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Multiple solutions for the coercive semilinear elliptic equations [☆]

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ABSTRACT

In this paper we study the semilinear elliptic equations

$$\begin{cases} -\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. By using the minimax methods, bifurcation methods, Conley index theory and Morse theory, we obtain six nontrivial solutions for the equations with coercive nonlinearities.

1. Introduction

The present paper deals with the existence and multiplicity of nontrivial weak solutions to semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = f(x,u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots$ be the sequence of distinct eigenvalues of $-\Delta$ with zero Dirichlet boundary condition in Ω. We make the following assumptions on f:

- (f_1) $f \in C^1(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, $f(x,0) = 0$ for all $x \in \overline{\Omega}$.
- (f_2) $f'_t(x,0) = \lambda \in \mathbb{R}$ for all $x \in \overline{\Omega}$.
- (f_3) There are $C > 0$ and $2 \leqslant p < 2^*$ such that $|f'_t(x,t)| \leqslant C(1 + |t|^{p-2})$ for all $x \in \overline{\Omega}$ and $t \in \mathbb{R}$, where $2^* = \frac{2N}{N-2}$ for $N \geqslant 3$ and $2^* = \infty$ for $N = 1, 2$.

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(f_4) There exist $\gamma < \lambda_1$ and $C_1 > 0$ such that

$$F(x,t) := \int_0^t f(x,s)ds \leqslant \frac{1}{2}\gamma t^2 + C_1 \quad \text{for all } x \in \overline{\Omega} \text{ and } t \in \mathbb{R}.$$

$(f_5)^{\pm}$ There exists $\xi > 0$ such that $\pm(f(x,t) - \lambda t)t > 0$ for all $0 < |t| \leqslant \xi$, $x \in \overline{\Omega}$.

Clearly, by (f_1) the equation (1.1) has the trivial solution for any $\lambda \in \mathbb{R}$. The existence of nontrivial solutions for the equation (1.1) is closely related to the position of the range of f' with respect to the spectrum of $-\Delta$. In the autonomous case, i.e. $f(x,u) \equiv f(u)$, Castro and Lazer in [9] found two nontrivial solutions for (1.1) if there is at least one eigenvalue λ_j between $f'(0)$ and $f'(\infty) := \lim_{|t|\to\infty} f(t)/t$ and $f'(t) < \lambda_{j+1}$ for all $t \in \mathbb{R}$. This result was extended by Chang in [10] and by Li and Willem in [19] using Morse theory. Also the results in [9] were extended by Castro and Cossio in [5] where at least four nontrivial solutions for (1.1) were obtained in the case when $f'(0) < \lambda_1$ and $f'(\infty) \in (\lambda_k, \lambda_{k+1})$ with $k \geqslant 2$ and $f'(t) \leqslant \alpha < \lambda_{k+1}$. The proofs in [5] were based on Lyapunov-Schmidt reduction arguments, the mountain pass theorem, and characterizations of the local degree of critical points. In an interesting paper [7], Castro, Cossio and Vélez proved the existence of six nontrivial solution in the case that $f'(0) \leqslant 0$, $tf''(t) > 0$ for $t \neq 0$ and $\lim_{|t|\to\infty} f'(t) \in (\lambda_k, \lambda_k + \epsilon)$ for $k \geqslant 3$ and $\epsilon > 0$ small. Hofer in [18] proved the existence of four nontrivial solutions of (1.1) using degree theory in the case that $f'(0) \in (\lambda_k, \lambda_{k+1})$ with $k \geqslant 2$ and $\limsup_{|t|\to\infty} f(t)/t < \lambda_1$. More recently, by the combinations of the cut-off technique, the mountain pass theorem and the degree theory, Castro, Cossio, Herrón and Vélez in [8] proved the existence of four nontrivial classical solutions for (1.1) in the case that $f'(0) \in (\lambda_k, \lambda_{k+1})$ with $k \geqslant 2$ and $f(t)/t \leqslant \gamma < \lambda_1$ for large $|t|$. One of the novelties of [8] was that there was no subcritical growth condition on the nonlinearity f. Motivated by [8],[15] and [22], the purpose of this paper is giving a lower bound on the number of nontrivial solutions under the assumptions (f_1)–(f_4) and $(f_5)^{\pm}$ with λ very close to a higher eigenvalue λ_{k+1} of $-\Delta$ for some $k \geqslant 2$. We note that (f_4) characterizes (1.1) as a coercive elliptic problem so that the associated energy functional is bounded from below. We also note that (f_4) includes the condition near infinity in [18] and the partial condition in [8] as special cases. Indeed the condition $\limsup_{|t|\to\infty} f(t)/t < \lambda_1$ in [18] implies (f_4) and $f(t)/t \leqslant \gamma < \lambda_1$ in [8] implies (f_4) for $t > 0$. The local sign condition similar to $(f_5)^{\pm}$ was introduced by Rabinowitz, Su and Wang in [22] to prove the existence of three nontrivial solutions for superlinear elliptic equations with saddle point structure at zero by using bifurcation theory, homological linking and Morse theory.

In this paper, by combining bifurcation analysis, Morse theory and Conley index theory, we will prove that the equation (1.1) has at least six nontrivial solutions, including two constant-sign solutions and two sign-changing solutions. It is well-known that, in comparison with the degree theory used in [8], critical groups provide both a finer structure and better estimate of the number of solutions. The Conley index we use in this paper can be regarded as an extension of both the Leray-Schauder degree and the critical groups.

The main results of the present paper are the following theorems.

