

RATIONAL TETRA-INNER FUNCTIONS AND THE SPECIAL VARIETY OF THE TETRABLOCK

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ABSTRACT. The set

$$\overline{\mathbb{E}} = \{x \in \mathbb{C}^3 : 1 - x_1z - x_2w + x_3zw \neq 0 \text{ whenever } |z| < 1, |w| < 1\}$$

is called the tetrablock and has intriguing complex-geometric properties. It is polynomially convex, nonconvex and starlike about 0. It has a group of automorphisms parametrised by $\text{Aut } \mathbb{D} \times \text{Aut } \mathbb{D} \times \mathbb{Z}_2$ and its distinguished boundary $b\overline{\mathbb{E}}$ is homeomorphic to the solid torus $\overline{\mathbb{D}} \times \mathbb{T}$. It has a special subvariety

$$\mathcal{R}_{\overline{\mathbb{E}}} = \{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : x_1x_2 = x_3\},$$

called *the royal variety* of $\overline{\mathbb{E}}$. $\mathcal{R}_{\overline{\mathbb{E}}}$ is a complex geodesic of $\overline{\mathbb{E}}$ and it is invariant under all automorphisms of $\overline{\mathbb{E}}$. We make use of these geometric properties of $\overline{\mathbb{E}}$ to develop an explicit structure theory for the rational maps from the unit disc \mathbb{D} to $\overline{\mathbb{E}}$ that map the unit circle \mathbb{T} to the distinguished boundary $b\overline{\mathbb{E}}$ of $\overline{\mathbb{E}}$. Such maps are called rational $\overline{\mathbb{E}}$ -inner functions. We call the points $\lambda \in \overline{\mathbb{D}}$ such that $x(\lambda) \in \mathcal{R}_{\overline{\mathbb{E}}}$ the *royal nodes* of x . We describe the construction of rational $\overline{\mathbb{E}}$ -inner functions of prescribed degree from the zeros of x_1 and x_2 and the royal nodes of x . The proof of this theorem is constructive: it gives an algorithm for the construction of a 3-parameter family of such functions x subject to the computation of Fejér-Riesz factorizations of certain non-negative functions on the circle. We show that, for each nonconstant rational $\overline{\mathbb{E}}$ -inner function x , either $x(\overline{\mathbb{D}}) \subseteq \mathcal{R}_{\overline{\mathbb{E}}} \cap \overline{\mathbb{E}}$ or $x(\overline{\mathbb{D}})$ meets $\mathcal{R}_{\overline{\mathbb{E}}}$ exactly $\deg(x)$ times. We study convex subsets of the set \mathcal{J} of all rational $\overline{\mathbb{E}}$ -inner functions and extreme points of \mathcal{J} . We show that whether a rational $\overline{\mathbb{E}}$ -inner function x is an extreme point of \mathcal{J} depends on how many royal nodes of x lie on \mathbb{T} .

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1. INTRODUCTION

Some unsolved problems in H^∞ control theory require a deep understanding of “inner mappings” from \mathbb{D} to certain domains in \mathbb{C}^d with $d > 1$. For example, a special case of the problem of robust stabilization under structured uncertainty, or the μ -synthesis problem [13, 14, 3, 5], leads naturally to a class of μ -synthesis domains. A typical member of this class of domains is

$$\mathbb{E} = \{x \in \mathbb{C}^3 : 1 - x_1 z - x_2 w + x_3 zw \neq 0 \text{ whenever } |z| \leq 1, |w| \leq 1\}$$

the *open tetrablock*, as was observed by Abouhajar, White and Young in [1]. The complex geometry of \mathbb{E} was further developed in [15, 16, 17, 20] and associated operator theory in [9, 8]. The solvability of the μ -synthesis problem connected to \mathbb{E} can be expressed in terms of the existence of rational inner functions from the open unit disc \mathbb{D} in the complex plane \mathbb{C} to the closure of \mathbb{E} subject to interpolation conditions [10].

Recall that a classical *rational inner function* is a rational map f from the unit disc \mathbb{D} to its closure $\overline{\mathbb{D}}$ with the property that f maps the unit circle \mathbb{T} into itself. See [12] for a survey of results, linking inner functions and operator theory. We denote the closure of \mathbb{E} by $\overline{\mathbb{E}}$ and we define a *rational $\overline{\mathbb{E}}$ -inner or (tetra-inner) function* to be a rational analytic map $x : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ such that x maps \mathbb{T} into the distinguished boundary $b\overline{\mathbb{E}}$ of $\overline{\mathbb{E}}$. Here, $b\overline{\mathbb{E}}$ is the smallest closed subset of $\overline{\mathbb{E}}$ on which every continuous function on $b\overline{\mathbb{E}}$ that is analytic in \mathbb{E} attains its maximum modulus. Of course, rational $\overline{\mathbb{E}}$ -inner functions have many similarities with rational inner functions. On the other hand, the complex geometry of $\overline{\mathbb{E}}$ is richer than that of $\overline{\mathbb{D}}$, and so rational $\overline{\mathbb{E}}$ -inner functions have some striking differences from rational inner functions.

Here are some points of difference between well-studied domains in \mathbb{C}^3 , such as the tridisc \mathbb{D}^3 and the Euclidean ball \mathbb{B}_3 on the one hand and \mathbb{E} on the other. Firstly, whereas \mathbb{D}^3 and \mathbb{B}_3 are homogeneous (so that the holomorphic automorphisms of these domains act transitively), \mathbb{E} is inhomogeneous [20].

Secondly, the distinguished boundary of \mathbb{E} differs markedly in its topological properties from those of \mathbb{D}^3 and \mathbb{B}_3 . The distinguished boundaries of \mathbb{D}^3 and \mathbb{B}_3 are the 3-dimensional torus and the 5-sphere respectively. They are smooth manifolds without boundary. The distinguished boundary $b\overline{\mathbb{E}}$ of \mathbb{E} is homeomorphic to the solid torus $\overline{\mathbb{D}} \times \mathbb{T}$, which has a boundary. For a rational $\overline{\mathbb{E}}$ -inner function x the curve $x(e^{it})$, $0 \leq t < 2\pi$, lies in $b\overline{\mathbb{E}}$ and may or may not touch the boundary of $b\overline{\mathbb{E}}$, and so the algebraic and geometric properties of x are dependent on that.

We call the set

$$(1.1) \quad \mathcal{R}_{\overline{\mathbb{E}}} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 x_2 = x_3\}$$

the *royal variety* of the tetrablock. The complex geodesic $\mathcal{R}_{\overline{\mathbb{E}}} \cap \mathbb{E}$ is invariant under the group of biholomorphic automorphisms of \mathbb{E} [20]. This paper shows that the variety $\mathcal{R}_{\overline{\mathbb{E}}}$ plays a central role in the function theory of \mathbb{E} . The intersection $\mathcal{R}_{\overline{\mathbb{E}}} \cap b\overline{\mathbb{E}}$ is exactly the

boundary of $b\overline{\mathbb{E}}$, that is, $\{(x_1, x_2, x_1x_2) \in \mathbb{C}^3 : |x_1| = |x_2| = 1\}$ [1, Theorem 7.1], which is homeomorphic to the 2-torus $\mathbb{T} \times \mathbb{T}$.

These geometric properties of \mathbb{E} lead to very interesting facts in the theory of rational $\overline{\mathbb{E}}$ -inner functions that do not have analogues in the theory of classical inner functions.

One of the main theorems of this paper is the following.

Theorem 1.1. *Let x be a nonconstant rational $\overline{\mathbb{E}}$ -inner function. Then either $x(\overline{\mathbb{D}}) \subseteq \mathcal{R}_{\overline{\mathbb{E}}} \cap \overline{\mathbb{E}}$ or $x(\overline{\mathbb{D}})$ meets $\mathcal{R}_{\overline{\mathbb{E}}}$ exactly $\deg(x)$ times.*

It is Theorem 5.12. Here $\deg(x)$ is the degree of x . In Subsection 4.2 we define $\deg(x)$ in a natural way by means of fundamental groups. In Proposition 4.14 we show that, for any rational $\overline{\mathbb{E}}$ -inner function $x = (x_1, x_2, x_3)$, $\deg(x)$ is equal to the degree $\deg(x_3)$ of the finite Blaschke product x_3 . The precise way of counting the number of times that $x(\overline{\mathbb{D}})$ meets $\mathcal{R}_{\overline{\mathbb{E}}}$ is also described in Section 5. We call the points $\lambda \in \overline{\mathbb{D}}$ such that $x(\lambda) \in \mathcal{R}_{\overline{\mathbb{E}}}$ the *royal nodes* of x and, for such λ , we call $x(\lambda)$ a *royal point* of x .

Another main result is the construction of rational $\overline{\mathbb{E}}$ -inner functions of prescribed degree from the zeros of x_1 and x_2 and the royal nodes of x . One can consider this result as an analogue of the expression for a finite Blaschke product in terms of its zeros. This result is proved in Theorem 5.17.

Theorem 1.2. *Let n be a positive integer. Suppose that $\alpha_1^1, \dots, \alpha_{k_1}^1 \in \overline{\mathbb{D}}$ and $\alpha_1^2, \dots, \alpha_{k_2}^2 \in \overline{\mathbb{D}}$, where $k_1 + k_2 = n$. Suppose that $\sigma_1, \dots, \sigma_n \in \overline{\mathbb{D}}$ are distinct from the points of the set $\{\alpha_j^i, j = 1, \dots, k_i, i = 1, 2\} \cap \mathbb{T}$. Then there exists a rational $\overline{\mathbb{E}}$ -inner function $x = (x_1, x_2, x_3) : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ such that*

- (1) *the zeros of x_1 in $\overline{\mathbb{D}}$, repeated according to multiplicity, are $\alpha_1^1, \dots, \alpha_{k_1}^1$;*
- (2) *the zeros of x_2 in $\overline{\mathbb{D}}$, repeated according to multiplicity, are $\alpha_1^2, \dots, \alpha_{k_2}^2$;*
- (3) *the royal nodes of x are $\sigma_1, \dots, \sigma_n \in \overline{\mathbb{D}}$, with repetition according to the multiplicity of the nodes.*

This function x can be constructed as follows. Let $t_+ > 0$ and let $t \in \mathbb{C} \setminus \{0\}$. Let R be defined by

$$R(\lambda) = t_+ \prod_{j=1}^n (\lambda - \sigma_j)(1 - \overline{\sigma_j}\lambda).$$

Let E_1 be defined by

$$E_1(\lambda) = t \prod_{j=1}^{k_1} (\lambda - \alpha_j^1) \prod_{j=1}^{k_2} (1 - \overline{\alpha_j^2}\lambda).$$

Then the following statements hold:

- (i) *There exists an outer polynomial D of degree at most n such that*

$$\lambda^{-n} R(\lambda) + |E_1(\lambda)|^2 = |D(\lambda)|^2$$

for all $\lambda \in \mathbb{T}$.

- (ii) *The function x defined by*

$$x = \left(\frac{E_1}{D}, \frac{E_1^{\sim n}}{D}, \frac{D^{\sim n}}{D} \right) \text{ where, for any polynomial } D, D^{\sim n}(\lambda) = \overline{\lambda^n D\left(\frac{1}{\lambda}\right)},$$

is a rational $\overline{\mathbb{E}}$ -inner function such that the degree of x is equal to n and conditions (1), (2) and (3) hold. The royal polynomial R_x of x is equal to R .

Here the royal polynomial of x is defined as $R_x = [D^{\sim n}D - E_1E_1^{\sim n}]$. The proof of this theorem is constructive: it gives an algorithm for the construction of a 3-parameter family of such functions x .

In Section 6 we study convex subsets of the set \mathcal{J} of all rational $\overline{\mathbb{E}}$ -inner functions and extremality. We show that the set \mathcal{J} is not convex. On the other hand, the subset of \mathcal{J} with a fixed inner function x_3 is convex (Theorem 6.5). Recall that the distinguished boundary of the tridisc \mathbb{D}^3 contain no line segments. Thus every inner function in the set of analytic functions $\text{Hol}(\mathbb{D}, \mathbb{D}^3)$ from \mathbb{D} to \mathbb{D}^3 is an extreme point of $\text{Hol}(\mathbb{D}, \mathbb{D}^3)$. However, this property is in clear contrast with the situation in the tetrablock. We show that whether a rational inner function x is an extreme point of \mathcal{J} depends on how many royal nodes of x lie on \mathbb{T} .

Theorem 1.3. *Let x be a rational $\overline{\mathbb{E}}$ -inner function and let x have n royal nodes where k of them are in \mathbb{T} . If $2k \leq n$, then x is not an extreme point of \mathcal{J} .*

The way of counting the number of royal nodes was introduced in Section 5. In Proposition 6.21 we provide a class of extreme functions of the set \mathcal{J} .

In [2] there is a construction of the general rational $\overline{\mathcal{E}}$ -inner function $x = (x_1, x_2, x_3)$ of degree n , in terms of different data, namely, the royal nodes of x and royal values of x . The algorithm [2] for the construction of x exploits a known construction of the finite Blaschke products of given degree which satisfy some interpolation conditions with the aid of a Pick matrix formed from the interpolation data.

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2. THE TETRABLOCK \mathbb{E}

Definition 2.1. [1] *The open tetrablock is the domain defined by*

$$\mathbb{E} = \{x \in \mathbb{C}^3 : 1 - x_1z - x_2w + x_3zw \neq 0 \text{ for } |z| \leq 1, |w| \leq 1\}.$$

Despite the fact that \mathbb{E} is not convex, its intersection with \mathbb{R}^3 is. It is proved in [1] that $\mathbb{E} \cap \mathbb{R}^3$ is the open tetrahedron with the vertices $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$ and $(-1, -1, 1)$. The following function plays an important role in the study of the tetrablock.

Definition 2.2. *For $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ and $z \in \mathbb{C}$ we define*

$$\Psi(z, x) = \frac{x_3z - x_1}{x_2z - 1}, \quad \text{whenever } x_2z \neq 1.$$

Remark 2.3. In the case that $x_3 = x_1x_2$, $z \in \mathbb{C}$,

$$\Psi(z, x) = \frac{x_1x_2z - x_1}{x_2z - 1} = \frac{x_1(x_2z - 1)}{x_2z - 1} = x_1.$$

Theorem 2.4. [1, Theorem 2.2] *Let $x \in \mathbb{C}^3$. The following are equivalent*

- (1) $x \in \mathbb{E}$;
- (2) $\|\Psi(\cdot, x)\|_{H^\infty} < 1$ and if $x_1x_2 = x_3$, then, in addition, $|x_2| < 1$;
- (3) $|x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3| < 1 - |x_3|^2$;
- (4) there exists a 2×2 matrix $A = [a_{ij}]$ such that $\|A\| < 1$ and $x = (a_{11}, a_{22}, \det(A))$;
- (5) $|x_3| < 1$ and there exist $\beta_1, \beta_2 \in \mathbb{C}$ such that $|\beta_1| + |\beta_2| < 1$ and

$$x_1 = \beta_1 + \overline{\beta_2}x_3, \quad x_2 = \beta_2 + \overline{\beta_1}x_3.$$

Theorem 2.5. [1, Theorem 2.4] *Let $x \in \mathbb{C}^3$. The following are equivalent*

- (1) $x \in \overline{\mathbb{E}}$;

- (2) $\|\Psi(., x)\|_{H^\infty} \leq 1$ and if $x_1 x_2 = x_3$, then, in addition, $|x_2| \leq 1$;
- (3) $|x_1 - \overline{x}_2 x_3| + |x_2 - \overline{x}_1 x_3| \leq 1 - |x_3|^2$ and if $|x_3| = 1$ then, in addition, $|x_1| \leq 1$;
- (4) there exists a 2×2 matrix $A = [a_{ij}]$ such that $\|A\| \leq 1$ and $x = (a_{11}, a_{22}, \det(A))$;
- (5) $|x_3| \leq 1$ and there exist $\beta_1, \beta_2 \in \mathbb{C}$ such that $|\beta_1| + |\beta_2| \leq 1$ and

$$x_1 = \beta_1 + \overline{\beta}_2 x_3, \quad x_2 = \beta_2 + \overline{\beta}_1 x_3.$$

2.1. The tetrablock and the μ_{Diag} -synthesis problem. The tetrablock is associated with the μ_{Diag} -synthesis problem from \mathbb{D} to $\mathbb{C}^{2 \times 2}$. The structured singular value in this case is defined by

$$(2.1) \quad \mu_{\text{Diag}}(A) = \frac{1}{\inf\{\|X\| : X \in \text{Diag}, \det(I - AX) = 0\}},$$

where

$$\text{Diag} := \left\{ \begin{bmatrix} z & 0 \\ 0 & w \end{bmatrix} : z, w \in \mathbb{C} \right\}.$$

We set $\mu_{\text{Diag}}(A) = 0$ if $(I - AX)$ is non-singular for all $X \in \text{Diag}$.

