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Finite-time blow-up and global boundedness for chemotaxis system with strong logistic dampening

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ABSTRACT

In the present study, we consider the chemotaxis system with logistic-type superlinear degradation

$$\begin{cases} \partial_t u_1 = \tau_1 \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v) + \lambda_1 u_1 - \mu_1 u_1^{k_1}, & x \in \Omega, \quad t > 0, \\ \partial_t u_2 = \tau_2 \Delta u_2 - \chi_2 \nabla \cdot (u_2 \nabla v) + \lambda_2 u_2 - \mu_2 u_2^{k_2}, & x \in \Omega, \quad t > 0, \\ 0 = \Delta v - \gamma v + \alpha_1 u_1 + \alpha_2 u_2, & x \in \Omega, \quad t > 0, \end{cases}$$

under the homogeneous Neumann boundary condition, where $\gamma > 0$, $\tau_i > 0$, $\chi_i > 0$, $\lambda_i \in \mathbb{R}$, $\mu_i > 0$, $\alpha_i > 0$ $(i = 1, 2)$. Consider an arbitrary ball $\Omega = B_R(0) \subset \mathbb{R}^n, n \geq 3, R > 0$, when $k_i > 1 (i = 1, 2)$, it is shown that for any parameter $\hat{k} = \max\{k_1, k_2\}$ satisfies

$$\hat{k} < \begin{cases} \frac{7}{6} & \text{if } \ n \in \{3, 4\}, \\ 1 + \frac{1}{2(n-1)} & \text{if } \ n \geq 5, \end{cases}$$

there exist nonnegative radially symmetric initial data under suitable conditions such that the corresponding solutions blow up in finite time in the sense that

$$\limsup_{t \nearrow T_{max}} \left(\|u_1(\cdot, t)\|_{L^\infty(\Omega)} + \|u_2(\cdot, t)\|_{L^\infty(\Omega)} \right) = \infty \quad \text{for some } \ 0 < T_{max} < \infty.$$

Furthermore, for any smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geq 1)$, when $k_i \geq 2 (i = 1, 2)$, we prove that the system admits a unique global bounded solution.

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1. Introduction

In many biological processes, such as pattern formation, embryo development, tumor invasion etc, the cells move towards the higher concentration of the chemical substance when they plunge into hunger, this phenomenon is referred to as chemotaxis. A well-known mathematical model for one-species and one-stimuli chemotaxis model was first proposed by Keller-Segel in [9] as follows:

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega, \quad t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega, \quad t > 0, \end{cases} \tag{1.1}$$

where $\tau \in \{0, 1\}$, $\chi > 0$ is called the chemotactic sensitivity, $u(x,t)$ denotes the cell density and $v(x,t)$ represents the chemical concentration. The classical Keller-Segel system (1.1) has been intensively studied about blow-up in finite or infinite time and global existence in time. When $\tau = 0$, $n = 2$, Nagai [13,14] found a critical mass $m_c > 0$ in a weakened sense for the parabolic-elliptic system, under the circumstance that $\int_\Omega u_0 \leq m_c$, the solution of system (1.1) is global bounded, otherwise, there exists a solution blows up either in finite or infinite time. When $\tau = 1$, the solutions of (1.1) are uniformly bounded-in-time for the one dimensional case (see [17]); however, in the higher-dimensional case $n \geq 2$, the solutions to (1.1) can blow up, for instance, in the case $n = 2$, it was shown that there is a critical value $C > 0$ ($C = \frac{8\pi}{\chi}$ in the radial setting or $C = \frac{4\pi}{\chi}$ in the other setting) such that, if $\|u_0\|_{L^1(\Omega)} < C$ then global solutions exist [15], and if $\|u_0\|_{L^1(\Omega)} > C$ then the corresponding solution blows up in finite time [6,8]; in the case $n \geq 3$, Horstmann and Winkler asserted possibility of existence of unbounded solutions [8]; and Winkler [32] showed that (1.1) possesses unbounded solutions with arbitrarily small total mass of cells, however, it remains to be seen whether the associated blow-up time is finite; latter, assume that $\Omega \subset \mathbb{R}^n (n \geq 3)$ is a ball, Winkler [33] obtained that there exists radially symmetric solution blowing up in finite time with proper initial conditions.

In recent years, an abundant number of elaborate models for taxis mechanisms under the influence of spontaneous proliferation and death is considered, the interaction between cross-diffusion and logistic kinetics plays an important role in population dynamics [7,19]. Especially, on account of the comparatively strong logistic diffusion, the chemotaxis system with growth terms is given by

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \lambda u - \mu u^k, & x \in \Omega, \quad t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega, \quad t > 0, \end{cases} \tag{1.2}$$

with $\tau \in \{0, 1\}$, $\chi > 0, \lambda \in \mathbb{R}, \mu \geq 0, k > 1, \Omega \subset \mathbb{R}^n, n \geq 1$. It is understood that the logistic growth term can prevent the occurrence of blow-up, such as, for the situation $\tau = 0, k = 2$, $\lambda \in \mathbb{R}$, let μ_c be a critical number satisfying $\mu_c \leq \frac{n-2}{n}$, in both case of $n \leq 2, \mu > 0$ and $n \geq 3$, μ is bigger than μ_c, no blow-up occurs; for $k > 2$, the same conclusion holds without any restriction on μ [24]. Apart from that, for the whole space situation, the parabolic-parabolic or the parabolic-elliptic analogue, relevant results which ensures global existence of solutions can also be found in [11,16,18,25–27,30,31]. Furthermore, the mathematical literature has identifies that the logistic growth term can prevent the occurrence of blow-up to several modifications of (1.2) [3,10,21,23,28,29]. So it is quite interesting to find out when this superlinear absorption mechanism expressed in the first equation of (1.2) confirms the possibility of aggregation phenomenon for models of type (1.2), however, only few results are obtained, and most of the results can be achieved only in high dimension case. Recently, in [34], Winkler generate the result to low-dimensional spatial settings (including the three-dimensional cases).

After the pioneering works above, the following two-species and one-stimuli chemotaxis model which is a generalized problem of Keller-Segel system draws more and more attention

$$\begin{cases} \partial_t u_1 = \tau_1 \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v), & x \in \Omega, \quad t > 0, \\ \partial_t u_2 = \tau_2 \Delta u_2 - \chi_2 \nabla \cdot (u_2 \nabla v), & x \in \Omega, \quad t > 0, \\ \tau \partial_t v = \Delta v - \gamma v + \alpha_1 u_1 + \alpha_2 u_2, & x \in \Omega, \quad t > 0. \end{cases} \tag{1.3}$$

When $\tau = 1$, based on a Lyapunov function, Li etc. constructed a finite time blow-up result for symmetric solutions of the two species Keller-Segel model in [12]. In the case $\tau = \gamma = 0$, on the whole space $\mathbb{R}^2$, the conditions for finite time blow-up and the existence of self-similar solutions to (1.3) were investigated [2]. And in [4], it has been proved that system (1.3) has a threshold curve that determines global existence or blow-up on $\mathbb{R}^2$. As for the parabolic-elliptic case (i.e. $\tau = 0$), the finite time blow-up of (1.3) was established in [1] on the whole space $\mathbb{R}^n (n \geq 3)$. Recently, for $\tau = 0, \gamma > 0$, $\Omega \subset \mathbb{R}^2$, Zhao et al. obtained the results for finite time blow-up and global boundedness [36]. Moreover, when $\Omega \subset \mathbb{R}^2$ is a disk, Espejo et al. showed the simultaneous finite time blow-up in a two-species model for chemotaxis [5].

We study herein a coupled system of the chemotaxis equations with strong logistic dampening

$$\begin{cases} \partial_t u_1 = \tau_1 \Delta u_1 - \chi_1 \nabla \cdot (u_1 \nabla v) + \lambda_1 u_1 - \mu_1 u_1^{k_1}, & x \in \Omega, \quad t > 0, \\ \partial_t u_2 = \tau_2 \Delta u_2 - \chi_2 \nabla \cdot (u_2 \nabla v) + \lambda_2 u_2 - \mu_2 u_2^{k_2}, & x \in \Omega, \quad t > 0, \\ 0 = \Delta v - \gamma v + \alpha_1 u_1 + \alpha_2 u_2, & x \in \Omega, \quad t > 0, \\ \frac{\partial u_1}{\partial \nu} = \frac{\partial u_2}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \Omega, \quad t > 0, \\ u_1(x,0) = u_{10}(x), u_2(x,0) = u_{20}(x), & x \in \Omega, \quad t > 0, \end{cases} \tag{1.4}$$

where $\gamma > 0$, $\tau_i > 0$, $\chi_i > 0$, $\lambda_i \in \mathbb{R}$, $\mu_i > 0$, $\alpha_i > 0$, $k_i > 1$ $(i = 1, 2)$ and $\Omega \subset \mathbb{R}^n$. Here $u_1(x,t)$ and $u_2(x,t)$ denote the density of two species, $v(x,t)$ stands for the concentration of the chemical substance.

So far, up to the best of our knowledge, in dimension $n \geq 3$, only few results of blow-up to the multi-species and single-stimuli K-S system with logistic resource have been developed. One of our aim in this paper is to derive the finite-time blow-up for (1.4) in higher dimension $n \geq 3$, our approach is motivated by the works on a one-specie and one-stimili Keller-Segel system with generalized logistic source in [34]. In addition, inspired by [24], for dimension $n \geq 1$, we find out that the solution of (1.4) is global bounded when $k_i \geq 2 (i = 1, 2)$. The technical obstacle of this paper lies in the fact that too many parameters in (1.4) are unknown, the chemoattractant is produced by both two species, and the strong logistic dampening complicates the computations, which needs the finer analysis and estimates.

To establish the blow-up result, let us assume that the initial data u_{10} and u_{20} satisfy

$$\begin{cases} 0 \leq u_{10} \in C^0(\bar{\Omega}) \text{ is radially symmetric}, \\ 0 \leq u_{20} \in C^0(\bar{\Omega}) \text{ is radially symmetric}. \end{cases} \tag{1.5}$$

Now we state the main theorem for blow-up in finite time of this paper.

Theorem 1.1. *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ with $n \geq 3$, $R > 0$, $\gamma > 0$, $\tau_i > 0$, $\chi_i > 0$, $\lambda_i \in \mathbb{R}$, $\mu_i > 0$, $\alpha_i > 0$ and $k_i > 1 (i = 1, 2)$. Suppose that*

$$\hat{k} = \max\{k_1, k_2\} < \begin{cases} \frac{7}{6} & \text{if } n \in \{3, 4\}, \\ 1 + \frac{1}{2(n-1)} & \text{if } n \geq 5. \end{cases} \tag{1.6}$$

Then for any choice of $L_i > 0$, $m_i > 0$ and $m_1^*, m_2^* \in (0, m_1) \cap (0, m_2)$, one can find $\bar{r}_i = \bar{r}_i(R, \lambda_i, \mu_i, k_i, L_i, m_i, m_i^*) \in (0, R)$, $i = 1, 2$, if the initial data $u_{i0} (i = 1, 2)$ satisfy the following assumptions

$$\int_\Omega u_{i0} \leq m_i \ \ \text{but} \int_{B_{\bar{r}_i}(0)} u_{i0} \geq m_i^* \tag{1.7}$$

as well as

$$u_{i0} \le L_i|x|^{-n(n-1)} \text{ for all } x \in \Omega, \tag{1.8}$$

the corresponding classical solution (u_1, u_2, v) *of* (1.4) *uniquely determined by the inclusions*

$$\begin{cases} (u_1,u_2) \in \left(C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max}))\right)^2, \\ v \in \cap_{q>n} L^\infty_{loc}([0,T_{\max}); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega}\times(0,T_{\max})), \end{cases}$$

blows up at the finite time $t = T_{max} \in (0,\infty)$ *in the sense that*

$$\limsup_{t\nearrow T_{max}} (\|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t)\|_{L^\infty(\Omega)}) = \infty. \tag{1.9}$$

Remark 1.1. In Theorem 1.1, we only give the blow-up result for (1.4) under the case $\mu_i > 0 (i = 1, 2)$. For the case $\lambda_i = \mu_i = 0(i = 1, 2)$, equations (1.4) become to (1.3), following the same method used in the proof of Theorem 1.1, it is easy to derive the finite time blow-up result, thus we don't discuss this case here. And in the previous reference [36], the finite time blow-up result to (1.3) was only derived under the dimension $n = 2$, we further obtain the blow-up result for higher dimension $n \ge 3$. Hence, the results of this paper supplement the previous works. However, for the solution (u_1, u_2), there is still an open problem about whether u_1, u_2 blow up simultaneously.