Theorem 1.1. *Assume (f_1)–$(f_5)^+$ hold and let $k \geqslant 2$ be fixed. Then there is $\delta > 0$ such that for $\lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1})$, equation (1.1) has at least six nontrivial solutions, in which one is positive, one is negative and two are sign-changing.*

Theorem 1.2. *Assume (f_1)–$(f_5)^-$ hold and let $k \geqslant 2$ be fixed. Then there is $\delta > 0$ such that for $\lambda \in (\lambda_{k+1}, \lambda_{k+1} + \delta)$, equation (1.1) has at least six nontrivial solutions, in which one is positive, one is negative and two are sign-changing.*

The paper is organized as follows. In Section 2 we get two constant sign solutions and one mountain pass solution. In Section 3, we prove the existence of two bifurcation solutions by giving information on their Morse indices and sign-changing property. In Section 4, we give the proofs of the main theorems with some comments.

2. Variational solutions

In this section, we will apply variational methods to find three nontrivial solutions of (1.1). The proofs of the existence of these solutions follow exactly the same ideas used by Castro, Cossio, Herrón and Vélez in [8] and these results were proven in [8] for a similar problem. For the sake of completeness we sketch out the proofs.

Denote by $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{1/2}$ the norm of $H_0^1(\Omega)$. For $k \geqslant 1$, $H_0^1(\Omega)$ has a orthogonal decomposition as follows,

$$H_0^1(\Omega) = E_k \oplus E_k^\perp, \quad \text{where} \quad E(\lambda_k) = \ker(-\Delta - \lambda_k), \quad E_k = \bigoplus_{i=1}^{k} E(\lambda_i). \tag{2.1}$$

Denote $\nu_k = \dim E(\lambda_k)$ and $\ell_k = \dim E_k$. We have that $\ell_{k+1} = \ell_k + \nu_{k+1}$. The energy functional Φ associated to the equation (1.1) is given by

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx, \quad u \in H_0^1(\Omega). \tag{2.2}$$

By (f_1) and (f_3), Φ is a well-defined C^2 functional on $H_0^1(\Omega)$ with derivatives

$$\langle \Phi'(u), \varphi\rangle = \int_\Omega \nabla u \nabla \varphi dx - \int_\Omega f(x,u)\varphi dx, \tag{2.3}$$

$$\langle \Phi''(u)\varphi, \psi\rangle = \int_\Omega \nabla \varphi \nabla \psi dx - \int_\Omega f_t'(x,u)\varphi\psi dx \tag{2.4}$$

where $u, \varphi, \psi \in H_0^1(\Omega)$. Thus the weak solutions of (1.1) correspond to the critical points of Φ in $H_0^1(\Omega)$, and are contained in

$$\mathcal{K}(\Phi) = \left\{u \in H_0^1(\Omega) : \ \Phi'(u) = 0\right\}.$$

Also denote

$$\Phi^c = \left\{u \in H_0^1(\Omega) : \Phi(u) \leqslant c\right\}, \quad \mathcal{K}_c(\Phi) = \left\{u \in \mathcal{K}(\Phi) : \ \Phi(u) = c\right\}.$$

Associated with the functional Φ, we introduce two truncated functionals $\Phi_\pm : H_0^1(\Omega) \to \mathbb{R}$ as

$$\Phi_\pm(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F_\pm(x,u)dx,$$

where $F_\pm(x,t) := \int_0^t f_\pm(x,s)ds$, and

$$f_\pm(x,t) = \begin{cases} f(x,t), & \pm t \geqslant 0, \\ 0, & \pm t < 0. \end{cases}$$

Under (f_1)–(f_3), $f_\pm$ is locally Lipschitz continuous on t for all $x \in \overline{\Omega}$ and then we have $\Phi_\pm \in C^{2-0}(H_0^1(\Omega), \mathbb{R})$ (see [1,14]). For $a \in \mathbb{R}$, let $a^+ = \max\{a, 0\}$, $a^- = \max\{-a, 0\}$, then for $u \in H_0^1(\Omega)$, $u = u^+ - u^-$ and $u^\pm \in H_0^1(\Omega)$.

First we verify the compactness of the functional Φ and compute the critical groups $C_q(\Phi, \infty)$ of Φ at infinity. This notation was introduced by Bartsch and Li in [2].

Lemma 2.1. *Assume that f satisfies (f_1), (f_3) and (f_4). Then*

(i) the functional Φ is coercive on $H_0^1(\Omega)$;
(ii) the functional Φ satisfies the Palais-Smale condition;
(iii) $C_q(\Phi, \infty) \cong \delta_{q,0}\mathbb{F}$.

Proof. (i) For $u \in H_0^1(\Omega)$, we have by (f_4) that

$$\Phi(u) \geqslant \frac{1}{2}\|u\|^2 - \frac{1}{2}\gamma\|u\|^2_{L^2(\Omega)} - C_1|\Omega|. \tag{2.5}$$

Since $\gamma < \lambda_1$, we have that $\Phi(u) \to \infty$ as $\|u\| \to \infty$. This proves that Φ is coercive.

(ii) Let $\{u_n\} \subset H_0^1(\Omega)$ be a Palais-Smale sequence i.e. $\Phi'(u_n) \to 0$ and $\Phi(u_n) \to c$ for some $c \in \mathbb{R}$ as $n \to \infty$. By the coerciveness of Φ, $\{u_n\}$ is bounded and then by [21, Proposition B.35] it contains a convergent subsequence.