Definition 2.6. We define the map $\pi : \mathbb{C}^{2 \times 2} \rightarrow \mathbb{C}^3$ for a matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ in $\mathbb{C}^{2 \times 2}$ to be

$$\pi(A) = (a_{11}, a_{22}, \det(A)),$$

and Σ to be

$$\Sigma := \{A \in \mathbb{C}^{2 \times 2} : \mu_{\text{Diag}}(A) < 1\},$$

where $\mu_{\text{Diag}}(A)$ is defined by equation (2.1).

Theorem 2.7. [1, Theorem 9.2] Suppose that $\lambda_1, \dots, \lambda_n \in \mathbb{D}$ are distinct points and $A_k = [a_{ij}^k] \in \Sigma$ are such that $a_{11}^k a_{22}^k \neq \det(A_k)$, $1 \leq k \leq n$. The following conditions are equivalent.

- (1) There exists an analytic function $F : \mathbb{D} \rightarrow \Sigma$ such that $F(\lambda_k) = A_k$, $1 \leq k \leq n$;
- (2) There exists an analytic function $\varphi : \mathbb{D} \rightarrow \mathbb{E}$ such that $\varphi(\lambda_k) = \pi(A_k)$, that is,

$$\varphi(\lambda_k) = (a_{11}^k, a_{22}^k, \det(A_k)), \quad k = 1, 2, \dots, n.$$

In the following theorem the authors give a necessary and sufficient condition for the solvability of a μ_{Diag} -synthesis problem by a rational $\overline{\mathbb{E}}$ -inner function.

Theorem 2.8. [10, Theorem 1.1 and Theorem 8.1] Let $\lambda_1, \dots, \lambda_n$ be distinct points in \mathbb{D} and let $A_k = [a_{ij}^k] \in \mathbb{C}^{2 \times 2}$ be such that $a_{11}^k a_{22}^k \neq \det(A_k)$, $1 \leq k \leq n$. Let

$$(x_1^k, x_2^k, x_3^k) = (a_{11}^k, a_{22}^k, \det(A_k)), \quad 1 \leq k \leq n.$$

The following two conditions are equivalent.

- (1) There exists an analytic 2×2 matrix function F in \mathbb{D} such that

$$F(\lambda_k) = A_k \quad \text{for } k = 1, \dots, n,$$

and

$$\mu_{\text{Diag}}(F(\lambda)) \leq 1 \quad \text{for all } \lambda \in \mathbb{D};$$

- (2) there exists a rational $\overline{\mathbb{E}}$ -inner function $x : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ such that

$$x(\lambda_k) = (x_1^k, x_2^k, x_3^k) \quad \text{for } k = 1, \dots, n.$$

Therefore, the understanding of rational $\overline{\mathbb{E}}$ -inner functions will be useful for such μ -synthesis problems.

2.2. The distinguished boundary of the tetrablock.

Theorem 2.9. [1, Theorem 2.9] $\overline{\mathbb{E}}$ is polynomially convex.

Therefore, there exists a distinguished boundary $b\overline{\mathbb{E}}$ of \mathbb{E} . Let $A(\mathbb{E})$ be the algebra of continuous scalar functions on $\overline{\mathbb{E}}$ that are holomorphic on \mathbb{E} endowed with the supremum norm. If there is a function $f \in A(\mathbb{E})$ and a point p in $\overline{\mathbb{E}}$ such that $f(p) = 1$ and $|f(x)| < 1$ for all $x \in \overline{\mathbb{E}} \setminus \{p\}$, then $p \in b\overline{\mathbb{E}}$ and is called a *peak point* of $\overline{\mathbb{E}}$, and the function f is called a *peaking function* for p .

Theorem 2.10. [1, Theorem 7.1] For $x \in \mathbb{C}^3$ the following are equivalent.

- (1) $x_1 = \overline{x}_2 x_3, |x_3| = 1$ and $|x_2| \leq 1$;
- (2) either $x_1 x_2 \neq x_3$ and $\Psi(\cdot, x)$ is an automorphism of \mathbb{D} or $x_1 x_2 = x_3$ and $|x_1| = |x_2| = |x_3| = 1$;
- (3) x is a peak point of $\overline{\mathbb{E}}$;
- (4) there exists a 2×2 unitary matrix U such that $x = \pi(U)$;
- (5) there exists a symmetric 2×2 unitary matrix U such that $x = \pi(U)$;
- (6) $x \in b\overline{\mathbb{E}}$;
- (7) $x \in \overline{\mathbb{E}}$ and $|x_3| = 1$.

Lemma 2.11. Let $x = (x_1, x_2, x_3) \in \mathbb{C}^3$. Then $x \in b\overline{\mathbb{E}}$ if and only if

$$x_2 = \overline{x}_1 x_3, \quad |x_3| = 1 \quad \text{and} \quad |x_1| \leq 1.$$

Proof. By Theorem 2.10 (1),

$$x \in b\overline{\mathbb{E}} \quad \Leftrightarrow \quad x_1 = \overline{x}_2 x_3, \quad |x_3| = 1 \quad \text{and} \quad |x_2| \leq 1.$$

Since $|x_3| = 1$ this implies $\overline{x}_3 x_3 = 1$. Now, since $x \in b\overline{\mathbb{E}}$,

$$x_1 = \overline{x}_2 x_3, \quad \text{and so} \quad \overline{x}_1 = x_2 \overline{x}_3.$$

Thus $\overline{x}_1 x_3 = x_2 \overline{x}_3 x_3 = x_2$. Note, by Theorem 2.5, $|x_1| \leq 1$.

Conversely, if

$$x_2 = \overline{x}_1 x_3, \quad |x_3| = 1 \quad \text{and} \quad |x_1| \leq 1$$

then, as in the previous steps, one can show that $x \in b\overline{\mathbb{E}}$. Therefore,

$$x \in b\overline{\mathbb{E}} \quad \text{if and only if} \quad x_2 = \overline{x}_1 x_3, \quad |x_3| = 1 \quad \text{and} \quad |x_2| \leq 1.$$

□

3. THE SYMMETRISED BIDISC AND Γ -INNER FUNCTIONS

In Section 4.1 we show that there exist useful relations between Γ -inner functions and $\overline{\mathbb{E}}$ -inner functions. Recall the definition of the symmetrised bidisc Γ .

Definition 3.1. The symmetrised bidisc is the set

$$\mathbb{G} \stackrel{\text{def}}{=} \{(z + w, zw) : |z| < 1, |w| < 1\},$$

and its closure is

$$\Gamma \stackrel{\text{def}}{=} \{(z + w, zw) : |z| \leq 1, |w| \leq 1\}.$$

In 1995 Jim Agler and Nicholas Young started the study of the symmetrised bidisc with the aim of solving a robust control problem in H^∞ control theory. Although, the aim has not yet been achieved, it turned out that the symmetrised bidisc has a rich structure and it has attracted the attention of specialists in the several complex variables and in operator theory.

We will use the co-ordinates (s, p) for points in the symmetrized bidisc \mathbb{G} , chosen to suggest ‘sum’ and ‘product’. The following result [4, Proposition 3.2] provides practical criteria for membership of \mathbb{G} , of the distinguished boundary $b\Gamma$ of Γ and of the topological boundary $\partial\Gamma$ of Γ .

Proposition 3.2. [4, Proposition 3.2] *Let (s, p) belong to \mathbb{C}^2 . Then*

(1) (s, p) belongs to \mathbb{G} if and only if

$$|s - \bar{s}p| < 1 - |p|^2;$$

(2) (s, p) belongs to Γ if and only if

$$|s| \leq 2 \quad \text{and} \quad |s - \bar{s}p| \leq 1 - |p|^2;$$

(3) (s, p) lies in $b\Gamma$ if and only if

$$|p| = 1, \quad |s| \leq 2 \quad \text{and} \quad s - \bar{s}p = 0;$$

(4) $(s, p) \in \partial\Gamma$ if and only if

$$|s| \leq 2 \quad \text{and} \quad |s - \bar{s}p| = 1 - |p|^2.$$

Γ -inner functions were defined and studied in [4].

Definition 3.3. A Γ -inner function is an analytic function $h : \mathbb{D} \rightarrow \Gamma$ such that the radial limit

$$(3.1) \quad \lim_{r \rightarrow 1^-} h(r\lambda)$$

exists and belongs to $b\Gamma$ for almost all $\lambda \in \mathbb{T}$ with respect to Lebesgue measure.

By Fatou’s Theorem, the limit (3.1) exists for almost all $\lambda \in \mathbb{T}$.

Definition 3.4. Let f be a polynomial of degree less than or equal to n , where $n \geq 0$. Then we define the polynomial $f^{\sim n}$ by

$$f^{\sim n}(\lambda) = \lambda^n \overline{f(1/\bar{\lambda})}.$$

The polynomial f^\vee is defined by

$$f^\vee(\lambda) = \overline{f(\bar{\lambda})}.$$

Remark 3.5. One can see that

(1)

$$f^{\sim n}(\lambda) = \lambda^n \overline{f(1/\bar{\lambda})} = \lambda^n f^\vee(1/\lambda).$$

(2) If f is a polynomial of degree k , then, for $n \geq k$, $(f^{\sim n})^{\sim n}(\lambda) = f(\lambda)$.

Algebraic and geometric aspects of rational Γ -inner functions were studied in [6]. We are going to use some results from the paper.

Proposition 3.6. [6, Proposition 2.2] *Let $h = (s, p)$ be a rational Γ -inner function of degree n . Then there exist polynomials E and D such that*

(1) $\deg(E), \deg(D) \leq n$,

(2) $E^{\sim n} = E$,

(3) $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$,

(4) $|E(\lambda)| \leq 2|D(\lambda)|$ on $\overline{\mathbb{D}}$,

(5) $s = \frac{E}{D}$ on $\overline{\mathbb{D}}$,

(6) $p = \frac{D^{\sim n}}{D}$ on $\overline{\mathbb{D}}$.

Furthermore, E_1 and D_1 is a second pair of polynomials satisfying conditions (1)–(6) if and only if there exists a nonzero $t \in \mathbb{R}$ such that

$$E_1 = tE \quad \text{and} \quad D_1 = tD.$$

Conversely, if E and D are polynomials which satisfy conditions (1), (2), (4), $D(\lambda) \neq 0$ on \mathbb{D} , and s and p are defined by equations (5) and (6), then $h = (s, p)$ is a rational Γ -inner function of degree less than or equal to n .

The royal variety \mathcal{R}_Γ of the symmetrised bidisc is

$$\mathcal{R}_\Gamma = \{(s, p) \in \mathbb{C}^2 : s^2 = 4p\}.$$

Definition 3.7. [6, Page 7] Let $h = (s, p)$ be a Γ -inner function of degree n . Let E and D be as in Proposition 3.6. The royal polynomial R_h of h is defined by

$$R_h(\lambda) = 4D(\lambda)D^{\sim n}(\lambda) - E(\lambda)^2.$$

Definition 3.8. [6, Definition 3.6] Let h be a rational Γ -inner function such that $h(\mathbb{D}) \not\subseteq \mathcal{R}_\Gamma \cap \Gamma$. Let R_h be the royal polynomial of h , and let σ be a zero of R_h of order ℓ . We define the multiplicity $\#\sigma$ of σ (as a royal node of h) by

$$\#\sigma = \begin{cases} \ell & \text{if } \sigma \in \mathbb{D}, \\ \frac{1}{2}\ell & \text{if } \sigma \in \mathbb{T}. \end{cases}$$

We define the type of h to be the ordered pair (n, k) , where n is the sum of the multiplicities of the royal nodes of h that lie in \mathbb{D} , and k is the sum of the multiplicities of the royal nodes of h that lie in \mathbb{T} . We define $\mathcal{R}_\Gamma^{n,k}$ to be the collection of rational Γ -inner functions h of type (n, k) .

Theorem 3.9. [6, Theorem 3.8] Let $h \in \mathcal{R}_\Gamma^{n,k}$ be nonconstant. Then $\deg(h) = n$.

4. RATIONAL $\overline{\mathbb{E}}$ -INNER FUNCTIONS

In this section we give a definition of the degree of a rational tetra-inner function x by means of the fundamental group π_1 . Recall that the rational inner functions on \mathbb{D} of degree n are exactly the finite Blaschke products of degree n . Similar to this description of rational inner functions on \mathbb{D} we give an algorithm for the construction of all rational $\overline{\mathbb{E}}$ -inner functions on \mathbb{D} in Theorem 4.15. In [6], the authors describe all rational Γ -inner functions (see Proposition 3.6). We use this description and the connection between Γ -inner functions and $\overline{\mathbb{E}}$ -inner functions to describe all rational $\overline{\mathbb{E}}$ -inner functions on \mathbb{D} .

4.1. Relations between $\overline{\mathbb{E}}$ -inner functions and Γ -inner functions.

Definition 4.1. An $\overline{\mathbb{E}}$ -inner or tetra-inner function is a map $f : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ that is analytic and is such that the radial limit

$$\lim_{r \rightarrow 1^-} f(r\lambda)$$

exists and belongs to $b\overline{\mathbb{E}}$ for almost all $\lambda \in \mathbb{T}$ with respect to Lebesgue measure.

Remark 4.2. Let $x : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ be a rational $\overline{\mathbb{E}}$ -inner function. Since x is rational and bounded on \mathbb{D} it has no poles in \mathbb{D} and hence x is continuous on $\overline{\mathbb{D}}$. Thus one can consider the continuous function

$$\tilde{x} : \mathbb{T} \rightarrow b\overline{\mathbb{E}}, \quad \text{where} \quad \tilde{x}(\lambda) = \lim_{r \rightarrow 1^-} x(r\lambda) \quad \text{for all } \lambda \in \mathbb{T}.$$

In future we will use the same notation x for both continuous functions x and \tilde{x} .

Lemma 4.3. *Let $x = (x_1, x_2, x_3)$ be an $\overline{\mathbb{E}}$ -inner function. Then*

- (1) $x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda)$, $|x_2(\lambda)| \leq 1$ and $|x_3(\lambda)| = 1$ for almost all $\lambda \in \mathbb{T}$;
- (2) x_3 is an inner function on \mathbb{D} .

