clearly, this implies that $\tilde{F}^{p+2}$ is the identity map. The restriction of $\tilde{F}$ at E_0 is exactly the map $h(y) = \frac{\beta_0}{\gamma_0 + y}$ extended to the projective line, so we can assert that $\tilde{F}^p[1:0]_{E_0} = [1:-\gamma_0]_{E_0}$ if and only if $h(y)$ is a $(p+2)$-periodic map. Hence, (a) is proved.

Following the same process, if $\tilde{F}^p[1:0]_{E_0} = [1:-\gamma_0]_{E_0}$, then we have to blow-up all the points $\tilde{F}^j[1:0]_{E_0}$ for $j = 0, 1, \ldots, p$. We refer to E_{0j} as the corresponding principal divisors and we obtain:

$$E_{00} \longrightarrow E_{01} \longrightarrow E_{02} \longrightarrow \cdots \longrightarrow E_{0p}. \tag{13}$$

We refer to $\tilde{F}$ as the map for this new variety and we find the image of S_0, which is the image of E_{0p}.

A point coordinate k in E_{00} is considered as $\lim_{t\to 0}[t:1:kt^2]$. Then, for any point in S_0 that differs from the indeterminacy points and for $t \sim 0$, we have:

$$F(t, x_1, x_2) \sim [x_2 t : (\alpha_1 x_1 + \alpha_2 x_2) x_2 : \beta_0 t^2] = \left[\frac{t}{\alpha_1 x_1 + \alpha_2 x_2} : 1 : \frac{\beta_0}{x_2(\alpha_1 x_1 + \alpha_2 x_2)} t^2\right].$$

We set $T := \frac{t}{\alpha_1 x_1 + \alpha_2 x_2}$ and this point resembles $\left[T : 1 : \frac{\beta_0(\alpha_1 x_1 + \alpha_2 x_2)}{x_2} T^2\right]$, i.e.,

$$\tilde{F}[0:x_1:x_2] = \frac{\beta_0(\alpha_1 x_1 + \alpha_2 x_2)}{x_2} \in E_{00}.$$

Now, we consider a point coordinate k in E_{0p}. This point is considered as $\lim_{t\to 0}[t:1:-\gamma_0 t + kt^2]$. Then, for $t \sim 0$:

$$F[t:1:-\gamma_0 t + kt^2] \sim [kt : \alpha_1 k : \beta_0] \to_{t\to 0} [0 : \alpha_1 k : \beta_0] \in S_0.$$

Hence, (13) can be completed and we obtain the cycle:

$$\hat{S}_0 \longrightarrow E_{00} \longrightarrow E_{01} \longrightarrow E_{02} \longrightarrow \cdots \longrightarrow E_{0p} \longrightarrow \hat{S}_0.$$

Now, $S_1 \twoheadrightarrow A_1 \in S_0$ and $\tilde{F}^{p+2}$ sends $\hat{S}_0$ to itself, so it is possible that for some $n \in \mathbb{N}$, $\tilde{F}^{n(p+2)}(A_1) = O_0$, which still is an indeterminacy point of $\tilde{F}$.

If this is not the case, then $\tilde{F}$ is AS. Let us compute the matrix of $\tilde{F}^*$. The Picard group of X is $Pic(X) =< \hat{L}, E_{00}, E_{01}, \ldots, E_{0p}, E_0 >$. In order to write $\hat{S}_0$ and $\hat{S}_1$ as a linear combination of their basis elements, we use the identity (6). For instance, $\pi^*(F^{-1}(L)) = \widehat{F^{-1}}(L) + \sum_{j=1}^{p} m_j E_{0j}$, where the multiplicities m_j are the order of vanishing for $F^{-1}(L)$ at the generic points of E_{0j}. If $\delta_0 x_0 + \delta_1 x_1 + \delta_2 x_2 = 0$ is the equation for a generic straight line L, then a calculation gives $\delta_0 F[t:1:wt+kt^2][1] + \delta_1 F[t:1:wt+kt^2][2] + \delta_2 F[t:1:wt+kt^2][3] = \delta_1 \alpha_1(\gamma_0 + w)t + o(t^2)$, which allows us to write $\pi^*(F^{-1}(L)) = \widehat{F^{-1}}(L) + \sum_{j=1}^{p-1} E_{0j} + 2E_{0p}$.