(iii) Since Φ is coercive and is weakly lower semicontinuous on $H_0^1(\Omega)$, Φ attains its global minima at some u_*:

$$\Phi(u_*) = \min_{u \in H_0^1(\Omega)} \Phi(u).$$

Take $b < \Phi(u_*)$. Then

$$C_q(\Phi, \infty) := H^q(H_0^1(\Omega), \Phi^b) \cong H^q(\{u_*\}, \emptyset) \cong \delta_{q,0}\mathbb{F}. \quad \square$$

Next we apply the cut-off techniques and the direct method of the calculus of variations to find nontrivial solutions of (1.1) with constant sign.

Theorem 2.2. *Assume that (f_1)–(f_4) hold with $\lambda > \lambda_1$. Then the equation (1.1) admits two constant sign solutions, one is strictly positive and the other is strictly negative in Ω, which are local minimizers of the energy functional Φ.*

Proof. It is easy to see from Lemma 2.1 that Φ_+ is coercive in $H_0^1(\Omega)$, and there exists $u_+ \in H_0^1(\Omega)$ s.t.

$$\Phi_+(u_+) = \min_{u \in H_0^1(\Omega)} \Phi_+(u). \tag{2.6}$$

In particular, $u_+ \in \mathcal{K}(\Phi_+)$, we claim that $u_+ \neq 0$. By (f_2) and $\lambda > \lambda_1$, we can find $\sigma > 0$ such that $F_+(x, u) > \frac{\lambda_1}{2}u^2$ for a.e. $x \in \Omega$ and all $0 < u \leqslant \sigma$. Let ϕ_1 be a positive eigenfunction associated to λ_1. Then for all $t > 0$ small enough we have $\|t\phi_1\|_\infty \leqslant \sigma$ and thus

$$\Phi_+(t\phi_1) = \frac{1}{2}\|t\phi_1\|^2 - \int_\Omega F_+(x, t\phi_1)dx = \int_\Omega \left(\frac{1}{2}\lambda_1 t^2\phi_1^2 - F_+(x, t\phi_1)\right) dx < 0. \tag{2.7}$$

By (2.6) we have $\Phi_+(u_+) < 0$, hence $u_+ \neq 0$. Take $\varphi = (u_+)^-$ in (2.3) with Φ replaced by Φ_+, we can get $(u_+)^- = 0$ and thus $u_+ = (u_+)^+ \geqslant 0$ a.e. in Ω. This implies that u_+ satisfies

$$\begin{cases} -\Delta u_+ = f(x, u_+) & x \in \Omega, \\ u_+ = 0 & x \in \partial\Omega. \end{cases} \tag{2.8}$$

By standard elliptic arguments we deduce $u_+ \in C_0^1(\bar{\Omega})$. Noting that there exists $0 < \theta < 1$ such that

$$|f(x, u_+) - f(x, 0)| = |f_t'(x, \theta u_+)| u_+ \leqslant \kappa u_+, \quad \text{uniformly in } \ x \in \bar{\Omega},$$

where $\kappa = \sup\limits_{x\in\bar{\Omega},\ 0\leqslant\xi\leqslant\|u_+\|_{C(\bar{\Omega})}} |f_t'(x, \xi)|$, we obtain $f(x, u_+) \geqslant -\kappa u_+$, i.e. it follows from (2.8) that

$$\begin{cases} -\Delta u_+ + \kappa u_+ \geqslant 0, & x \in \Omega, \\ u_+ \geqslant 0, & x \in \Omega. \end{cases}$$

It follows from [4, Theorem 3] and $u_+ \neq 0$ that there is some $\epsilon > 0$ such that

$$u_+(x) \geqslant \epsilon \ \mathrm{dist}(x, \partial\Omega) > 0 \quad \text{in } \Omega. \tag{2.9}$$

Thus if $u \in C_0^1(\bar{\Omega})$ and $\|u - u_+\|_{C^1} \leqslant \epsilon$ then

$$u \geqslant 0 \quad \text{in} \quad \Omega.$$

So, for such a $u \in C_0^1(\bar{\Omega})$, there holds

$$\Phi(u_+) = \Phi_+(u_+) \leqslant \Phi_+(u) = \Phi(u), \tag{2.10}$$

we see that u_+ is a local minimizer of Φ in the C^1 topology. Hence by [4, Theorem 1] it is a local minimizer of Φ in the H_0^1 topology. Therefore $u_+ \in \mathcal{K}(\Phi)$, and u_+ is a positive solution of (1.1).

Similarly, we find another local minimizer $u_- \in H_0^1(\Omega)$ of Φ, which turns out to be a negative solution of (1.1). □

Next we find the third variational solution as a mountain pass point of Φ.

Theorem 2.3. *Assume that (f_1)–(f_4) hold with $\lambda > \lambda_1$. Then (1.1) admits a solution $\tilde{u}$ of mountain pass type differing from $u_\pm$.*

Proof. By Lemma 2.1, Φ is coercive and satisfies (PS) condition. Next we check that the functional Φ has a mountain pass geometry (see [21, Theorem 2.2]).

Without loss of generality, we assume that $\Phi(u_+) \geqslant \Phi(u_-)$ and u_+ is a strict local minimizer of Φ. Clearly, there exists $r \in (0, \|u_+ - u_-\|)$ such that $\Phi(u) > \Phi(u_+)$ for all $u \in B_r(u_+) \setminus \{u_+\} \subset H_0^1(\Omega)$. Moreover, there holds

$$\eta_r := \inf_{u\in\partial B_r(u_+)} \Phi(u) > \Phi(u_+).$$

Otherwise, we could find a sequence $\{u_n\} \subset \partial B_r(u_+)$ satisfying $\Phi(u_n) \to \Phi(u_+)$ and $\Phi'(u_n) \to 0$ as $n \to \infty$ (see [20, Corollary 5.12]). Since Φ verifies the (PS) condition, there exists $\bar{u} \in \partial B_r(u_+)$ such that $u_n \to \bar{u}$ in $H_0^1(\Omega)$, a contradiction. Thus Φ possesses the mountain pass theorem geometry.

