J. Math. Anal. Appl. 491 (2020) 124270

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Random sampling in reproducing kernel subspaces of $L^p(\mathbb{R}^n)$

Dhiraj Patel, Sivananthan Sampath *

Department of Mathematics, Indian Institute of Technology Delhi, New Delhi-110016, India

A R T I C L E I N F O

Article history:
Received 14 October 2019
Available online 29 May 2020
Submitted by U. Stadtmueller

Keywords:
Idempotent operator
Reproducing kernel space
Random sampling
p-frame
Reconstruction algorithm

A B S T R A C T

In this paper, we study random sampling in a reproducing kernel space V, which is the range of an idempotent integral operator. Under certain decay condition on the integral kernel, we show that any element in V can be approximated by an element in a finite-dimensional subspace of V. Moreover, we prove with overwhelming probability that random points uniformly distributed over a cube C is a stable set of sampling for the set of functions concentrated on C. Further, we discuss a reconstruction algorithm for functions in a finite-dimensional subspace of V from its random samples.

1. Introduction

Sampling problem deals with finding a discrete sample set which allows to convert an analog signal into a digital signal and vice-versa without losing any information. Of course, the problem is well-posed only when we impose some conditions on signals (functions). For example, the Shannon-sampling theorem states that if $f \in PW_{[-\frac{1}{2},\frac{1}{2}]}(\mathbb{R})$, the space of functions in $L^2(\mathbb{R})$ whose Fourier transform is compactly supported in $[-\frac{1}{2},\frac{1}{2}]$, then f can be reconstructed by its uniform sample values $\{f(k) : k \in \mathbb{Z}\}$ and its reconstruction is given by

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\frac{\sin \pi(x-k)}{\pi(x-k)}.$$

Further, it is well-known by Kadec's 1/4-theorem that if $X = \{x_k : |x_k - k| \le L < \frac{1}{4}\}$, then every $f \in PW_{[-\frac{1}{2},\frac{1}{2}]}(\mathbb{R})$ can be reconstructed from its sample values on X. Mathematically, the sampling problem can be stated as follows.

Given a closed subspace V of $L^p(\mathbb{R}^n)$, find a countable set $\Gamma \subset \mathbb{R}^n$ such that

$$A\|f\|^p_{L^p(\mathbb{R}^n)} \le \sum_{\gamma\in\Gamma} |f(\gamma)|^p \le B\|f\|^p_{L^p(\mathbb{R}^n)} \quad \text{for all } f \in V, \tag{1.1}$$

* Corresponding author.
E-mail addresses: dpatel.iitd@gmail.com (D. Patel), siva@maths.iitd.ac.in (S. Sampath).

https://doi.org/10.1016/j.jmaa.2020.124270
0022-247X/

for some $A, B > 0$. This is equivalent to say that the sampling operator $S : f \mapsto (f(\gamma))_{\gamma\in\Gamma}$ from the space V into $\ell^p(\Gamma)$ is continuous and the corresponding inverse operator $S^{-1} : Range(S) \to V$ is also continuous. Hence any $f \in V$ can be reconstructed from its sample values on Γ, and the set Γ is called a stable set of sample, or simply stable sampling for the space V. For the detailed study of sampling problem refer to [6,1,16].

For the Paley-Wiener space $PW_{[a,b]}(\mathbb{R})$, the set of stable sampling is characterized by Beurling density condition. However, a similar characterization is not true in higher dimension. In particular, the sufficient Beurling density condition for stable sampling in $\mathbb{R}^n$ $(n \geq 2)$ does not hold, see [16, Section 5.7]. At the same time, the Paley-Wiener theorem is still valid for the convex spectra in $\mathbb{R}^n$, but for $n \geq 2$ the zero set of a function in $PW_\Omega(\mathbb{R}^n)$ is analytic manifold, where Ω is a convex subset of $\mathbb{R}^n$. Hence the classical result on the density of zeros of an entire function of exponential type is no longer valid. So the non-uniform sampling in higher dimension is still difficult to solve. These difficulties motivate to study the sampling problem in probabilistic framework.

Random sampling method has been used frequently in the field of image processing [9], learning theory [17,10], and compressed sensing [12]. Bass and Gröchenig [2] studied random sampling for multivariate trigonometric polynomial. Later, Candés, Romberg, and Tao [7,8] investigated the reconstruction of sparse trigonometric polynomial from a few random samples. Smale and Zhou [19] studied the function reconstruction error from its random samples satisfying (1.1) in a reproducing kernel Hilbert space.

Note that for “nice” functions f, the sample value $f(\gamma)$ is close to 0 when γ is large value. Therefore, the sample value may not significantly contribute to sampling inequality for large sample points. Moreover, it is shown by Bass and Gröchenig [3] that for each random samples identically and uniformly distributed over each cube $k + [0, 1]^n$ in $\mathbb{R}^n$, the sampling inequality (1.1) fails almost surely for Paley-Wiener space. For these reasons, they considered random sample points from the compact set $C_R = \left[-\frac{R}{2}, \frac{R}{2}\right]^n$ and proved that the sampling inequality (1.2) hold for the functions concentrated on C_R with high probability, see [3,4]. Then the result was generalized by Führ and Xian [13] for finitely generated shift-invariant subspace V of $L^2(\mathbb{R}^n)$, and a further generalization for L^p-norm was studied by Yang and Wei [22,20]. Later, Yang and Tao [21] investigated random sampling for the space of continuous functions with bounded derivative.

In this paper, we study random sampling on a closed subspace V of $L^p(\mathbb{R}^n)$ which is defined as the image of an idempotent integral operator. More precisely, we derive the random sampling inequality for the set $V^\star(R, \delta) = \left\{f \in V : (1 - \delta)\|f\|^p_{L^p(\mathbb{R}^n)} \leq \int_{[-\frac{R}{2},\frac{R}{2}]^n} |f(x)|^p dx\right\}$, and prove the following main theorem.

Theorem 1.1. *Assume that $\{x_j : j \in \mathbb{N}\}$ is a sequence of i.i.d. random variables that are uniformly distributed over the cube $C_R = \left[-\frac{R}{2}, \frac{R}{2}\right]^n$ and $0 < \mu < 1-\delta$. Then there exist $a, b > 0$ such that the sampling inequality*

$$\frac{r}{R^n}(1 - \mu - \delta)\|f\|^p_{L^p(\mathbb{R}^n)} \leq \sum_{j=1}^{r} |f(x_j)|^p \leq \frac{r}{R^n}(1 + \mu)\|f\|^p_{L^p(\mathbb{R}^n)} \tag{1.2}$$

holds for every $f \in V^\star(R, \delta)$ with the probability at least $1 - 2a \exp\left(-\frac{b}{pk^{p-1}R^n}\frac{r\mu^2}{12+\mu}\right)$, where $k = \sup_{x\in C_R} \|K(x, \cdot)\|_{L^{p'}(\mathbb{R}^n)}$ and $b = \min\{\frac{2^{\frac{4}{\log 2}-10+\frac{n+1}{n+2}}(\log 2)^4}{(n+2)^4}, \frac{3}{4k}\}$. For large R and sufficiently large sample size r, the constant $a = \exp(MR^n)$ with M depending on the dimension n.

The paper is organized as follows. In Section 2, we introduce our hypothesis space V and prove that any element in V can be approximated by a finite-dimensional subspace of V with respect to $\|\cdot\|_{L^p(C_R)}$. Further, we show that the set of functions in $V^\star(R, \delta)$ with unit norm is totally bounded and estimate the number of open balls which covers the set. In Section 3, we define independent random variables with respect to given random samples, and then using Bernstein's inequality we prove the main result. In the end, a reconstruction algorithm is discussed for functions in finite-dimensional subspace of V from its random samples.