Proof. (1) By the definition of $\overline{\mathbb{E}}$ -inner function

$$x(\lambda) = (x_1(\lambda), x_2(\lambda), x_3(\lambda)) \in b\overline{\mathbb{E}}, \quad \text{for almost every } \lambda \in \mathbb{T}$$

and, by Theorem 2.10,

$$x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda), \quad |x_3(\lambda)| = 1 \quad \text{and} \quad |x_2(\lambda)| \leq 1 \quad \text{for almost all } \lambda \in \mathbb{T}.$$

(2) Since

$$x_3 : \mathbb{D} \rightarrow \overline{\mathbb{D}} \quad \text{and,} \quad \text{for almost all } \lambda \in \mathbb{T}, \quad |x_3(\lambda)| = 1,$$

x_3 is an inner function. □

Remark 4.4. Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$ -inner function. By Lemma 4.3, x_3 is an inner function on \mathbb{D} , and so x_3 is a finite Blaschke product.

In [9] the author shows that there is a relation between points in the symmetrised bidisc and the tetrablock as follows.

Lemma 4.5. [9, Lemma 3.2] *A point $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ belongs to the tetrablock if and only if the pair $(x_1 + zx_2, zx_3)$ is in the symmetrised bidisc \mathbb{G} for every $z \in \mathbb{T}$.*

Proof. By Proposition 3.2 (1), $(s, p) \in \mathbb{G}$ if and only if

$$(4.1) \quad |s - \overline{s}p| < 1 - |p|^2.$$

Suppose that $x = (x_1, x_2, x_3) \in \mathbb{E}$, $s_z = x_1 + zx_2$ and $p_z = zx_3$.

$$\begin{aligned} |s_z - \overline{s_z}p_z| &= |x_1 + zx_2 - \overline{(x_1 + zx_2)}zx_3| \\ &= |x_1 - \overline{x_2}x_3 + z(x_2 - \overline{x_1}x_3)| \\ &\leq |x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3|, \quad \text{since } |z| = 1, \\ &< 1 - |x_3|^2 = 1 - |p_z|^2, \quad \text{by Theorem 2.4.} \end{aligned}$$

Hence $(s_z, p_z) \in \mathbb{G}$.

Conversely, let $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ and, for $z \in \mathbb{T}$, let

$$(4.2) \quad s_z = x_1 + zx_2 \quad \text{and} \quad p_z = zx_3.$$

Suppose for all $z \in \mathbb{T}$, we have $(s_z, p_z) \in \mathbb{G}$. We want to show that $x = (x_1, x_2, x_3) \in \mathbb{E}$. Let us prove that

$$|x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3| < 1 - |x_3|^2.$$

By assumption for all $z \in \mathbb{T}$, $|s_z - \overline{s_z}p_z| < 1 - |x_3|^2$. By equations (4.2), we have

$$(4.3) \quad |x_1 - \overline{x_2}x_3 + z(x_2 - \overline{x_1}x_3)| < 1 - |x_3|^2, \quad \text{for all } z \in \mathbb{T}.$$

Let

$$\begin{cases} z = e^{i\theta} & \theta \in (0, 2\pi]; \\ w_1 = x_1 - \overline{x_2}x_3 = |w_1|e^{i\theta_1} & \theta_1 \in (0, 2\pi]; \\ w_2 = x_2 - \overline{x_1}x_3 = |w_2|e^{i\theta_2} & \theta_2 \in (0, 2\pi]. \end{cases}$$

Now substitute z , w_1 and w_2 in inequality (4.3)

$$||w_1|e^{i\theta_1} + e^{i\theta}(|w_2|e^{i\theta_2})| < 1 - |x_3|^2.$$

This implies that

$$||w_1|e^{i\theta_1} + |w_2|e^{i(\theta+\theta_2)}| < 1 - |x_3|^2, \quad \text{for all } e^{i\theta}.$$

We can choose θ such that $\theta + \theta_2 = \theta_1$, that is, $\theta = \theta_1 - \theta_2$. Hence

$$|w_1|e^{i\theta_1} + |w_2|e^{i\theta_1} = |e^{i\theta_1}|(|w_1| + |w_2|) = |w_1| + |w_2| = |x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3| < 1 - |x_3|^2.$$

By Theorem 2.4, $(x_1, x_2, x_3) \in \mathbb{E}$. \square

Lemma 4.6. *A point $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ belongs to the closed tetrablock if and only if for every $a \in \overline{\mathbb{D}}$, $(ax_1 + \overline{a}x_2, x_3) \in \Gamma$.*

Proof. Suppose $x = (x_1, x_2, x_3) \in \overline{\mathbb{E}}$. Consider $(s_a, p_a) = (ax_1 + \overline{a}x_2, x_3)$. By Proposition 3.2 (2), $(s, p) \in \Gamma$ if and only if

$$(4.4) \quad |s - \overline{s}p| \leq 1 - |p|^2 \text{ and } |s| \leq 2.$$

$$\begin{aligned} |s_a - \overline{s_a}p_a| &= |ax_1 + \overline{a}x_2 - \overline{(ax_1 + \overline{a}x_2)}x_3| \\ &= |a(x_1 - \overline{x_2}x_3) + \overline{a}(x_2 - \overline{x_1}x_3)| \\ &\leq |a(x_1 - \overline{x_2}x_3)| + |\overline{a}(x_2 - \overline{x_1}x_3)|, \\ &\leq |x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3|, \quad \text{since } |a| \leq 1, \\ (4.5) \quad &\leq 1 - |x_3|^2, \quad \text{by Theorem 2.5.} \end{aligned}$$

Thus, $|s_a - \overline{s_a}p_a| \leq 1 - |x_3|^2 = 1 - |p_a|^2$ and $|s_a| = |ax_1 + \overline{a}x_2| \leq 2$. Hence $(s_a, p_a) \in \Gamma$.

Conversely, let $x = (x_1, x_2, x_3) \in \mathbb{C}^3$. Suppose for every $a \in \overline{\mathbb{D}}$, we have $(s_a, p_a) \in \Gamma$ where

$$(4.6) \quad s_a = ax_1 + \overline{a}x_2 \quad \text{and} \quad p_a = x_3.$$

By equations (4.5) and (4.6), we have

$$(4.7) \quad |s_a - \overline{s_a}p_a| = |a(x_1 - \overline{x_2}x_3) + \overline{a}(x_2 - \overline{x_1}x_3)| \leq 1 - |p_a|^2, \quad \text{for all } a \in \overline{\mathbb{D}}.$$

Take $a \in \mathbb{T}$, then

$$\begin{cases} a = e^{i\theta} & \theta \in (0, 2\pi]; \\ w_1 = x_1 - \overline{x_2}x_3 = |w_1|e^{i\theta_1} & \theta_1 \in (0, 2\pi]; \\ w_2 = x_2 - \overline{x_1}x_3 = |w_2|e^{i\theta_2} & \theta_2 \in (0, 2\pi]. \end{cases}$$

Substitute a , w_1 and w_2 into inequality (4.7), to get

$$\begin{aligned} |a(x_1 - \overline{x_2}x_3) + \overline{a}(x_2 - \overline{x_1}x_3)| &= |e^{i\theta}|w_1|e^{i\theta_1} + e^{-i\theta}|w_2|e^{i\theta_2}| = ||w_1|e^{i(\theta+\theta_1)} + |w_2|e^{i(\theta_2-\theta)}| \\ &\leq 1 - |x_3|^2, \end{aligned}$$

for every $\theta \in (0, 2\pi]$. Now choose $\theta = \frac{\theta_2 - \theta_1}{2}$ to get

$$\begin{aligned} |x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3| &= |w_1| + |w_2| \\ &= |e^{i(\frac{\theta_2+\theta_1}{2})}|(|w_1| + |w_2|) \\ &= ||w_1|e^{i(\frac{\theta_2+\theta_1}{2})} + |w_2|e^{i(\frac{\theta_2+\theta_1}{2})}| \\ &= ||w_1|e^{i(\frac{\theta_2-\theta_1}{2}+\theta_1)} + |w_2|e^{i(\theta_2-\frac{\theta_2-\theta_1}{2})}| \leq 1 - |x_3|^2. \end{aligned}$$

Therefore $x = (x_1, x_2, x_3) \in \overline{\mathbb{E}}$. \square

Lemma 4.7. *Let $s, p \in \mathbb{C}$ be such that $|s| \leq 2$ and $|p| \leq 1$. The pair (s, p) belongs to Γ if and only if $(\frac{1}{2}s, \frac{1}{2}s, p) \in \overline{\mathbb{E}}$.*

Proof. By Theorem 2.5,

$$\left(\frac{1}{2}s, \frac{1}{2}s, p\right) \in \overline{\mathbb{E}} \Leftrightarrow \left|\frac{1}{2}s - \frac{1}{2}\overline{s}p\right| + \left|\frac{1}{2}s - \frac{1}{2}\overline{s}p\right| \leq 1 - |p|^2.$$

Thus

$$\begin{aligned} \left(\frac{1}{2}s, \frac{1}{2}s, p\right) \in \overline{\mathbb{E}} &\Leftrightarrow 2\left|\frac{1}{2}s - \frac{1}{2}\overline{s}p\right| \leq 1 - |p|^2 \\ &\Leftrightarrow |s - \overline{s}p| \leq 1 - |p|^2. \end{aligned}$$

By assumption $|s| \leq 2$, hence by Proposition 3.2 (2),

$$\left(\frac{1}{2}s, \frac{1}{2}s, p\right) \in \overline{\mathbb{E}} \Leftrightarrow (s, p) \in \Gamma.$$

□

Lemma 4.8. *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$ -inner function. Then*

- (1) $h_1(\lambda) = (x_1(\lambda) + x_2(\lambda), x_3(\lambda))$, for $\lambda \in \mathbb{D}$, is a rational Γ -inner function;
- (2) $h_2(\lambda) = (ix_1(\lambda) - ix_2(\lambda), x_3(\lambda))$, for $\lambda \in \mathbb{D}$, is a rational Γ -inner function.

Proof. (1) By Lemma 4.5, for all $\lambda \in \mathbb{D}$, $x(\lambda) \in \mathbb{E}$ implies that

$$(x_1(\lambda) + x_2(\lambda), x_3(\lambda)) \in \mathbb{G}.$$

Consider $h_1 = (s_1, p_1)$ where

$$s_1(\lambda) = x_1(\lambda) + x_2(\lambda) \quad \text{and} \quad p_1(\lambda) = x_3(\lambda), \quad \text{for } \lambda \in \mathbb{D}.$$

It is obvious that h_1 is a rational function from \mathbb{D} to \mathbb{G} . By assumption, x is an $\overline{\mathbb{E}}$ -inner function. Thus $x(\lambda) \in b\overline{\mathbb{E}}$ for almost every $\lambda \in \mathbb{T}$. By Theorem 2.10 and Lemma 2.11, for almost all $\lambda \in \mathbb{T}$,

$$(4.8) \quad x_2(\lambda) = \overline{x_1(\lambda)}x_3(\lambda), \quad x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda), \quad |x_3(\lambda)| = 1 \quad \text{and} \quad |x_2(\lambda)| \leq 1.$$

It is clear that

$$|p_1(\lambda)| = |x_3(\lambda)| = 1 \quad \text{for } \lambda \in \mathbb{T},$$

and, for almost all $\lambda \in \mathbb{T}$,

$$\begin{aligned} |s_1(\lambda)| &= |x_1(\lambda) + x_2(\lambda)| \\ &\leq |x_1(\lambda)| + |x_2(\lambda)| \leq 2. \end{aligned}$$

Since, for almost all $\lambda \in \mathbb{T}$, $x_2(\lambda) = \overline{x_1(\lambda)}x_3(\lambda)$, we have

$$\begin{aligned} \overline{s_1(\lambda)}p_1(\lambda) &= [\overline{x_1(\lambda)} + \overline{x_2(\lambda)}]x_3(\lambda) \\ &= \overline{x_1(\lambda)}x_3(\lambda) + \overline{x_2(\lambda)}x_3(\lambda), \quad \text{by equations (4.8),} \\ &= x_1(\lambda) + x_2(\lambda) = s_1(\lambda). \end{aligned}$$

Hence $s_1(\lambda) = \overline{s_1(\lambda)}p_1(\lambda)$ for almost every $\lambda \in \mathbb{T}$. Therefore, by Proposition 3.2 (3), h_1 is a rational Γ -inner function.

(2) Following the same steps as (1), let $h_2(\lambda) = (s_2(\lambda), p_2(\lambda))$, where

$$s_2(\lambda) = ix_1(\lambda) - ix_2(\lambda) \quad \text{and} \quad p_2(\lambda) = x_3(\lambda), \quad \lambda \in \mathbb{D}.$$

By Lemma 4.6, h_2 is rational function from \mathbb{D} to \mathbb{G} . Since x is an $\overline{\mathbb{E}}$ -inner function, $x(\lambda) \in b\overline{\mathbb{E}}$ for almost all $\lambda \in \mathbb{T}$. By Proposition 3.2, to prove that h_2 is a rational Γ -inner function we need to show that

$$|p_2(\lambda)| = 1, \quad |s_2(\lambda)| \leq 2 \quad \text{and} \quad s_2(\lambda) = \overline{s_2(\lambda)}p_2(\lambda) \quad \text{for almost every } \lambda \in \mathbb{T}.$$

By Theorem 2.10, for almost all $\lambda \in \mathbb{T}$, $|p(\lambda)| = |x_3(\lambda)| = 1$ and

$$|s_2(\lambda)| \leq |ix_1(\lambda)| + |ix_2(\lambda)| \leq 2.$$

By Lemma 2.11, $x_2(\lambda) = \overline{x_1(\lambda)}x_3(\lambda)$ for almost all $\lambda \in \mathbb{T}$. Hence, for almost all $\lambda \in \mathbb{T}$,

$$\begin{aligned} \overline{s_2(\lambda)}p_2(\lambda) &= [\overline{ix_1(\lambda)} - \overline{ix_2(\lambda)}]x_3(\lambda) \\ &= i(\overline{x_2(\lambda)})x_3(\lambda) - i(\overline{x_1(\lambda)})x_3(\lambda), \quad \text{by equations (4.8),} \\ &= ix_1(\lambda) - ix_2(\lambda) = s_2(\lambda). \end{aligned}$$

Hence $s_2(\lambda) = \overline{s_2(\lambda)}p_2(\lambda)$ for almost every $\lambda \in \mathbb{T}$. Therefore, by Proposition 3.2, h_2 is a rational Γ -inner function. \square

Lemma 4.9. *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$ -inner function. Then*

$$x_1(\lambda) = x_2^\vee(1/\lambda)x_3(\lambda) \quad \text{for all } \lambda \in \mathbb{C} \setminus \{0\}.$$

Proof. By Theorem 2.10, for all $\lambda \in \mathbb{T}$,

$$x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda).$$

For $\lambda \in \mathbb{T}$, we have $|\lambda| = 1$, that is, $\lambda\overline{\lambda} = 1$, and so

$$\overline{x_2(\lambda)} = x_2^\vee(\overline{\lambda}) = x_2^\vee(\frac{1}{\lambda}).$$

Therefore, for all $\lambda \in \mathbb{T}$,

$$x_1(\lambda) = x_2^\vee(1/\lambda)x_3(\lambda).$$

Since x_1, x_2, x_3 are rational functions,

$$x_1(\lambda) = x_2^\vee(1/\lambda)x_3(\lambda) \quad \text{for all } \lambda \in \mathbb{C} \setminus \{0\}.$$

\square

Proposition 4.10. *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$ -inner function*

- (1) *If $a \in \mathbb{C} \cup \{\infty\}$ is a pole of x_3 of multiplicity $k \geq 0$ and $\frac{1}{a}$ is a zero of x_2 of multiplicity $\ell \geq 0$, then a is a pole of x_1 of multiplicity at least $k - \ell$.*
- (2) *If $a \in \mathbb{C} \cup \{\infty\}$ is a pole of x_1 of multiplicity $k \geq 1$, then a is a pole of x_3 of multiplicity at least k .*

Proof. (1) By Lemma 4.9, we have

$$(4.9) \quad x_1(\lambda) = x_2^\vee(1/\lambda)x_3(\lambda) \quad \text{for } \lambda \in \mathbb{C} \setminus \{0\}.$$

Since x_3 is a rational inner function, x_3 cannot have any pole in $\overline{\mathbb{D}}$. Hence $|a| > 1$ and so $|\frac{1}{a}| < 1$. We know that x_2^\vee is analytic in \mathbb{D} , so $\frac{1}{a}$ cannot be a pole of x_2^\vee . By equation (4.9),

$$(\lambda - a)^{k-\ell-1}x_1(\lambda) = (\lambda - a)^{k-\ell-1}x_2^\vee(1/\lambda)x_3(\lambda).$$

Take the limit for both sides as λ goes to a :

$$\lim_{\lambda \rightarrow a} (\lambda - a)^{k-\ell-1}x_1(\lambda) = \lim_{\lambda \rightarrow a} (\lambda - a)^{k-\ell-1}x_2^\vee(1/\lambda)x_3(\lambda).$$

The right hand side goes to ∞ , therefore x_1 has a pole of multiplicity at least $k - \ell$ at a .