Remark 1.2. Actually, in the indicated range of $k_i (i = 1, 2)$, there exist abundant initial data $u_{i0}(i = 1, 2)$ which entail the blow-up. Since a similar method in the proof of Corollary 1.2 in [34] can be applied to (1.4), from which, for any $u_{i0} \in C^0(\bar{\Omega})(i = 1, 2)$, one can find two sequences $\{u^l_{10}\}, \{u^l_{20}\} \subset C^0(\bar{\Omega})(l \in \mathbb{N})$ of radial initial data fulfill $u^l_{i0} \to u_{i0}$ in $L^1(\Omega)$ as $l \to \infty$, and for each $\{u^l_{i0}\}$ the corresponding solution of (1.4) exhibits a finite-time explosion phenomenon in the sense of Theorem 1.1, which ensures blow-up phenomenon throughout a large number of initial data.

Next, to further understand the relationship between the global boundedness and blow-up of solutions, we discuss the global boundedness of solutions to (1.4) for $\Omega \subset \mathbb{R}^n (n \ge 1)$.

Theorem 1.2. *Let* $\Omega \subset \mathbb{R}^n (n \ge 1)$ *be a smooth bounded domain, and* $\gamma > 0$, $\tau_i > 0$, $\chi_i > 0$, $\lambda_i \in \mathbb{R}$, $\mu_i > 0$, $\alpha_i > 0$. *If either* $k_i > 2(i = 1, 2)$, *or* $k_1 = k_2 = 2$ *and*

$$\mu_1 > \frac{n-2}{n}[\chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1], \qquad \mu_2 > \frac{n-2}{n}[\chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2],$$

then for any $u_{10}, u_{20} \in C^0(\bar{\Omega})$, *problem* (1.4) *possesses a unique global classical solution, which is uniformly bounded in the sense that*

$$\|u_1\|_{L^\infty} + \|u_2\|_{L^\infty} \le \overline{M} \text{ for all } t > 0, \tag{1.10}$$

with some constant $\overline{M}$ *that is independent of* t.

Remark 1.3. In our Theorem 1.2, we generate the boundedness result for the one-specie case considered in [24]. Especially, when $\chi_1 = \chi_2 = \chi$, $\alpha_1 = \alpha_2 = 1$, we find out that both the lower bounds of μ_1 and μ_2 are equal to $\frac{3(n-2)}{n}\chi$, which is about the same to the case in [24]. In addition, we only give the global boundedness result for $k_i \ge 2(i = 1, 2)$, but when $n = 3$, the coefficients $\frac{7}{6} \le k_i < 2(i = 1, 2)$, $\frac{7}{6} \le k_1 < 2 \le k_2$ ($\frac{7}{6} \le k_2 < 2 \le k_1$), or when $n \ge 4$, the coefficients $1 + \frac{1}{2(n-1)} \le k_i < 2(i = 1, 2)$,

$1+\frac{1}{2(n-1)} \leq k_1 < 2 \leq k_2$ $(1+\frac{1}{2(n-1)} \leq k_2 < 2 \leq k_1)$, whether the solution will blow up or not, all these remain to be solved.

The outline of this paper is as follows. In Section 2, the basic statements concerning local well-posedness and some important inequalities of solutions for (1.4) are derived, also, a new system under the new variables are constructed. And Section 3 is devoted to establishing the pointwise upper bounds for u_1, u_2 and v. Then the crucial estimates and technical tools to determine the finite time blow-up result are discussed in Section 4, and a proof of Theorem 1.1 is given here. For the last section, we establish the global bounded solution to (1.4) for $\Omega \subset \mathbb{R}^n (n \geq 1)$.

2. Preliminary

Our goal in this section is to collect the following local well-posedness result and some important inequalities of solutions for (1.4), moreover, we denote some new variables to transfer the original equations (1.4) to a new system.

Lemma 2.1. *Let $n \geq 1$, $R > 0$, $\gamma > 0$, $\tau_i > 0$, $\chi_i > 0$, $\lambda_i \in \mathbb{R}$, $\mu_i > 0$, $\alpha_i > 0$ and $k_i > 1$ $(i = 1, 2)$. Then for any (u_{10}, u_{20}) satisfying (1.5), there exist a maximal $T_{\max} \in (0, \infty]$ and unique radially symmetric nonnegative functions*

$$\begin{cases} (u_1, u_2) \in \left(C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))\right)^2, \\ v \in \cap_{q>n} L^\infty_{loc}([0, T_{\max}); W^{1,q}(\Omega)) \cap C^{2,0}(\overline{\Omega} \times (0, T_{\max})), \end{cases}$$

that solve (1.4) classically, either $T_{\max} = \infty$,

$$\text{or } ||u_1(\cdot, t)||_{L^\infty(\Omega)} + ||u_2(\cdot, t)||_{L^\infty(\Omega)} \to \infty \quad \text{as } t \to T_{\max}.$$

Moreover, the solution (u_1, u_2, v) satisfies

$$\int_\Omega u_i(x, t)dx \leq M_i, \quad i = 1, 2, \quad \text{for all} \quad t \in (0, T_{max}), \tag{2.1}$$

$$\int_\Omega v(x, t)dx \leq \frac{\alpha_1}{\gamma} M_1 + \frac{\alpha_2}{\gamma} M_2 \quad \text{for all} \quad t \in (0, T_{max}), \tag{2.2}$$

where $M_1 = \overline{C}_1 + \int_\Omega u_{10}dx$, $M_2 = \overline{C}_2 + \int_\Omega u_{20}dx$, $\overline{C}_1, \overline{C}_2$ are given below.

Proof. In light of a straightforward fixed point argument and the strong maximum principle, we can obtain the local existence, uniqueness of classical solutions for (1.4), we omit the detailed proof since a similar approach of the result is adapted in [34] for closely related problems. To obtain (2.1) and (2.2), by integrating the first two equations of system (1.4) in space, we have

$$\frac{d}{dt}\int_\Omega u_1 dx = \lambda_1 \int_\Omega u_1 dx - \mu_1 \int_\Omega u_1^{k_1} dx \quad \text{for all } t \in (0, T_{max})$$

and

$$\frac{d}{dt}\int_\Omega u_2 dx = \lambda_2 \int_\Omega u_2 dx - \mu_2 \int_\Omega u_2^{k_2} dx \quad \text{for all } t \in (0, T_{max}),$$

then we obtain

$$\frac{d}{dt}\int_\Omega u_1dx + \int_\Omega u_1dx = (\lambda_1+1)\int_\Omega u_1dx - \mu_1\int_\Omega u_1^{k_1}dx \quad \text{for all } t\in(0,T_{max})$$

and

$$\frac{d}{dt}\int_\Omega u_2dx + \int_\Omega u_2dx = (\lambda_2+1)\int_\Omega u_2dx - \mu_2\int_\Omega u_2^{k_2}dx \quad \text{for all } t\in(0,T_{max}).$$

In light of the Young inequality, it transpires that

$$\frac{d}{dt}\int_\Omega u_1dx + \int_\Omega u_1dx \le \overline{C}_1, \qquad \frac{d}{dt}\int_\Omega u_2dx + \int_\Omega u_2dx \le \overline{C}_2 \quad \text{for all } t\in(0,T_{max}),$$

with $\overline{C}_1,\overline{C}_2>0$, hence (2.1) can be immediately obtained. Then integrating the third one, we find

$$\gamma\int_\Omega vdx = \alpha_1\int_\Omega u_1dx + \alpha_2\int_\Omega u_2dx,$$

which implies (2.2). □

For some u_{10},u_{20} satisfying (1.5), we denote the corresponding local solution of (1.4) as $(u_1,u_2,v)=(u_1(r,t),u_2(r,t),v(r,t))$, and we set

$$A_1(s,t) := \int_0^{s^{\frac{1}{n}}}\rho^{n-1}u_1(\rho,t)d\rho, \quad s\in[0,R^n],\ t\in(0,T_{max}), \tag{2.3}$$

$$A_2(s,t) := \int_0^{s^{\frac{1}{n}}}\rho^{n-1}u_2(\rho,t)d\rho, \quad s\in[0,R^n],\ t\in(0,T_{max}) \tag{2.4}$$

and

$$B(s,t) := \int_0^{s^{\frac{1}{n}}}\rho^{n-1}v(\rho,t)d\rho, \quad s\in[0,R^n],\ t\in(0,T_{max}). \tag{2.5}$$

Then

$$A_{1s}(s,t) = \frac{1}{n}u_1(s^{\frac{1}{n}},t), A_{1ss}(s,t) = \frac{1}{n^2}s^{\frac{1}{n}-1}u_{1r}(s^{\frac{1}{n}},t) \quad s\in(0,R^n),\ t\in(0,T_{max}), \tag{2.6}$$

$$A_{2s}(s,t) = \frac{1}{n}u_2(s^{\frac{1}{n}},t), A_{2ss}(s,t) = \frac{1}{n^2}s^{\frac{1}{n}-1}u_{2r}(s^{\frac{1}{n}},t) \quad s\in(0,R^n),\ t\in(0,T_{max}), \tag{2.7}$$

$$B_s(s,t) = \frac{1}{n}v(s^{\frac{1}{n}},t), B_{ss}(s,t) = \frac{1}{n^2}s^{\frac{1}{n}-1}v_r(s^{\frac{1}{n}},t) \quad s\in(0,R^n),\ t\in(0,T_{max}). \tag{2.8}$$

Now we derive a new system under the independent variables (s,t). First, it follows from the third equation that

$$r^{n-1}\cdot v_r(r,t) = \gamma B(r^n,t) - (\alpha_1A_1(r^n,t)+\alpha_2A_2(r^n,t)) \quad \text{for all } s\in(0,R^n),\ t\in(0,T_{max}). \tag{2.9}$$

Next, by a straightforward calculation, we can transfer system (1.4) to the following system

$$\begin{cases} A_{1t} = \tau_1 n^2 s^{2-\frac{2}{n}} A_{1ss} + n\chi_1 A_{1s}(\alpha_1 A_1 + \alpha_2 A_2) - n\gamma\chi_1 A_{1s} B + \lambda_1 A_1 - n^{k_1-1}\mu_1 \int_0^s A_{1s}^{k_1}(\sigma,t)d\sigma, \\ A_{2t} = \tau_2 n^2 s^{2-\frac{2}{n}} A_{2ss} + n\chi_2 A_{2s}(\alpha_1 A_2 + \alpha_2 A_2) - n\gamma\chi_2 A_{2s} B + \lambda_2 A_2 - n^{k_2-1}\mu_2 \int_0^s A_{2s}^{k_2}(\sigma,t)d\sigma \\ n^2 s^{2-\frac{2}{n}} B_{ss} = \gamma B - (\alpha_1 A_1 + \alpha_2 A_2), \end{cases} \tag{2.10}$$

for all $s \in (0, R^n)$ $t \in (0, T_{max})$.