Now, from $\pi^*(F^{-1}(L)) = 2\hat{L}$, we obtain $\tilde{F}^*(\hat{L}) = 2\hat{L} - \sum_{j=1}^{p-1} E_{0j} - 2E_{0p}$. By proceeding in this manner, we find that the matrix of $\tilde{F}^*$ is:

$$\begin{pmatrix} 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & 0 & 0 & \dots & 1 & 0 \\ -2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & -1 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Some simple calculations show that the characteristic polynomial of this matrix is $(-z)^p\,(z-1)^2$. Hence, the sequence of degrees satisfies $d_{n+p+2} = 2d_{n+p+1} - d_{n+p}$ and its behavior depends on the initial conditions, i.e., on the first terms $d_1, d_2, \dots, d_{p+2}$. Thus, if $h(y)$ is k-periodic, then $k = p+2$ and $f^k(x,y)[2] = y$, which implies that $d_k = d_{k-1}$. The first degrees are $2, 3, 4, \dots, k, k$ from $d_{n+k} = 2d_{n+k-1} - d_{n+k-2}$, so we find that $d_n = k$ for all $n \geq k$. We still need to prove that the condition $\tilde{F}^{n(p+2)}(A_1) = O_0$ is equivalent to $1 + \alpha_1^k + \alpha_1^{2k} + \cdots + \alpha_1^{nk} = 0$ for $k = p+2$. Therefore, by considering the terms for the maximum degree of $f^k(x,y)$, (see (15) below), we find that:

$$\tilde{F}^k[0 : x_1 : x_2] = [0 : \alpha_1^{k-1}(\alpha_1 x_1 + x_2) : x_2],$$

and thus:

$$\tilde{F}^{nk}[0 : 0 : 1] = [0 : \alpha_1^{k-1}(1 + \alpha_1^k + \alpha_1^{2k} + \cdots + \alpha_1^{(n-1)k}) : 1].$$

Therefore, $\tilde{F}^{nk}[0 : 0 : 1] = O_1 = [0 : 1 : -\alpha_1]$ if and only if:

$$1 + \alpha_1^k + \alpha_1^{2k} + \cdots + \alpha_1^{nk} = 0. \tag{14}$$

Statement (b) is now proved. To prove (c), we simply compute $f^{(n+1)k}$. In this case, $\alpha_1 \neq 1$, so we can consider (a translation may be performed if necessary) that $\alpha_0 = 0$. Now, the expression for f^k is:

$$f^k(x,y) = \left(\alpha_1^k x + \alpha_1^{k-1} y + \alpha_1^{k-2} h(y) + \alpha_1^{k-3} h^2(y) + \cdots + \alpha_1 h^{k-2}(y) + h^{k-1}(y), y\right) \tag{15}$$

Hence:

$$f^{(n+1)k}(x,y) = \left(\alpha_1^{(n+1)k} x + (1 + \alpha_1^k + \alpha_1^{2k} + \cdots + \alpha_1^{nk})(\alpha_1^{k-1} y + \alpha_1^{k-2} h(y) + h^{k-1}(y)), y\right).$$

Then, condition (14) implies that $\alpha_1^{(n+1)k} = 1$, so condition (14) is satisfied and f is a $(n+1)k$-periodic map, and thus the sequence of degrees is also $(n+1)k$-periodic.

Next, we prove (ii). First, $\gamma_1 = 0$ implies that $\gamma_2 \neq 0$, and from $\gamma_1\beta_2 - \gamma_2\beta_1 = 0$, we obtain $\beta_1 = 0$. By performing a translation on y and renaming the coefficients, we obtain Equation (11). This map is very simple so we can prove the result based on the behavior of d_n using elementary arguments. We observe that the first component of $f^k(x,y)$ is $a_k x + b_k$ for certain a_k, b_k. The second components are simply the iterates of $h(y) = \frac{\beta_0}{\gamma_0 + y}$, which is a one-dimensional Möbius map. We claim that if $h(y)$ is not a periodic map, then $h^k(y)$ is a Möbius map with non-constant denominator for all $k \in \mathbb{N}$, and that the denominators of $h^i(y)$ and $h^j(y)$ are also different for $i \neq j$. From this claim, we can deduce that when $h(y)$ is not a periodic map, then $d_n = 2$ for all $n \in \mathbb{N}$. In addition, when $h(y)$ is a k-periodic map, then the sequence of degrees is $d_n = 2$ for all n, which is not a multiple of k, and $d_n = 1$ when n is a multiple of k.