2. Assumption and covering number

Let T be an idempotent integral operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ defined by

$$Tf(x) := \int_{\mathbb{R}^n} K(x,y)f(y)dy \ \text{ satisfy } T^2 = T, \tag{2.1}$$

where $1 \leq p < \infty$, and the integral kernel K is symmetric and satisfy regularity condition

$$\lim_{\varepsilon \to 0} \left\| \sup_{z \in \mathbb{R}^n} |osc_\varepsilon(K)(\cdot + z, z)| \right\|_{L^1(\mathbb{R}^n)} = 0, \tag{2.2}$$

with decay

$$|K(x,y)| \leq \frac{C}{(1 + \|x - y\|_1)^\alpha}, \qquad \alpha > \frac{n}{p'} + n + 1 \text{ and } C > 0, \tag{2.3}$$

where $\|x\|_1 := \sum_{i=1}^{n} |x(i)|,\ x := (x(1), x(2), \cdots, x(n)) \in \mathbb{R}^n,\ \frac{1}{p} + \frac{1}{p'} = 1$, and

$$osc_\varepsilon(K)(x,y) = \sup_{x', y' \in [-\varepsilon, \varepsilon]^n} |K(x + x', y + y') - K(x,y)|.$$

Under these assumptions, we see that $\sup_{x \in \mathbb{R}^n} \|K(x,\cdot)\|_{L^1(\mathbb{R}^n)}$ exists and the associated integral operator T is bounded. Moreover, the kernel K satisfy the off-diagonal decay condition

$$\| \sup_{z \in \mathbb{R}^n} |K(\cdot + z, z)| \|_{L^1(\mathbb{R}^n)} < \infty. \tag{2.4}$$

A reproducing kernel Banach space is a Banach space M of functions on a set Ω such that the point evaluation functional $f \mapsto f(x)$ is continuous for each $x \in \Omega$ i.e., for every $x \in \Omega$, there exists $C_x > 0$, such that $|f(x)| \leq C_x \|f\|$, for all $f \in M$. Let us consider the space $V := Range(T)$. It is easily verifiable that the space V is closed and reproducing kernel subspace of $L^p(\mathbb{R}^n)$.

Originally, Nashed and Sun [15] proposed the space V as a general model to study the sampling inequality (1.1). They showed that if the integral kernel K satisfies off-diagonal decay condition (2.4) and regularity condition (2.2), then there exists a discrete stable sampling for the space V. For more details about the space V, we refer the reader to [15].

Note that the sampling inequality (1.2) is true for $f \in V^\star(R,\delta)$ if and only if it is true for $f \in V^\star(R,\delta)$ with $\|f\|_{L^p(\mathbb{R}^n)} = 1$. Hence it is enough to prove the Theorem 1.1 for the set

$$V(R,\delta) = \left\{ f \in V \ : \ (1 - \delta) \leq \int_{C_R} |f(x)|^p dx \ \text{ and } \|f\|_{L^p(\mathbb{R}^n)} = 1 \right\}.$$

The "key step" of Theorem 1.1 is to prove $V(R,\delta)$ is totally bounded with respect to $\|\cdot\|_{L^\infty(C_R)}$. In the previous works, Bass and Gröchenig [3] used spectral decomposition of truncated Fourier transform on band-limited functions and eigenvalue decay condition of prolate spheroidal wave functions. Yang and Wei [22] considered shift-invariant space generated by a compactly supported function, which implies $V(R,\delta)$ is a subset of finite-dimensional space. In [13], Führ and Xian calculated the maximum number of eigenvalues of some self-adjoint operator which are greater than $\frac{1}{2}$ and used a similar method as in [4]. For finitely generated shift-invariant space [20], Yang assumed a fixed decay on each generator and approximated any function in $V(R,\delta)$ by a function in some finite-dimensional subspace of V. In this paper, the considered space generalizes the existing model spaces. Moreover, the function space need not have finite generators. The key idea is to use the existence of stable sampling set for the space V, which was proved in [15]. This allows us to represent any functions in V via some frame sequence, and then we estimate the decay of frame sequence using decay property of the integral kernel. Using these estimates, we are able to show that $V(R,\delta)$ is totally bounded with respect to $\|\cdot\|_{L^\infty(C_R)}$.

A collection of points $U=\{u : u\in\mathbb{R}^n\}$ is *relatively separated* if

$$\beta=\inf_{\substack{u,u'\in U\\ u\neq u'}}\|u-u'\|_\infty>0,$$

and β is called *gap* of the set U.

As the kernel K of the integral operator T defined in (2.1) satisfies off-diagonal decay condition (2.4) and regularity condition (2.2), then from [15, Theorem A.2.] there exist a relatively separated set $\Gamma=\{\gamma : \gamma\in\mathbb{R}^n\}$ with positive gap $\eta\,(<\frac{2}{n})$, and two families $\Phi:=\{\phi_\gamma\}_{\gamma\in\Gamma}\subseteq L^p(\mathbb{R}^n)$ and $\tilde{\Phi}:=\{\tilde{\phi}_\gamma\}_{\gamma\in\Gamma}\subseteq L^{p'}(\mathbb{R}^n)$ such that for any $f\in V$ can be written as

$$f(x)=\sum_{\gamma\in\Gamma}\langle f,\tilde{\phi}_\gamma\rangle\phi_\gamma(x), \tag{2.5}$$

where for each $\gamma\in\Gamma$, ϕ_γ is given by

$$\phi_\gamma(x)=\eta^{-\frac{n}{p}}\int\limits_{C_\eta}K(\gamma+z,x)dz,\quad x\in\mathbb{R}^n, \tag{2.6}$$

and $\{\tilde{\phi}_\gamma : \gamma\in\Gamma\}$ forms p-frame for V, i.e. there exist $A,B>0$ such that

$$A\|f\|^p_{L^p(\mathbb{R}^n)}\leq\sum_{\gamma\in\Gamma}|\langle f,\tilde{\phi}_\gamma\rangle|^p\leq B\|f\|^p_{L^p(\mathbb{R}^n)},\qquad\text{for all } f\in V. \tag{2.7}$$

By (2.5) and (2.7), any f in V is of the form $\sum\limits_{\gamma\in\Gamma}c_\gamma\phi_\gamma$ for some $(c_\gamma)\in\ell^p$. Now, our interest is to approximate any function in V by an element in a finite-dimensional space. In the following lemma, given f in V, we determine the sufficient condition on real number N such that the truncated series $\sum\limits_{\gamma\in\Gamma\cap C_N}c_\gamma\phi_\gamma$ is close to f.

Before we move to the lemma, we define the subspace V_N of V by

$$V_N=\Big\{\sum_{\gamma\in\Gamma\cap[-\frac{N}{2},\frac{N}{2}]^n}c_\gamma\phi_\gamma : c_\gamma\in\mathbb{R}\Big\},$$

where $N>R+\frac{2}{n}$ be a positive real.

Lemma 2.1. *For a given $\epsilon>0$ and $f\in V$, choose $N>R+\frac{2}{n}+\frac{2}{n}\left[\frac{4^nB^{(p'-1)}\|f\|^{p'}_{L^p(\mathbb{R}^n)}C^{p'}R^{n(p'-1)}}{w_\alpha\epsilon^{p'}}\right]^{\frac{1}{\alpha p'-n}}$, then $\|f-\sum\limits_{\gamma\in\Gamma\cap[-\frac{N}{2},\frac{N}{2}]^n}\langle f,\tilde{\phi}_\gamma\rangle\phi_\gamma\|_{L^p(C_R)}<\epsilon$.*

Proof. Given $f \in V$ we consider $f_N \in V_N$ by

$$f_N(x) = \sum_{\gamma \in \Gamma \cap [-\frac{N}{2}, \frac{N}{2}]^n} \langle f, \tilde{\phi}_\gamma \rangle \phi_\gamma(x). \tag{2.8}$$

Let $x \in C_R$, then by (2.5), (2.8) and (2.7) we have

$$\begin{aligned}
|f(x) - f_N(x)| &= \Big| \sum_{\gamma \in \Gamma \setminus [-\frac{N}{2}, \frac{N}{2}]^n} \langle f, \tilde{\phi}_\gamma \rangle \phi_\gamma(x) \Big| \\
&\leq \Big(\sum_{\gamma \in \Gamma \setminus [-\frac{N}{2}, \frac{N}{2}]^n} |\langle f, \tilde{\phi}_\gamma \rangle|^p \Big)^{\frac{1}{p}} \Big(\sum_{\gamma \in \Gamma \setminus [-\frac{N}{2}, \frac{N}{2}]^n} |\phi_\gamma(x)|^{p'} \Big)^{\frac{1}{p'}} \\
&\leq B^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)} \Big(\sum_{\gamma \in \Gamma \setminus [-\frac{N}{2}, \frac{N}{2}]^n} |\phi_\gamma(x)|^{p'} \Big)^{\frac{1}{p'}}.
\end{aligned}$$

In order to estimate the upper bound of the series, we derive the bound for ϕ_γ using the conditions (2.6) and (2.3).