Then denoting $\varrho_n := n|B_1(0)|$, due to the fact that u_1, u_2 and v are nonnegative, it is easy to see that

$$0 = A_1(0,t) \le A_1(s,t) \le A_1(R^n,t) = \frac{1}{\varrho_n}\int_\Omega u_1(\cdot,t)dx \ \text{ for all } s \in (0,R^n), \ \ t \in (0,T_{max}), \tag{2.11}$$

$$0 = A_2(0,t) \le A_2(s,t) \le A_2(R^n,t) = \frac{1}{\varrho_n}\int_\Omega u_2(\cdot,t)dx \ \text{ for all } s \in (0,R^n), \ \ t \in (0,T_{max}), \tag{2.12}$$

$$0 = B(0,t) \le B(s,t) \le B(R^n,t) = \frac{1}{\varrho_n}\int_\Omega v(\cdot,t)dx \ \text{ for all } s \in (0,R^n), \ \ t \in (0,T_{max}) \tag{2.13}$$

as well as

$$A_{1s}(s,t), \ A_{2s}(s,t), \ B_s(s,t) \ge 0 \ \text{ for all } s \in (0,R^n), \ \ t \in (0,T_{max}). \tag{2.14}$$

3. The pointwise upper bounds for u_1, u_2 and v

Our attention in this section is turned to establishing the pointwise upper bounds for u_1, u_2 and v, which play a significant part in establishing the differential inequality introduced in section 4. First, we derive the pointwise bounds for v in the following lemma with the aid of the method in [34].

Lemma 3.1. *Let $n \ge 3$, $R > 0$, $\gamma > 0$, $\tau_i > 0$, $\chi_i > 0$, $\lambda_i \in \mathbb{R}$, $\mu_i > 0$, $\alpha_i > 0$ and $k_i > 1$ $(i = 1, 2)$. Fix any $m_1, m_2 > 0$, suppose that (1.5) holds with $\int_\Omega u_{10}dx \le m_1$, $\int_\Omega u_{20}dx \le m_2$, then there exists $\hat{C} > 0$ such that*

$$|v_r(r,t)| \le \hat{C} r^{1-n} \ \textit{ for all } \ r \in (0,R) \textit{ and } t \in (0,\hat{T}_{max}) \tag{3.1}$$

and

$$|v(r,t)| \le \hat{C} r^{2-n} \ \textit{ for all } \ r \in (0,R) \textit{ and } t \in (0,\hat{T}_{max}), \tag{3.2}$$

where $\hat{T}_{max} = \min\{1, T_{max}\}$.

Proof. From the previous estimates (2.1) and (2.2) in Lemma 2.1 we have

$$\int_\Omega u_i(x,t)dx \le M_i, \ \ i = 1,2, \ \text{ for all } \ t \in (0,\hat{T}_{max}) \tag{3.3}$$

and

$$\int_\Omega v(x,t)dx \le \frac{\alpha_1}{\gamma} M_1 + \frac{\alpha_2}{\gamma} M_2 \ \text{ for all } \ t \in (0,\hat{T}_{max}), \tag{3.4}$$

with $M_1 = \overline{C}_1 + \int_\Omega u_{10}dx$, $M_2 = \overline{C}_2 + \int_\Omega u_{20}dx$ defined in Lemma 2.1, which entails that

$$\int_\Omega u_i(x,t)dx,\ \int_\Omega v(x,t)dx \le C_0 \ \text{ for all } \ i=1,2,\ t\in(0,\hat{T}_{max}), \tag{3.5}$$

where $C_0 := \max_{i=1,2}\{M_i, \frac{\alpha_1}{\gamma}M_1 + \frac{\alpha_2}{\gamma}M_2\}$, then it directly follows from (2.11), (2.12) and (2.13) that

$$0 \le A_1(s,t), A_2(s,t), B(s,t) \le \frac{C_0}{\varrho_n} \ \text{ for all } \ s\in(0,R^n) \text{ and } t\in(0,\hat{T}_{max}), \tag{3.6}$$

along with (2.9), we deduce

$$r^{n-1}v_r(r,t) \le \gamma B \le \gamma\frac{C_0}{\varrho_n} \ \text{ for all } \ r\in(0,R) \text{ and } t\in(0,\hat{T}_{max}), \tag{3.7}$$

$$r^{n-1}v_r(r,t) \ge -(\alpha_1 A_1 + \alpha_2 A_2) \ge -(\alpha_1+\alpha_2)\frac{C_0}{\varrho_n} \ \text{ for all } \ r\in(0,R) \text{ and } t\in(0,\hat{T}_{max}), \tag{3.8}$$

which directly implies (3.1). Notice that for all $r_0\in(0,R)$, $r\in(0,R)$ and $t\in(0,\hat{T}_{max})$,

$$|v(r,t)-v(r_0,t)| = |\int_{r_0}^{r} v_r(\rho,t)d\rho| \le \frac{\gamma C_0}{(n-2)\varrho_n}|r^{2-n}-r_0^{2-n}|,$$

and applying (3.5), one has

$$\min_{r_0\in[\frac{R}{2},R]} v(r_0,t) \le \frac{C_0}{|\Omega\setminus B_{\frac{R}{2}}(0)|} \ \text{ for all } \ t\in(0,\hat{T}_{max}),$$

thus (3.2) holds. □

Next, based on the bound for $|v_r|$ in Lemma 3.1, following a result on pointwise bounds for radial solutions to heat equation, we obtain a upper bound for u.

Lemma 3.2. *Let $n\ge3$, $R>0$, $\gamma>0$, $\tau_i>0$, $\chi_i>0$, $\lambda_i\in\mathbb{R}$, $\mu_i>0$, $\alpha_i>0$ and $k_i>1$ $(i=1,2)$. Fix any $m_1,m_2>0$, $L_1,L_2>0$, suppose that (1.5) holds with*

$$\int_\Omega u_{10}dx \le m_1,\ \int_\Omega u_{20}dx \le m_2 \tag{3.9}$$

and

$$u_{10}(r) \le L_1 r^{-n(n-1)},\ u_{20}(r) \le L_2 r^{-n(n-1)} \ \textit{for all } r\in(0,R), \tag{3.10}$$

then for any $\epsilon>0$, there exists $M>0$ such that

$$u_i \le M r^{-n(n-1)-\epsilon},\ \ i=1,2, \ \textit{for all } \ r\in(0,R) \textit{ and } t\in(0,\hat{T}_{max}), \tag{3.11}$$

where $\hat{T}_{max}=\min\{1,T_{max}\}$.

Proof. For any $\epsilon>0$, choosing $p>n$ sufficiently large, it is found that

$$\delta := n(n-1)+\epsilon > \frac{n(n-1)p}{p-n}, \tag{3.12}$$

on account of (3.1) in Lemma 3.1, there appears the relation

$$\begin{aligned}\int_{\Omega} |x|^{(n-1)p}|\nabla v(x,t)|^p dx &= \varrho_n \int_0^R r^{(n-1)(p+1)}|v_r(r,t)|^p dr \\ &\le \hat{C}^p \varrho_n \int_0^R r^{n-1} dr = \frac{\hat{C}^p \varrho_n R^n}{n} \text{ for all } t \in (0, \hat{T}_{max}).\end{aligned} \tag{3.13}$$

Here we denote $U_i(x,t) := e^{-\lambda_i t} u_i(x,t)$, $i = 1, 2$, for $(x,t) \in \overline{\Omega} \times [0, T_{max})$, it follows from (1.4) that

$$\begin{aligned}U_{it} &= e^{-\lambda_i t} u_{it} - \lambda_i e^{-\lambda_i t} u_i \le \{\tau_i \Delta U_i - \chi_i \nabla \cdot (U_i \nabla v) + \lambda_i U_i - \mu_i e^{-\lambda_i t} u_i^{k_i}\} - \lambda_i U_i \\ &\le \Delta U_i - \nabla \cdot (U_i \nabla v) \text{ in } \Omega \times (0, T_{max}),\end{aligned}$$

where U_i satisfies $\frac{\partial U_i}{\partial \nu} = 0$ on $\partial\Omega \times (0, T_{max})$ and $\int_\Omega U_i(\cdot, 0) = \int_\Omega u_i(\cdot, 0) \le m_i$. Thanks to (3.9), (3.10), (3.12) and (3.13), it is applicable to using Theorem 1.1 in [35] to obtain that

$$U_i(x,t) \le C_1 |x|^{-\delta} \text{ for all } x \in \Omega \text{ and } t \in (0, \hat{T}_{max})$$

with $C_1 > 0$, which concludes the proof of this Lemma. □

4. Proof of Theorem 1.1

In this section, we aim at proving the blow-up result in Theorem 1.1, given a solution (u_1, u_2, v) of (1.4), we choose suitable $s_0 \in (0, R^n)$, $\beta \in (1 - \frac{2}{n}, 1)$. Inspired by [34], we define $\Phi(t) : [0, T_{max}) \to R$ by

$$\Phi(t) := \Phi_1(t) + \Phi_2(t) = \int_0^{s_0} s^{-\beta}(s_0 - s) A_1(s,t) ds + \int_0^{s_0} s^{-\beta}(s_0 - s) A_2(s,t) ds \text{ for all } t \in [0, \hat{T}_{max}), \tag{4.1}$$

it is clear that Φ is well-defined and belongs to $C^0([0, T_{max})) \cap C^1((0, T_{max}))$, since u_i, $u_{it} (i = 1, 2)$ are continuous in $\overline{\Omega} \times [0, T_{max})$ and in $\overline{\Omega} \times (0, T_{max})$ respectively and the map $(0, s_0) \ni s \mapsto s^{-\beta}(s_0 - s)$ is integrable for $s_0 \in (0, R^n)$, $\beta \in (1 - \frac{2}{n}, 1)$. The cornerstone of our approach is based on the following differential inequality

$$\Phi'(t) \ge d_1 s_0^{\beta-3} \Phi^2(t) - d_2 s_0^{\frac{2}{n}+1-\beta} \text{ for all } t \in [0, \hat{T}_{max}), \tag{4.2}$$

where $d_1, d_2 > 0$. To obtain (4.2), our analysis depends on the following time evolution function

$$\begin{aligned}\varphi(t) &= \varphi_1(t) + \varphi_2(t) \\ &= \int_0^{s_0} s^{-\beta}(s_0 - s) A_1(s,t) A_{1s}(s,t) ds + \int_0^{s_0} s^{-\beta}(s_0 - s) A_2(s,t) A_{2s}(s,t) ds \text{ for all } t \in (0, \hat{T}_{max}).\end{aligned} \tag{4.3}$$

At the first step, we find the relationship between Φ and $\varphi(t)$ as follows.

Lemma 4.1. *Let $n \ge 3$, $R > 0$, $\gamma > 0$, $\tau_i > 0$, $\chi_i > 0$, $\lambda_i \in \mathbb{R}$, $\mu_i > 0$, $\alpha_i > 0$ and $k_i > 1$ $(i = 1, 2)$. Suppose (1.5) holds, then for any $\beta \in (0, 2)$ and $s_0 \in (0, R^n)$, there exists $C_2 = C_2(\beta) > 0$ such that*

$$\varphi(t) \ge C_2 s_0^{\beta-3} \cdot \Phi(t)^2 \text{ for all } t \in (0, \hat{T}_{max}). \tag{4.4}$$

Proof. In light of the Lemma 4.2 in [34], we have

$$A_i(s) \le \sqrt{2} \cdot s^{\frac{\beta}{2}} (s_0 - s)^{-\frac{1}{2}} \cdot \left\{ \int_0^{s_0} \sigma^{-\beta} (s_0 - \sigma) A_i(\sigma) A_{is}(\sigma) d\sigma \right\}^{\frac{1}{2}}, \quad i = 1, 2 \text{ for all } s \in (0, s_0), \tag{4.5}$$

using the fact that $\frac{\beta}{2} < 1$, a simple computation shows that

$$\begin{aligned} \int_0^{s_0} s^{-\beta} (s_0 - s) A_i ds &\le s_0 \int_0^{s_0} s^{-\beta} A_i ds \\ &\le \sqrt{2} \cdot \left\{ \int_0^{s_0} s^{-\beta} (s_0 - s) A_i A_{is} ds \right\}^{\frac{1}{2}} \cdot s_0 \int_0^{s_0} s^{-\frac{\beta}{2}} (s_0 - s)^{-\frac{1}{2}} ds \\ &= \sqrt{2} \cdot \left\{ \int_0^{s_0} s^{-\beta} (s_0 - s) A_i A_{is} ds \right\}^{\frac{1}{2}} \cdot s_0^{\frac{3-\beta}{2}} \mathcal{B}(1 - \frac{\beta}{2}, \frac{1}{2}), \ i = 1, 2, \text{ for all } \ t \in (0, T_{max}), \end{aligned} \tag{4.6}$$

where $\mathcal{B}$ is Euler's Beta function. Hence we obtain

$$\varphi_i \ge \frac{1}{2} \left(\mathcal{B}(1 - \frac{\beta}{2}, \frac{1}{2}) \right)^{-2} \cdot s_0^{\beta - 3} \Phi_i^2, \ i = 1, 2, \text{ for all } \ t \in (0, T_{max}), \tag{4.7}$$

which directly implies (4.3). $\square$

At the second step, we enter into a set of calculations to show (4.2).