Now suppose that ∞ is a pole of x_3 of multiplicity k and 0 is a zero of x_2 of multiplicity ℓ . By equation (4.9), for all $\lambda \in \mathbb{C} \setminus \{0\}$, we have

$$x_1(\frac{1}{\lambda}) = x_2^\vee(\lambda)x_3(\frac{1}{\lambda}).$$

Multiply both sides by $\frac{\lambda^{k-1}}{\lambda^\ell}$ to obtain the equation

$$(4.10) \quad \frac{\lambda^{k-1}}{\lambda^\ell}x_1(\frac{1}{\lambda}) = \frac{\lambda^{k-1}}{\lambda^\ell}x_2^\vee(\lambda)x_3(\frac{1}{\lambda}).$$

Since x_2^\vee is analytic at 0 and has a zero of multiplicity $\ell > 0$ at 0, we have

$$\lim_{\lambda \rightarrow 0} \frac{x_2^\vee(\lambda)}{\lambda^\ell} = c, \quad \text{where } c \in \mathbb{C} \setminus \{0\}.$$

Since by assumption, $x_3(\lambda)$ has a pole of multiplicity k at ∞ ,

$$\lim_{\lambda \rightarrow 0} \lambda^{k-1} x_3\left(\frac{1}{\lambda}\right) = \infty.$$

Hence by equation (4.10),

$$\lim_{\lambda \rightarrow 0} \lambda^{k-\ell-1} x_1\left(\frac{1}{\lambda}\right) = \infty.$$

It follows that $x_1\left(\frac{1}{\lambda}\right)$ has a pole of multiplicity at least $k - \ell$ at 0. That is, $x_1(\lambda)$ has a pole of multiplicity at least $k - \ell$ at ∞ .

(2) Let $a \in \mathbb{C}$ be a pole of x_1 of multiplicity $k \geq 1$. Then $|a| > 1$. This implies $|\frac{1}{a}| < 1$. Therefore x_2^\vee is analytic at $\frac{1}{a}$. Now

$$\lim_{\lambda \rightarrow a} (\lambda - a)^{k-1} x_1(\lambda) = \infty.$$

Thus a is a pole of x_3 of multiplicity at least k .

If ∞ is a pole of x_1 of multiplicity $k \geq 1$. Then 0 is a pole of $x_1\left(\frac{1}{\lambda}\right)$ of multiplicity k , that is,

$$\lim_{\lambda \rightarrow 0} \lambda^{k-1} x_1\left(\frac{1}{\lambda}\right) = \infty.$$

By relation (4.10),

$$\lambda^{k-1} x_1\left(\frac{1}{\lambda}\right) = \lambda^{k-1} x_2^\vee(\lambda) x_3\left(\frac{1}{\lambda}\right).$$

Since x_2^\vee is analytic at 0, 0 cannot be a pole of x_2^\vee and thus

$$\lim_{\lambda \rightarrow 0} x_2^\vee(\lambda) = x_2^\vee(0).$$

Therefore

$$\lim_{\lambda \rightarrow 0} \lambda^{k-1} x_3\left(\frac{1}{\lambda}\right) = \infty.$$

This completes the proof that x_3 has a pole of multiplicity at least k at ∞ . \square

4.2. The degree of a rational $\overline{\mathbb{E}}$ -inner function. Let us define the notion of the degree $\deg(x)$ of a rational $\overline{\mathbb{E}}$ -inner function x by means of fundamental groups.

Definition 4.11. *The degree $\deg(x)$ of a rational $\overline{\mathbb{E}}$ -inner function x is defined to be $x_*(1)$, where $x_* : \mathbb{Z} = \pi_1(\mathbb{T}) \rightarrow \pi_1(b\overline{\mathbb{E}})$ is the homomorphism of fundamental groups induced by x when x is regarded as a continuous map from \mathbb{T} to $b\overline{\mathbb{E}}$.*

We will assume that $\deg(x)$ is a non-negative integer.

Lemma 4.12. *$b\overline{\mathbb{E}}$ is homotopic to \mathbb{T} and $\pi_1(b\overline{\mathbb{E}}) = \mathbb{Z}$.*

Proof. The maps

$$\begin{aligned} f : b\overline{\mathbb{E}} &\rightarrow \mathbb{T}, \text{ defined by } f(x_1, x_2, x_3) = x_3, \\ g : \mathbb{T} &\rightarrow b\overline{\mathbb{E}}, \text{ defined by } g(z) = (0, 0, z), \end{aligned}$$

satisfy

$$(g \circ f)(x_1, x_2, x_3) = g(f(x_1, x_2, x_3)) = g(x_3) = (0, 0, x_3)$$

and

$$(f \circ g)(z) = f(0, 0, z) = z,$$

that is, $f \circ g = \text{id}_{\mathbb{T}}$. If $(x_1, x_2, x_3) \in b\overline{\mathbb{E}}$ and $0 \leq t \leq 1$, then $(tx_1, tx_2, x_3) \in b\overline{\mathbb{E}}$. Let $I = [0, 1]$. Consider the map

$$h : b\overline{\mathbb{E}} \times I \rightarrow b\overline{\mathbb{E}},$$

which is defined by

$$h(x_1, x_2, x_3, t) = (tx_1, tx_2, x_3).$$

One can see that

$$h(x_1, x_2, x_3, 0) = (0x_1, 0x_2, x_3) = (0, 0, x_3) = (g \circ f)(x_1, x_2, x_3) \text{ and}$$

$$h(x_1, x_2, x_3, 1) = (1x_1, 1x_2, x_3) = (x_1, x_2, x_3) = \text{id}_{b\overline{\mathbb{E}}}(x_1, x_2, x_3).$$

Therefore h defines a homotopy between $g \circ f$ and $\text{id}_{b\overline{\mathbb{E}}}$, that is, $g \circ f \simeq \text{id}_{b\overline{\mathbb{E}}}$. Hence $b\overline{\mathbb{E}}$ is homotopically equivalent to \mathbb{T} and it follows that $\pi_1(b\overline{\mathbb{E}}) = \pi_1(\mathbb{T}) = \mathbb{Z}$. \square

Lemma 4.13. *Let B be a finite Blaschke product. Then the degree of B is equal to $B_*(1)$.*

Proof. Since B is a finite Blaschke product, it can be written as

$$B(\lambda) = e^{i\theta} \prod_{j=1}^N \frac{\lambda - \alpha_j}{1 - \overline{\alpha_j}\lambda}, \quad \text{where } \alpha_j \in \mathbb{D}, j = 1 \dots N, \text{ and } \theta \in [0, 2\pi).$$

One can consider the map, $B : \mathbb{T} \rightarrow \mathbb{T}$, and

$$B_* : \pi_1(\mathbb{T}) = \mathbb{Z} \rightarrow \pi_1(\mathbb{T}) = \mathbb{Z}.$$

Now $1 \in \pi_1(\mathbb{T})$ is the homotopy class of $\text{id}_{\mathbb{T}}$ and $B_*(1)$ is equal to the homotopy class of $B \circ \text{id}_{\mathbb{T}} = B$, when B is regarded as a continuous map from \mathbb{T} to \mathbb{T} . Therefore $B_*(1) = n(\gamma, a)$, where $n(\gamma, a)$ is the winding number of γ about a , which lies inside $\gamma = \{B(e^{it}) : 0 \leq t \leq 2\pi\}$. Thus, one can see that

$$\begin{aligned} n(\gamma, a) &= \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z - a} \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{B'(z)dz}{B(z) - a}. \end{aligned}$$

By the Argument Principle, [7, Theorem 18], the integral

$$\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{B'(z)}{B(z) - a} dz,$$

is equal to the number of zeros of B in \mathbb{D} . It is clear that B has N zeros, counting multiplicities, and has degree N . Therefore the number of zeros of B is equal to the winding number of γ about a , and it is equal to N . \square

Proposition 4.14. *For any rational $\overline{\mathbb{E}}$ -inner function $x = (x_1, x_2, x_3)$, $\deg(x)$ is the degree $\deg(x_3)$ (in the usual sense) of the finite Blaschke product x_3 .*

Proof. Since x is a rational $\overline{\mathbb{E}}$ -inner function, x_3 is an inner function, and so x_3 is a finite Blaschke product. Two $\overline{\mathbb{E}}$ -inner functions $x = (x_1, x_2, x_3)$ and $y = (0, 0, x_3)$ are homotopic if there exists a continuous mapping $f : \mathbb{T} \times I \rightarrow b\overline{\mathbb{E}}$ such that

$$f(\lambda, 0) = y(\lambda) \quad \text{and} \quad f(\lambda, 1) = x(\lambda), \quad \lambda \in \mathbb{T}.$$

Let

$$x^t(\lambda) = (tx_1(\lambda), tx_2(\lambda), x_3(\lambda)) \text{ for } \lambda \in \mathbb{D} \text{ and } t \in [0, 1].$$

Since $x(\lambda) \in b\overline{\mathbb{E}}$, for all $\lambda \in \mathbb{T}$, by Theorem 2.10 (1),

$$x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda) \quad \text{and} \quad |x_3(\lambda)| = 1.$$

Hence for all $\lambda \in \mathbb{T}$,

$$tx_1(\lambda) = \overline{tx_2(\lambda)}x_3(\lambda).$$

Therefore,

$$x^t(\lambda) = (tx_1(\lambda), tx_2(\lambda), x_3(\lambda)) \in b\overline{\mathbb{E}} \quad \text{for } \lambda \in \mathbb{T}.$$

Hence x^t is a homotopy between $x = x^1$ and $(0, 0, x_3) = x^0$.

It follows that the homomorphism

$$x_* : \pi_1(\mathbb{T}) = \mathbb{Z} \rightarrow \pi_1(b\overline{\mathbb{E}}) = \mathbb{Z}$$

coincides with $(x^0)_* = (0, 0, x_3)_*$. By Lemma 4.13, $(x_3)_*(1) = \deg x_3$, since x_3 is a finite Blaschke product. Therefore $(0, 0, x_3)_*(1)$ is the degree of the finite Blaschke product x_3 . \square

4.3. Description of rational $\overline{\mathbb{E}}$ -inner functions.

Theorem 4.15. *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$ -inner function of degree n . Then there exist polynomials E_1, E_2, D such that*

- (1) $\deg(E_1), \deg(E_2), \deg(D) \leq n$,
- (2) $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$,
- (3) $x_3 = \frac{D^{\sim n}}{D}$ on $\overline{\mathbb{D}}$,
- (4) $x_1 = \frac{E_1}{D}$ on $\overline{\mathbb{D}}$,
- (5) $x_2 = \frac{E_2}{D}$ on $\overline{\mathbb{D}}$,
- (6) $|E_i(\lambda)| \leq |D(\lambda)|$ on $\overline{\mathbb{D}}$, for $i = 1, 2$,
- (7) $E_1(\lambda) = E_2^{\sim n}(\lambda)$, for $\lambda \in \overline{\mathbb{D}}$.

Conversely, if E_1, E_2 and D satisfy conditions (1), (6) and (7), $D(\lambda) \neq 0$ on \mathbb{D} and x_1, x_2 and x_3 are defined by equations (3)–(5), then $x = (x_1, x_2, x_3)$ is a rational $\overline{\mathbb{E}}$ -inner function of degree at most n .

Furthermore, a triple of polynomials E_1^1, E_2^1 and D^1 satisfies relations (1)–(7) if and only if there exists a real number $t \neq 0$ such that

$$E_1^1 = tE_1, \quad E_2^1 = tE_2 \quad \text{and} \quad D^1 = tD.$$

Proof. By assumption $x = (x_1, x_2, x_3)$ is a rational $\overline{\mathbb{E}}$ -inner function. By Lemma 4.8 (1), $h_1 = (s, p)$ where $s = x_1 + x_2, p = x_3$ is a rational Γ -inner function. Since $x_3 : \mathbb{D} \rightarrow \mathbb{D}$ is an inner function, it is a finite Blaschke product and, by [4, Corollary 6.10], it can be written in the form

$$x_3(\lambda) = c \frac{\lambda^k D^{\sim(n-k)}(\lambda)}{D(\lambda)},$$

where $|c| = 1, 0 \leq k \leq n$ and D is a polynomial of degree $n - k$ such that $D(0) = 1$. By Proposition 3.6, there exist polynomials E, D such that

- (1) $\deg(E), \deg(D) \leq n$,
- (2) $E^{\sim n} = E$,
- (3) $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$,
- (4) $|E(\lambda)| \leq 2|D(\lambda)|$ on $\overline{\mathbb{D}}$,
- (5) $s = \frac{E}{D}$ on $\overline{\mathbb{D}}$,
- (6) $p = \frac{D^{\sim n}}{D}$ on $\overline{\mathbb{D}}$.

Hence

$$(4.11) \quad x_1 + x_2 = s = \frac{E}{D} \quad \text{and} \quad x_3 = p = \frac{D^{\sim n}}{D}.$$

By Lemma 4.8 (2), $h_2 = (s_2, p_2)$, where $s_2 = ix_1 - ix_2$, $p_2 = x_3 = p_1$ is a rational Γ -inner function. By Proposition 3.6, for $h_2 = (s_2, p_2)$, there exist polynomials G, D such that

- (1) $\deg(G), \deg(D) \leq n$,
- (2) $G^{\sim n} = G$,
- (3) $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$,
- (4) $|G(\lambda)| \leq 2|D(\lambda)|$ on $\overline{\mathbb{D}}$,
- (5) $s_2 = ix_1 - ix_2 = \frac{G}{D}$ on $\overline{\mathbb{D}}$,
- (6) $p_2 = x_3 = \frac{D^{\sim n}}{D}$ on $\overline{\mathbb{D}}$.