$$\begin{aligned}
|\phi_\gamma(x)| &\leq \eta^{-\frac{n}{p}} \int_{C_\eta} \frac{C}{(1 + \|\gamma + z - x\|_1)^\alpha} dz \\
&\leq \eta^{-\frac{n}{p}} \int_{C_\eta} \frac{C}{(1 + \|\gamma - x\|_1 - \|z\|_1)^\alpha} dz \\
&\leq \eta^{-\frac{n}{p}} \int_{C_\eta} \frac{C}{(1 - \frac{n\eta}{2} + \|\gamma - x\|_1)^\alpha} dz \\
&\leq \eta^{\frac{n}{p'}} \frac{C}{(1 - \frac{n\eta}{2} + \|\gamma - x\|_1)^\alpha}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{\gamma \in \Gamma \setminus [-\frac{N}{2}, \frac{N}{2}]^n} |\phi_\gamma(x)|^{p'} &= \left(\frac{2}{\eta}\right)^n \sum_{\gamma \in \Gamma \setminus [-\frac{N}{2}, \frac{N}{2}]^n} |\phi_\gamma(x)|^{p'} \left(\frac{\eta}{2}\right)^n \\
&\leq \left(\frac{2}{\eta}\right)^n \sum_{\gamma \in \Gamma \setminus [-\frac{N}{2}, \frac{N}{2}]^n} \frac{\eta^n C^{p'}}{(1 - \frac{n\eta}{2} + \|x - \gamma\|_1)^{\alpha p'}} \left(\gamma - \gamma + \frac{\eta}{2}\right)^n \\
&\leq 2^n C^{p'} \sum_{\gamma \in \Gamma \setminus [-\frac{N}{2}, \frac{N}{2}]^n} \int_{B_{\frac{\eta}{2}}(\gamma)} \frac{dy}{(1 - \frac{n\eta}{2} + \|x - y\|_1)^{\alpha p'}},
\end{aligned}$$

where $B_{\frac{\eta}{2}}(\gamma) = \{y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n : \gamma \leq y_i \leq \gamma + \frac{\eta}{2},\ 1 \leq i \leq n\}$ is cube of length $\frac{\eta}{2}$ containing γ. Since each cube $B_{\frac{\eta}{2}}(\gamma)$ are disjoint, we get

$$\begin{aligned}
\sum_{\gamma \in \Gamma \setminus [-\frac{N}{2}, \frac{N}{2}]^n} |\phi_\gamma(x)|^{p'} &\leq 2^n C^{p'} \int_{\mathbb{R}^n \setminus [-\frac{N-\eta}{2}, \frac{N-\eta}{2}]^n} \frac{dy}{(1 - \frac{n\eta}{2} + \|x - y\|_1)^{\alpha p'}} \\
&\leq 2^n C^{p'} \int_{\mathbb{R}^n \setminus [-\frac{N-\eta-R}{2}, \frac{N-\eta-R}{2}]^n} \frac{dy}{(1 - \frac{n\eta}{2} + \|y\|_1)^{\alpha p'}}
\end{aligned}$$

$$
\begin{aligned}
&\leq 2^n C^{p'} \times 2^n \int\limits_{[\frac{N-\eta-R}{2},\infty)^n} \frac{dy}{\|y\|_1^{\alpha p'}} \\
&= 4^n C^{p'} \frac{1}{w_\alpha \left(\frac{N-\eta-R}{2}n\right)^{\alpha p'-n}},
\end{aligned}
$$

where $w_\alpha = (\alpha p' - 1)(\alpha p' - 2)\cdots(\alpha p' - n)$.
Hence,

$$
|f(x) - f_N(x)| \leq B^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)} \left(4^n C^{p'} \frac{1}{w_\alpha \left(\frac{N-\eta-R}{2}n\right)^{\alpha p'-n}}\right)^{\frac{1}{p'}}
$$

$$
|f(x) - f_N(x)| \leq 4^{\frac{n}{p'}} B^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)} \frac{C}{w_\alpha^{\frac{1}{p'}} \left(\frac{N-R}{2}n - 1\right)^{(\alpha - \frac{n}{p'})}} \quad \text{as } \eta < \frac{2}{n} \tag{2.9}
$$

Therefore,

$$
\begin{aligned}
\|f - f_N\|^p_{L^p(C_R)} &\leq 4^{\frac{np}{p'}} B \|f\|^p_{L^p(\mathbb{R}^n)} C^p \frac{R^n}{w_\alpha^{\frac{p}{p'}} \left(\frac{N-R}{2}n - 1\right)^{(\alpha - \frac{n}{p'})p}} \\
&< \epsilon
\end{aligned}
$$

whenever, $N > R + \frac{2}{n} + \frac{2}{n}\left[\frac{4^n B^{(p'-1)} \|f\|^{p'}_{L^p(\mathbb{R}^n)} C^{p'} R^{n(p'-1)}}{w_\alpha \epsilon^{p'}}\right]^{\frac{1}{\alpha p' - n}}$. $\square$

Lemma 2.2. *If $f \in V(R, \delta)$, then $\|f\|_{L^\infty(C_R)} \leq D\|f\|_{L^p(C_R)}$, where $D = \frac{\sup\limits_{x \in C_R} \|K(x,\cdot)\|_{L^{p'}(\mathbb{R}^n)}}{(1-\delta)^{\frac{1}{p}}}$.*

Proof. Since V is the range of an idempotent integral operator, we have

$$
f(x) = \int\limits_{\mathbb{R}^n} f(y)K(x,y)dy, \qquad \text{for all } f \in V,\ x \in \mathbb{R}^n.
$$

Then,

$$
\begin{aligned}
|f(x)| &\leq \int\limits_{\mathbb{R}^n} |f(y)||K(x,y)|dy \\
&\leq \|f\|_{L^p(\mathbb{R}^n)} \|K(x,\cdot)\|_{L^{p'}(\mathbb{R}^n)} \\
\|f\|_{L^\infty(C_R)} &\leq \sup_{x \in C_R} \|K(x,\cdot)\|_{L^{p'}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{2.10}
$$

Let $f \in V(R, \delta)$, then we have $(1 - \delta)\|f\|^p_{L^p(\mathbb{R}^n)} \leq \|f\|^p_{L^p(C_R)}$.

Therefore,

$$
\|f\|_{L^\infty(C_R)} \leq \frac{\sup\limits_{x \in C_R} \|K(x,\cdot)\|_{L^{p'}(\mathbb{R}^n)}}{(1-\delta)^{\frac{1}{p}}} \|f\|_{L^p(C_R)}. \quad \square
$$

The following result is a well-known bound for the number of open balls of fixed radius to cover a closed ball in a finite-dimensional space, see [11].

Lemma 2.3. *Let X be a Banach space of dimension s and $\overline{B(0;r)}$ denotes the closed ball of radius r centered at the origin. Then minimum number of open balls of radius ω to cover $\overline{B(0;r)}$ is bounded by $\left(\frac{2r}{\omega}+1\right)^s$.*

Lemma 2.4. *The set $V(R,\delta)$ is totally bounded with respect to $\|\cdot\|_{L^\infty(C_R)}$.*

Proof. Let $\epsilon > 0$ and $f \in V(R,\delta)$ be given. Then by Lemma 2.1 and Lemma 2.2 there exists f_N in a finite-dimensional subspace V_N of V such that $\|f-f_N\|_{L^\infty(C_R)} < \frac{\epsilon}{2}$.

Let $\overline{B(0;D+\frac{\epsilon}{2})}$ be a closed ball in V_N with respect to $\|\cdot\|_{L^\infty(C_R)}$. We know that $\overline{B(0;D+\frac{\epsilon}{2})}$ is totally bounded, and let $\mathcal{A}(\epsilon)$ be the finite collection of $\frac{\epsilon}{2}$-net for $\overline{B(0;D+\frac{\epsilon}{2})}$.

Since $\|f\|_{L^\infty(C_R)} \le D$ and $\|f-f_N\|_{L^\infty(C_R)} < \frac{\epsilon}{2}$, we get $f_N \in \overline{B(0;D+\frac{\epsilon}{2})}$. This implies that there exists $\tilde{f} \in \mathcal{A}(\epsilon)$ such that $\|f_N-\tilde{f}\|_{L^\infty(C_R)} < \frac{\epsilon}{2}$, and hence $\|f-\tilde{f}\|_{L^\infty(C_R)} < \epsilon$. Therefore, the finite set $\mathcal{A}(\epsilon)$ forms an ϵ-net for $V(R,\delta)$. $\square$

Remark 2.5.

1. In the above lemma, we choose $f_N \in V_N$ such that $\|f-f_N\|_{L^\infty(C_R)} < \frac{\epsilon}{2}$, and $N > R + \frac{2}{n} + \frac{2}{n}\left[\frac{4^n B^{(p'-1)}(2C)^{p'}}{w_\alpha \epsilon^{p'}}\right]^{\frac{1}{\alpha p'-n}}$.