Lemma 4.2. *Let $n \ge 3$, $R > 0$, $\gamma > 0$, $\tau_i > 0$, $\chi_i > 0$, $\lambda_i \in \mathbb{R}$, $\mu_i > 0$, $\alpha_i > 0$, $\beta \in (1 - \frac{2}{n}, 1)$ and $k_i > 1$ $(i = 1, 2)$ be such that*

$$(n-1)(k_i - 1) < \frac{\beta}{2}, \ i = 1, 2. \tag{4.8}$$

Fix any $m_1, m_2 > 0$,$L_1, L_2 > 0$, suppose (1.5) holds with

$$\int_\Omega u_{10} dx \le m_1, \ \int_\Omega u_{20} dx \le m_2 \tag{4.9}$$

and

$$u_{10}(r) \le L_1 r^{-n(n-1)}, \ u_{20}(r) \le L_2 r^{-n(n-1)} \ \textit{for all } r \in (0, R), \tag{4.10}$$

then for any $\epsilon > 0$ and $s_0 \in (0, R^n) \cap (0, 1)$, the function $\Phi(t)$ defined in (4.1) satisfies

$$\begin{aligned} \Phi'(t) &= \Phi_1'(t) + \Phi_2'(t) \\ &\ge C_3 n \varphi - C_4 s_0^{\frac{3-\beta}{2} - \frac{2}{n}} (\varphi_1^{\frac{1}{2}} + \varphi_2^{\frac{1}{2}}) - C_4 s_0 \int_0^{s_0} s^{-\beta - 1} (A_1 + A_2) B ds \\ &\quad - C_4 s_0^{\frac{3-\beta}{2}} (\varphi_1^{\frac{1}{2}} + \varphi_2^{\frac{1}{2}}) - C_4 s_0^{-(n-1)(\hat{k} - 1) + \frac{3-\beta}{2} - \epsilon} (\varphi_1^{\frac{1}{2}} + \varphi_2^{\frac{1}{2}}) \ \textit{for all } t \in (0, \hat{T}_{max}), \end{aligned} \tag{4.11}$$

with $\varphi_i (i = 1, 2)$ *defined in* (4.3), $C_3, C_4 > 0$, $\hat{k} = \max\{k_1, k_2\}$.

Proof. Note that $\Phi(t) \in C^0([0, T_{max})) \cap C^1((0, T_{max}))$ and differentiate Φ with respect to t, from (2.10) one obtains

$$\begin{aligned}
\Phi'(t) = & \sum_{i=1}^{2} \tau_i n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\beta}(s_0 - s) A_{iss} ds + \sum_{i=1}^{2} n\chi_i \int_0^{s_0} s^{-\beta}(s_0 - s) A_{is}(\alpha_1 A_1 + \alpha_2 A_2) ds \\
& - \left(\sum_{i=1}^{2} n\gamma\chi_i \int_0^{s_0} s^{-\beta}(s_0 - s) B A_{is} ds \right) + \sum_{i=1}^{2} \lambda_i \int_0^{s_0} s^{-\beta}(s_0 - s) A_i ds \\
& - \left(\sum_{i=1}^{2} n^{k_i - 1} \mu_i \int_0^{s_0} s^{-\beta}(s_0 - s) \cdot [\int_0^{s} A_{is}^{k_i}(\sigma, t) d\sigma] ds \right) \\
= & I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) \text{ for all } t \in (0, \hat{T}_{max}).
\end{aligned} \tag{4.12}$$

For $I_1(t)$ in (4.12), applying two integrations by parts, we find $d_1 > 0$ such that

$$\begin{aligned}
& \sum_{i=1}^{2} \tau_i n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\beta}(s_0 - s) A_{iss} ds \\
= & -\sum_{i=1}^{2} \tau_i n^2 (2 - \frac{2}{n} - \beta) \int_0^{s_0} s^{1-\frac{2}{n}-\beta}(s_0 - s) A_{is} ds \\
& + \sum_{i=1}^{2} \left(\tau_i n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\beta} A_{is} ds + \tau_i n^2 s^{2-\frac{2}{n}-\beta}(s_0 - s) A_{is} \Big|_0^{s_0} \right) \\
\geq & -\sum_{i=1}^{2} \tau_i n^2 (2 - \frac{2}{n} - \beta) \int_0^{s_0} s^{1-\frac{2}{n}-\beta}(s_0 - s) A_{is} ds \\
= & -\sum_{i=1}^{2} \left(\tau_i n^2 (2 - \frac{2}{n} - \beta)(\beta - 1 + \frac{2}{n}) \int_0^{s_0} s^{-\beta-\frac{2}{n}}(s_0 - s) A_i ds + \tau_i n^2 (2 - \frac{2}{n} - \beta) \int_0^{s_0} s^{1-\beta-\frac{2}{n}} A_i ds \right) \\
\geq & -\sum_{i=1}^{2} \tau_i n^2 (2 - \frac{2}{n} - \beta)(\beta + \frac{2}{n}) s_0 \int_0^{s_0} s^{-\beta-\frac{2}{n}} A_i ds \\
\geq & -d_1 s_0 \int_0^{s_0} s^{-\beta-\frac{2}{n}} (A_1 + A_2) ds \text{ for all } t \in (0, \hat{T}_{max}),
\end{aligned} \tag{4.13}$$

where we use the fact that $A_i, A_{is} (i = 1, 2)$ are nonnegative, and $2 - \frac{2}{n} - \beta, \beta - 1 + \frac{2}{n}$ are positive. It follows from (4.5) and (4.13) that

$$
\begin{aligned}
& - d_1 s_0 \int_0^{s_0} s^{-\beta-\frac{2}{n}}(A_1+A_2)ds \\
& \geq -\sqrt{2} d_1 s_0 \cdot \left(\sum_{i=1}^{2} \left\{ \int_0^{s_0} s^{-\beta}(s_0-s)A_i A_{is} ds \right\}^{\frac{1}{2}} \right) \cdot \int_0^{s_0} s^{-\frac{\beta}{2}-\frac{2}{n}}(s_0-s)^{-\frac{1}{2}} ds \\
& = -\sqrt{2} d_1 s_0^{\frac{3-\beta}{2}-\frac{2}{n}} \mathcal{B}\left(1-\frac{\beta}{2}-\frac{2}{n}, \frac{1}{2}\right) \cdot \left(\sum_{i=1}^{2} \left\{ \int_0^{s_0} s^{-\beta}(s_0-s)A_i A_{is} ds \right\}^{\frac{1}{2}} \right) \quad \text{for all} \quad t \in (0, \hat{T}_{max}),
\end{aligned}
\tag{4.14}
$$

since the assumption $\beta \leq 2 - \frac{4}{n}$ ensures that Euler's Beta function $\mathcal{B}$ is well-defined, therefore

$$
\sum_{i=1}^{2} \tau_i n^2 \int_0^{s_0} s^{2-\frac{2}{n}-\beta}(s_0-s)A_{iss} ds \geq -d_1' s_0^{\frac{3-\beta}{2}-\frac{2}{n}} \sum_{i=1}^{2} \left\{ \int_0^{s_0} s^{-\beta}(s_0-s)A_i A_{is} ds \right\}^{\frac{1}{2}}
\tag{4.15}
$$

for all $t \in (0, \hat{T}_{max})$, with $d_1' > 0$.

As for $I_2(t)$, due to the fact that A_1, A_2 is nonnegative, we obtain

$$
\begin{aligned}
& \sum_{i=1}^{2} n\chi_i \int_0^{s_0} s^{-\beta}(s_0-s)A_{is}(\alpha_1 A_1 + \alpha_2 A_2) ds \\
& \geq \min\{\alpha_1, \alpha_2\} \cdot \min\{\chi_1, \chi_2\} n \int_0^{s_0} s^{-\beta}(s_0-s)(A_1 A_{1s} + A_2 A_{2s}) ds.
\end{aligned}
\tag{4.16}
$$

To estimate $I_3(t)$, in view of the nonnegativity of A_1, A_2 and B, from the integration by parts we obtain

$$
\begin{aligned}
& - \left(\sum_{i=1}^{2} n\gamma\chi_i \int_0^{s_0} s^{-\beta}(s_0-s) B A_{is} ds \right) \\
& = \sum_{i=1}^{2} n\gamma\chi_i \int_0^{s_0} s^{-\beta}(s_0-s) B_s A_i ds - n\gamma \left(\sum_{i=1}^{2} \chi_i \int_0^{s_0} s^{-\beta} A_i B ds \right) \\
& \quad - n\beta\gamma \left(\sum_{i=1}^{2} \chi_i \int_0^{s_0} s^{-\beta-1}(s_0-s) A_i B ds \right) \\
& \geq -n\beta\gamma s_0 \left(\sum_{i=1}^{2} \chi_i \int_0^{s_0} s^{-\beta-1} A_i B ds \right) - n\gamma s_0 \left(\sum_{i=1}^{2} \chi_i \int_0^{s_0} s^{-\beta-1} A_i B ds \right) \\
& \geq -d_2 s_0 \int_0^{s_0} s^{-\beta-1}(A_1+A_2) B ds,
\end{aligned}
\tag{4.17}
$$

with $d_2 > 0$.