To prove the claim, we consider N_k and D_k with $h^k(z) = \frac{N_k}{D_k}$, and we see that if we do not perform simplifications, then $N_{k+1} = \beta_0 D_k$ and $D_{k+1} = \gamma_0 D_k + N_k$. Let $p_k, q_k \in \mathbb{C}$ such that $D_k = p_k + q_k z$. Then, $D_{k+2} - \gamma_0 D_{k+1} - \beta_0 D_k = 0$, which implies that $q_{k+2} - \gamma_0 q_{k+1} - \beta_0 q_k = 0$. The claim follows by analyzing this linear recurrence with constant coefficients and considering that this sequence is k-periodic if and only if $\left(\frac{\lambda_2}{\lambda_1}\right)^k = 1$, where λ_1, λ_2 are the two different roots of $\lambda^2 - \gamma_0 \lambda - \beta_0 = 0$ ([9]). □

Proposition 9. *We consider the birational mappings:*

$$f(x,y) = \left(\alpha_0 + \alpha_1 x + y, \frac{\beta_0}{\gamma_0 + y}\right), \ \alpha_1 \neq 0 \neq \beta_0. \tag{16}$$

Then, the following hold.

(a) If the one-dimensional mapping $h(y) := \frac{\beta_0}{\gamma_0 + y}$ is not a periodic map, then $f(x,y)$ has the unique invariant fibration $V_1(x,y) = y$.

(b) If $h(y)$ is a k-periodic map and $1 + \alpha_1^k + \alpha_1^{2k} + \cdots + \alpha_1^{nk} \neq 0$ for all $n \in \mathbb{N}$, then $f(x,y)$ is integrable to

$$H_1(x,y) = y + h(y) + h(h(y)) + \cdots + h^{k-1}(y)$$

as a first integral and it also has a second invariant fibration $V_2(x,y)$:

(b_1) If $\alpha_1^k \neq 1$, we can assume that $\alpha_0 = 0$ and thus $V_2(x,y) =$

$$(\alpha_1^k - 1)x + \alpha_1^{k-1} y + \alpha_1^{k-2} h(y) + \alpha_1^{k-3} h^2(y) + \cdots + \alpha_1 h^{k-2}(y) + h^{k-1}(y) \tag{17}$$

satisfies $V_2(f(x,y)) = \alpha_1 V_2(x,y)$.

(b_2) If $\alpha_1^k = 1$ but $\alpha_1 \neq 1$, we can assume that $\alpha_0 = 0$ and thus $V_2(x,y) =$

$$\frac{kx + (k-1)\alpha_1^{k-1} y + (k-2)\alpha_1^{k-2} h(y) + (k-3)\alpha_1^{k-3} h(h(y)) + \cdots + 2\alpha_1^2 h^{k-3}(y) + \alpha_1 h^{k-2}(y)}{\alpha_1^{k-1} y + \alpha_1^{k-2} h(y) + \alpha_1^{k-3} h(h(y)) + \cdots + \alpha_1 h^{k-2}(y) + h^{k-1}(y)}$$

satisfies $V_2(f(x,y)) = V_2(x,y) + 1$.

(b_3) If $\alpha_1 = 1$, then:

$$V_2(x,y) = \frac{kx + (k-1)y + (k-2)h(y) + (k-3)h(h(y)) + \cdots + 2h^{k-3}(y) + h^{k-2}(y)}{k\alpha_0 + y + h(y) + h(h(y)) + \cdots + h^{k-2}(y) + h^{k-1}(y)}$$

satisfies $V_2(f(x,y)) = V_2(x,y) + 1$.

(c) If $h(y)$ is a k-periodic map and $1 + \alpha_1^k + \alpha_1^{2k} + \cdots + \alpha_1^{nk} = 0$ for some $n \in \mathbb{N}$, then $f(x,y)$ has a second first integral $H_2(x,y)$, which can be given by $H_2(x,y)) = V_2^{(n+1)k}(x,y)$ and $V_2(x,y)$ is defined by (17).

The proofs are straightforward. We only note that finding the fibrations requires that we consider the combinations of $x, y, h(y), h(h(y)), \ldots, h^{k-1}(y)$ or their quotients.