   In particular, we select $N = R + 2 + \frac{2}{n}\left[\frac{4^n B^{(p'-1)}(2C)^{p'}}{w_\alpha \epsilon^{p'}}\right]^{\frac{1}{\alpha p'-n}}$, then dimension of V_N is bounded by

$$\begin{aligned} N^n N_0(\Gamma) &= N_0(\Gamma)\left[R+2+\frac{2}{n}\left(\frac{4^n B^{(p'-1)}(2C)^{p'}}{w_\alpha \epsilon^{p'}}\right)^{\frac{1}{\alpha p'-n}}\right]^n \\ &\le 2^n N_0(\Gamma)\Big[(R+2)^n + C_1 \epsilon^{-\frac{np'}{\alpha p'-n}}\Big] := d_\epsilon, \end{aligned}$$

   where $N_0(\Gamma) = \sup_{k\in\mathbb{Z}^n} \left(k+\left[-\frac{1}{2},\frac{1}{2}\right]^n\right) \cap \Gamma$, and $C_1 = \left(\frac{2}{n}\right)^n \left(\frac{4^n B^{(p'-1)}(2C)^{p'}}{w_\alpha}\right)^{\frac{n}{\alpha p'-n}}$.
2. If $N(\epsilon)$ denotes the number of elements in $\mathcal{A}(\epsilon)$, i.e. the minimum number of open balls of radius $\frac{\epsilon}{2}$ covers for $\overline{B(0;D+\frac{\epsilon}{2})}$ with respect to $\|\cdot\|_{L^\infty(C_R)}$, then from Lemma 2.3 we have

$$N(\epsilon) \le \left(1+\frac{4D+2\epsilon}{\epsilon}\right)^{d_\epsilon} = \exp\left(d_\epsilon \log\left(3+\frac{4D}{\epsilon}\right)\right) \le \exp\left(d_\epsilon \log\left(\frac{8D}{\epsilon}\right)\right).$$

3. Random sampling

In this section, we define independent random variables on V through random samples and estimate their variance and bound. Later, we use Bernstein's inequality to prove the sampling inequality for the set of functions in $V(R,\delta)$ with high probability.

Let $\{x_j : j \in \mathbb{N}\}$ be a sequence of i.i.d. random variables uniformly distributed over C_R. For every $f \in V$, we introduce the random variable

$$Z_j(f) = |f(x_j)|^p - \frac{1}{R^n}\int_{C_R} |f(x)|^p dx. \tag{3.1}$$

Then $\{Z_j(f)\}_{j\in\mathbb{N}}$ is a sequence of independent random variable with expectation $\mathbb{E}[Z_j(f)] = 0$.

Lemma 3.1. *Let $f,g \in V$ with $\|f\|_{L^p(\mathbb{R}^n)} = \|g\|_{L^p(\mathbb{R}^n)} = 1$ and $j \in \mathbb{N}$. Then the following inequalities hold:*

(i) $VarZ_j(f) \leq \frac{1}{R^n}\Big(\sup\limits_{x\in C_R} \|K(x,\cdot)\|_{L^{p'}(\mathbb{R}^n)}\Big)^p$,

(ii) $\|Z_j(f)\|_\infty \leq \Big(\sup\limits_{x\in C_R} \|K(x,\cdot)\|_{L^{p'}(\mathbb{R}^n)}\Big)^p$,

(iii) $Var(Z_j(f)-Z_j(g)) \leq \frac{2p}{R^n}\Big(\sup\limits_{x\in C_R} \|K(x,\cdot)\|_{L^{p'}(\mathbb{R}^n)}\Big)^{p-1}\|f-g\|_{L^\infty(C_R)}$,

(iv) $\|Z_j(f)-Z_j(g)\|_\infty \leq p\Big(\sup\limits_{x\in C_R} \|K(x,\cdot)\|_{L^{p'}(\mathbb{R}^n)}\Big)^{p-1}\|f-g\|_{L^\infty(C_R)}$.

Proof. (i) For random variable $Z_j(f)$ with $\mathbb{E}(Z_j(f)) = 0$ and by (2.10),

$$\begin{aligned}
VarZ_j(f) &= \mathbb{E}\left([|f(x_j)|^p - \mathbb{E}(|f(x_j)|^p)]^2\right) \\
&= \mathbb{E}(|f(x_j)|^{2p}) - [\mathbb{E}(|f(x_j)|^p)]^2 \\
&\leq \mathbb{E}(|f(x_j)|^{2p}) \\
&= \frac{1}{R^n}\int\limits_{C_R} |f(x)|^{2p}dx \\
&\leq \frac{1}{R^n}\|f\|^p_{L^p(C_R)}\|f\|^p_{L^\infty(C_R)} \\
&\leq \frac{1}{R^n}\Big(\sup_{x\in C_R}\|K(x,\cdot)\|_{L^{p'}(\mathbb{R}^n)}\Big)^p.
\end{aligned}$$

(ii) Since $f \in V$ with $\|f\|_{L^p(\mathbb{R}^n)} = 1$ and by (2.10), we obtain

$$\begin{aligned}
\|Z_j(f)\|_\infty &= \sup_{\omega\in\Omega}\Big||f(x_j(\omega))|^p - \frac{1}{R^n}\int\limits_{C_R}|f(x)|^pdx\Big| \\
&\leq \max\left\{\|f\|^p_{L^\infty(C_R)}, \frac{1}{R^n}\|f\|^p_{L^p(C_R)}\right\} \\
&= \|f\|^p_{L^\infty(C_R)} \leq \Big(\sup_{x\in C_R}\|K(x,\cdot)\|_{L^{p'}(\mathbb{R}^n)}\Big)^p.
\end{aligned}$$

(iii) Using the same method as in (i), we get

$$\begin{aligned}
Var(Z_j(f)-Z_j(g)) &= \mathbb{E}\left([|f(x_j)|^p - |g(x_j)|^p]^2\right) - (\mathbb{E}(|f(x_j)|^p - |g(x_j)|^p))^2 \\
&\leq \frac{1}{R^n}\int\limits_{C_R}(|f(x)|^p - |g(x)|^p)^2dx \\
&\leq \frac{1}{R^n}\int\limits_{C_R}\big||f(x)|^p - |g(x)|^p\big|(|f(x)|^p + |g(x)|^p)dx \\
&\leq \frac{1}{R^n}\||f|^p - |g|^p\|_{L^\infty(C_R)}\left(\|f\|^p_{L^p(C_R)} + \|g\|^p_{L^p(C_R)}\right) \\
&\leq \frac{2}{R^n}\|(|f|-|g|)(|f|^{p-1} + |f|^{p-2}|g| + \cdots + |f||g|^{p-2} + |g|^{p-1})\|_{L^\infty(C_R)} \\
&\leq \frac{2}{R^n}p\max\left\{\|f\|_{L^\infty(C_R)}, \|g\|_{L^\infty(C_R)}\right\}^{p-1}\|f-g\|_{L^\infty(C_R)} \\
&\leq \frac{2p}{R^n}\Big(\sup_{x\in C_R}\|K(x,\cdot)\|_{L^{p'}(\mathbb{R}^n)}\Big)^{p-1}\|f-g\|_{L^\infty(C_R)}.
\end{aligned}$$

(iv) The estimate follows similarly from (ii).

$$
\begin{aligned}
\|Z_j(f)-Z_j(g)\|_\infty &= \sup_{\omega\in\Omega}\Big||f(x_j(\omega))|^p-|g(x_j(\omega))|^p-\frac{1}{R^n}\Big(\int\limits_{C_R}(|f(x)|^p-|g(x)|^p)dx\Big)\Big| \\
&\leq \max\Big\{\||f|^p-|g|^p\|_{L^\infty(C_R)}, \frac{1}{R^n}\||f|^p-|g|^p\|_{L^1(C_R)}\Big\} \\
&= \||f|^p-|g|^p\|_{L^\infty(C_R)} \\
&\leq p\Big(\sup_{x\in C_R}\|K(x,\cdot)\|_{L^{p'}(\mathbb{R}^n)}\Big)^{p-1}\|f-g\|_{L^\infty(C_R)}.
\end{aligned}
$$

The last inequality follows from the estimation in (iii). □

In the rest of the paper, we denote $k=\sup_{x\in C_R}\|K(x,\cdot)\|_{L^{p'}(\mathbb{R}^n)}$. The following Bernstein's inequality plays an important role in Theorem 1.1.