As for I_4, we let $\lambda_{i-} := \max\{0, -\lambda_i\}$, it is clear that

$$
\sum_{i=1}^{2} \lambda_i \int_0^{s_0} s^{-\beta}(s_0-s) A_i ds \geq -d_3 \int_0^{s_0} s^{-\beta}(s_0-s)(A_1+A_2) ds,
\tag{4.18}
$$

with $d_3 = \lambda_{1-} + \lambda_{2-}$. Then it follows from (4.6) that

$$\begin{aligned}
&\sum_{i=1}^{2} \lambda_i \int_0^{s_0} s^{-\beta}(s_0 - s) A_i ds \\
&\geq -d_3' s_0^{\frac{3-\beta}{2}} \left(\left\{ \int_0^{s_0} s^{-\beta}(s_0 - s) A_1 A_{1s} ds \right\}^{\frac{1}{2}} + \left\{ \int_0^{s_0} s^{-\beta}(s_0 - s) A_2 A_{2s} ds \right\}^{\frac{1}{2}} \right).
\end{aligned} \tag{4.19}$$

The estimate for I_5 is much more delicate. With the aid of the Fubini theorem and $\beta < 1$, we derive

$$\begin{aligned}
&- \left(\sum_{i=1}^{2} n^{k_i - 1} \mu_i \int_0^{s_0} s^{-\beta}(s_0 - s) \cdot [\int_0^{s} A_{is}^{k_i}(\sigma, t) d\sigma] ds \right) \\
&= - \left(\sum_{i=1}^{2} n^{k_i - 1} \mu_i \int_0^{s_0} [\int_\sigma^{s_0} s^{-\beta}(s_0 - s) ds] \cdot A_{is}^{k_i}(\sigma, t) d\sigma \right) \\
&\geq - \left(\sum_{i=1}^{2} n^{k_i - 1} \mu_i \int_0^{s_0} [(s_0 - \sigma) \int_0^{s_0} s^{-\beta} ds] \cdot A_{is}^{k_i}(\sigma, t) d\sigma \right) \\
&= - \left(\sum_{i=1}^{2} \frac{n^{k_i - 1} \mu_i}{1 - \beta} s_0^{1-\beta} \int_0^{s_0} (s_0 - s) A_{is}^{k_i}(s, t) ds \right).
\end{aligned} \tag{4.20}$$

Now we want to derive the upper bound for $A_{is}^{k_i - 1}(s, t)$. Let $\hat{k} = \max\{k_1, k_2\}$, for any choice of $\epsilon > 0$, in view of (4.8), there exists $\eta > 0$ sufficiently small such that

$$\frac{\eta}{n}(\hat{k} - 1) \leq \min\{\epsilon, 1\} \tag{4.21}$$

and

$$\frac{\eta}{n}(\hat{k} - 1) + (n - 1)(\hat{k} - 1) < \frac{\beta}{2}. \tag{4.22}$$

Relying on Lemma 3.2, for $\eta > 0$ we have

$$u_i \leq M r^{-n(n-1)-\eta}, \quad i = 1, 2, \quad \text{for all} \quad r \in (0, R) \text{ and } t \in (0, \hat{T}_{max}),$$

then for any $s \in (0, 1) \cap (0, R^n)$, we find $d_4 > 0$ such that

$$A_{is}^{k_i - 1}(s, t) = \left(\frac{u_i(s^{\frac{1}{n}}, t)}{n} \right)^{k_i - 1} \leq d_4 s^{-\frac{\eta}{n}(\hat{k}-1)-(n-1)(\hat{k}-1)}, \quad i = 1, 2, \quad \text{for all} \quad t \in (0, \hat{T}_{max}). \tag{4.23}$$

By (4.20) and (4.23), using the integration by parts, it is thereby inferred that

$$
\begin{aligned}
&-\left(\sum_{i=1}^{2} n^{k_i-1}\mu_i \int_0^{s_0} s^{-\beta}(s_0-s)\cdot[\int_0^{s} A_{is}^{k_i}(\sigma,t)d\sigma]ds\right)\\
&\geq -\sum_{i=1}^{2}\frac{n^{k_i-1}\mu_i}{1-\beta}s_0^{1-\beta}\int_0^{s_0}(s_0-s)A_{is}^{k_i}(s,t)ds\\
&\geq -\sum_{i=1}^{2}\frac{n^{k_i-1}\mu_i d_4}{1-\beta}s_0^{1-\beta}\int_0^{s_0} s^{-\frac{\eta}{n}(\hat{k}-1)-(n-1)(\hat{k}-1)}(s_0-s)A_{is}ds\\
&= -\sum_{i=1}^{2}[\frac{\eta}{n}(\hat{k}-1)+(n-1)(\hat{k}-1)]\frac{n^{k_i-1}\mu_i d_4}{1-\beta}s_0^{1-\beta}\int_0^{s_0} s^{-\frac{\eta}{n}(\hat{k}-1)-(n-1)(\hat{k}-1)-1}(s_0-s)A_i ds \qquad (4.24)\\
&\quad -\sum_{i=1}^{2}\frac{n^{k_i-1}\mu_i d_4}{1-\beta}s_0^{1-\beta}\int_0^{s_0} s^{-\frac{\eta}{n}(\hat{k}-1)-(n-1)(\hat{k}-1)}A_i ds\\
&\geq -\sum_{i=1}^{2}[(n-1)(\hat{k}-1)+2]\frac{n^{k_i-1}\mu_i d_4}{1-\beta}s_0^{2-\beta}\int_0^{s_0} s^{-\frac{\eta}{n}(\hat{k}-1)-(n-1)(\hat{k}-1)-1}A_i ds\\
&\geq -d_5 s_0^{2-\beta}\int_0^{s_0} s^{-\frac{\eta}{n}(\hat{k}-1)-(n-1)(\hat{k}-1)-1}(A_1+A_2)ds,
\end{aligned}
$$

with $d_5>0$. Moreover, we employ (4.5) to estimate

$$
\begin{aligned}
&-d_5 s_0^{2-\beta}\int_0^{s_0} s^{-\frac{\eta}{n}(\hat{k}-1)-(n-1)(\hat{k}-1)-1}(A_1+A_2)ds\\
&\geq -\sqrt{2}d_5 s_0^{2-\beta}\left(\left\{\int_0^{s_0} s^{-\beta}(s_0-s)A_1A_{1s}ds\right\}^{\frac{1}{2}}+\left\{\int_0^{s_0} s^{-\beta}(s_0-s)A_2A_{2s}ds\right\}^{\frac{1}{2}}\right)\\
&\quad\cdot\int_0^{s_0} s^{-\frac{\eta}{n}(\hat{k}-1)-(n-1)(\hat{k}-1)-1+\frac{\beta}{2}}(s_0-s)^{-\frac{1}{2}}ds\\
&\geq -\sqrt{2}d_5 s_0^{-(n-1)(\hat{k}-1)+\frac{3-\beta}{2}-\frac{\eta}{n}(\hat{k}-1)}\left(\left\{\int_0^{s_0} s^{-\beta}(s_0-s)A_1A_{1s}ds\right\}^{\frac{1}{2}}+\left\{\int_0^{s_0} s^{-\beta}(s_0-s)A_2A_{2s}ds\right\}^{\frac{1}{2}}\right)\\
&\quad\cdot\mathcal{B}\left(\frac{\beta}{2}-(n-1)(\hat{k}-1)-\frac{\eta}{n}(\hat{k}-1),\frac{1}{2}\right),
\end{aligned}
\tag{4.25}
$$

it follows from (4.22) that $\frac{\beta}{2}-(n-1)(\hat{k}-1)-\frac{\eta}{n}(\hat{k}-1)$ is positive, which implies that $d_6=\mathcal{B}\left(\frac{\beta}{2}-(n-1)(\hat{k}-1)-\frac{\eta}{n}(\hat{k}-1),\frac{1}{2}\right)$ is well-defined and finite. And on account of (4.21), (4.24) and (4.25), we can deduce that

$$
\begin{aligned}
&-\left(\sum_{i=1}^{2} n^{k_i-1}\mu_i \int_0^{s_0} s^{-\beta}(s_0-s)\cdot[\int_0^{s} A_{is}^{k_i}(\sigma,t)d\sigma]ds\right) \\
&\geq -d_7 s_0^{-(n-1)(\hat{k}-1)+\frac{3-\beta}{2}-\epsilon}\left(\left\{\int_0^{s_0} s^{-\beta}(s_0-s)A_1A_{1s}ds\right\}^{\frac{1}{2}}+\left\{\int_0^{s_0} s^{-\beta}(s_0-s)A_2A_{2s}ds\right\}^{\frac{1}{2}}\right),
\end{aligned}
\tag{4.26}
$$

with $d_7=\sqrt{2}d_5d_6R^{[n\epsilon-\eta(\hat{k}-1)]}$.

Thus a combination of (4.15)-(4.19) and (4.26) yield

$$
\begin{aligned}
\Phi'(t) &= \Phi_1'(t)+\Phi_2'(t) \\
&\geq \min\{\alpha_1,\alpha_2\}\cdot\min\{\chi_1,\chi_2\}n\int_0^{s_0} s^{-\beta}(s_0-s)(A_1A_{1s}+A_2A_{2s})ds \\
&\quad - d_1' s_0^{\frac{3-\beta}{2}-\frac{2}{n}}\sum_{i=1}^{2}\left\{\int_0^{s_0} s^{-\beta}(s_0-s)A_iA_{is}ds\right\}^{\frac{1}{2}} - d_2 s_0\int_0^{s_0} s^{-\beta-1}(A_1+A_2)Bds \\
&\quad - d_3' s_0^{\frac{3-\beta}{2}}\left(\left\{\int_0^{s_0} s^{-\beta}(s_0-s)A_1A_{1s}ds\right\}^{\frac{1}{2}}+\left\{\int_0^{s_0} s^{-\beta}(s_0-s)A_2A_{2s}ds\right\}^{\frac{1}{2}}\right) \\
&\quad - d_7 s_0^{-(n-1)(\hat{k}-1)+\frac{3-\beta}{2}-\epsilon}\left(\left\{\int_0^{s_0} s^{-\beta}(s_0-s)A_1A_{1s}ds\right\}^{\frac{1}{2}}+\left\{\int_0^{s_0} s^{-\beta}(s_0-s)A_2A_{2s}ds\right\}^{\frac{1}{2}}\right) \\
&= \min\{\alpha_1,\alpha_2\}\cdot\min\{\chi_1,\chi_2\}n\varphi - d_1' s_0^{\frac{3-\beta}{2}-\frac{2}{n}}(\varphi_1^{\frac{1}{2}}+\varphi_2^{\frac{1}{2}}) - d_2 s_0\int_0^{s_0} s^{-\beta-1}(A_1+A_2)Bds \\
&\quad - d_3' s_0^{\frac{3-\beta}{2}}(\varphi_1^{\frac{1}{2}}+\varphi_2^{\frac{1}{2}}) - d_7 s_0^{-(n-1)(\hat{k}-1)+\frac{3-\beta}{2}-\epsilon}(\varphi_1^{\frac{1}{2}}+\varphi_2^{\frac{1}{2}})
\end{aligned}
\tag{4.27}
$$

for all $t\in(0,\hat{T}_{max})$, $s\in(0,1)\cap(0,R^n)$, it then turns out that (4.11) holds. □

For the purpose of obtaining (4.2), we still need to deal with the term $C_4 s_0\int_0^{s_0} s^{-\beta-1}(A_1+A_2)Bds$, a crucial idea is to utilize φ to bound B. To achieve this, we recall the following Lemma which plays an important role in proving Lemma 4.4, the proof of Lemma 4.3 is similar to [34], and we omit it here for brevity.

Lemma 4.3. *Let $\vartheta\in(1,2)$ and $l\in(0,1)$. Then for any $s_0>0$, there exists $C=C(\vartheta,l)>0$ such that*

$$
\int_0^{s}\int_\sigma^{s_0}\tau^{-\vartheta}(s_0-\tau)^{-l}d\tau d\sigma \leq Cs_0^{-l}s^{2-\vartheta} \text{ for all } s\in(0,s_0). \tag{4.28}
$$

Lemma 4.4. *Let $n\geq 3$, $R>0$, $\gamma>0$, $\tau_i>0$, $\chi_i>0$, $\lambda_i\in\mathbb{R}$, $\mu_i>0$, $\alpha_i>0$ and $k_i>1$ $(i=1,2)$. Assume that $\beta\in(0,2-\frac{4}{n})\cap(0,1)$, $u_{i0}(i=1,2)$ satisfies (1.5), $\int_\Omega u_i\leq m_i(i=1,2)$ for any $m_i>0$, then for all $s_0\in(0,R^n)$, one can find $J>0$ such that*

$$
-C_4 s_0\int_0^{s_0} s^{-\beta-1}(A_1+A_2)Bds \geq -Js_0^{1+\frac{2}{n}-\beta} - Js_0^{\frac{n}{2}}\varphi \tag{4.29}
$$

for all $s \in (0, s_0)$ *and* $t \in (0, \hat{T}_{max})$.