Remark 10. Assuming that hypothesis (b) holds, and since d_n is a bounded sequence and $f(x,y)$ is not a periodic map, then from [6], we know that $f(x,y)$ is birationally equivalent to either $(x,y) \to (ax, by)$ where a is a root of unity and b is not, or to $(x,y) \to (ax, y+1)$. The fibrations encountered in (b) allow us to construct these conjugations. In fact, when $V_2(f(x,y)) = \alpha_1 V_2(x,y)$, we are considering the first case, whereas we are considering the second case when $V_2(f(x,y)) = V_2(x,y) + 1$.

The invariant fibrations and first integrals that correspond to the mappings while satisfying (ii) in Theorem 8 appear simple after a suitable affine change of the coordinates. The next proposition gives this information.

Proposition 11. *We consider the birational mappings:*

$$f(x,y)=\left(\alpha_0+\alpha_1 x,\frac{\beta_0}{\gamma_0+y}\right)\ ,\ \alpha_1\neq 0\neq\beta_0. \tag{18}$$

These mappings preserve the two generically transverse invariant foliations $V_1(x,y)=x$ and $V_2(x,y)=y$. Furthermore:

(a) If $h(y)=\frac{\beta_0}{\gamma_0+y}$ is k-periodic, then

$$H_1(x,y)=y+h(y)+h(h(y))+\cdots+h^{k-1}(y)$$

is a first integral of $f(x,y)$.

(b) If $m(x):=\alpha_0+\alpha_1 x$ is p-periodic, then

$$H_2(x,y)=x+m(x)+m(m(x))+\cdots+m^{p-1}(x)$$

is a first integral of $f(x,y)$.

(c) If $h(y)$ and $m(x)$ are k-periodic, then $f(x,y)$ is a k-periodic mapping with two independent first integrals $H_1(x,y)$ and $H_2(x,y)$ with $p=k$.

4.3. Case where $(\beta\gamma)_{12}=0$ with $\gamma_2=0$

If $\gamma_2=0$, we know that $\gamma_1\neq 0$ and from $(\beta\gamma)_{12}=0$, we obtain $\beta_2=0$. In addition, $\alpha_2\neq 0$ if $f(x,y)$ does not depend only on x.

Theorem 12. *We consider the birational mappings:*

$$f(x_1,x_2)=\left(\alpha_0+\alpha_1 x_1+\alpha_2 x_2,\frac{\beta_0+\beta_1 x_1}{\gamma_0+\gamma_1 x_1}\right),\ (\gamma_1,\alpha_2)\neq(0,0). \tag{19}$$

(a) If we assume that $\alpha_1\neq 0$, then the dynamical degree of F is $\delta(F)=\frac{1+\sqrt{5}}{2}$ and $d_{n+2}=d_n+d_{n+1}$.

(b) If we assume that $\alpha_1=0$, then after an affine change of the coordinates, $f(x,y)$ takes the form:

$$f(x_1,x_2)=\left(x_2,\frac{\beta_0}{\gamma_0+x_1}\right), \tag{20}$$

and the dynamical degree of F is $\delta(F)=1$. Furthermore:

(b_1) If $h(z):=\frac{\beta_0}{\gamma_0+z}$ is not a periodic map, then $d_n=2$ for all $n\in\mathbb{N}$.

(b_2) If $h(z)$ is a k-periodic map, then d_n is a 2k-periodic sequence.

Proof. To prove (a), we observe that $S_1\twoheadrightarrow A_1=O_0=[0:0:1]$ and $F(A_0)=[0:\alpha_1\gamma_1:0]=A_0\notin\mathcal{I}(F)$. Thus, we must blow-up $A_1=[0:0:1]$ to obtain E_1. Then, $\tilde{F}$ sends $S_1\to E_1\twoheadrightarrow[0:1:0]=A_0$. Since $A_1\in S_1$, $\pi^*(S_1)=\hat{S}_1+E_1$ and the matrix of $\widetilde{F}^*$ is:

$$\begin{pmatrix}2 & 1\\ -1 & -1\end{pmatrix}. \tag{21}$$

Then, the characteristic polynomial associated with F is $z^2 - z - 1$. Hence, the dynamical degree is $\delta(F) = \delta^*$ and $d_{n+2} = d_{n+1} + d_n$ for all $n \in \mathbb{N}$.