Theorem 3.2 *(Bernstein's inequality [5]). Let Y_j, $j=1,2,\cdots,r$ be a sequence of bounded, independent random variable with $\mathbb{E}Y_j=0$, $VarY_j\leq\sigma^2$, and $\|Y_j\|_\infty\leq M$ for $j=1,2,\cdots,r$. Then*

$$
P\Big(\Big|\sum_{j=1}^r Y_j\Big|\geq\lambda\Big)\leq 2\exp\Big(-\frac{\lambda^2}{2r\sigma^2+\frac{2}{3}M\lambda}\Big). \tag{3.2}
$$

Theorem 3.3. *Let $\{x_j : j\in\mathbb{N}\}$ be a sequence of i.i.d. random variables that are drawn uniformly from $C_R=[-R/2,R/2]^n$. Then there exist constants $a,b>0$ depending on n, R, and δ such that*

$$
P\Big(\sup_{f\in V(R,\delta)}\Big|\sum_{j=1}^r Z_j(f)\Big|\geq\lambda\Big)\leq 2a\exp\Big(-\frac{b}{pk^{p-1}}\frac{\lambda^2}{12rR^{-n}+\lambda}\Big). \tag{3.3}
$$

Proof. The proof follows from the similar idea of Bass and Gröchenig [3]. To determine the required probability, we use Bernstein's Inequality (3.2) repeatedly on independent random variable Z_j. We prove the result in the following steps:

Step 1: Let $f\in V(R,\delta)$. By Lemma 2.4 we can construct a sequence $\{f_l\}_{l\in\mathbb{N}}$ such that $f_l\in\mathcal{A}(2^{-l})$ and $\|f-f_l\|_{L^\infty(C_R)}<2^{-l}$. Then we write

$$
Z_j(f)=Z_j(f_1)+\sum_{l=2}^\infty\Big(Z_j(f_l)-Z_j(f_{l-1})\Big). \tag{3.4}
$$

Indeed, $s_m(f)=Z_j(f_1)+\sum_{l=2}^m(Z_j(f_l)-Z_j(f_{l-1}))=Z_j(f_m)$ and

$$
\begin{aligned}
\|Z_j(f)-Z_j(f_m)\|_\infty &\leq pk^{p-1}\|f-f_m\|_{L^\infty(C_R)} \\
&\to 0 \quad as\ m\to\infty.
\end{aligned}
$$

Now consider the events

$$
\begin{aligned}
\mathcal{E} &= \Big\{\sup_{f\in V(R,\delta)}\Big|\sum_{j=1}^r Z_j(f)\Big|\geq\lambda\Big\}, \\
\mathcal{E}_1 &= \Big\{\exists\ f_1\in\mathcal{A}\Big(\frac{1}{2}\Big)\ :\ \Big|\sum_{j=1}^r Z_j(f_1)\Big|\geq\frac{\lambda}{2}\Big\},
\end{aligned}
$$

and for $l \geq 2$

$$\mathcal{E}_l = \Big\{ \exists\ f_l \in \mathcal{A}(2^{-l}) \text{ and } f_{l-1} \in \mathcal{A}(2^{-l+1}) \text{ with } \|f_l - f_{l-1}\|_{L^\infty(C_R)} \leq 3 \cdot 2^{-l}\ :\ \Big|\sum_{j=1}^{r} (Z_j(f_l) - Z_j(f_{l-1}))\Big| \geq \frac{\lambda}{2l^2} \Big\}.$$

Claim. *If* $\sup_{f \in V(R,\delta)} \Big|\sum_{j=1}^{r} Z_j(f)\Big| \geq \lambda$, *i.e.* $P(\mathcal{E}) > 0$ *then one of the events* $\mathcal{E}_l$ *hold for* $l \geq 1$, *i.e.* $\mathcal{E} \subseteq \bigcup_{l=1}^{\infty} \mathcal{E}_l$.

Suppose for all $l \geq 1$, $P(\mathcal{E}_l) = 0$, then for $f \in V(R, \delta)$ and (3.4) we get

$$\begin{aligned} \Big|\sum_{j=1}^{r} Z_j(f)\Big| &\leq \Big|\sum_{j=1}^{r} Z_j(f_1)\Big| + \sum_{l=2}^{\infty} \Big|\sum_{j=1}^{r} (Z_j(f_l) - Z_j(f_{l-1}))\Big| \\ &< \frac{\lambda}{2} + \sum_{l=2}^{\infty} \frac{\lambda}{2l^2} = \frac{\pi^2}{12}\lambda < \lambda. \end{aligned}$$

This is a contradiction.

Step 2: We compute bound for the probability of the event $\mathcal{E}_1$. Using Bernstein's inequality (3.2) for the sequence of independent random variable $Z_j(f_1)$, and the results in Lemma 3.1 (i) & (ii), we get

$$\begin{aligned} P\Big(\Big|\sum_{j=1}^{r} Z_j(f_1)\Big| \geq \frac{\lambda}{2}\Big) &\leq 2\exp\Big(- \frac{\frac{\lambda^2}{4}}{2rR^{-n}k^p + \frac{1}{3}k^p\lambda}\Big) \\ &= 2\exp\left(-\frac{3}{4k^p}\frac{\lambda^2}{6rR^{-n} + \lambda}\right). \end{aligned}$$

Therefore,

$$P(\mathcal{E}_1) \leq 2N\left(\frac{1}{2}\right)\exp\left(-\frac{3}{4k^p}\frac{\lambda^2}{6rR^{-n} + \lambda}\right). \tag{3.5}$$

Step 3: The bound of the probability of the event $\mathcal{E}_l$ can be found in a similar way as in Step 2. From (3.2) and Lemma 3.1 (iii) & (iv), we have

$$\begin{aligned} &P\Big(\Big|\sum_{j=1}^{r} (Z_j(f_l) - Z_j(f_{l-1}))\Big| \geq \frac{\lambda}{2l^2}\Big) \\ &\leq 2\exp\Big(- \frac{\frac{\lambda^2}{4l^4}}{4rpR^{-n}k^{p-1}\|f_l - f_{l-1}\|_{L^\infty(C_R)} + \frac{1}{3}pk^{p-1}\|f_l - f_{l-1}\|_{L^\infty(C_R)}\frac{\lambda}{l^2}}\Big) \\ &\leq 2\exp\Big(- \frac{1}{4l^4}\frac{\lambda^2}{(4rR^{-n} + \frac{\lambda}{3l^2})pk^{p-1}3 \cdot 2^{-l}}\Big) \\ &\leq 2\exp\Big(- \frac{2^l}{4l^4}\frac{\lambda^2}{pk^{p-1}(12rR^{-n} + \lambda)}\Big). \end{aligned}$$

Hence,

$$P(\mathcal{E}_l) \leq 2N(2^{-l})N(2^{-l+1})\exp\Big(- \frac{2^l}{4l^4}\frac{\lambda^2}{pk^{p-1}(12rR^{-n} + \lambda)}\Big) \qquad l \geq 2. \tag{3.6}$$

In the view of the fact in Remark 2.5 that $N(\epsilon)$ is bounded and

$$N(\epsilon) \leq \exp\left(2^n N_0(\Gamma)\left[(R+2)^n + C_1\epsilon^{-\frac{np'}{\alpha p'-n}}\right]\log\frac{8D}{\epsilon}\right),$$

we have

$$\begin{aligned}N(2^{-l}) &\leq \exp\left(2^n N_0(\Gamma)\left[(R+2)^n + C_1 2^{\frac{lnp'}{\alpha p'-n}}\right]\log 2^{l+3}D\right)\\ &\leq \exp\left(2^n N_0(\Gamma)\left[(R+2)^n + C_1 2^{\frac{lnp'}{\alpha p'-n}}\right]\left[(l+3)\log 2+\log D\right]\right)\end{aligned}$$

and similarly,

$$\begin{aligned}N(2^{-l+1}) &\leq \exp\left(2^n N_0(\Gamma)\left[(R+2)^n + C_1 2^{\frac{(l-1)np'}{\alpha p'-n}}\right]\left[(l+2)\log 2+\log D\right]\right)\\ &\leq \exp\left(2^n N_0(\Gamma)\left[(R+2)^n + C_1 2^{\frac{lnp'}{\alpha p'-n}}\right]\left[(l+2)\log 2+\log D\right]\right).\end{aligned}$$

Therefore,

$$N(2^{-l})N(2^{-l+1}) \leq \exp\left(2^n N_0(\Gamma)\left[(R+2)^n + C_1 2^{\frac{lnp'}{\alpha p'-n}}\right]\left[(2l+5)\log 2+2\log D\right]\right).$$