Therefore, by equation (5),

$$(4.12) \quad x_1 - x_2 = -\frac{iG}{D}.$$

By relation (4.11),

$$(4.13) \quad x_1 + x_2 = \frac{E}{D}.$$

Add equations (4.12) and (4.13) to get

$$x_1 = \frac{\frac{1}{2}(E - iG)}{D}.$$

Substitution of x_1 in equation (4.13) gives

$$x_2 = \frac{\frac{1}{2}(E + iG)}{D}.$$

Define the polynomials E_1 and E_2 by

$$E_1 = \frac{1}{2}(E - iG), \quad E_2 = \frac{1}{2}(E + iG).$$

Since the degrees of both polynomials E, G are at most n , $\deg(E_1), \deg(E_2) \leq n$. Thus, for $\lambda \in \overline{\mathbb{D}}$,

$$x_1(\lambda) = \frac{E_1(\lambda)}{D(\lambda)} \quad \text{and} \quad x_2(\lambda) = \frac{E_2(\lambda)}{D(\lambda)}.$$

Since x is an $\overline{\mathbb{E}}$ -inner function, for $\lambda \in \overline{\mathbb{D}}$,

$$\begin{aligned} |x_1(\lambda)| &\leq 1 & \text{and} & & |x_2(\lambda)| &\leq 1, \\ \text{and so} & & |E_1(\lambda)| &\leq |D(\lambda)| & \text{and} & & |E_2(\lambda)| &\leq |D(\lambda)|. \end{aligned}$$

Hence $|E_i(\lambda)| \leq |D(\lambda)|$ on $\overline{\mathbb{D}}$, where $i = 1, 2$. Therefore conditions (1)–(6) of Theorem 4.15 are satisfied.

By assumption, x is a rational $\overline{\mathbb{E}}$ -inner function. Thus, for all $\lambda \in \mathbb{T}$,

$$\begin{aligned}
 x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda) &\Leftrightarrow \frac{E_1(\lambda)}{D(\lambda)} = \frac{\overline{E_2(\lambda)}}{\overline{D(\lambda)}} \times \frac{D^{\sim n}(\lambda)}{D(\lambda)} \\
 &\Leftrightarrow \frac{E_1(\lambda)}{D(\lambda)} = \frac{\overline{E_2(\lambda)}}{D^\vee(1/\lambda)} \times \frac{\lambda^n D^\vee(1/\lambda)}{D(\lambda)}, \text{ since } \overline{D(\lambda)} = D^\vee(\overline{\lambda}) = D^\vee(1/\lambda). \\
 &\Leftrightarrow \frac{E_1(\lambda)}{D(\lambda)} = \frac{\lambda^n \overline{E_2(\lambda)}}{D(\lambda)} \\
 &\Leftrightarrow \frac{E_1(\lambda)}{D(\lambda)} = \frac{E_2^{\sim n}(\lambda)}{D(\lambda)} \\
 (4.14) \quad &\Leftrightarrow E_1(\lambda) = E_2^{\sim n}(\lambda).
 \end{aligned}$$

Hence $E_1(\lambda) = E_2^{\sim n}(\lambda)$ for all $\lambda \in \mathbb{T}$, and therefore on $\overline{\mathbb{D}}$. Thus equation (7) of Theorem 4.15 is proved.

Let us prove the converse statement. Let E_1, E_2 and D satisfy relations (1), (6) and (7) of Theorem 4.15 and $D(\lambda) \neq 0$ on \mathbb{D} , and x_1, x_2, x_3 be defined by equations (3)–(5), that is,

$$x_1 = \frac{E_1}{D}, \quad x_2 = \frac{E_2}{D} \quad \text{and} \quad x_3 = \frac{D^{\sim n}}{D}.$$

Let us show that $x = (x_1, x_2, x_3)$ is a rational $\overline{\mathbb{E}}$ -inner function. By Theorem 2.10, we have to prove that $x : \mathbb{D} \rightarrow \mathbb{E}$ and the following conditions are satisfied.

- (1) $|x_3(\lambda)| = 1$ for almost all λ on \mathbb{T} , that is, x_3 is inner,
- (2) $|x_2| \leq 1$ on $\overline{\mathbb{D}}$,
- (3) $x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda)$ for almost all $\lambda \in \mathbb{T}$.

(1) Firstly, if D has no zeros on the unit circle, then D and $D^{\sim n}$ have no common factor. Therefore, $x_3(\lambda) = \frac{D^{\sim n}(\lambda)}{D(\lambda)}$ maps \mathbb{T} to \mathbb{T} . Hence, x_3 is an inner function and

$$\deg(x_3) = \deg\left(\frac{D^{\sim n}}{D}\right) = \max\{\deg(D^{\sim n}), \deg(D)\} = n.$$

Second case: if D has the zeros a_1, \dots, a_ℓ on \mathbb{T} then D and $D^{\sim n}$ have the common factor $\prod_{i=1}^\ell (\lambda - a_i)$ and hence $x_3 = \frac{D^{\sim n}}{D}$ is inner and

$$\deg(x_3) = \deg\left(\frac{D^{\sim n}}{D}\right) \leq n - \ell.$$

(2) By assumption (6),

$$|E_2(\lambda)| \leq |D(\lambda)| \quad \text{for all } \lambda \in \overline{\mathbb{D}}.$$

This implies $|\frac{E_2(\lambda)}{D(\lambda)}| \leq 1$ and hence $|x_2(\lambda)| \leq 1$.

(3) By assumption (7), $E_1(\lambda) = E_2^{\sim n}(\lambda)$, for almost all $\lambda \in \mathbb{T}$ and by the equality (4.14), $x_1(\lambda) = \overline{x_2(\lambda)}x_3(\lambda)$, for almost all $\lambda \in \mathbb{T}$.

Let us show that $x = (x_1, x_2, x_3) = \left(\frac{E_1}{D}, \frac{E_2}{D}, \frac{D^{\sim n}}{D}\right)$ **maps** \mathbb{D} **to** \mathbb{E} , that is,
 $x(\lambda) = (x_1(\lambda), x_2(\lambda), x_3(\lambda)) \in \mathbb{E}$ for all $\lambda \in \mathbb{D}$. By Theorem 2.5, for $\lambda \in \mathbb{D}$,

$$x(\lambda) \in \overline{\mathbb{E}} \Leftrightarrow \|\Psi(\cdot, x(\lambda))\|_{H^\infty} \leq 1,$$

where $\Psi(z, x) = \frac{x_3 z - x_1}{x_2 z - 1}$. Note that, for every $z \in \mathbb{D}$,

$$\begin{aligned} \Psi(z, x) : \mathbb{D} &\rightarrow \mathbb{C} \\ &: \lambda \rightarrow \Psi(z, x(\lambda)) \end{aligned}$$

is analytic on \mathbb{D} because $x_i, i = 1, 2, 3$, are analytic functions on \mathbb{D} , and $|x_2(\lambda)| \leq 1$ and $x_2(\lambda)z \neq 1$ for all $\lambda \in \mathbb{D}$. We have shown above that, for almost all $\lambda \in \mathbb{T}$, $x(\lambda) \in b\overline{\mathbb{E}}$. Thus, by Theorem 2.10 (2),

$$x(\lambda) \in b\overline{\mathbb{E}} \text{ if and only if } \Psi(\cdot, x(\lambda)) \text{ is an automorphism of } \mathbb{D}.$$

By the maximum principle, for all $z, \lambda \in \mathbb{D}$, $|\Psi(z, x(\lambda))| < 1$. Thus by Theorem 2.10, $x(\lambda) \in \mathbb{E}$ for all $\lambda \in \mathbb{D}$.

Suppose that t is a nonzero real number and

$$E_1^1 = tE_1, \quad E_2^1 = tE_2 \quad \text{and} \quad D^1 = tD.$$

Then it is clear that E_1^1, E_2^1 and D^1 satisfy conditions (1)–(7). Conversely, let E_1^1, E_2^1 and D^1 be a second triple that satisfies relations (1)–(7). Then

$$(4.15) \quad x_1 = \frac{E_1}{D} = \frac{E_1^1}{D^1} \quad \text{on } \overline{\mathbb{D}},$$

$$(4.16) \quad x_2 = \frac{E_2}{D} = \frac{E_2^1}{D^1} \quad \text{on } \overline{\mathbb{D}},$$

$$(4.17) \quad x_3 = \frac{D^{\sim n}}{D} = \frac{D^{1 \sim n}}{D^1} \quad \text{on } \overline{\mathbb{D}}.$$

Suppose that $D(\lambda) = a_0 + a_1\lambda + \dots + a_k\lambda^k$ where $a_0 \neq 0$ and $k \leq n$. Then

$$\begin{aligned} D^{\sim n}(\lambda) &= \lambda^n \overline{D(1/\overline{\lambda})} \\ &= \lambda^n \left(\overline{a_0 + \frac{a_1}{\lambda} + \dots + \frac{a_k}{\lambda^k}} \right) \\ &= \overline{a_0}\lambda^n + \overline{a_1}\lambda^{n-1} + \dots + \overline{a_k}\lambda^{n-k}. \end{aligned}$$

Thus, for all $\lambda \in \mathbb{D}$,

$$x_3 = \frac{D^{\sim n}(\lambda)}{D(\lambda)} = \frac{\lambda^{n-k}(\overline{a_0}\lambda^k + \overline{a_1}\lambda^{k-1} + \dots + \overline{a_k})}{a_0 + a_1\lambda + \dots + a_k\lambda^k}.$$

Therefore, x_3 has a zero of multiplicity $(n - k)$ at 0, has k poles in \mathbb{C} , counting multiplicity, and has degree n . Hence the poles of x_3 in $\{z \in \mathbb{C} : |z| > 1\}$, n and k are determined by x_3 . Thus polynomials D and D^1 have the same degree k and the same finite number of zeros in $\{z \in \mathbb{C} : |z| > 1\}$, counting multiplicity. Hence there exists $t \in \mathbb{C}, t \neq 0$ where

$$(4.18) \quad D^1 = tD \quad \text{on } \overline{\mathbb{D}}.$$

By equality (4.17), for $\lambda \in \overline{\mathbb{D}}$

$$x_3 = \frac{D^{\sim n}}{D} = \frac{D^{1 \sim n}}{D^1} = \frac{\bar{t}D^{\sim n}}{tD}$$

Thus $t = \bar{t}$, and so, $t \in \mathbb{R} \setminus \{0\}$. By the equalities (4.15) and (4.18)

$$x_1 = \frac{E_1}{D} = \frac{E_1^1}{D^1} = \frac{E_1^1}{tD}, \quad \text{on } \overline{\mathbb{D}}.$$

This implies that $E_1^1 = tE_1$. By the equalities (4.16) and (4.18)

$$x_2 = \frac{E_2}{D} = \frac{E_2^1}{D^1} = \frac{E_2^1}{tD}, \quad \text{on } \overline{\mathbb{D}}.$$

Thus $E_2^1 = tE_2$. □

Remark 4.16. For a fixed polynomial D of degree n , the set of polynomials E_1 satisfying the conditions of Theorem 4.15 is a subset of a real vector space of dimension $2n + 2$. Hence the set of rational $\overline{\mathbb{E}}$ -inner functions of degree n with $x_3 = \frac{D^{\sim n}}{D}$ is a subset of a $(2n + 2)$ -dimensional real space of rational functions.

Lemma 4.17. *Let*

$$x = (x_1, x_2, x_3) = \left(\frac{E_1}{D}, \frac{E_2}{D}, \frac{D^{\sim n}}{D} \right)$$

be a rational $\overline{\mathbb{E}}$ -inner function. Then, for $\lambda \in \mathbb{T}$,

$$|E_1(\lambda)| = |E_2(\lambda)|, \quad \text{and so} \quad |x_1(\lambda)| = |x_2(\lambda)|.$$

Proof. By Theorem 4.15 (7), for all $\lambda \in \mathbb{T}$,

$$E_1(\lambda) = E_2^{\sim n}(\lambda) = \lambda^n \overline{E_2(1/\bar{\lambda})}.$$

Thus, since $\lambda\bar{\lambda} = 1$,

$$\begin{aligned} |E_1(\lambda)| &= |\lambda^n \overline{E_2(1/\bar{\lambda})}| \\ &= |E_2(1/\bar{\lambda})| = |E_2(\lambda)|. \end{aligned}$$

Therefore, for all $\lambda \in \mathbb{T}$,

$$|x_1(\lambda)| = |x_2(\lambda)|.$$

□

Example 4.18. Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$ -inner function such that $x_3(\lambda) = \lambda$. Clearly,

$$\deg(x) = \deg(x_3) = 1.$$

By Theorem 4.15, there exist polynomials E_1, E_2, D such that

$$\begin{aligned} D(\lambda) &= 1, \quad \deg(E_1) \leq 1, \quad \deg(E_2) \leq 1, \\ E_1(\lambda) &= E_2^{\sim n}(\lambda), \quad |E_i(\lambda)| \leq |D(\lambda)| = 1, \quad i = 1, 2, \quad \text{for all } \lambda \in \overline{\mathbb{D}}, \end{aligned}$$

and

$$x = \left(\frac{E_1}{D}, \frac{E_2}{D}, \frac{D^{\sim 1}}{D} \right).$$

Therefore the function

$$x(\lambda) = (a_1 + a_2\lambda, \bar{a}_2 + \bar{a}_1\lambda, \lambda)$$

is rational $\overline{\mathbb{E}}$ -inner for $a_1, a_2 \in \overline{\mathbb{D}}$ such that

$$|a_1 + a_2\lambda| \leq 1 \quad \text{and} \quad |\bar{a}_2 + \bar{a}_1\lambda| \leq 1 \quad \text{for all } \lambda \in \overline{\mathbb{D}}.$$

In particular, one can choose $a_1 = 1$ and $a_2 = 0$ to get the rational $\overline{\mathbb{E}}$ -inner function

$$x(\lambda) = (1, \lambda, \lambda).$$

Example 4.19. $\overline{\mathbb{E}}$ -inner functions

Suppose that $\mathbb{B}_{2 \times 2} = \{A \in \mathbb{C}^{2 \times 2} : \|A\| < 1\}$. Let us construct an analytic map from the open unit disc \mathbb{D} to $\mathbb{B}_{2 \times 2}$. Consider nonconstant inner functions $\varphi, \psi \in H^\infty(\mathbb{D})$ and the diagonal matrix

$$h(\lambda) = \begin{bmatrix} \varphi(\lambda) & 0 \\ 0 & \psi(\lambda) \end{bmatrix} \quad \text{for } \lambda \in \mathbb{D}.$$

Note $\|h(\lambda)\| = \max\{|\varphi(\lambda)|, |\psi(\lambda)|\} < 1$ for $\lambda \in \mathbb{D}$ and $h : \mathbb{D} \rightarrow \mathbb{B}_{2 \times 2}$ is analytic. By Theorem 2.4, for all $\lambda \in \mathbb{D}$,

$$(\varphi(\lambda), \psi(\lambda), \det h(\lambda)) \in \mathbb{E},$$

and $\varphi(\lambda)\psi(\lambda) = \det h(\lambda)$. Recall that such points are called triangular points of \mathbb{E} . However, we are seeking more interesting and general examples. To get such examples we make use of the singular value decomposition.