Proof. First, we aim at finding the relationship between B and φ. It follows from (2.9) that

$$r^{n-1} \cdot v_r(r,t) \geq -(\alpha_1 A_1(r^n,t) + \alpha_2 A_2(r^n,t)) \quad \text{for all } s \in (0, R^n), \ \ t \in (0, \hat{T}_{max}),$$

whence according to (2.8) we find

$$B_{ss}(s,t) = \frac{1}{n^2} s^{\frac{1}{n}-1} v_r(s^{\frac{1}{n}}, t) \geq -\frac{1}{n^2} s^{\frac{2}{n}-2}(\alpha_1 A_1(s,t) + \alpha_2 A_2(s,t)) \ \text{ for all } \ r \in (0,R) \text{ and } t \in (0, \hat{T}_{max}), \tag{4.30}$$

and (2.8) along with Lemma 3.1 entails that

$$B_s(s,t) \leq \frac{\hat{C}}{n} s^{\frac{2}{n}-1} \ \text{ for all } \ s \in (0, s_0) \text{ and } t \in (0, \hat{T}_{max}), \tag{4.31}$$

and thereupon applying (4.5) and the above two equations, there appears the relationship

$$\begin{aligned}
B(s,t) &= \int_0^s B_s(\sigma,t)d\sigma = \int_0^s \left(B_s(s_0,t) - \int_\sigma^{s_0} B_{ss}(\tau,t)d\tau \right) d\sigma \\
&\leq \frac{\hat{C}}{n} s_0^{\frac{2}{n}-1} s + \frac{1}{n^2} \int_0^s \int_\sigma^{s_0} \tau^{\frac{2}{n}-2} (\alpha_1 A_1(\tau,t) + \alpha_2 A_2(\tau,t)) d\tau d\sigma \\
&\leq \frac{\hat{C}}{n} s_0^{\frac{2}{n}-1} s + \frac{\sqrt{2}}{n^2} \int_0^s \int_\sigma^{s_0} \tau^{\frac{2}{n}+\frac{\beta}{2}-2} (s_0-\tau)^{-\frac{1}{2}} d\tau d\sigma \\
&\quad \cdot \left(\alpha_1 \left\{ \int_0^{s_0} s^{-\beta}(s_0-s) A_1 A_{1s} ds \right\}^{\frac{1}{2}} + \alpha_2 \left\{ \int_0^{s_0} s^{-\beta}(s_0-s) A_2 A_{2s} ds \right\}^{\frac{1}{2}} \right)
\end{aligned}$$

for all $s \in (0, s_0)$ and $t \in (0, \hat{T}_{max})$. Moreover, note that $\beta < 2 - \frac{4}{n}$, which implies

$$-1 = \frac{2}{n} + \frac{2-\frac{4}{n}}{2} - 2 > \frac{2}{n} + \frac{\beta}{2} - 2 > -2.$$

Hence we employ Lemma (4.3) to find c_1 such that

$$\int_0^s \int_\sigma^{s_0} \tau^{\frac{2}{n}+\frac{\beta}{2}-2} (s_0-\tau)^{-\frac{1}{2}} d\tau d\sigma \leq c_1 s_0^{-\frac{1}{2}} s^{\frac{2}{n}+\frac{\beta}{2}} \ \text{ for all } s \in (0, s_0).$$

This readily yields

$$B(s,t) \leq c_2 s_0^{\frac{2}{n}-1} s + c_2 s_0^{-\frac{1}{2}} s^{\frac{2}{n}+\frac{\beta}{2}} \cdot (\varphi_1^{\frac{1}{2}} + \varphi_2^{\frac{1}{2}}) \ \text{ for all } \ s \in (0, s_0) \text{ and } t \in (0, \hat{T}_{max}) \tag{4.32}$$

Therefore, upon (4.5) there exists $J > 0$ such that

$$
\begin{aligned}
&C_4 s_0 \int_0^{s_0} s^{-\beta-1}(A_1+A_2)B ds \\
&\le c_2 C_4 s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\beta}(A_1+A_2)ds + c_2 C_4(\varphi_1^{\frac{1}{2}}+\varphi_2^{\frac{1}{2}}) s_0^{\frac{1}{2}} \int_0^{s_0} s^{\frac{2}{n}-\frac{\beta}{2}-1}(A_1+A_2)ds \\
&\le 2\frac{C_0}{\varrho_n} c_2 C_4 s_0^{\frac{2}{n}} \int_0^{s_0} s^{-\beta}ds + \sqrt{2} c_2 C_4 (\varphi_1^{\frac{1}{2}}+\varphi_2^{\frac{1}{2}})^2 s_0^{\frac{1}{2}} \int_0^{s_0} s^{\frac{2}{n}-1}(s_0-s)^{-\frac{1}{2}}ds \\
&= 2\frac{C_0}{\varrho_n} c_2 C_4 \frac{1}{1-\beta} s_0^{1+\frac{2}{n}-\beta} + \sqrt{2} c_2 C_4 (\varphi_1^{\frac{1}{2}}+\varphi_2^{\frac{1}{2}})^2 s_0^{\frac{n}{2}} \mathcal{B}(\frac{2}{n},\frac{1}{2}) \\
&\le J s_0^{1+\frac{2}{n}-\beta} + J s_0^{\frac{n}{2}}(\varphi_1+\varphi_2) \quad \text{for all } t \in (0,\hat{T}_{max}). \quad \square
\end{aligned}
\tag{4.33}
$$

Lemma 4.5. *Let $n \ge 3$, $R > 0$, $\gamma > 0$, $\tau_i > 0$, $\chi_i > 0$, $\lambda_i \in \mathbb{R}$, $\mu_i > 0$, $\alpha_i > 0$ and $k_i > 1$ $(i = 1, 2)$ satisfies*

$$
\hat{k} = \max\{k_1, k_1\} < \begin{cases} \frac{7}{6} & \text{if } n \in \{3,4\}, \\ 1 + \frac{1}{2(n-1)} & \text{if } n \ge 5. \end{cases} \tag{4.34}
$$

For any $m_i, L_i > 0 (i = 1, 2)$, suppose that (1.5) holds with

$$
\int_\Omega u_{10} dx \le m_1, \quad \int_\Omega u_{20} dx \le m_2 \tag{4.35}
$$

and

$$
u_{10}(r) \le L_1 r^{-n(n-1)}, \quad u_{20}(r) \le L_2 r^{-n(n-1)} \quad \text{for all } r \in (0,R), \tag{4.36}
$$

then for any $\beta \in (1-\frac{2}{n}, 1) \cap (0, 2-\frac{4}{n})$ and $(0,R^n) \cap (0,1) \ni s^* = s^*(R,\lambda_i,m_i) \le (\frac{C_3 n}{8J})^{\frac{2}{n}}$ (C_3, J are defined in Lemma 4.2 and Lemma 4.4 respectively), there exists $\mathfrak{C}_i = \mathfrak{C}_i(R,\mu_i,k_i,L_i,\lambda_i,m_i) (i = 1,2)$ such that

$$
\Phi'(t) \ge \mathfrak{C}_1 s_0^{\beta-3}\Phi^2(t) - \mathfrak{C}_2 s_0^{\frac{2}{n}+1-\beta} \quad \text{for all } s_0 \in (0,s^*) \text{ and } t \in (0,\hat{T}_{max}). \tag{4.37}
$$

Proof. First, we claim that

$$
(n-1)(\hat{k}-1) < \frac{\beta}{2} \quad \text{for any } \beta \in (1-\frac{2}{n},1) \cap (0, 2-\frac{4}{n}). \tag{4.38}
$$

Indeed, in view of (4.34), we find that

$$
(\hat{k}-1)(n-1) < \begin{cases} \frac{1}{3} & \text{if } n = 3, \\ \frac{1}{2} & \text{if } n \ge 4. \end{cases} \tag{4.39}
$$

Therefore, the above equation and $\beta \in (1-\frac{2}{n}, 1) \cap (0, 2-\frac{4}{n})$ indicate that (4.38) holds.

Next, choosing $\epsilon > 0$ such that

$$
2\epsilon \le 1 - \frac{2}{n}, \tag{4.40}
$$

a combination of Lemma 4.2 and Lemma 4.4 yield

$$\begin{aligned}
\Phi'(t) &= \Phi_1'(t) + \Phi_2'(t) \\
&\geq (C_3 n - J s_0^{\frac{n}{2}})\varphi - C_4 s_0^{\frac{3-\beta}{2}-\frac{2}{n}}(\varphi_1^{\frac{1}{2}} + \varphi_2^{\frac{1}{2}}) - J s_0^{1+\frac{2}{n}-\beta} \\
&\quad - C_4 s_0^{\frac{3-\beta}{2}}(\varphi_1^{\frac{1}{2}} + \varphi_2^{\frac{1}{2}}) - C_4 s_0^{-(n-1)(\hat{k}-1)+\frac{3-\beta}{2}-\epsilon}(\varphi_1^{\frac{1}{2}} + \varphi_2^{\frac{1}{2}}) \\
&\geq \frac{7C_3 n}{8}\varphi - \sqrt{2}C_4 s_0^{\frac{3-\beta}{2}-\frac{2}{n}}(\varphi_1+\varphi_2)^{\frac{1}{2}} - \sqrt{2}C_4 s_0^{\frac{3-\beta}{2}}(\varphi_1+\varphi_2)^{\frac{1}{2}} \\
&\quad - \sqrt{2}C_4 s_0^{-(n-1)(\hat{k}-1)+\frac{3-\beta}{2}-\epsilon}(\varphi_1+\varphi_2)^{\frac{1}{2}} - J s_0^{1+\frac{2}{n}-\beta} \quad \text{for all } s_0 \in (0, s^*),
\end{aligned} \tag{4.41}$$

where the choice of s^* guarantees that $J s_0^{\frac{n}{2}} \leq J s^{*\frac{n}{2}} \leq \frac{C_3 n}{8}$. And relying on the young inequality $ab \leq \frac{nC_3}{8}a^2 + \frac{2}{nC_3}b^2$, we infer that

$$\begin{aligned}
\Phi'(t) &\geq \frac{7C_3 n}{8}\varphi - \sqrt{2}C_4 s_0^{\frac{3-\beta}{2}-\frac{2}{n}}\varphi^{\frac{1}{2}} - \sqrt{2}C_4 s_0^{\frac{3-\beta}{2}}\varphi^{\frac{1}{2}} \\
&\quad - \sqrt{2}C_4 s_0^{-(n-1)(\hat{k}-1)+\frac{3-\beta}{2}-\epsilon}\varphi^{\frac{1}{2}} - J s_0^{1+\frac{2}{n}-\beta} \\
&\geq \frac{7C_3 n}{8}\varphi - \left(\frac{nC_3}{8}\varphi + \frac{4}{nC_3}C_4^2 s_0^{3-\beta-\frac{4}{n}}\right) - \left(\frac{nC_3}{8}\varphi + \frac{4}{nC_3}C_4^2 s_0^{3-\beta}\right) \\
&\quad - \left(\frac{nC_3}{8}\varphi + \frac{4}{nC_3}C_4^2 s_0^{-2(n-1)(\hat{k}-1)+3-\beta-2\epsilon}\right) - J s_0^{1+\frac{2}{n}-\beta} \\
&= \frac{C_3 n}{2}\varphi - \frac{4}{nC_3}C_4^2\left(s_0^{3-\beta-\frac{4}{n}} + s_0^{3-\beta} + s_0^{-2(n-1)(\hat{k}-1)+3-\beta-2\epsilon}\right) - J s_0^{1+\frac{2}{n}-\beta} \\
&= \frac{C_3 n}{2}\varphi - \left[\frac{4}{nC_3}C_4^2\left(s_0^{2-\frac{6}{n}} + s_0^{2-\frac{2}{n}} + s_0^{-2(n-1)(\hat{k}-1)+2-\frac{2}{n}-2\epsilon}\right) + J\right] s_0^{1+\frac{2}{n}-\beta},
\end{aligned} \tag{4.42}$$

then using the fact that $2\epsilon \leq 1 - \frac{2}{n}$, $n \geq 3$, $\beta \leq \min\{1, 2 - \frac{4}{n}\}$ and $(n-1)(\hat{k}-1) \leq \frac{\beta}{2} \leq \frac{1}{2}$, it turns out that

$$\begin{aligned}
&\frac{4}{nC_3}C_4^2\left(s_0^{2-\frac{6}{n}} + s_0^{2-\frac{2}{n}} + s_0^{-2(n-1)(\hat{k}-1)+2-\frac{2}{n}-2\epsilon}\right) + J \\
&\leq \frac{4}{nC_3}C_4^2\left(R^{(2-\frac{6}{n})n} + R^{(2-\frac{2}{n})n} + R^{[-2(n-1)(\hat{k}-1)+2-\frac{2}{n}-2\epsilon]n}\right) + J.
\end{aligned} \tag{4.43}$$