Since $(\alpha - \frac{n}{p'}) > (n+1)$,

$$\begin{aligned}P(\mathcal{E}_l) &\leq 2\exp\left(2^n N_0(\Gamma)\left[(R+2)^n + C_1 2^{\frac{lnp'}{\alpha p'-n}}\right]\left[(2l+5)\log 2+2\log D\right]\right.\\ &\qquad\left. - \frac{2^l}{4l^4}\frac{\lambda^2}{pk^{p-1}(12rR^{-n}+\lambda)}\right)\\ &\leq 2\exp\left(2^n N_0(\Gamma)\left[(R+2)^n + C_1 2^{\frac{ln}{n+1}}\right]\left[(2l+5)\log 2+2\log D\right]\right.\\ &\qquad\left. - \frac{2^l}{4l^4}\frac{\lambda^2}{pk^{p-1}(12rR^{-n}+\lambda)}\right)\\ &= 2\exp\left[2^{\frac{n+1}{n+2}l}\left(2^n N_0(\Gamma)\left[(R+2)^n 2^{-\frac{n+1}{n+2}l} + C_1 2^{-\frac{l}{(n+1)(n+2)}}\right]\left[(2l+5)\log 2+2\log D\right]\right.\right.\\ &\qquad\left.\left. - \frac{2^{\frac{l}{n+2}}}{4l^4}\frac{\lambda^2}{pk^{p-1}(12rR^{-n}+\lambda)}\right)\right]\\ &= 2\exp\left[2^{\frac{n+1}{n+2}l}\left(2^n N_0(\Gamma)\left[(R+2)^n(2l+5)2^{-\frac{n+1}{n+2}l}\log 2 + 2(R+2)^n 2^{-\frac{n+1}{n+2}l}\log D\right.\right.\right.\\ &\qquad\left. + C_1(2l+5)2^{-\frac{l}{(n+1)(n+2)}}\log 2 + 2C_1 2^{-\frac{l}{(n+1)(n+2)}}\log D\right]\\ &\qquad\left.\left. - \frac{2^{\frac{l}{n+2}}}{4l^4}\frac{\lambda^2}{pk^{p-1}(12rR^{-n}+\lambda)}\right)\right]\\ &\leq 2\exp\left[2^{\frac{n+1}{n+2}l}\left(2^n N_0(\Gamma)\left[9(R+2)^n 2^{-\frac{2(n+1)}{n+2}}\log 2 + 2(R+2)^n 2^{-\frac{2(n+1)}{n+2}}\log D\right.\right.\right.\\ &\qquad\left. + C_1 2(n+1)(n+2)2^{-\frac{2(n+1)(n+2)-5\log 2}{2(n+1)(n+2)\log 2}} + C_1 2^{1-\frac{2}{(n+1)(n+2)}}\log D\right]\end{aligned}$$

$$- \frac{2^{\frac{4}{\log 2}}}{4\left[\frac{4(n+2)}{\log 2}\right]^4} \frac{\lambda^2}{pk^{p-1}(12rR^{-n}+\lambda)}\Big)\Big].$$

Let

$$c_1 = \frac{2^{\frac{4}{\log 2}-10}(\log 2)^4}{(n+2)^4},$$
$$c_2 = 2^n N_0(\Gamma)\Big[9(R+2)^n 2^{-\frac{2(n+1)}{n+2}}\log 2 + 2(R+2)^n 2^{-\frac{2(n+1)}{n+2}}\log D$$
$$+ C_1 2(n+1)(n+2)2^{-\frac{2(n+1)(n+2)-5\log 2}{2(n+1)(n+2)\log 2}} + C_1 2^{1-\frac{2}{(n+1)(n+2)}}\log D\Big],$$
$$\phi = \frac{\lambda^2}{pk^{p-1}(12rR^{-n}+\lambda)}.$$

Then $P(\mathcal{E}_l) \le 2\exp\left(-2^{\frac{n+1}{n+2}l}(c_1\phi - c_2)\right)$, for λ large enough such that $c_1\phi - c_2 > 0$.

Step 4: Since $\mathcal{E} \subseteq \bigcup_{l=1}^{\infty} \mathcal{E}_l$, we have

$$P(\mathcal{E}) \le \sum_{l=1}^{\infty} P(\mathcal{E}_l). \tag{3.7}$$

The series $\sum_{l=2}^{\infty} P(\mathcal{E}_l) \le \sum_{l=2}^{\infty} 2\exp\left(-2^{\frac{n+1}{n+2}l}(c_1\phi - c_2)\right)$, and a further upper bound can be obtained by the fact $\sum_{l=2}^{\infty} e^{-u^l v} \le \frac{1}{uv\log u}e^{-uv}$.

Therefore,

$$\sum_{l=2}^{\infty} P(\mathcal{E}_l) \le 2 \times \frac{1}{2^{\frac{n+1}{n+2}}(c_1\phi - c_2)\log 2^{\frac{n+1}{n+2}}}\exp\left(-2^{\frac{n+1}{n+2}}(c_1\phi - c_2)\right)$$
$$= \frac{2^{\frac{1}{n+2}}(n+2)}{(n+1)(c_1\phi - c_2)\log 2}\exp\left(-2^{\frac{n+1}{n+2}}(c_1\phi - c_2)\right).$$

Choose λ large enough such that $(c_1\phi - c_2) \ge \frac{2^{\frac{1}{n+2}}(n+2)}{(n+1)\log 2}$.

Then

$$\sum_{l=2}^{\infty} P(\mathcal{E}_l) \le e^{2^{\frac{n+1}{n+2}}c_2}\exp\{-2^{\frac{n+1}{n+2}}c_1\phi\}$$
$$\le e^{2^{\frac{n+1}{n+2}}c_2}\exp\Big(-2^{\frac{n+1}{n+2}}c_1\frac{\lambda^2}{pk^{p-1}(12rR^{-n}+\lambda)}\Big). \tag{3.8}$$

Let $a_1 = \max\left\{e^{2^{\frac{n+1}{n+2}}c_2}, N\left(\frac{1}{2}\right)\right\}$ and $b = \min\left\{2^{\frac{n+1}{n+2}}c_1, \frac{3}{4k}\right\}$. Now from (3.7), (3.5), and (3.8) we have

$$P(\mathcal{E}) \le 2a_1\exp\Big(-\frac{b}{pk^{p-1}}\frac{\lambda^2}{12rR^{-n}+\lambda}\Big).$$

Now we compute a bound for the constant a_1: consider

$$\exp\left(2^{\frac{n+1}{n+2}}c_2\right) \leq \exp\left(2^{n+1}N_0(\Gamma)\big[(R+2)^n(9\log 2+2\log D)+4C_1(n+1)(n+2)+2C_1\log D\big]\right)$$
$$\leq \exp\left(2^{n+1}N_0(\Gamma)\big[5(R+2)^n\log 4D+4C_1(n+1)(n+2)\log 4D\big]\right)$$

and since $\alpha > \frac{n}{p'}+n+1$

$$N\Big(\frac{1}{2}\Big) \leq \exp\left(2^n N_0(\Gamma)\big[(R+2)^n+2C_1\big]\log 16D\right).$$

Also

$$C_1 = \left(\frac{2}{n}\right)^n\left(\frac{4^n B^{(p'-1)}(2C)^{p'}}{w_\alpha}\right)^{\frac{n}{\alpha p'-n}}$$
$$\leq \left(\frac{4^n B^{(p'-1)}(2C)^{p'}}{w_\alpha}\right)^{\frac{1}{p'+1}}$$
$$\leq 4^{\frac{n}{p'+1}} \times 2BCw_\alpha^{-\frac{1}{p'+1}}$$

Hence, for $R \geq 2$, and $M = 2^{n+1}N_0(\Gamma)\big[2^n 5 + 4^{\frac{n}{p'+1}+\frac{3}{2}}BC(n+1)(n+2)w_\alpha^{-\frac{1}{p'+1}}\big]\log 16D$, then $a_1 \leq \exp(MR^n) := a$. This completes the proof. $\square$

Proof of Theorem 1.1. As mentioned in Section 2, it is enough to prove the result for the set $V(R,\delta)$. Put $\lambda = \frac{r\mu}{R^n}$, then

$$\mathcal{E}^c = \Big\{\sup_{f\in V(R,\delta)}\Big|\sum_{j=1}^{r} Z_j(f)\Big| \leq \frac{r\mu}{R^n}\Big\}.$$

Let $\{x_j\}$ be a random sample set such that the event $\mathcal{E}^c$ is true, then

$$\Big|\sum_{j=1}^{r}|f(x_j)|^p - \frac{r}{R^n}\int_{C_R}|f(x)|^p dx\Big| \leq \frac{r\mu}{R^n} \quad \forall\ f \in V(R,\delta)$$

$$\frac{r}{R^n}\int_{C_R}|f(x)|^p dx - \frac{r\mu}{R^n} \leq \sum_{j=1}^{r}|f(x_j)|^p \leq \frac{r}{R^n}\int_{C_R}|f(x)|^p dx + \frac{r\mu}{R^n} \tag{3.9}$$

$$\frac{r}{R^n}(1-\delta) - \frac{r\mu}{R^n} \leq \frac{r}{R^n}\int_{C_R}|f(x)|^p dx - \frac{r\mu}{R^n} \leq \sum_{j=1}^{r}|f(x_j)|^p \leq \frac{r}{R^n}\int_{C_R}|f(x)|^p dx + \frac{r\mu}{R^n} \leq \frac{r(1+\mu)}{R^n}$$

$$\frac{r}{R^n}(1-\delta-\mu) \leq \sum_{j=1}^{r}|f(x_j)|^p \leq \frac{r(1+\mu)}{R^n}.$$

Hence random sample $\{x_j\}$ satisfy the above sampling inequality with probability

$$P(\mathcal{E}^c) = 1 - P(\mathcal{E})$$
$$\geq 1 - 2a\exp\Big(-\frac{b}{pk^{p-1}}\frac{\big(\frac{r\mu}{R^n}\big)^2}{12rR^{-n}+\frac{r\mu}{R^n}}\Big)$$
$$P(\mathcal{E}^c) \geq 1 - 2a\exp\left(-\frac{b}{pk^{p-1}R^n}\frac{r\mu^2}{12+\mu}\right). \tag{3.10}$$

This completes the proof. $\square$

Remark 3.4.