Let U, V be 2×2 unitary matrices. Then $h_1 : \mathbb{D} \rightarrow \mathbb{C}^{2 \times 2}$ defined by

$$h_1 = UhV$$

maps \mathbb{D} to $\mathbb{B}_{2 \times 2}$. For example, if

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \quad \text{and} \quad V = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

then U is unitary and we obtain

$$\begin{aligned} h_1(\lambda) &= Uh(\lambda)I \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \varphi(\lambda) & 0 \\ 0 & \psi(\lambda) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} \varphi(\lambda) & \psi(\lambda) \\ -\varphi(\lambda) & \psi(\lambda) \end{bmatrix}. \end{aligned}$$

Define $x : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ by $x = \pi \circ h_1$, where π is given in Definition 2.6. Then, for $\lambda \in \mathbb{D}$,

$$\begin{aligned} x(\lambda) &= \pi(h_1(\lambda)) \\ &= \pi\left(\frac{1}{\sqrt{2}} \begin{bmatrix} \varphi(\lambda) & \psi(\lambda) \\ -\varphi(\lambda) & \psi(\lambda) \end{bmatrix}\right) \\ &= \left(\frac{\varphi(\lambda)}{\sqrt{2}}, \frac{\psi(\lambda)}{\sqrt{2}}, \varphi(\lambda)\psi(\lambda)\right). \end{aligned}$$

Note that this $x(\lambda)$ is not a triangular point unless either $\varphi(\lambda) = 0$ or $\psi(\lambda) = 0$.

Let us show that this function x is $\overline{\mathbb{E}}$ -inner. By Theorem 2.10(1), for $\lambda \in \mathbb{T}$, since φ, ψ are inner functions,

$$\begin{aligned} \overline{x_2(\lambda)}x_3(\lambda) &= \overline{\left(\frac{\psi(\lambda)}{\sqrt{2}}\right)}\varphi(\lambda)\psi(\lambda) = \frac{\varphi(\lambda)}{\sqrt{2}}\overline{\psi(\lambda)}\psi(\lambda) \\ &= \frac{\varphi(\lambda)}{\sqrt{2}}|\psi(\lambda)|^2 = \frac{\varphi(\lambda)}{\sqrt{2}} = x_1(\lambda). \end{aligned}$$

Since $|\psi(\lambda)| < 1$ for $\lambda \in \mathbb{D}$, this implies that $\left|\frac{\psi(\lambda)}{\sqrt{2}}\right| < 1$. Thus $|x_2(\lambda)| < 1$. Finally, for $\lambda \in \mathbb{T}$, since φ, ψ are inner functions,

$$\begin{aligned} |x_3(\lambda)| &= |\varphi(\lambda)\psi(\lambda)| \\ &= |\varphi(\lambda)||\psi(\lambda)| = 1. \end{aligned}$$

Therefore x is an $\overline{\mathbb{E}}$ -inner function.

Remark 4.20. In the previous example if we choose the functions φ and ψ to be in the Schur class but not to be inner functions then one can check that we obtain an analytic function $x : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ which is not an $\overline{\mathbb{E}}$ -inner function.

Proposition 4.21. *Let (s, p) be a Γ -inner function. Then $x = (\frac{s}{2}, \frac{s}{2}, p)$ is an $\overline{\mathbb{E}}$ -inner function.*

Proof. By Lemma 4.7, for every $\lambda \in \mathbb{D}$, $x(\lambda) = (\frac{s}{2}(\lambda), \frac{s}{2}(\lambda), p(\lambda)) \in \overline{\mathbb{E}}$. It is easy to see that x is in the set of analytic functions $\text{Hol}(\mathbb{D}, \overline{\mathbb{E}})$ from \mathbb{D} to the tetrablock $\overline{\mathbb{E}}$. By Proposition 3.2 (3), for almost all $\lambda \in \mathbb{T}$,

$$|p(\lambda)| = 1, \quad |s(\lambda)| \leq 2 \quad \text{and} \quad s(\lambda) - \overline{s(\lambda)}p(\lambda) = 0.$$

Thus, for almost all $\lambda \in \mathbb{T}$,

$$|p(\lambda)| = 1, \quad \frac{s(\lambda)}{2} = \frac{\overline{s(\lambda)}}{2}p(\lambda) \quad \text{and} \quad \frac{|s(\lambda)|}{2} \leq 1.$$

Hence x is $\overline{\mathbb{E}}$ -inner. □

See [4] for many examples of Γ -inner functions.

4.4. Superficial $\overline{\mathbb{E}}$ -inner functions. In this subsection, we study $\overline{\mathbb{E}}$ -inner functions x such that $x(\lambda)$ lies in the topological boundary $\partial\overline{\mathbb{E}}$ of $\overline{\mathbb{E}}$ for all $\lambda \in \mathbb{D}$.

The topological boundary of \mathbb{E} is denoted by $\partial\mathbb{E}$.

Lemma 4.22. [1] Let $x = (x_1, x_2, x_3) \in \mathbb{C}^3$. Then $x \in \partial\mathbb{E}$ if and only if

$$|x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3| = 1 - |x_3|^2, \quad |x_1| \leq 1 \quad \text{and} \quad |x_2| \leq 1.$$

Here we show that, for any inner function x_3 and $\beta_1, \beta_2 \in \mathbb{C}$ such that $|\beta_1| + |\beta_2| = 1$, the function $x = (\beta_2 + \overline{\beta_1}x_3, \beta_1 + \overline{\beta_2}x_3, x_3)$ is $\overline{\mathbb{E}}$ -inner and has the property that it maps \mathbb{D} to $\partial\overline{\mathbb{E}}$. Recall the definition of superficial function in the set of analytic functions $\text{Hol}(\mathbb{D}, \Gamma)$ from \mathbb{D} to the symmetrised bidisc Γ from [4].

Definition 4.23. *An analytic function $h : \mathbb{D} \rightarrow \Gamma$ is superficial if $h(\mathbb{D}) \subset \partial\Gamma$.*

One can define a similar notion for functions in $\text{Hol}(\mathbb{D}, \overline{\mathbb{E}})$.

Definition 4.24. *An analytic function $x : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ is superficial if $x(\mathbb{D}) \subset \partial\overline{\mathbb{E}}$.*

We also consider relations between superficial Γ -inner functions and superficial $\overline{\mathbb{E}}$ -inner functions.

Proposition 4.25. [4, Proposition 8.3] *A Γ -inner function h is superficial if and only if there is an $\omega \in \mathbb{T}$ and an inner function p such that $h = (\omega p + \overline{\omega}, p)$.*

Proposition 4.26. *An analytic function $x : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ such that*

$$x(\lambda) = (\beta_1 + \overline{\beta_2}x_3(\lambda), \beta_2 + \overline{\beta_1}x_3(\lambda), x_3(\lambda)), \quad \lambda \in \mathbb{D},$$

where x_3 is an inner function and $|\beta_1| + |\beta_2| = 1$ is $\overline{\mathbb{E}}$ -inner and superficial.

Proof. By Lemma 4.22, we need to show that, for $\lambda \in \mathbb{D}$,

$$x(\lambda) = (\beta_1 + \overline{\beta_2}x_3(\lambda), \beta_2 + \overline{\beta_1}x_3(\lambda), x_3(\lambda)).$$

is in $\partial\mathbb{E}$. Here

$$x_1(\lambda) = \beta_1 + \overline{\beta_2}x_3(\lambda), \quad x_2(\lambda) = \beta_2 + \overline{\beta_1}x_3(\lambda).$$

Note, for $\lambda \in \mathbb{D}$,

$$\begin{aligned} (4.19) \quad |(x_1 - \overline{x_2}x_3)(\lambda)| &= \left| \beta_1 + \overline{\beta_2}x_3(\lambda) - \overline{(\beta_2 + \overline{\beta_1}x_3(\lambda))}x_3(\lambda) \right| \\ &= \left| \beta_1(1 - |x_3(\lambda)|^2) \right|. \end{aligned}$$

We also have

$$\begin{aligned} (4.20) \quad |(x_2 - \overline{x_1}x_3)(\lambda)| &= \left| \beta_2 + \overline{\beta_1}x_3(\lambda) - \overline{(\beta_1 + \overline{\beta_2}x_3(\lambda))}x_3(\lambda) \right| \\ &= \left| \beta_2(1 - |x_3(\lambda)|^2) \right|. \end{aligned}$$

Note that by equations (4.19) and (4.20), for all $\lambda \in \mathbb{D}$,

$$\begin{aligned} |(x_1 - \overline{x_2}x_3)(\lambda)| + |(x_2 - \overline{x_1}x_3)(\lambda)| &= \left| \beta_1(1 - |x_3(\lambda)|^2) \right| + \left| \beta_2(1 - |x_3(\lambda)|^2) \right| \\ &= (|\beta_1| + |\beta_2|)(1 - |x_3(\lambda)|^2) = 1 - |x_3(\lambda)|^2. \end{aligned}$$

By Theorem 2.5 and Lemma 4.22, for $\lambda \in \mathbb{D}$, the point

$$x(\lambda) = (x_1(\lambda), x_2(\lambda), x_3(\lambda))$$

lies in $\partial\mathbb{E}$. Let us check that x is $\overline{\mathbb{E}}$ -inner. Clearly, for almost all $\lambda \in \mathbb{T}$,

$$\begin{aligned} \overline{x_2(\lambda)}x_3(\lambda) &= \overline{(\beta_2 + \overline{\beta_1}x_3(\lambda))}x_3(\lambda) \\ &= \overline{\beta_2}x_3(\lambda) + \beta_1\overline{x_3(\lambda)}x_3(\lambda) \\ &= \beta_1 + \overline{\beta_2}x_3(\lambda) = x_1(\lambda). \end{aligned}$$

We also have, for almost all $\lambda \in \mathbb{T}$,

$$|x_2(\lambda)| = |\beta_2 + \overline{\beta_1}x_3(\lambda)| \leq |\beta_2| + |\beta_1x_3(\lambda)| = |\beta_2| + |\beta_1| = 1.$$

Since x_3 is inner, for almost all $\lambda \in \mathbb{T}$, $|x_3(\lambda)| = 1$. Therefore $x(\lambda) \in b\overline{\mathbb{E}}$, for almost all $\lambda \in \mathbb{T}$, and hence x is $\overline{\mathbb{E}}$ -inner. \square

Lemma 4.27. *Let $x : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ be such that $x(\lambda) = (\beta_1 + \overline{\beta_2}x_3(\lambda), \beta_2 + \overline{\beta_1}x_3(\lambda), x_3(\lambda))$, where x_3 is a non-constant rational inner function and $|\beta_1| + |\beta_2| = 1$. Then $\Psi_\omega(x(\lambda)) = k$ for all $\lambda \in \mathbb{D}$, where*

$$\omega = \frac{\overline{\beta_2}}{|\beta_2|}, \quad k = \frac{\beta_1}{|\beta_1|} \quad \text{on } \mathbb{T}.$$

Proof. By definition,

$$\begin{aligned} \Psi_\omega(x(\lambda)) &= \frac{x_3(\lambda)\omega - x_1(\lambda)}{x_2(\lambda)\omega - 1} \\ &= \frac{x_3(\lambda)\omega - (\beta_1 + \overline{\beta_2}x_3(\lambda))}{(\beta_2 + \overline{\beta_1}x_3(\lambda))\omega - 1} \\ &= \frac{x_3(\lambda)\omega - \beta_1 - \overline{\beta_2}x_3(\lambda)}{\beta_2\omega + \overline{\beta_1}\omega x_3(\lambda) - 1}. \end{aligned}$$

Thus, for all $\lambda \in \mathbb{D}$,

$$\begin{aligned} \Psi_\omega(x(\lambda)) = k &\Leftrightarrow \frac{x_3(\lambda)\omega - \beta_1 - \overline{\beta_2}x_3(\lambda)}{\beta_2\omega + \overline{\beta_1}\omega x_3(\lambda) - 1} = k \\ &\Leftrightarrow x_3(\lambda)\omega - \beta_1 - \overline{\beta_2}x_3(\lambda) = k[\beta_2\omega + \overline{\beta_1}\omega x_3(\lambda) - 1] \\ &\Leftrightarrow x_3(\lambda)[\omega - \overline{\beta_2} - k\overline{\beta_1}\omega] + [k - \beta_1 - k\beta_2\omega] = 0. \end{aligned}$$

Since x_3 is a nonconstant rational inner function, this statement is equivalent to

$$\omega - \overline{\beta_2} - k\overline{\beta_1}\omega = 0 \quad \text{and} \quad k - \beta_1 - k\beta_2\omega = 0.$$

Multiply both sides of the first equation by $\overline{\omega}$ and the second equation by \overline{k} . We get

$$(4.21) \quad \begin{cases} \overline{\beta_1}k + \overline{\beta_2}\overline{\omega} = 1 \\ \beta_1\overline{k} + \beta_2\omega = 1. \end{cases}$$

Since $|\beta_1| + |\beta_2| = 1$, it is easy to see that

$$\omega = \frac{\overline{\beta_2}}{|\beta_2|} \quad \text{and} \quad k = \frac{\beta_1}{|\beta_1|}$$

satisfy equation (4.21), and so

$$\Psi_\omega(x(\lambda)) = k \quad \text{for all } \lambda \in \mathbb{D}.$$

□

Lemma 4.28. *For any inner function $x_3 : \mathbb{D} \rightarrow \overline{\mathbb{D}}$, there are $x_1, x_2 : \mathbb{D} \rightarrow \overline{\mathbb{D}}$ such that the function $x : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ defined by $x = (x_1, x_2, x_3)$ is a superficial $\overline{\mathbb{E}}$ -inner function, but $h = (x_1 + x_2, x_3) : \mathbb{D} \rightarrow \Gamma$ is not a superficial Γ -inner function.*

Proof. By Proposition 4.26, for any $\beta_1, \beta_2 \in \mathbb{C}$ such that $|\beta_1| + |\beta_2| = 1$, the function $x : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ defined by

$$x = (\beta_1 + \overline{\beta_2}x_3, \beta_2 + \overline{\beta_1}x_3, x_3)$$

is a superficial $\overline{\mathbb{E}}$ -inner function. By Proposition 4.25, $h : \mathbb{D} \rightarrow \Gamma$ is superficial if and only if there exists an $\omega \in \mathbb{T}$ such that $h = (\omega p + \overline{\omega}, p)$. Note that, for $x_1 = \beta_1 + \overline{\beta_2}x_3$

and $x_2 = \beta_2 + \overline{\beta_1}x_3$,

$$\begin{aligned} h(\lambda) = (x_1 + x_2, x_3)(\lambda) &= \left(\beta_1 + \overline{\beta_2}x_3(\lambda) + \beta_2 + \overline{\beta_1}x_3(\lambda), x_3(\lambda) \right) \\ &= \left((\overline{\beta_1} + \overline{\beta_2})x_3(\lambda) + (\beta_1 + \beta_2), x_3(\lambda) \right), \quad \lambda \in \mathbb{D}. \end{aligned}$$

One can see that there are some $\beta_1, \beta_2 \in \mathbb{C}$ with $|\beta_1| + |\beta_2| = 1$, but $\beta_1 + \beta_2 \notin \mathbb{T}$. For example, take

$$\beta_1 = i\frac{1}{2}, \quad \beta_2 = -i\frac{1}{2}.$$

Then $|\beta_1| + |\beta_2| = \frac{1}{2} + \frac{1}{2} = 1$, but $\beta_1 + \beta_2 = i\frac{1}{2} - i\frac{1}{2} = 0 \notin \mathbb{T}$. Thus, h is not a superficial Γ -inner function for $\beta_1 = \frac{i}{2}$ and $\beta_2 = \frac{-i}{2}$. \square

5. THE CONSTRUCTION OF RATIONAL $\overline{\mathbb{E}}$ -INNER FUNCTIONS

The formula for a Blaschke product is an explicit representation of a rational inner function in terms of its zeros and one other parameter (a unimodular complex number). In this chapter we aim to find a comparable representation for rational $\overline{\mathbb{E}}$ -inner functions. The first question is: what is the tetrablock analogue of the zeros of an inner function? We shall show that one satisfactory choice consists of the royal nodes of an $\overline{\mathbb{E}}$ -inner function x together with the zeros of x_1 and x_2 . We construct a rational $\overline{\mathbb{E}}$ -inner function x from its royal nodes and the zeros of x_1 and x_2 . We show that there exists a 3-parameter family of rational $\overline{\mathbb{E}}$ -inner functions with prescribed zero sets of x_1 , x_2 and prescribed royal nodes. We also prove that a nonconstant rational $\overline{\mathbb{E}}$ -inner function x of degree n either maps \mathbb{D} to the royal variety of $\overline{\mathbb{E}}$ or $x(\mathbb{D})$ meets the royal variety exactly n times.