In addition, going back to Lemma 4.1 we obtain

$$\varphi(t) \geq C_2 s_0^{\beta-3} \cdot \Phi(t)^2 \ \text{ for all } t \in (0, \hat{T}_{max}). \tag{4.44}$$

Thus, (4.42)-(4.44) establish (4.37). □

Proof of Theorem 1.1. Choosing

$$\overline{r}_1 = \overline{r}_2 := \left(\frac{s_0}{4}\right)^{\frac{1}{n}} \in (0, R). \tag{4.45}$$

And for any $m_1^*, m_2^* \in (0, m_1) \cap (0, m_2)$, we select $s_0 = s_0(R, \lambda_i, \mu_i, k_i, L_i, m_i, m_1^*, m_2^*) \in (0, s^*)$ sufficiently small such that

$$s_0^{\frac{2}{n}} \leq \frac{2^{2\beta-7}\mathfrak{C}_1(m_1^* + m_2^*)^2}{\mathfrak{C}_2 \varrho_n^2} \tag{4.46}$$

and

$$s_0 \le \frac{\mathfrak{C}_1(m_1^* + m_2^*)}{2^{7-\beta}\varrho_n}, \tag{4.47}$$

where $s^*, \mathfrak{C}_1, \mathfrak{C}_2$ are defined in Lemma 4.5. Fix any u_{i0} complying with (1.5) as well as with (1.7) and (1.8), let (u_1, u_2, v) be a corresponding classical solution of (1.4) in $\Omega \times (0, T_{max})$ satisfies the assumptions in Lemma 2.1.

To obtain the blow-up result, it suffices to show that $T_{max} \le \frac{1}{2}$. Assume, on the contrary, $T_{max} > \frac{1}{2}$. It follows from (4.45) and the second restriction in (1.7) that the correspondingly transformed variables in (2.3) and (2.4) satisfy

$$A_i(s,0) \ge A_i(\frac{s_0}{4}, 0) \ge \frac{m_i^*}{\varrho_n} \ \text{ for all } \ s \in (\frac{s_0}{4}, R^n).$$

From the above equation, for the function $\Phi(t)$ introduced in (4.1), we have

$$\begin{aligned}\Phi(0) = \Phi_1(0) + \Phi_2(0) &= \int_0^{s_0} s^{-\beta}(s_0 - s)(A_1(s,0) + A_2(s,0))ds \\ &\ge \int_{\frac{s_0}{4}}^{\frac{s_0}{2}} (\frac{s_0}{2})^{-\beta} \cdot \frac{s_0}{2} \cdot \frac{(m_1^* + m_2^*)}{\varrho_n} ds \\ &= \frac{2^{\beta-3}(m_1^* + m_2^*)}{\varrho_n} s_0^{2-\beta},\end{aligned} \tag{4.48}$$

where $\beta \in (1 - \frac{2}{n}, 1) \cap (0, 2 - \frac{4}{n})$. On account of Lemma 4.5, it is deduced that

$$\Phi'(t) \ge \mathfrak{C}_1 s_0^{\beta-3}\Phi^2(t) - \mathfrak{C}_2 s_0^{\frac{2}{n}+1-\beta} \ \text{ for all } \ t \in (0, \hat{T}_{max}). \tag{4.49}$$

Now we claim that

$$\mathfrak{C}_1 s_0^{\beta-3}\Phi^2(t) - \mathfrak{C}_2 s_0^{\frac{2}{n}+1-\beta} \ge \frac{\mathfrak{C}_1}{2} s_0^{\beta-3}\Phi^2(t), \tag{4.50}$$

upon the ODE comparison argument to (4.49), to get (4.50) we only need prove

$$\mathfrak{C}_1 s_0^{\beta-3}\Phi^2(0) - \mathfrak{C}_2 s_0^{\frac{2}{n}+1-\beta} \ge \frac{\mathfrak{C}_1}{2} s_0^{\beta-3}\Phi^2(0),$$

namely, we need prove

$$\frac{\mathfrak{C}_1 s_0^{\beta-3}\Phi^2(0)}{\mathfrak{C}_2 s_0^{\frac{2}{n}+1-\beta}} \ge 2, \tag{4.51}$$

whence according to (4.48), we find

$$\frac{\mathfrak{C}_1 s_0^{\beta-3}\Phi^2(0)}{\mathfrak{C}_2 s_0^{\frac{2}{n}+1-\beta}} \ge \frac{2^{2\beta-6}\mathfrak{C}_1(m_1^* + m_2^*)^2}{\mathfrak{C}_2\varrho_n^2} s_0^{-\frac{2}{n}},$$

therefore, the above inequality and (4.46) entail (4.51), hence the claim is proved, then we obtain

$$\Phi'(t) \ge \frac{\mathfrak{C}_1}{2} s_0^{\beta-3}\Phi^2(t) \ \text{ for all } \ t \in (0, \hat{T}_{max}), \tag{4.52}$$

and integrating the above equation over $(0,t)$ we have

$$\frac{\mathfrak{C}_1}{2}s_0^{\beta-3}t \leq -\frac{1}{\Phi} + \frac{1}{\Phi(0)} \leq \frac{1}{\Phi(0)} \text{ for all } t \in (0, \hat{T}_{max}),$$

once more by means of (4.47) and (4.48), it turns out that

$$t \leq \frac{2}{\mathfrak{C}_1}s_0 \cdot \frac{\varrho_n}{2^{\beta-3}(m_1^* + m_2^*)} \leq \frac{1}{8} \text{ for all } t \in (0, \hat{T}_{max}),$$

this thus contradicts the assumption $T_{max} > \frac{1}{2}$. In consequence, from Lemma 2.1 and the finity of T_{max}, we obtain the blow-up result directly. □

5. Global bounded solution for $k_i \geq 2(i = 1, 2)$

In this section, we are devoted to showing the boundedness of solutions in Theorem 1.2, to do this, we first show the L^P estimates for $u_i(i = 1, 2)$ when $k_1, k_2 > 2$ as follows.

Lemma 5.1. *Let $\Omega \subset \mathbb{R}^n(n \geq 1)$ be a smooth bounded domain, $k_i > 2(i = 1, 2)$, then for any $u_{10}, u_{20} \in C^0(\overline{\Omega}), P > 1$, there exists $\mathcal{C} = \mathcal{C}(P, \|u_{10}(t)\|_{L^P}, \|u_{20}(t)\|_{L^P})$ satisfying*

$$\|u_i(t)\|_{L^P(\Omega)} \leq \mathcal{C},\ i = 1, 2, \quad \text{for all } t \in (0, T_{max}). \tag{5.1}$$

Proof. Multiplying the first equation in (1.4) by u_1^{P-1} and integrating over Ω, we have

$$\begin{aligned}\frac{1}{P}\frac{d}{dt}\int_\Omega u_1^P dx + \tau_1(P-1)\int_\Omega u_1^{P-2}|\nabla u_1|^2 dx &= \chi_1(P-1)\int_\Omega u_1^{P-1}\nabla u_1 \cdot \nabla v dx \\ &+ \lambda_1 \int_\Omega u_1^P dx - \mu_1 \int_\Omega u_1^{P+k_1-1} dx,\end{aligned} \tag{5.2}$$

and we multiply the third equation in (1.4) by u_1^P to obtain

$$P\int_\Omega u_1^{P-1}\nabla u_1 \cdot \nabla v dx = -\gamma \int_\Omega u_1^P v + \alpha_1 \int_\Omega u_1^{P+1} dx + \alpha_2 \int_\Omega u_1^P u_2 dx. \tag{5.3}$$

Inserting the above equation into (5.2), it follows from the Young inequality that

$$\begin{aligned}&\frac{1}{P}\frac{d}{dt}\int_\Omega u_1^P dx + \tau_1(P-1)\int_\Omega u_1^{P-2} \cdot |\nabla u_1|^2 dx \\ &= \frac{\chi_1(P-1)}{P}\left(-\gamma\int_\Omega u_1^P v dx + \alpha_1 \int_\Omega u_1^{P+1} dx + \alpha_2 \int_\Omega u_1^P u_2 dx\right) + \lambda_1 \int_\Omega u_1^P dx - \mu_1 \int_\Omega u_1^{P+k_1-1} dx \\ &\leq \frac{\chi_1(\alpha_1+\alpha_2)(P-1)}{P}\int_\Omega u_1^{P+1} dx + \frac{\chi_1\alpha_2(P-1)}{P}\int_\Omega u_2^{P+1} dx + \lambda_1 \int_\Omega u_1^P dx - \mu_1 \int_\Omega u_1^{P+k_1-1} dx.\end{aligned} \tag{5.4}$$

Similarly, for u_2, we find

$$\begin{aligned}&\frac{1}{P}\frac{d}{dt}\int_\Omega u_2^P dx+\tau_2(P-1)\int_\Omega u_2^{P-2}\cdot|\nabla u_2|^2dx\\&\le\frac{\chi_2(\alpha_1+\alpha_2)(P-1)}{P}\int_\Omega u_2^{P+1}dx+\frac{\chi_2\alpha_1(P-1)}{P}\int_\Omega u_1^{P+1}dx+\lambda_2\int_\Omega u_2^P dx-\mu_2\int_\Omega u_2^{P+k_2-1}dx.\end{aligned}\tag{5.5}$$

Then by a combination of (5.4) and (5.5), it transpires that

$$\begin{aligned}&\frac{1}{P}\frac{d}{dt}(\int_\Omega u_1^P dx+\int_\Omega u_2^P dx)+\tau_1(P-1)\int_\Omega u_1^{P-2}\cdot|\nabla u_1|^2dx+\tau_2(P-1)\int_\Omega u_2^{P-2}\cdot|\nabla u_2|^2dx\\&\le\frac{[\chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1](P-1)}{P}\int_\Omega u_1^{P+1}dx+\frac{[\chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2](P-1)}{P}\int_\Omega u_2^{P+1}dx\\&+\lambda_1\int_\Omega u_1^P dx-\mu_1\int_\Omega u_1^{P+k_1-1}dx+\lambda_2\int_\Omega u_2^P dx-\mu_2\int_\Omega u_2^{P+k_2-1}dx.\end{aligned}\tag{5.6}$$

Due to the fact that $k_1, k_2 > 2$, we obtain $P + k_i - 1 > P + 1 > P$, thus the Young inequality yields

$$\frac{1}{P}\frac{d}{dt}(\int_\Omega u_1^P dx+\int_\Omega u_2^P dx)\le-\mathcal{C}_1\int_\Omega u_1^P dx-\mathcal{C}_1\int_\Omega u_2^P dx+\mathcal{C}_2,\tag{5.7}$$

with $\mathcal{C}_1, \mathcal{C}_2 > 0$, which implies (5.1). □

Next, our purpose is to prove the L^P estimates for $u_i (i = 1, 2)$ when $k_1 = k_2 = 2$. To achieve this, a crucial step is to derive the L^{P_0} bound for u_i under the assumption that P_0 is larger than 1 but less than some constant.