1. From (3.10) one can make the probability close to 1 by taking a sufficiently large sample size.
2. The sampling inequality (1.2) is true for sufficiently large $\lambda = \frac{r\mu}{R^n}$ such that $(c_1\phi - c_2) \geq \frac{2^{\frac{1}{n+2}}(n+2)}{(n+1)\log 2}$, i.e.

$$r \geq \frac{pk^{p-1}R^n(12+\mu)}{c_1\mu^2}\Big[\frac{2^{\frac{1}{n+2}}(n+2)}{(n+1)\log 2} + c_2\Big] = \mathcal{O}(R^{2n}).$$

Example. Let $\phi(x,y) = \frac{2}{\sqrt{3}}\max\{1-2|x|-2|y|, 0\}$ and $supp(\phi) \subseteq \big[-\frac{1}{2}, \frac{1}{2}\big]^2$.

Consider $\Lambda = \{(\alpha,\beta)\} \subset \mathbb{R}^2$ be a relatively separated countable collection of points with gap greater than or equal to 1.

Now for each distinct $(\alpha_1,\beta_1), (\alpha_2,\beta_2) \in \Lambda$, we have

$$\int_{\mathbb{R}^2} \phi(x-\alpha_1, y-\beta_1)\phi(x-\alpha_2, y-\beta_2)\, dxdy = 0$$

and

$$\sum_{(\alpha,\beta)\in\Lambda} \sup_{x,y\in[-\frac{1}{2},\frac{1}{2}]} |\phi(x-\alpha, y-\beta)|^2 = \frac{4}{3} < \infty.$$

This implies the space $V_\phi := \Big\{\sum_{(\alpha,\beta)\in\Lambda} c_{\alpha,\beta}\phi(\cdot-\alpha, \cdot-\beta) : c = (c_{\alpha,\beta}) \in \ell^2(\Lambda)\Big\} \subseteq L^2(\mathbb{R}^2)$ is an image of an idempotent integral operator and the kernel defined by

$$K((x_1,y_1),(x_2,y_2)) = \sum_{(\alpha,\beta)\in\Lambda} \phi(x_1-\alpha, y_1-\beta)\phi(x_2-\alpha, y_2-\beta),$$

is symmetric and satisfy (2.2), see [15]. Further, it is easy to show that the kernel K satisfies the decay condition (2.3). Indeed, we observe that

$$|\phi(x,y)| \leq e^{-(x^2+y^2)},$$

so we get

$$\begin{aligned}
|K((x_1,y_1),(x_2,y_2))| &\leq \sum_{(\alpha,\beta)\in\Lambda} |\phi(x_1-\alpha, y_1-\beta)\phi(x_2-\alpha, y_2-\beta)| \\
&\leq \frac{4}{3}\sum_{(\alpha,\beta)\in\Lambda} e^{-((x_1-\alpha)^2+(y_1-\beta)^2)}e^{-((x_2-\alpha)^2+(y_2-\beta)^2)} \\
&\leq \frac{4}{3}\sum_{(\alpha,\beta)\in\Lambda} e^{-\frac{1}{4}((3x_1^2-2x_1x_2+3x_2^2)+(3y_1^2-2y_1y_2+3y_2^2))}e^{-2(\alpha-\frac{x_1+x_2}{2})^2-2(\beta-\frac{y_1+y_2}{2})^2} \\
&\leq \frac{4}{3}e^{-\frac{1}{4}\|(x_1,y_1)-(x_2,y_2)\|_2^2}\int_{\mathbb{R}^2} e^{-2(z_1-\frac{x_1+x_2}{2})^2-2(z_2-\frac{y_1+y_2}{2})^2}\, dz_1dz_2 \\
&\leq \frac{\pi}{3}e^{-\frac{1}{4}\|(x_1,y_1)-(x_2,y_2)\|_2^2}.
\end{aligned}$$

Since the collection $\{\phi(\cdot - \alpha, \cdot - \beta) : (\alpha, \beta) \in \Lambda\}$ is an orthonormal basis for $V(\phi)$, it follows that it is a tight frame for V_ϕ. Therefore, the frame bound constant B is equal to 1.

For given $\epsilon > 0$, the sampling inequality (1.2) holds with minimum probability $1 - \epsilon$ if

$$2a \exp\left(-\frac{b}{pk^{p-1}R^2}\frac{r\mu^2}{12+\mu}\right) < \epsilon.$$

Here $k = \sup_{x \in C_R} \|K(x, \cdot)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{2\pi\sqrt{\pi}}{3}$ and $b = \frac{2^{\frac{3}{4}+\frac{4}{\log 2}}(\log 2)^4}{2^{18}} > \frac{(\log 2)^4}{2^4}$.

Thus

$$\begin{aligned} &\exp\left(MR^2 - \frac{(\log 2)^4}{2^4} \times \frac{3}{4\pi\sqrt{\pi}R^2} \times \frac{r\mu^2}{12+\mu}\right) < \frac{\epsilon}{2}, \\ \implies &\frac{(\log 2)^4}{2^4} \times \frac{3}{4\pi\sqrt{\pi}R^2} \times \frac{r\mu^2}{12+\mu} - MR^2 > \log\frac{2}{\epsilon}, \\ \implies &r > \frac{64\pi\sqrt{\pi}(12+\mu)}{3\mu^2(\log 2)^4} R^2\left(MR^2 + \log\frac{2}{\epsilon}\right). \end{aligned}$$

Therefore, if we choose the number of sample r satisfying the above inequality, then the stable set of sampling is true for the set $V_\phi(R, \delta)$ with high probability.

4. Reconstruction algorithm

As mentioned in [3], the set $V(R, \delta)$ is neither a subspace nor a convex set, so we cannot employ frame or projection algorithms for function reconstruction. At the same time, we observe from Lemma 2.1 and 2.2 that the finite-dimensional space V_N is an approximation of $V(R, \delta)$. Thus we discuss a reconstruction algorithm for functions in V_N from its random sample (see [20]).

Let N be a fixed positive integer and

$$V_N^\star = \{f \in V_N : \|f\|_{L^p(\mathbb{R}^n)} = 1\}.$$

Then $V_N^\star$ is totally bounded with respect to $\|\cdot\|_{L^\infty(C_R)}$ and the number of open balls of radius ϵ that covers $V_N^\star$ is bounded by $\exp\left(N_0(\Gamma)N^n \log\frac{8D}{\epsilon}\right)$. Now we follow the same lines of proof of Theorem 3.3 to show that

$$P\Big(\sup_{f \in V_N^\star}\Big|\sum_{j=1}^{r} Z_j(f)\Big| \geq \lambda\Big) \leq 2a_N \exp\Big(-\frac{b}{pk^{p-1}}\frac{\lambda^2}{12rR^{-n}+\lambda}\Big), \tag{4.1}$$

where the constant $a_N = \exp(M_1 N^n)$ with $M_1 = 2^{2-\frac{2(n+1)}{n+2}} N_0(\Gamma) \log 16D$.

Consequently, we derive stable set of sampling sets for the finite-dimensional space V_N with some additional assumption on the frame sequence $\{\phi_\gamma\}$ of V.

Theorem 4.1. *Let $\{x_j : j = 1, \dots, r\}$ be a sequence of i.i.d. random variables uniformly drawn from C_R and suppose the set of functions $\{\phi_\gamma\}_{\gamma \in \Gamma \cap C_N}$ is linearly independent over C_R, i.e. there exists a constant $\sigma > 0$ such that for all $c = \{c_\gamma\}_{\gamma \in \Gamma \cap C_N}$,*

$$\Big\|\sum_{\gamma \in \Gamma \cap C_N} c_\gamma \phi_\gamma\Big\|_{L^p(C_R)}^p \geq \sigma \sum_{\gamma \in \Gamma \cap C_N} |c_\gamma|^p.$$

Then for $\mu \in (0, \sigma A)$

$$\frac{r}{R^n}(\sigma A - \mu)\|f\|^p_{L^p(\mathbb{R}^n)} \leq \sum_{j=1}^{r} |f(x_j)|^p \leq \frac{r}{R^n}(1+\mu)\|f\|^p_{L^p(\mathbb{R}^n)} \tag{4.2}$$

holds for every $f \in V_N$ *with probability* $1 - 2a_N \exp\Big(-\frac{b}{pk^{p-1}}\frac{\lambda^2}{12rR^{-n}+\lambda}\Big)$.