5.1. The royal polynomial of an $\overline{\mathbb{E}}$ -inner function. We define the *royal variety* for $\overline{\mathbb{E}}$ to be

$$\mathcal{R}_{\overline{\mathbb{E}}} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1x_2 = x_3\}.$$

It was shown in [20] that $\mathcal{R}_{\overline{\mathbb{E}}} \cap \mathbb{E}$ is the orbit of $\{(0, 0, 0)\}$ under the group of biholomorphic automorphisms of \mathbb{E} . By Theorem 4.15, for a rational $\overline{\mathbb{E}}$ -inner function $x = (x_1, x_2, x_3)$, there are polynomials E_1, E_2, D such that

$$x_1 = \frac{E_1}{D}, \quad x_2 = \frac{E_2}{D}, \quad x_3 = \frac{D^{\sim n}}{D}.$$

Thus, for $\lambda \in \overline{\mathbb{D}}$,

$$(x_3 - x_1x_2)(\lambda) = \left[\frac{D^{\sim n}}{D} - \frac{E_1 E_2}{D^2} \right](\lambda).$$

The *royal polynomial* of the rational $\overline{\mathbb{E}}$ -inner function x is defined to be

$$\begin{aligned} R_x(\lambda) &= D^2(\lambda) \left[\frac{D^{\sim n}}{D} - \frac{E_1 E_2}{D^2} \right](\lambda) \\ &= [D^{\sim n} D - E_1 E_2](\lambda). \end{aligned}$$

Definition 5.1. [6, Definition 3.4] *We say a polynomial f is n -symmetric if $\deg(f) \leq n$ and $f^{\sim n} = f$.*

Definition 5.2. [6, Definition 3.4] *For any $E \subset \mathbb{C}$, the number of zeros of f in E , counted with multiplicities, is denoted by $\text{ord}_E(f)$ and $\text{ord}_0(f)$ means the same as $\text{ord}_{\{0\}}(f)$.*

Proposition 5.3. *Let x be a rational $\overline{\mathbb{E}}$ -inner function of degree n and let R_x be the royal polynomial of x . Then, for $\lambda \in \mathbb{T}$,*

- (1) $\lambda^{-n}R_x(\lambda) = |D(\lambda)|^2 - |E_2(\lambda)|^2$ and
- (2) $\lambda^{-n}R_x(\lambda) = |D(\lambda)|^2 - |E_1(\lambda)|^2$.

Proof. (1) For $\lambda \in \mathbb{T}$,

$$\begin{aligned}
 \lambda^{-n}R_x(\lambda) &= \lambda^{-n}[D^{\sim n}D - E_1E_2](\lambda) \\
 &= \lambda^{-n}[\lambda^n \overline{D(1/\overline{\lambda})}D(\lambda) - E_2^{\sim n}(\lambda)E_2(\lambda)], \quad \text{since } E_1(\lambda) = E_2^{\sim n}(\lambda) \text{ on } \mathbb{T} \\
 &= \lambda^{-n}[\lambda^n \overline{D(\lambda)}D(\lambda) - \lambda^n \overline{E_2(1/\overline{\lambda})}E_2(\lambda)], \quad \text{since } E_2(1/\overline{\lambda}) = E_2(\lambda) \text{ on } \mathbb{T} \\
 &= \lambda^{-n}[\lambda^n \overline{D(\lambda)}D(\lambda) - \lambda^n \overline{E_2(\lambda)}E_2(\lambda)] \\
 (5.1) \quad &= |D(\lambda)|^2 - |E_2(\lambda)|^2.
 \end{aligned}$$

(2) Since x is rational $\overline{\mathbb{E}}$ -inner function, by Lemma 4.17,

$$(5.2) \quad |E_1(\lambda)| = |E_2(\lambda)| \quad \text{for } \lambda \in \mathbb{T}.$$

By equations (5.1) and (5.2),

$$\lambda^{-n}R_x(\lambda) = |D(\lambda)|^2 - |E_1(\lambda)|^2 \quad \text{for } \lambda \in \mathbb{T}.$$

□

Lemma 5.4. *Let E_1 and E_2 be two polynomials such that $\deg E_1, \deg E_2 \leq n$. Then*

$$E_1(\lambda) = E_2^{\sim n}(\lambda) \text{ for all } \lambda \in \mathbb{T} \quad \text{if and only if} \quad E_2(\lambda) = E_1^{\sim n}(\lambda) \text{ for all } \lambda \in \mathbb{T}.$$

Proof. Suppose that $E_1(\lambda) = E_2^{\sim n}(\lambda)$ for all $\lambda \in \mathbb{T}$. Then by definition,

$$E_1(\lambda) = E_2^{\sim n}(\lambda) = \lambda^n \overline{E_2(1/\overline{\lambda})}, \quad \lambda \in \mathbb{T}.$$

Therefore, for all $\lambda \in \mathbb{T}$,

$$\begin{aligned}
 E_1(\lambda) = \lambda^n \overline{E_2(1/\overline{\lambda})} \text{ for all } \lambda \in \mathbb{T} &\Leftrightarrow (1/\lambda^n)E_1(\lambda) = \overline{E_2(1/\overline{\lambda})} \text{ for all } \lambda \in \mathbb{T} \\
 &\Leftrightarrow (1/\overline{\lambda})^n \overline{E_1(\lambda)} = E_2(1/\overline{\lambda}) \text{ for all } \lambda \in \mathbb{T} \\
 &\Leftrightarrow \lambda^n \overline{E_1(1/\overline{\lambda})} = E_2(\lambda) \text{ for all } \lambda \in \mathbb{T} \\
 &\Leftrightarrow E_1^{\sim n}(\lambda) = E_2(\lambda) \text{ for all } \lambda \in \mathbb{T}.
 \end{aligned}$$

The converse is obvious.

□

Definition 5.5. *A nonzero polynomial R is n -balanced if*

- (1) $\deg(R) \leq 2n$,
- (2) R is $2n$ -symmetric, and
- (3) $\lambda^{-n}R(\lambda) \geq 0$ for all $\lambda \in \mathbb{T}$.

For completeness, we shall say that zeros of the zero polynomial have infinite order.

Proposition 5.6. *Let x be a rational $\overline{\mathbb{E}}$ -inner function of degree n and let R_x be the royal polynomial of x . Then R_x is $2n$ -symmetric, $\lambda^{-n}R_x(\lambda) \geq 0$ for all $\lambda \in \mathbb{T}$, and the zeros of R_x on \mathbb{T} have even order or infinite order.*

Proof. To show that R_x is $2n$ -symmetric we have to prove that $R_x^{\sim 2n}(\lambda) = R_x(\lambda)$, for $\lambda \in \mathbb{T}$. Recall that

$$R_x(\lambda) = D^{\sim n}(\lambda)D(\lambda) - E_1(\lambda)E_2(\lambda), \quad \text{where } x = \left(\frac{E_1}{D}, \frac{E_2}{D}, \frac{D^{\sim n}}{D} \right),$$

By Theorem 4.15 (7) and Lemma 5.4, for $\lambda \in \mathbb{T}$,

$$E_1(\lambda) = E_2^{\sim n}(\lambda) = \lambda^n \overline{E_2(1/\bar{\lambda})}, \quad E_2(\lambda) = E_1^{\sim n}(\lambda) = \lambda^n \overline{E_1(1/\bar{\lambda})}.$$

Now

$$\begin{aligned} R_x^{\sim 2n}(\lambda) &= \lambda^{2n} \overline{R_x(1/\bar{\lambda})} \\ &= \lambda^{2n} \overline{[D^{\sim n}(1/\bar{\lambda})D(1/\bar{\lambda}) - E_1(1/\bar{\lambda})E_2(1/\bar{\lambda})]} \\ &= \lambda^n \overline{D^{\sim n}(1/\bar{\lambda})} \lambda^n \overline{D(1/\bar{\lambda})} - \lambda^n \overline{E_1(1/\bar{\lambda})} \lambda^n \overline{E_2(1/\bar{\lambda})} \\ &= D(\lambda)D^{\sim n}(\lambda) - E_2(\lambda)E_1(\lambda) = R_x(\lambda). \end{aligned}$$

Hence R_x is $2n$ -symmetric.

Clearly, if $x(\overline{\mathbb{D}}) \subseteq \mathcal{R}_{\overline{\mathbb{E}}}$, the royal polynomial R_x is identically zero. Thus the zeros of R_x on \mathbb{T} have infinite order.

On the other hand, if $x(\overline{\mathbb{D}}) \not\subseteq \mathcal{R}_{\overline{\mathbb{E}}}$, by Proposition 5.3, for $\lambda \in \mathbb{T}$,

$$(5.3) \quad \lambda^{-n} R_x(\lambda) = |D(\lambda)|^2 - |E_2(\lambda)|^2.$$

By Theorem 4.15 (6),

$$(5.4) \quad |D|^2 - |E_2|^2 \geq 0 \quad \text{on } \mathbb{T}.$$

By equations (5.3) and (5.4), $\lambda^{-n} R_x(\lambda) \geq 0$ on \mathbb{T} . By the Fejér-Riesz theorem [18, Section 53], there exists an analytic polynomial $P(\lambda) = \sum_{i=0}^n b_i \lambda^i$ of degree n such that P is outer and

$$\lambda^{-n} R_x(\lambda) = |P(\lambda)|^2 \quad \text{for all } \lambda \in \mathbb{T}.$$

Hence if $\sigma \in \mathbb{T}$ is a zero of R_x , then σ is a zero of even order. Therefore in the case $x(\overline{\mathbb{D}}) \not\subseteq \mathcal{R}_{\overline{\mathbb{E}}}$, the zeros of R_x that lie in \mathbb{T} have even order. \square

Lemma 5.7. *Let x be a rational $\overline{\mathbb{E}}$ -inner function of degree n . Then the royal polynomial R_x of x is either n -balanced or identically zero.*

Proof. If $x(\overline{\mathbb{D}}) \subseteq \mathcal{R}_{\overline{\mathbb{E}}}$ then, by the definition of the royal variety,

$$x_1(\lambda)x_2(\lambda) = x_3(\lambda) \quad \text{for all } \lambda \in \overline{\mathbb{D}}.$$

Thus

$$E_1(\lambda)E_2(\lambda) = D(\lambda)D^{\sim n}(\lambda) \quad \text{for all } \lambda \in \overline{\mathbb{D}}.$$

Therefore the royal polynomial R_x is identically zero.

If $x(\overline{\mathbb{D}}) \not\subseteq \mathcal{R}_{\overline{\mathbb{E}}}$ then, by Proposition 5.6, the royal polynomial R_x of x is $2n$ -symmetric and $\lambda^{-n} R_x(\lambda) \geq 0$ for all $\lambda \in \mathbb{T}$. Clearly, $\deg(R_x) \leq 2n$. Hence R_x is n -balanced. \square

5.2. Rational $\overline{\mathbb{E}}$ -inner functions of type (n, k) .

Definition 5.8. *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$ -inner function such that $x(\overline{\mathbb{D}}) \not\subseteq \mathcal{R}_{\overline{\mathbb{E}}}$. Let R_x be the royal polynomial of x . Let σ be a zero of R_x of order ℓ . We define the multiplicity $\#\sigma$ of σ (as a royal node of x) by*

$$\#\sigma = \begin{cases} \ell & \text{if } \sigma \in \mathbb{D}, \\ \frac{1}{2}\ell & \text{if } \sigma \in \mathbb{T}. \end{cases}$$

We define the type of x to be the ordered pair (n, k) , where n is the sum of the multiplicities of the royal nodes of x that lie in $\overline{\mathbb{D}}$, and k is the sum of the multiplicities of the royal nodes of x that lie in \mathbb{T} .

Definition 5.9. *Let $\mathcal{R}^{n,k}$ denote the collection of rational $\overline{\mathbb{E}}$ -inner functions of type (n, k) .*

Remark 5.10. [6, Equations (3.2) and (3.3)] For any m -symmetric polynomial f , the following two relations hold

(1)

$$\deg(f) = m - \text{ord}_0(f).$$

(2) Since f is m -symmetric, if $\alpha \in \mathbb{D} \setminus \{0\}$ is a zero of f , then $\frac{1}{\alpha}$ is also a zero of f . Thus

$$\text{ord}_0(f) + 2\text{ord}_{\mathbb{D} \setminus \{0\}}(f) + \text{ord}_{\mathbb{T}}(f) = \deg(f).$$

Theorem 5.11. Let $x \in \mathcal{R}^{n,k}$ be nonconstant. Then the degree of x is equal to n .

Proof. Let R_x be the royal polynomial of x . By assumption $x \in \mathcal{R}^{n,k}$ and is nonconstant. Hence $n \geq 1$ and $x(\overline{\mathbb{D}}) \not\subseteq \mathcal{R}_{\overline{\mathbb{E}}}$. Thus R_x is not identically zero. By Proposition 5.6, R_x is $2\deg(x)$ -symmetric. By Remark 5.10 (1) and (2), it follows that

$$\deg(R_x) = 2\deg(x) - \text{ord}_0(R_x)$$

and

$$\text{ord}_0(R_x) + 2\text{ord}_{\mathbb{D} \setminus \{0\}}(R_x) + \text{ord}_{\mathbb{T}}(R_x) = \deg(R_x).$$

Substitute the first equation in the second equation,

$$\text{ord}_0(R_x) + 2\text{ord}_{\mathbb{D} \setminus \{0\}}(R_x) + \text{ord}_{\mathbb{T}}(R_x) = 2\deg(x) - \text{ord}_0(R_x),$$

which implies that

$$2\text{ord}_0(R_x) + 2\text{ord}_{\mathbb{D} \setminus \{0\}}(R_x) + \text{ord}_{\mathbb{T}}(R_x) = 2\deg(x).$$

Therefore, by Definition 5.9,

$$n = \text{ord}_0(R_x) + \text{ord}_{\mathbb{D} \setminus \{0\}}(R_x) + \frac{1}{2}\text{ord}_{\mathbb{T}}(R_x) = \deg(x).$$

□

Theorem 5.12. Let x be a nonconstant rational $\overline{\mathbb{E}}$ -inner function. Then either $x(\overline{\mathbb{D}}) \subseteq \mathcal{R}_{\overline{\mathbb{E}}}$ or $x(\overline{\mathbb{D}})$ meets $\mathcal{R}_{\overline{\mathbb{E}}}$ exactly $\deg(x)$ times.