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^n (n \ge 1)$ be a smooth bounded domain and $k_1 = k_2 = 2$. Assume that*

$$P_0\in\left(1,\frac{\chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1}{[\chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1-\mu_1]_+}\right)\cap\left(1,\frac{\chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2}{[\chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2-\mu_2]_+}\right),$$

then for any $u_{10}, u_{20} \in C^0(\overline{\Omega})$, there exists $\mathcal{C}_0 = \mathcal{C}_0(P_0, \|u_{10}(t)\|_{L^{P_0}}, \|u_{20}(t)\|_{L^{P_0}})$ satisfying

$$\|u_i(t)\|_{L^{P_0}(\Omega)}\le\mathcal{C}_0,\ i=1,2\quad\text{for all } t\in(0,T_{max}).\tag{5.8}$$

Proof. Following the same step in Lemma 5.1, testing the first two equations in (1.4) by $u_1^{P_0-1}, u_2^{P_0-1}$, and testing the third equation in (1.4) by $u_i^{P_0} (i = 1, 2)$ respectively, one obtains

$$\begin{aligned}&\frac{1}{P_0}\frac{d}{dt}\left(\int_\Omega u_1^{P_0}dx+\int_\Omega u_2^{P_0}dx\right)+\tau_1(P_0-1)\int_\Omega u_1^{P_0-2}\cdot|\nabla u_1|^2dx+\tau_2(P_0-1)\int_\Omega u_2^{P_0-2}\cdot|\nabla u_2|^2dx\\&\le-\left(\mu_1-\frac{[\chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1](P_0-1)}{P_0}\right)\int_\Omega u_1^{P_0+1}dx+\lambda_1\int_\Omega u_1^{P_0}dx\\&-\left(\mu_2-\frac{[\chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2](P_0-1)}{P_0}\right)\int_\Omega u_2^{P_0+1}dx+\lambda_2\int_\Omega u_2^{P_0}dx.\end{aligned}\tag{5.9}$$

On account of the fact that

$$P_0 \in \left(1, \frac{\chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1}{[\chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1-\mu_1]_+}\right) \cap \left(1, \frac{\chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2}{[\chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2-\mu_2]_+}\right),$$

it is thereby inferred that $\mu_1 - \frac{[\chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1](P_0-1)}{P_0}$ and $\mu_2 - \frac{[\chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2](P_0-1)}{P_0}$ are positive, and in view of the Young inequality, a simple computation shows that

$$\frac{1}{P_0}\frac{d}{dt}(\int_\Omega u_1^{P_0}dx + \int_\Omega u_2^{P_0}dx) \le \mathcal{C}_3(\int_\Omega u_1^{P_0}dx + \int_\Omega u_2^{P_0}dx) + \mathcal{C}_4, \tag{5.10}$$

thus we obtain (5.8). □

Relying on Lemma 5.2, now we give the L^P estimates for $u_i (i = 1, 2)$.

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and $k_1 = k_2 = 2$. Suppose that*

$$\mu_1 > \frac{n-2}{n}[\chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1] \tag{5.11}$$

and

$$\mu_2 > \frac{n-2}{n}[\chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2], \tag{5.12}$$

then for any $P > 1, u_{10}, u_{20} \in C^0(\overline{\Omega})$, one can find $\mathcal{C}' = \mathcal{C}'(P, \|u_1(t)\|_{L^P}, \|u_2(t)\|_{L^P})$ such that

$$\|u_i(t)\|_{L^P(\Omega)} \le \mathcal{C}',\ i = 1, 2 \quad \textit{for all } t \in (0, T_{max}). \tag{5.13}$$

Proof. Recalling (5.9) in Lemma 5.2

$$\begin{aligned} &\frac{1}{P}\frac{d}{dt}(\int_\Omega u_1^P dx + \int_\Omega u_2^P dx) + \tau_1(P-1)\int_\Omega u_1^{P-2}\cdot|\nabla u_1|^2 dx + \tau_2(P-1)\int_\Omega u_2^{P-2}\cdot|\nabla u_2|^2 dx \\ \le &\left(\frac{[\chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1](P-1)}{P} - \mu_1\right)\int_\Omega u_1^{P+1}dx + \lambda_1\int_\Omega u_1^P dx \\ &+ \left(\frac{[\chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2](P-1)}{P} - \mu_2\right)\int_\Omega u_2^{P+1}dx + \lambda_2\int_\Omega u_2^P dx. \end{aligned} \tag{5.14}$$

1. For the case $\mu_1 \ge \chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1, \mu_2 \ge \chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2$. In light of the same proof as in Lemma 5.2, we can obtain (5.13).
2. For the case $\mu_1 < \chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1, \mu_2 < \chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2$. Let P_0 be as in Lemma 5.2, it follows from (5.11) and (5.12) that

$$\frac{n}{2} < \frac{\chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1}{\chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1-\mu_1}, \tag{5.15}$$

$$\frac{n}{2} < \frac{\chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2}{\chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2-\mu_2}, \tag{5.16}$$

thus we can choose P_0 satisfying $P > P_0 > \frac{n}{2}$. Denoting $w_i = u_i^{\frac{P}{2}}$, by substituting u_i by w_i in (5.14) we derive

$$\begin{aligned}&\frac{1}{P}\frac{d}{dt}(\int_\Omega w_1^2dx+\int_\Omega w_2^2dx)+\frac{4\tau_1(P-1)}{P^2}\int_\Omega|\nabla w_1|^2dx+\frac{4\tau_2(P-1)}{P^2}\int_\Omega|\nabla w_2|^2dx\\&\le \mathcal{C}_5\int_\Omega w_1^{\frac{2(P+1)}{P}}dx+\lambda_1\int_\Omega w_1^2dx+\mathcal{C}_5\int_\Omega w_2^{\frac{2(P+1)}{P}}dx+\lambda_2\int_\Omega w_2^2dx,\end{aligned}\tag{5.17}$$

where $\mathcal{C}_5=\max\left\{1,\frac{[\chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1](P-1)}{P}-\mu_1,\frac{[\chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2](P-1)}{P}-\mu_2\right\}$. In view of Lemma 5.2, there exists positive $\mathcal{C}_6=\mathcal{C}_6(P_0,\|u_{10}(t)\|_{L^{P_0}},\|u_{20}(t)\|_{L^{P_0}})$ such that $\|u_i(t)\|_{L^{P_0}(\Omega)}\le\mathcal{C}_6$, $i=1,2,$ for all $t\in(0,T_{max})$, then it is deduced that

$$\|w_i\|_{L^{\frac{2P_0}{P}}(\Omega)}=\left(\int_\Omega u_i^{P_0}\right)^{\frac{P}{2P_0}}\le\mathcal{C}_6^{\frac{p}{2}},\tag{5.18}$$

and applying the Gagliardo-Nirenberg, there appears the relation

$$\|w_i\|_{L^p(\Omega)}\le\overline{C}\|w_i\|_{L^q}^{1-a}\cdot\|w_i\|^a_{W^{1,2}(\Omega)},\tag{5.19}$$

with $p=\frac{2P+2}{P},q=\frac{2P_0}{P},a=\frac{\frac{1}{q}-\frac{1}{p}}{\frac{1}{q}-\frac{1}{2}+\frac{1}{n}}=\frac{\frac{nP}{2P_0}-\frac{nP}{2P+2}}{1-\frac{n}{2}+\frac{nP}{2P_0}}$. Since $P>P_0>\frac{n}{2}>\frac{n-2}{2}$, we obtain $a\in(0,1)$. Hence (5.18), (5.19) and the Poincaré inequality yield the estimate

$$\begin{aligned}\int_\Omega u_i^{P+1}dx=\int_\Omega w_i^{\frac{2(P+1)}{P}}dx=\|w_i\|^{\frac{2P+2}{P}}_{L^{\frac{2P+2}{P}}}&\le\overline{C}\|w_i\|^{(1-a)\frac{2P+2}{P}}_{L^{\frac{2P_0}{P}}}\cdot\|w_i\|^{a\cdot\frac{2P+2}{P}}_{W^{1,2}(\Omega)}\\&\le\mathcal{C}_7(\|u_{10}\|_{L^{P_0}},\|u_{20}\|_{L^{P_0}})\left(1+\|\nabla w_i\|^{a\cdot\frac{2P+2}{P}}_{L^2}\right),\end{aligned}\tag{5.20}$$

due to the fact that $P>P_0>\frac{n}{2}$, which entails

$$a\cdot\frac{2P+2}{P}-2=\frac{2P+2}{P}\cdot\frac{\frac{nP}{2P_0}-\frac{nP}{2P+2}}{1-\frac{n}{2}+\frac{nP}{2P_0}}-2=\frac{\frac{n}{P_0}-2}{1-\frac{n}{2}+\frac{nP}{2P_0}}<0,\tag{5.21}$$

then it follows from (5.20) and (5.21) that

$$\int_\Omega u_i^{P+1}dx=\int_\Omega w_i^{\frac{2(P+1)}{P}}dx\le\epsilon\int_\Omega|\nabla w_i|^2dx+C(\epsilon,P,\|u_{10}\|_{L^{P_0}},\|u_{20}\|_{L^{P_0}}),\tag{5.22}$$

where $\epsilon>0$ is sufficiently small and satisfies

$$\mathcal{C}_5\epsilon<\min\left\{\frac{4\tau_1(P-1)}{P^2},\frac{4\tau_2(P-1)}{P^2}\right\}.$$

Thus by (5.17), we see that

$$\frac{d}{dt}\left(\int_\Omega w_1^2dx+\int_\Omega w_2^2dx\right)+\mathcal{C}_8\int_\Omega|\nabla w_1|^2dx+\mathcal{C}_8\int_\Omega|\nabla w_2|^2dx\le\lambda_1\int_\Omega w_1^2dx+\lambda_2\int_\Omega w_2^2dx,\tag{5.23}$$

with $\mathcal{C}_8>0$. Again by (5.22) and the Young inequality, we find $\mathcal{C}_9,\mathcal{C}_{10},\mathcal{C}_{11},\mathcal{C}_{12}>0$ such that

$$\frac{d}{dt}\left(\int_\Omega w_1^2 dx + \int_\Omega w_2^2 dx\right) \le -\mathcal{C}_9 \int_\Omega w_1^{\frac{2(P+1)}{P}} dx - \mathcal{C}_9 \int_\Omega w_2^{\frac{2(P+1)}{P}} dx + \lambda_1 \int_\Omega w_1^2 dx + \lambda_2 \int_\Omega w_2^2 dx + \mathcal{C}_{10}$$
$$\le -\mathcal{C}_{11} \int_\Omega w_1^2 dx - \mathcal{C}_{11} \int_\Omega w_2^2 dx + \mathcal{C}_{12},$$

that is

$$\frac{d}{dt}\left(\int_\Omega u_1^P dx + \int_\Omega u_2^P dx\right) \le -\mathcal{C}_{11} \int_\Omega u_1^P dx - \mathcal{C}_{11} \int_\Omega u_2^P dx + \mathcal{C}_{12},$$

it then turns out that (5.13) holds.

3. For the case $\mu_1 \ge \chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1, \mu_2 < \chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2$ or the case $\mu_1 < \chi_1(\alpha_1+\alpha_2)+\chi_2\alpha_1, \mu_2 \ge \chi_2(\alpha_1+\alpha_2)+\chi_1\alpha_2$. On account of a similar proof as above, we can obtain (5.13), which ends the proof. □

Proof of Theorem 1.2. The boundedness of u_1 and u_2 in $\Omega \times (0, T_{max})$ can be immediately obtained by the consequence of Lemma 5.1, Lemma 5.3 and the well known Moser-type iterations [20,22], therefore in light of Lemma 2.1, we have $T_{max} = \infty$. □

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