Now set $\Gamma = \{\sigma \in C([0, 1], H_0^1(\Omega)) : \ \sigma(0) = u_+,\ \sigma(1) = u_-\}$, and define

$$c := \inf_{\sigma\in\Gamma} \max_{t\in[0,1]} \Phi(\sigma(t)).$$

By standard minimax arguments, we have $c \geqslant \eta_r$ and there exists $\tilde{u} \in \mathcal{K}_c(\Phi)$. By $\Phi(\tilde{u}) = c \geqslant \eta_r > \Phi(u_+) \geqslant \Phi(u_-)$ we have that $\tilde{u} \neq u_\pm$. $\square$

We note that the conclusion of Theorem 2.2 is valid in the case that $f'_t(x,0) \equiv \lambda_1$ and $2F(x,t) \geqslant \lambda_1 t^2$ for small $|t|$ and all $x \in \bar{\Omega}$. We also note that in the situation of $\lambda_1 < \lambda < \lambda_2$ the solution $\tilde{u}$ obtained in Theorem 2.3 may not be different from 0.

3. Bifurcation solutions

In this section, we prove the existence of bifurcation solutions of (1.1) and then give the information on Morse indices and the sign-changing property (see [6]) for the bifurcation solutions. We have the following results.

Theorem 3.1. *Let $k \geqslant 1$ be fixed. Assume that f satisfies (f_1)–(f_3) and $(f_5)^+$ (or $(f_5)^-$, respectively). Then there is $\delta > 0$ such that for every $\lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1})$ (or $\lambda \in (\lambda_{k+1}, \lambda_{k+1} + \delta)$, respectively), the equation (1.1) has at least two nontrivial solutions $u^i_\lambda (i = 1, 2)$. Furthermore, there hold the following conclusions.*

(i) $u^i_\lambda \to 0$ *in* $C^1_0(\overline{\Omega})$ *as* $\lambda \to \lambda_{k+1}$, $(i = 1, 2)$.

(ii) The Morse index $m(u^i_\lambda)$ and the nullity $n(u^i_\lambda)$ of Φ at u^i_λ $(i = 1, 2)$ satisfy

$$m(u^i_\lambda) \geqslant \ell_k, \quad n(u^i_\lambda) \leqslant \nu_{k+1}, \quad \text{for all } \ 0 < |\lambda - \lambda_{k+1}| < \delta. \tag{3.1}$$

(iii) The bifurcation solutions u^i_λ $(i = 1, 2)$ are sign-changing.

Proof. There is $\delta_1 > 0$ small such that for any λ satisfying $0 < |\lambda - \lambda_{k+1}| < \delta_1$, the existence of two solutions u^i_λ $(i = 1, 2)$ of the equation (1.1) has been proved in Rabinowitz, Su and Wang [22, Proposition 2.3] by applying the well-known Rabinowitz's bifurcation theorem (see [21, Theorem 11.35]). In fact, the conditions $(f_5)^\pm$ ensure the validity of the second case of [21, Theorem 11.35]).

Since $u^i_\lambda \in H^1_0(\Omega)$ $(i = 1, 2)$ are bifurcation solutions near the bifurcation point $(\lambda_{k+1}, 0) \in \mathbb{R} \times H^1_0(\Omega)$ of the equation (1.1), it is known that $u^i_\lambda \to 0 (i = 1, 2)$ in $H^1_0(\Omega)$ as $\lambda \to \lambda_{k+1}$. By standard elliptic regularity arguments we have the conclusion (i).

By (f_1), (f_2) and (i), for given constants λ_* and λ^* satisfying $\lambda_k < \lambda_* < \lambda_{k+1} < \lambda^* < \lambda_{k+2}$, there exist $0 < \tau \leqslant \xi$ and a corresponding $\delta_2 > 0$, such that for all $\lambda \in (\lambda_{k+1} - \delta_2, \lambda_{k+1} + \delta_2)$,

$$\|u^i_\lambda\|_{C^1_0(\overline{\Omega})} \leqslant \tau \leqslant \xi, \tag{3.2}$$

$$\lambda_k < \lambda_* \leqslant f'_t(x, u^i_\lambda(x)) \leqslant \lambda^* < \lambda_{k+2}, \quad \text{for all } x \in \overline{\Omega}. \tag{3.3}$$

By (f_3) we have (2.4). Then for $v \in E_k \setminus \{0\}$, we have

$$\langle \Phi''(u^i_\lambda)v, v\rangle = \int_\Omega |\nabla v|^2 dx - \int_\Omega f'_t(x, u^i_\lambda) v^2 dx \leqslant \left(1 - \frac{\lambda_*}{\lambda_k}\right) \|v\|^2 < 0. \tag{3.4}$$

For $w \in E^\perp_{k+1} \setminus \{0\}$, we have

$$\langle \Phi''(u^i_\lambda)w, w\rangle = \int_\Omega |\nabla w|^2 dx - \int_\Omega f'_t(x, u^i_\lambda) w^2 dx \geqslant \left(1 - \frac{\lambda^*}{\lambda_{k+2}}\right) \|w\|^2 > 0. \tag{3.5}$$

Take $\delta = \min\{\delta_1, \delta_2\}$. The conclusion (ii) follows from (3.4) and (3.5).