Next, we prove (b). When $\alpha_1 = 0$, (19) can be transformed into (20) via the conjugation

$$\psi(x,y) = \left(\frac{1}{\gamma_1}x + \frac{\alpha_0\gamma_1 + \alpha_2\beta_1}{\gamma_1}, \frac{1}{\gamma_1\alpha_2}y + \frac{\beta_1}{\gamma_1}\right).$$

From (20), we find that $f(f(x,y)) = (h(x), h(y))$ and generally:

$$f^{2n}(x,y) = (h^n(x), h^n(y)) \ , \ f^{2n+1}(x,y) = \big(h^n(y), h^{n+1}(x)\big)\,. \tag{22}$$

Based on the same arguments given above, if h is not periodic, then $d_n = 2$ for all $n \in \mathbb{N}$. If h is k-periodic, then $f^{2k}(x,y) = (x,y)$, and from (22), we find that $d_n = 2$ for all $n \in \mathbb{N}$ such that it is not a multiple of $2k$ and $d_n = 1$ for all $n \in \mathbb{N}$ such that it is a multiple of $2k$. In all cases, the dynamical degree of F is $\delta(F) = 1$. $\square$

Proposition 13. *We consider the family of mappings:*

$$f(x,y) = \left(y, \frac{\beta_0}{\gamma_0 + x}\right).$$

Then:

(a) If $\gamma_0^2 + 4\,\beta_0 \neq 0$, let p and q be the two different roots of $z^2 - \gamma_0\, z - \beta_0 = 0$, and let m such that $m^2 = q/p$, then $f(x,y)$ preserves the generically transverse fibrations:

$$H_1(x,y) = \frac{m^2\,p^2 + mpx + p(m^2 - m + 1)y + xy}{(x+p)\,(y+p)},$$
$$H_2(x,y) = \frac{m^2\,p^2 - mpx + p(m^2 + m + 1)y + xy}{(x+p)\,(y+p)}$$

with $H_1(f(x,y)) = mH_1(x,y)\,,\ H_2(f(x,y)) = -mH_2(x,y)$. Furthermore, $f(x,y)$ is $2k$-periodic if and only if $m^{2k} = 1$, and in this case, $H_1^{2k}(x,y)$ and $H_2^{2k}(x,y)$ are two independent first integrals of $f(x,y)$.

(b) If $\gamma_0^2 + 4\,\beta_0 = 0$, then it preserves the two generically transverse fibrations:

$$K_1(x,y) = \frac{\gamma_0^2 - 2\gamma_0\, x + 6\gamma_0\, y + 4\,x\,y}{(2\,x + \gamma_0)\,(2\,y + \gamma_0)}\,,\ K_2(x,y) = \frac{2\ \gamma_0\,(x + y + \gamma_0)}{(2\,x + \gamma_0)\,(2\,y + \gamma_0)},$$

with $K_1(f(x,y)) = -K_1(x,y)\,,\ K_2(f(x,y)) = K_2(x,y) + 1$. Furthermore, $f(x,y)$ is integrable to $W(x,y) = (K_1(x,y))^2$ as the first integral.

Proof. When $\gamma_0^2 + 4\,\beta_0 \neq 0$, some calculations show that in fact, $H_1(f(x,y)) = mH_1(x,y)$ and $H_2(f(x,y)) = -mH_2(x,y)$. Furthermore, $H_1(x,y), H_2(x,y)$ are generically transverse because the determinant of the Jacobian of $H_1(x,y), H_2(x,y)$ is

$$-\frac{2p^2m(m^2-1)}{(p+x)^2(p+y)^2},$$

which is different from zero (if this is not the case, $m^2 = 1$, and this occurs if and only if $p = q$, which contradicts $\gamma_0^2 + 4\,\beta_0 \neq 0$).

In addition, when $\gamma_0^2 + 4\,\beta_0 = 0$, the determinant of the Jacobian of $K_1(x,y), K_2(x,y)$ is different from zero because it is equal to:

$$\frac{16c^2}{(2y+c)^2(2x+c)^2}.$$

Finally, $W(x,y) = (K_1(x,y))^2$ is a first integral of $f(x,y)$ because $W(f(x,y)) = (K_1(f(x,y)))^2 = (-K_1(x,y))^2 = W(x,y)$. □

Remark 14. Simple computations show that when $\gamma_0^2 + 4\,\beta_0 \neq 0$, f is birationally conjugated to $(mx, -my)$ via the conjugation $\varphi(x,y) = (H_1(x,y), H_2(x,y))$, and that when $\gamma_0^2 + 4\,\beta_0 = 0$, f is birationally conjugated to $(-x, y+1)$ via $\psi(x,y) = (K_1(x,y), K_2(x,y))$.

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