Proof. Let $\lambda = \frac{r\mu}{R^n}$, then for all $f \in V_N \setminus \{0\}$

$$\Big|\sum_{j=1}^{r} Z_j\Big(\frac{f}{\|f\|_{L^p(\mathbb{R}^n)}}\Big)\Big| \leq \frac{r\mu}{R^n}$$

with probability at least $1 - 2a_N \exp\Big(-\frac{b}{pk^{p-1}}\frac{\lambda^2}{12rR^{-n}+\lambda}\Big)$, i.e. with same probability bound

$$\Big|\sum_{j=1}^{r} Z_j(f)\Big| \leq \frac{r\mu}{R^n}\|f\|^p_{L^p(\mathbb{R}^n)} \quad \forall f \in V_N$$

$$\Big|\sum_{j=1}^{r} |f(x_j)|^p - \frac{r}{R^n}\|f\|^p_{L^p(C_R)}\Big| \leq \frac{r\mu}{R^n}\|f\|^p_{L^p(\mathbb{R}^n)}$$

$$\frac{r}{R^n}(\|f\|^p_{L^p(C_R)} - \mu\|f\|^p_{L^p(\mathbb{R}^n)}) \leq \sum_{j=1}^{r} |f(x_j)|^p \leq \frac{r}{R^n}(\|f\|^p_{L^p(C_R)} + \mu\|f\|^p_{L^p(\mathbb{R}^n)})$$

Since $\{\phi_\gamma\}_{\gamma \in \Gamma \cap C_N}$ are linearly independent over C_R and every $f \in V_N$ satisfy (2.7), then

$$\|f\|^p_{L^p(C_R)} \geq \sigma A\|f\|^p_{L^p(\mathbb{R}^n)}$$

This completes the proof. □

In the following theorem, we give a reconstruction algorithm for finite-dimensional space. We follow the ideas of Yang [20] where the reconstruction algorithm is discussed for finite-dimensional shift-invariant space from random samples. A similar idea can also be found in [14,18].

Theorem 4.2. *Under the assumptions of Theorem 4.1, there exists a set of reconstruction functions* $(S_j(x))_{j=1}^r$ *such that for all* $f \in V_N$*,*

$$f(x) = \sum_{j=1}^{r} f(x_j)S_j(x)$$

holds with probability at least $1 - 2a_N \exp\Big(-\frac{b}{pk^{p-1}R^n}\frac{r\mu^2}{12+\mu}\Big)$.

Proof. Let $f = \sum\limits_{\gamma \in \Gamma \cap C_N} c_\gamma \phi_\gamma$ be an arbitrary function in V_N and $(x_j, f(x_j))_{j=1}^r$ be a given random data.

Then we have a system of linear equations

$$f(x_j) = \sum_{\gamma \in \Gamma \cap C_N} c_\gamma \phi_\gamma(x_j) \quad 1 \leq j \leq r$$

with the unknown coefficient $c = (c_\gamma)^T_{\gamma_{\gamma\in\Gamma\cap C_N}}$ (need to be determined). This linear system can be rewritten as $Uc = b$, where U is a rectangular matrix with entry $U_{j,\gamma} = \phi_\gamma(x_j)$ and $b = (f(x_j))^T_{1\le j\le r}$ is a column matrix.

Now by (4.2) and (2.7)

$$\|Uc\|^p_{\ell^p(\Gamma\cap C_N)} = \sum_{j=1}^{r} |f(x_j)|^p \ge \frac{r(\sigma A-\mu)}{BR^n}\|c\|^p_{\ell^p(\Gamma\cap C_N)} \quad \forall c \in \ell^p(\Gamma\cap C_N). \tag{4.3}$$

This implies U^TU is invertible and $c = (U^TU)^{-1}U^Tb$.

Define $\Theta(x) = (\phi_\gamma(x))^T_{\gamma\in\Gamma\cap C_N}$ and $(S_j(x))^T_{1\le j\le r} = U(U^TU)^{-1}\Theta$. Then we have the following reconstruction formula

$$f(x) = \sum_{j=1}^{r} f(x_j)S_j(x), \quad \forall\, x \in \mathbb{R}^n$$

with probability at least $1 - 2a_N \exp\left(-\frac{b}{pk^{p-1}R^n}\frac{r\mu^2}{12+\mu}\right)$, as the relation (4.3) valid with the same probability bound. $\square$

Acknowledgments

We thank the anonymous reviewers for meticulously reading the manuscript, and giving us valuable comments and suggestions which helped us to improve the quality of the paper. The first author acknowledges Council of Scientific and Industrial Research, India for the financial support with file no. 09/086(1309)/2018-EMR-I. The second author acknowledges the financial support through project no. CRG/2019/002412 funded by the Department of Science and Technology, Government of India.

References

[1] A. Aldroubi, K. Gröchenig, Nonuniform sampling and reconstruction in shift-invariant spaces, SIAM Rev. 43 (4) (2001) 585–620.

[2] R.F. Bass, K. Gröchenig, Random sampling of multivariate trigonometric polynomials, SIAM J. Math. Anal. 36 (3) (2005) 773–795.

[3] R.F. Bass, K. Gröchenig, Random sampling of bandlimited functions, Isr. J. Math. 177 (1) (2010) 1–28.

[4] R.F. Bass, K. Gröchenig, Relevant sampling of band-limited functions, Ill. J. Math. 57 (1) (2013) 43–58.

[5] G. Bennett, Probability inequalities for the sum of independent random variables, J. Am. Stat. Assoc. 57 (297) (1962) 33–45.

[6] P.L. Butzer, R.L. Stens, Sampling theory for not necessarily band-limited functions: a historical overview, SIAM Rev. 34 (1) (1992) 40–53.

[7] E.J. Candès, J. Romberg, T. Tao, Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inf. Theory 52 (2) (2006) 489–509.

[8] E.J. Candès, J.K. Romberg, T. Tao, Stable signal recovery from incomplete and inaccurate measurements, Commun. Pure Appl. Math. 59 (8) (2006) 1207–1223. A Journal Issued by the Courant Institute of Mathematical Sciences.

[9] S.H. Chan, T. Zickler, Y.M. Lu, Monte Carlo non-local means: random sampling for large-scale image filtering, IEEE Trans. Image Process. 23 (8) (2014) 3711–3725.

[10] F. Cucker, S. Smale, On the mathematical foundations of learning, Bull. Am. Math. Soc. 39 (1) (2002) 1–49.

[11] F. Cucker, D.X. Zhou, Learning Theory: an Approximation Theory Viewpoint, vol. 24, Cambridge University Press, 2007.

[12] Y. Eldar, Compressed sensing of analog signals in shift-invariant spaces, IEEE Trans. Signal Process. 57 (8) (2009) 2986–2997.

[13] H. Führ, J. Xian, Relevant sampling in finitely generated shift-invariant spaces, J. Approx. Theory 240 (2019) 1–15.

[14] K. Gröchenig, H. Schwab, Fast local reconstruction methods for nonuniform sampling in shift-invariant spaces, SIAM J. Matrix Anal. Appl. 24 (4) (2003) 899–913.

[15] M.Z. Nashed, Q. Sun, Sampling and reconstruction of signals in a reproducing kernel subspace of $L^p(\mathbb{R}^d)$, J. Funct. Anal. 258 (7) (2010) 2422–2452.

[16] A.M. Olevskii, A. Ulanovskii, Functions with Disconnected Spectrum, vol. 65, American Mathematical Soc., 2016.

[17] T. Poggio, S. Smale, The mathematics of learning: dealing with data, Not. Am. Math. Soc. 50 (5) (2003) 537–544.

[18] R. Radha, S. Sivananthan, Local reconstruction of a function from a non-uniform sampled data, Appl. Numer. Math. 59 (2) (2009) 393–403.
[19] S. Smale, D.-X. Zhou, Shannon sampling and function reconstruction from point values, Bull. Am. Math. Soc. 41 (3) (2004) 279–305.
[20] J. Yang, Random sampling and reconstruction in multiply generated shift-invariant spaces, Anal. Appl. 17 (02) (2019) 323–347.
[21] J. Yang, X. Tao, Random sampling and approximation of signals with bounded derivatives, J. Inequal. Appl. 2019 (1) (2019) 107.
[22] J. Yang, W. Wei, Random sampling in shift invariant spaces, J. Math. Anal. Appl. 398 (1) (2013) 26–34.