Now we follow a similar argument as in [21, Remark 5.19] to prove conclusion (iii). Since $u_\lambda^i (i = 1, 2)$ satisfies (1.1), multiplying both sides of (1.1) by ϕ_1 and integrating over Ω, there holds

$$\int_\Omega f(x, u_\lambda^i)\phi_1 dx = \int_\Omega (-\Delta u_\lambda^i)\phi_1 dx = \int_\Omega (-\Delta\phi_1)u_\lambda^i dx = \lambda_1 \int_\Omega \phi_1 u_\lambda^i dx.$$

Consequently

$$\int_\Omega (f(x, u_\lambda^i) - \lambda u_\lambda^i)\phi_1 dx = (\lambda_1 - \lambda) \int_\Omega \phi_1 u_\lambda^i dx. \tag{3.6}$$

On the one hand, if u_λ^i is positive (negative, *resp.*) in Ω, by $\phi_1 > 0$ in Ω, the right-hand side of (3.6) is negative (positive, *resp.*). On the other hand, it follows from (3.2) and $(f_5)^+$ that the left-hand side of (3.6) is positive (negative, *resp.*), a contradiction. In the case of $(f_5)^-$ the argument is similar. The proof is complete. □

Remark 3.2. We conclude this section by some remarks. *(1)* The sign-changing property of bifurcation solutions of (1.1) in this paper is new. *(2)* The conclusion (ii) of Theorem 3.1 indicates an essential property in terms of Morse indices of bifurcation solutions that emanate from trivial solution branch. *(3)* For any solution u of (1.1) with $\|u\|_{C^1}$ small enough, it holds that the Morse index $m(u)$ and the nullity $n(u)$ of Φ at u satisfying $m(u) \geqslant \ell_k$ and $n(u) \leqslant \nu_{k+1}$. Furthermore, the information on Morse indices of bifurcation solutions will provide a way to distinguish them from the solutions obtained by variational methods.

4. Proof of the main results

In this section, we will complete the proof of Theorems 1.1 and 1.2. Let us keep in mind all the assumptions on f in the main theorems. Then up to now we have six solutions of (1.1), in which, $u_\pm$ and $\tilde{u}$ from Theorems 2.2 and 2.3 are variational solutions; $u_\lambda^i (i = 1, 2)$ from Theorem 3.1 are bifurcation solutions and one is the trivial solution 0. What we need to do is that to distinguish these solutions. It is well-known that critical group is a very useful tool in distinguishing the known critical points of a differential functional obtained by other ways.

Recall that the q-th critical group of Φ at its isolated critical point u is defined as

$$C_q(\Phi, u) := H^q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}).$$

Here $c = \Phi(u)$ and $H^q(A, B)$ is the q-th relative singular cohomology group of the topological pair (A, B) with coefficients in a field $\mathbb{F}$ (see [11]).

We first compute the critical groups of the functional Φ defined by (2.2) at 0. We have

Lemma 4.1. *Assume that f satisfies (f_1)–(f_3). Then*

(i) For $\lambda \in (\lambda_k, \lambda_{k+1})$, it holds that $C_q(\Phi, 0) \cong \delta_{q,\ell_k}\mathbb{F}$.
(ii) For $\lambda \in (\lambda_{k+1}, \lambda_{k+2})$, it holds that $C_q(\Phi, 0) \cong \delta_{q,\ell_{k+1}}\mathbb{F}$.
(iii) If $\lambda = \lambda_{k+1}$ and $(f_5)^\pm$ holds, then 0 is an isolated solution of (1.1). Furthermore,

$$\text{(a) } C_q(\Phi, 0) \cong \delta_{q,\ell_{k+1}}\mathbb{F} \quad \textit{if } (f_5)^+ \textit{ holds}; \quad \text{(b) } C_q(\Phi, 0) \cong \delta_{q,\ell_k}\mathbb{F} \quad \textit{if } (f_5)^- \textit{ holds}.$$

Proof. In the cases (i) or (ii), 0 is a nondegenerate critical point of Φ with Morse index ℓ_k or ℓ_{k+1}, the conclusions then follow from [11, Theorem 4.1,Chapter II].

In the case (iii), $\lambda = \lambda_{k+1}$, then 0 is a degenerate critical point of Φ with Morse index ℓ_k and nullity ν_{k+1}. We need to prove that 0 is an isolated solution of (1.1) under $(f_5)^{\pm}$. Let $(f_5)^+$ hold. Assume that 0 is not an isolated solution of (1.1). Then there exists a sequence $\{u_n\} \subset H_0^1(\Omega) \setminus \{0\}$ such that $u_n \to 0$ in $H_0^1(\Omega)$ as $n \to \infty$, and for each $n \in \mathbb{N}$, u_n satisfies

$$\begin{cases} -\Delta u_n = f(x, u_n) & x \in \Omega, \\ u_n = 0 & x \in \partial\Omega. \end{cases} \tag{4.1}$$

By standard elliptic regularity arguments we have $u_n \to 0$ in $C_0^1(\overline{\Omega})$. Thus there holds that $\|u_n\|_{C^1} < \xi$ for all $n > N_1$ for some $N_1 \in \mathbb{N}$. We consider the weighted linear eigenvalue problem

$$\begin{cases} -\Delta v = \mu \eta_n(x) v & x \in \Omega, \\ v = 0 & x \in \partial\Omega, \end{cases} \tag{4.2}$$

where $\eta_n(x) = \frac{f(x,u_n(x))}{u_n(x)}$ for $u_n(x) \neq 0$ and $\eta_n(x) = \lambda_{k+1}$ for $u_n(x) = 0$. For $n > N_1$, it follows from (f_2) and $(f_5)^+$ that $\lambda_{k+2} > \eta_n(x) \geqslant \lambda_{k+1}$ holds in Ω. Let $\sigma(\eta_n) \subset \mathbb{R}^+$ be the set of all eigenvalues of (4.2). Then by (4.1) we have $1 \in \sigma(\eta_n)$ with u_n being an associated eigenfunction. By the unique continuation property for eigenfunctions of the linear elliptic eigenvalue problem (recall that $\{x \in \Omega : u_n(x) = 0\}$ has zero measure) we have $\lambda_{k+2} > \eta_n(x) > \lambda_{k+1}$ for a.e. $x \in \Omega$. Thus by the monotonicity of the weighted eigenvalue problem [16, Proposition 1.12A] we have

$$\mu_{k+1}(\eta_n) < \mu_{k+1}(\lambda_{k+1}) = 1 = \mu_{k+2}(\lambda_{k+2}) < \mu_{k+2}(\eta_n),$$

which contradicts to the fact that 1 is an eigenvalue of (4.2). The case $(f_5)^-$ is proved in the same way.

When $(f_5)^+$ (or $(f_5)^-$) holds, one can verify that Φ has a local linking structure at 0 with respect to $H_0^1(\Omega) = E_{k+1} \oplus E_{k+1}^{\perp}$ (or $H_0^1(\Omega) = E_k \oplus E_k^{\perp}$) (see [23, Lemma 3.6]). Therefore [23, Proposition 2.2] can be applied. The proof is complete. □

Now we use critical groups to distinguish the solutions found in previous sections. By Theorem 2.2, the equation (1.1) has two nontrivial solutions $u_{\pm}$ with $\Phi(u_{\pm}) < 0$. Since $u_{\pm}$ are local minimizers of Φ, their critical groups read as (see [4,11])

$$C_q(\Phi, u_{\pm}) \cong \delta_{q,0}\mathbb{F} \quad \text{for all} \quad q \in \mathbb{Z}. \tag{4.3}$$

By Theorem 2.3, the equation (1.1) has a solution $\tilde{u}$ which is a mountain pass point of Φ. Thus a standard argument involving Kato-Hess Theorem and [11, Theorem 1.6 in Chapter II] shows that

$$C_q(\Phi, \tilde{u}) \cong \delta_{q,1}\mathbb{F} \quad \text{for all} \quad q \in \mathbb{Z}. \tag{4.4}$$

By the Gromoll-Meyer Theorem [17] and Theorem 3.1(ii), the critical groups of Φ at the two bifurcation solutions $u_\lambda^i (i = 1, 2)$ read as

$$C_q(\Phi, u_\lambda^i) \cong 0, \quad \forall\, q \notin [\ell_k, \ell_{k+1}],\ \forall\, 0 < |\lambda - \lambda_{k+1}| < \delta, \quad i = 1, 2. \tag{4.5}$$

As $k \geqslant 2$ is fixed, we have $\ell_k \geqslant 2$. It follows from (4.3), (4.4), (4.5) and Lemma 4.1 (i) or (ii) that $\tilde{u} \neq 0$ and $\{u_\lambda^1, u_\lambda^2\}$ are different from $\{u_{\pm}, \tilde{u}\}$.

We end the proofs of Theorems 1.1 and 1.2 by finding the sixth nontrivial solution of (1.1) via the Conley index theory. For the reader's convenience we recall briefly some concepts and basic results from [13,12]. This part is almost a copy from [15] with the notations used here.

Let X be a Banach space, $\Phi \in C^1(X, \mathbb{R})$ satisfy (PS) and η be an associated pseudo-gradient flow. A subset S of the critical point set $\mathcal{K}(\Phi)$ of Φ is said be *a dynamically isolated critical set* if there exist a closed neighborhood O of S and regular values $\alpha < \beta$ such that

$$\mathrm{cl}(\tilde{O}) \cap \mathcal{K}(\Phi) \cap \Phi^{-1}[\alpha, \beta] = S, \quad \tilde{O} = \cup_{t\in\mathbb{R}}\eta(t, O).$$

We call (O, α, β) an isolating triple for S and

$$C_q(\Phi, S) = H^q(\Phi^\beta \cap \tilde{O}_+, \Phi^\alpha \cap \tilde{O}_+), \quad \tilde{O}_+ = \cup_{t\geqslant 0}\eta(t, O)$$

the critical groups for S.

A pair (W, W^-) is called a Gromoll-Meyer pair for S with respect to η if the following conditions hold: (i) W is a closed neighborhood of S with mean value property (MVP) that for any $u \in X$ and $t_1 < t_2$, $\eta(t_i, u) \in O (i = 1, 2)$ implies $\eta([t_1, t_2], u) \subset O$; (ii) W^- is an exit set for W, i.e. for each $u_0 \in W$ and $t_1 > 0$ such that $\eta(t_1, u_0) \notin W$, there exists $t_0 \in [0, t_1)$ such that $\eta([0, t_0], u_0) \subset W$ and $\eta(t_0, u_0) \in W^-$; (iii) W^- is closed and is a union of a finite number of submanifolds that are transversal to η.

The Gromoll-Meyer pair (W, W^-) for a critical set S_Φ of Φ is stable under C^1 perturbation on W, i.e. if (W, W^-) is a Gromoll-Meyer pair for Φ, $\|\Phi - \Psi\|_{C^1(W)} < \epsilon$ for $\epsilon > 0$ small, and both Φ and Ψ satisfy (PS), then there is a pseudo-gradient flow ξ of Ψ for which (W, W^-) is still a Gromoll-Meyer pair for any critical set S_Ψ for Ψ such that $W \cap \mathcal{K}(\Psi) = S_\Psi$ (see [13, Theorem 3.4]).

It is known from [13] that if (O, α, β) is an isolating triple for S and O satisfies (MVP) then

$$[S] := \{u \in X : \ \omega(u) \cup \omega^*(u) \subset S\} = \cap_{t\in\mathbb{R}}\eta(t, O)$$

is an isolated invariant set with respect to η where $\omega(u)$ and $\omega^*(u)$ are the ω- and ω^*- limit sets of u. For an isolated critical point u_0 of Φ that is located on an isolated critical level, the singleton $S = \{u_0\}$ is a dynamically isolated critical set and $[S] = S = \{u_0\}$.

A neighborhood U of an invariant set A with respect to η is called isolating if U is closed and $\cap_{|t|\leqslant T}\eta(t, U) \subset \mathrm{int}(U)$ for some $T > 0$ and $\cap_{t\in\mathbb{R}}\eta(t, U) = A$. In particular, if (O, α, β) is an isolated triple for a dynamically isolated critical point set S, then $U = O_\alpha^\beta = \tilde{O} \cap f^{-1}[\alpha, \beta]$ is an isolating neighborhood of $[S]$ (see [13]). By [13, Proposition 3.2], for any MVP closed neighborhood U of $[S]$ satisfying $U \subset O_\alpha^\beta$ where (O, α, β) is an isolating triple for S, there exists a Gromoll-Meyer pair (W, W^-) for S such that $W \subset \mathrm{int}(U)$. In particular, $W = O_\alpha^\beta$ and $W^- = W \cap \Phi^{-1}(\alpha)$ is a Gromoll-Meyer pair for S.

According to Benci [3], a pair (N, N_0) of closed subsets of X is called *an index pair* for an isolating neighborhood U if

(1) There exists $T > 0$ such that $\cap_{|t|\leqslant T}\eta\left(t, \mathrm{cl}(N \setminus N_0)\right) \subset \mathrm{int}(N \setminus N_0)$.
(2) N_0 is *positively invariant with respect to* N, i.e., for any $t > 0$ and for all $u \in N_0$, $\eta([0, t], u) \subset N$ implies that $\eta([0, t], u) \subset N_0$.
(3) N_0 is an exit set for N, i.e., for all $u \in N$ and $t_1 > 0$ such that $\eta(t_1, u) \notin N$, there exists $t_0 \in [0, t_1)$ such that $\eta([0, t_0], u) \subset N$ and $\eta(t_0, u) \in N_0$.
(4) $\mathrm{cl}(N \setminus N_0) \subset U$ and there exists $T > 0$ such that $\cap_{|t|\leqslant T}\eta(t, U) \subset \mathrm{cl}(N \setminus N_0)$.

The *Conley index* of the isolating neighborhood of U is defined to be the Alexander-Spanier cohomology $\bar{H}^*(N, N_0)$. It is well known that the Conley index is a *topological invariant* for isolating neighborhoods, i.e., if (N, N_0) and (N', N_0') are two index pairs for U, then $\bar{H}^*(N, N_0) = \bar{H}^*(N', N_0')$. Conley index enjoys the continuation property.

It is proved in [13] that *if (W, W^-) is a Gromoll-Meyer pair for a dynamically isolated critical set S satisfying $W \subset O_\alpha^\beta$, then (W, W^-) is a Conley index pair for any isolated neighborhood U of $[S]$ satisfying*

$W \subset U \subset O_\alpha^\beta$. Therefore if U is any isolating neighborhood for $[S]$ such that $U \subset O_\alpha^\beta$, then for any Conley index pair (N, N_0) of U, we have $\bar{H}^*(N, N_0) = C_*(\Phi, S)$.

We are ready to finish the proof of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Denote by $\Phi_{\lambda_{k+1}}$ the energy functional Φ in the case $\lambda = \lambda_{k+1}$ in (f_2), and $\mathcal{K}(\Phi_{\lambda_{k+1}}) = \{u \in H_0^1(\Omega) : \Phi'_{\lambda_{k+1}}(u) = 0\}$. We consider the functional $\Phi_{\lambda_{k+1}}$ with $k \geqslant 2$ be fixed and regard the functional Φ as a perturbation of $\Phi_{\lambda_{k+1}}$ for $\lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1})$ and $\delta > 0$ small. By $(f_5)^+$ and Lemma 4.1 we see that 0 is an isolated critical point of $\Phi_{\lambda_{k+1}}$. Thus by definition ([13]), $S = \{0\}$ is a dynamically isolated critical set of $\Phi_{\lambda_{k+1}}$.

Let $B_\rho(0) = \{u \in H_0^1(\Omega) : \|u\| < \rho\}$ be an isolated neighborhood of 0 such that $\mathcal{K}(\Phi_{\lambda_{k+1}}) \cap B_\rho(0) = \{0\}$, the special choice of ρ will be given below. Define $O = \overline{B_\rho(0)}$, and let α, β be regular values of $\Phi_{\lambda_{k+1}}$ satisfying

$$\alpha < \inf_O \Phi_{\lambda_{k+1}} < \sup_O \Phi_{\lambda_{k+1}} < \beta.$$

By Lemma 2.1(ii), $\Phi_{\lambda_{k+1}}$ satisfies (PS), (O, α, β) is an isolating triple for $S = \{0\}$ (see [13,12]). Define

$$W = O_\alpha^\beta = \tilde{O} \cap \Phi_{\lambda_{k+1}}^{-1}[\alpha, \beta], \qquad W^- = W \cap \Phi_{\lambda_{k+1}}^{-1}(\alpha),$$

where $\tilde{O} = \bigcup_{t\in\mathbb{R}} \vartheta(t, O)$ and ϑ is the negative gradient flow of $\Phi_{\lambda_{k+1}}$. Then (W, W^-) is a bounded Gromoll-Meyer pair for $S = \{0\}$, and there holds by [11, Theorem 5.2, Chapter I] and Lemma 4.1(iii)(a) that

$$H^q(W, W^-) = C_q(\Phi_{\lambda_{k+1}}, 0) \cong \delta_{q,\ell_{k+1}}\mathbb{F}, \quad \text{for all } q \in \mathbb{Z}. \tag{4.6}$$

Let (N, N_0) be the Conley index pair of an isolating neighborhood $U = O_\alpha^\beta$, by the topological invariance of Conley index and the fact that a Gromoll-Meyer pair is also a Conley index pair (see [13,12]), we have

$$\bar{H}^q(N, N_0) = H^q(W, W^-) \cong \delta_{q,\ell_{k+1}}\mathbb{F}.$$

Besides, for $\delta > 0$ small enough and $\lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1})$, (W, W^-) is also a Gromoll-Meyer pair for $\mathcal{K}(\Phi) \cap W$ (see [13,12]).

Claim: $\delta > 0$ can be chosen so small that for $\rho > 0$ small but fixed,

$$u_\pm, \tilde{u} \notin O, \quad u_\lambda^i \in O,\ i = 1, 2, \quad \text{for any } \lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1}). \tag{4.7}$$

First, for $\rho > 0$ small enough, we have $u_\pm, \tilde{u} \notin O$. Otherwise, Theorem 3.1(ii) and Remark 3.2(3) imply that $C_0(\Phi, u_\pm) \cong 0$ and $C_1(\Phi, \tilde{u}) \cong 0$, which contradict to (4.3) and (4.4). Second, fix such a $\rho > 0$, then by the bifurcation nature of solution u_λ^i, for $\delta > 0$ small and $\lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1})$, $\|u_\lambda^i\| < \rho$, i.e. $u_\lambda^i \in O$, $i = 1, 2$. It must be $u_\lambda^i \in N$, while $u_\pm$ and $\tilde{u}$ outside N for all $\lambda \in (\lambda_{k+1} - \delta, \lambda_{k+1})$ due to the definition of the Conley index pair.

Suppose that $\mathcal{K}(\Phi) \setminus \{0\} = \{u_\pm, u_\lambda^i, \tilde{u}\}$, we can construct a larger index pair (N', N_0') for $\mathcal{K}(\Phi)$, which contains the Conley index pair (N, N_0) and the known critical points $u_\pm$ and $\tilde{u}$ outside N. By Lemma 2.1, Φ is coercive and bounded from blow. It follows that $(H_0^1(\Omega), \emptyset)$ is also an index pair for $\mathcal{K}(\Phi)$. Therefore

$$\bar{H}^*(N', N_0') = \bar{H}^*(H_0^1(\Omega), \emptyset) = \delta_{*,0}\mathbb{F}. \tag{4.8}$$

The Morse relation gives $(-1)^0 = (-1)^{\ell_{k+1}} + (-1)^0 + (-1)^0 + (-1)^1$, which is a contradiction. Therefore Φ has at least one more nontrivial critical point which is also outside N. $\square$

Proof of Theorem 1.2. In this case, (4.6) becomes

$$H^q(W, W^-) = C_q(\Phi_{\lambda_{k+1}}, 0) \cong \delta_{q,\ell_k}\mathbb{F}, \tag{4.9}$$

since $\ell_k \geqslant 2$, the Morse relation gives the contradiction. □

By Lemma 2.1, (4.3), (4.4) and Lemma 4.1(iii), applying the Morse theory, we can obtain the following theorem on the existence of four nontrivial solutions of (1.1) in the case that the trivial solution 0 is a degenerate critical point of Φ. We note that this theorem extends the known results mentioned in the introduction.

Theorem 4.2. *Assume (f_1)–$(f_5)^{\pm}$ hold and $\lambda = \lambda_{k+1}$ with $k \geqslant 2$ being fixed. Then the equation (1.1) has at least four nontrivial solutions, in which one is positive, one is negative.*

Remark 4.3. In Theorem 4.2, $(f_5)^{\pm}$ can be weaken as

$(f_5')^{\pm}$ There exists $\xi > 0$ such that $\pm(2F(x,t) - \lambda t^2) \geqslant 0$ for all $0 < |t| \leqslant \xi$ and $x \in \overline{\Omega}$.

The conclusions of Theorems 1.1–1.2 and Theorem 4.2 are still valid when (f_4) is replaced by

$(f_4)'$ there holds that

$$\lim_{|t|\to\infty} \frac{2F(x,t)}{t^2} = \lambda_1, \quad \lim_{|t|\to\infty} (2F(x,t) - \lambda_1 t^2) = -\infty, \quad \text{uniformly in } x \in \overline{\Omega}.$$

Notice that $(f_4)'$ means the equation (1.1) is resonant near infinity at λ_1 from the left side.

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