Proof. Suppose that x is a nonconstant rational $\overline{\mathbb{E}}$ -inner function. Then either, $x(\overline{\mathbb{D}}) \subseteq \mathcal{R}_{\overline{\mathbb{E}}}$ and the royal polynomial R_x of x is identically zero, or by Theorem 5.11, $x(\overline{\mathbb{D}})$ meets $\mathcal{R}_{\overline{\mathbb{E}}}$ exactly $\deg(x)$ times. □

Lemma 5.13. [6, Lemma 4.4] Let R be a nonzero polynomial and let n be a positive integer. For $\sigma \in \overline{\mathbb{D}}$, let the polynomial Q_σ be defined by the formula

$$Q_\sigma(\lambda) = (\lambda - \sigma)(1 - \overline{\sigma}\lambda).$$

The polynomial R is n -balanced if and only if there are points $\sigma_1, \sigma_2, \dots, \sigma_n \in \overline{\mathbb{D}}$ and $t_+ > 0$ such that

$$R(\lambda) = t_+ \prod_{j=1}^n Q_{\sigma_j}(\lambda), \quad \lambda \in \mathbb{C}.$$

Proposition 5.14. Let x be a rational $\overline{\mathbb{E}}$ -inner function. Suppose the royal nodes of x are $\sigma_1, \dots, \sigma_n$, with repetition according to the multiplicity of the royal nodes as described in Definition 5.8. Then the royal polynomial R_x of x , up to a positive multiple, is

$$(5.5) \quad R_x(\lambda) = \prod_{j=1}^n Q_{\sigma_j}(\lambda).$$

Proof. By Lemma 5.7, R_x is n -balanced. This implies that, by Lemma 5.13, there exists $t_+ > 0$ and $\eta_1, \dots, \eta_n \in \overline{\mathbb{D}}$ such that

$$R_x(\lambda) = t_+ \prod_{j=1}^n Q_{\eta_j}(\lambda).$$

By Definition 5.8, the list η_1, \dots, η_n coincides, up to a permutation, with the list $\sigma_1, \dots, \sigma_n$. Therefore R_x is given, up to a positive multiple, by equation (5.5). \square

Before we proceed to the next theorem on the construction of a tetra-inner function x from the zeros of x_1 and x_2 and royal nodes of x , let us prove the following elementary lemma.

Lemma 5.15. *Let E_1 and E_2 be polynomials of degree at most n such that $E_1(\lambda) = E_2^{\sim n}(\lambda)$, for $\lambda \in \overline{\mathbb{D}}$. Let $\alpha_1^1, \dots, \alpha_{k_1}^1$ be the zeros of E_1 in $\overline{\mathbb{D}}$, and let $\alpha_1^2, \dots, \alpha_{k_2}^2$ be the zeros of E_2 in $\overline{\mathbb{D}}$, where $k_1 + k_2 = n$. Then*

$$E_1(\lambda) = t \prod_{j=1}^{k_1} (\lambda - \alpha_j^1) \prod_{j=1}^{k_2} (1 - \overline{\alpha_j^2} \lambda),$$

where $t \in \mathbb{C} \setminus \{0\}$.

Proof. Since $\alpha_1^1, \dots, \alpha_{k_1}^1 \in \overline{\mathbb{D}}$ and $\alpha_1^2, \dots, \alpha_{k_2}^2 \in \overline{\mathbb{D}}$, where $k_1 + k_2 = n$, are the zeros of E_1 and E_2 respectively, we have

$$(5.6) \quad E_1(\lambda) = (\lambda - \alpha_1^1) \dots (\lambda - \alpha_{k_1}^1) p_1(\lambda)$$

and

$$E_2(\lambda) = (\lambda - \alpha_1^2) \dots (\lambda - \alpha_{k_2}^2) p_2(\lambda).$$

where the polynomials $p_1(\lambda)$ and $p_2(\lambda)$ do not vanish in $\overline{\mathbb{D}}$.

Since $E_1(\lambda) = \lambda^n \overline{E_2(1/\overline{\lambda})}$ on $\overline{\mathbb{D}}$, we have

$$(5.7) \quad \begin{aligned} E_1(\lambda) &= \overline{\lambda^n (1/\overline{\lambda} - \alpha_1^2) \dots (1/\overline{\lambda} - \alpha_{k_2}^2) p_2(1/\overline{\lambda})} \\ &= \lambda^{n-k_2} (1 - \overline{\alpha_1^2} \lambda) \dots (1 - \overline{\alpha_{k_2}^2} \lambda) \overline{p_2(1/\overline{\lambda})}. \end{aligned}$$

Since $\deg E_1 \leq n$ and $k_1 + k_2 = n$, equations (5.6) and (5.7) implies that E_1 can be written in the form

$$\begin{aligned} E_1(\lambda) &= t_1 (\lambda - \alpha_1^1) \dots (\lambda - \alpha_{k_1}^1) (1 - \overline{\alpha_1^2} \lambda) \dots (1 - \overline{\alpha_{k_2}^2} \lambda) \\ &= t_1 \prod_{j=1}^{k_1} (\lambda - \alpha_j^1) \prod_{j=1}^{k_2} (1 - \overline{\alpha_j^2} \lambda), \quad \lambda \in \overline{\mathbb{D}}, \end{aligned}$$

for some $t_1 \in \mathbb{C} \setminus \{0\}$, and

$$E_2(\lambda) = t_2 \prod_{j=1}^{k_2} (\lambda - \alpha_j^2) \prod_{j=1}^{k_1} (1 - \overline{\alpha_j^1} \lambda) \quad \lambda \in \overline{\mathbb{D}},$$

for some $t_2 \in \mathbb{C} \setminus \{0\}$. Since $E_2(\lambda) = \lambda^n \overline{E_1(1/\overline{\lambda})}$,

$$\begin{aligned} \lambda^n \overline{E_1(1/\overline{\lambda})} &= \lambda^n \overline{\left(t_1 \prod_{j=1}^{k_1} (1/\overline{\lambda} - \alpha_j^1) \prod_{j=1}^{k_2} (1 - \overline{\alpha}_j^2 1/\overline{\lambda}) \right)} \\ &= \lambda^n \overline{t_1} \prod_{j=1}^{k_1} (1/\lambda - \overline{\alpha}_j^1) \prod_{j=1}^{k_2} (1 - \alpha_j^2 1/\lambda) \\ &= \overline{t_1} \prod_{j=1}^{k_1} (1 - \overline{\alpha}_j^1 \lambda) \prod_{j=1}^{k_2} (\lambda - \alpha_j^2) \\ &= E_2(\lambda) = t_2 \prod_{j=1}^{k_2} (\lambda - \alpha_j^2) \prod_{j=1}^{k_1} (1 - \overline{\alpha}_j^1 \lambda), \quad \lambda \in \overline{\mathbb{D}}, \end{aligned}$$

and so $t_2 = \overline{t_1}$. □

Remark 5.16. For the polynomials E_1 and E_2 from Lemma 5.15, if $\alpha \in \mathbb{D} \setminus \{0\}$ and α is a zero of E_1 then $\frac{1}{\overline{\alpha}}$ is a zero of E_2 .

Theorem 5.17. Suppose that $\alpha_1^1, \dots, \alpha_{k_1}^1 \in \overline{\mathbb{D}}$ and $\alpha_1^2, \dots, \alpha_{k_2}^2 \in \overline{\mathbb{D}}$, where $k_1 + k_2 = n$. Suppose that $\sigma_1, \dots, \sigma_n \in \overline{\mathbb{D}}$ are distinct from the points of the set $\{\alpha_j^i, j = 1, \dots, k_i, i = 1, 2\} \cap \mathbb{T}$. Then there exists a rational $\overline{\mathbb{E}}$ -inner function $x = (x_1, x_2, x_3) : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ such that

- (1) the zeros of x_1 in $\overline{\mathbb{D}}$, repeated according to multiplicity, are $\alpha_1^1, \dots, \alpha_{k_1}^1$;
- (2) the zeros of x_2 in $\overline{\mathbb{D}}$, repeated according to multiplicity, are $\alpha_1^2, \dots, \alpha_{k_2}^2$;
- (3) the royal nodes of x are $\sigma_1, \dots, \sigma_n \in \overline{\mathbb{D}}$, with repetition according to the multiplicity of the nodes.

Such a function x can be constructed as follows. Let $t_+ > 0$ and let $t \in \mathbb{C} \setminus \{0\}$. Let R be defined by

$$R(\lambda) = t_+ \prod_{j=1}^n (\lambda - \sigma_j)(1 - \overline{\sigma}_j \lambda), \quad \text{and}$$

let E_1 be defined by

$$E_1(\lambda) = t \prod_{j=1}^{k_1} (\lambda - \alpha_j^1) \prod_{j=1}^{k_2} (1 - \overline{\alpha}_j^2 \lambda).$$

Then the following statements hold.

- (i) There exists an outer polynomial D of degree at most n such that

$$(5.8) \quad \lambda^{-n} R(\lambda) + |E_1(\lambda)|^2 = |D(\lambda)|^2$$

for all $\lambda \in \mathbb{T}$.

- (ii) The function x defined by

$$x = (x_1, x_2, x_3) = \left(\frac{E_1}{D}, \frac{E_1^{\sim n}}{D}, \frac{D^{\sim n}}{D} \right)$$

is a rational $\overline{\mathbb{E}}$ -inner function such that the degree of x is equal to n and conditions (1), (2) and (3) hold. The royal polynomial of x is R .

Proof. (i) By Lemma 5.13, R is n -balanced, and so $\lambda^{-n}R(\lambda) \geq 0$ for all $\lambda \in \mathbb{T}$. Therefore

$$\lambda^{-n}R(\lambda) + |E_1(\lambda)|^2 \geq 0 \quad \text{for all } \lambda \in \mathbb{T}.$$

By the Fejér-Riesz theorem, there exists an outer polynomial D of degree at most n such that

$$(5.9) \quad \lambda^{-n}R(\lambda) + |E_1(\lambda)|^2 = |D(\lambda)|^2 \quad \text{for all } \lambda \in \mathbb{T}.$$

(ii) By hypothesis

$$\{\sigma_j : 1 \leq j \leq n\} \cap \left(\{\alpha_j^i : 1 \leq j \leq k_i, i = 1, 2\} \cap \mathbb{T} \right) = \emptyset.$$

Thus $\lambda^{-n}R(\lambda)$ and $|E_1(\lambda)|^2$ are non-negative trigonometric polynomials on \mathbb{T} with no common zero. Therefore

$$\lambda^{-n}R(\lambda) + |E_1(\lambda)|^2 > 0 \quad \text{on } \mathbb{T}.$$

The outer polynomial D is defined by equality (5.9) for all $\lambda \in \mathbb{T}$. Hence D has no zero on \mathbb{T} , and so D and $D^{\sim n}$ have no common factor. Thus

$$\deg(x_3) = \deg\left(\frac{D^{\sim n}}{D}\right) = \max\{\deg(D), \deg(D^{\sim n})\} = n.$$

Since

$$\begin{aligned} \lambda^{-n}R(\lambda) &\geq 0 \quad \text{for all } \lambda \in \mathbb{T}, \\ |D(\lambda)|^2 &= \lambda^{-n}R(\lambda) + |E_1(\lambda)|^2 \geq |E_1(\lambda)|^2 \end{aligned}$$

for all $\lambda \in \mathbb{T}$. It follows that

$$|D(\lambda)| \geq |E_1(\lambda)|, \quad \text{for all } \lambda \in \mathbb{T}.$$

Since $D(\lambda) \neq 0$ on $\overline{\mathbb{D}}$, the function $\frac{E_1}{D}$ is analytic in a neighbourhood of $\overline{\mathbb{D}}$. By the Maximum Modulus Principle, we have

$$\frac{|E_1(\lambda)|}{|D(\lambda)|} \leq 1 \quad \text{for all } \lambda \in \overline{\mathbb{D}}.$$

Therefore, by the converse of Theorem 4.15, since conditions (1), (2), (6) and (7) are satisfied, the function

$$x(\lambda) = \left(\frac{E_1(\lambda)}{D(\lambda)}, \frac{E_1^{\sim n}(\lambda)}{D(\lambda)}, \frac{D^{\sim n}(\lambda)}{D(\lambda)} \right) \quad \text{for } \lambda \in \mathbb{D},$$

is a rational $\overline{\mathbb{E}}$ -inner function such that $\deg(x) = n$. The royal polynomial of x is defined by

$$R_x(\lambda) = D(\lambda)D^{\sim n}(\lambda) - E_1(\lambda)E_2(\lambda), \quad \lambda \in \mathbb{D},$$

where $E_2(\lambda) = E_1^{\sim n}(\lambda)$, $\lambda \in \mathbb{D}$. By Proposition 5.3, for all $\lambda \in \mathbb{T}$,

$$\lambda^{-n}R_x(\lambda) = |D(\lambda)|^2 - |E_1(\lambda)|^2.$$

Therefore, by equation (5.8),

$$\lambda^{-n}R_x(\lambda) = \lambda^{-n}R(\lambda) \quad \text{for all } \lambda \in \mathbb{T},$$

where $E_2(\lambda) = E_1^{\sim n}(\lambda)$ for $\lambda \in \mathbb{D}$. Thus $R_x = R$, that is, the royal polynomial of x is equal to R . \square

Remark 5.18. (1) By the Fejér-Riesz theorem, there exists an outer polynomial D satisfying equation (5.8), but to find algebraically such D can be difficult for large n .
 (2) The solution D is only identified up to a multiplication by $\overline{\omega} \in \mathbb{T}$. Thus if we replace D by $\overline{\omega}D$ we obtain a new solution

$$x = \left(\omega \frac{E_1}{D}, \omega \frac{E_1^{\sim n}}{D}, \omega^2 \frac{D^{\sim n}}{D} \right).$$

Example 5.19. Let $n = 1$, $\alpha_1^2 = \frac{1}{2}$ and $\sigma_1 = 0$. Let us construct a rational $\overline{\mathbb{E}}$ -inner function $x = (x_1, x_2, x_3) : \mathbb{D} \rightarrow \overline{\mathbb{E}}$ such that α_1^2 is a zero of x_2 and σ_1 is a royal node of x .

As in Theorem 5.17, for $\lambda \in \mathbb{T}$, let

$$\begin{aligned} R(\lambda) &= t_+ \lambda, & t_+ \text{ is a positive real number.} \\ E_1(\lambda) &= t(1 - \tfrac{1}{2}\lambda), & t \in \mathbb{C} \setminus \{0\}. \end{aligned}$$

The equation (5.8) for the polynomial D is the following, for all $\lambda \in \mathbb{T}$,

$$\begin{aligned} (5.10) \quad |D(\lambda)|^2 &= \lambda^{-1} R(\lambda) + |E_1(\lambda)|^2 \\ &= \overline{\lambda} t_+ \lambda + |t(1 - \tfrac{1}{2}\lambda)|^2 \\ &= t_+ + \tfrac{5}{4}|t|^2 - \tfrac{|t|^2}{2}\lambda - \tfrac{|t|^2}{2}\overline{\lambda}. \end{aligned}$$

Since the degree of D is at most 1, $D(\lambda) = a_1 + a_2\lambda$, where $a_1, a_2 \in \mathbb{C}$ and $\lambda \in \mathbb{T}$,

$$\begin{aligned} (5.11) \quad D(\lambda)\overline{D(\lambda)} &= |a_1 + a_2\lambda|^2 \\ &= (a_1 + a_2\lambda)(\overline{a_1} + \overline{a_2}\overline{\lambda}) = |a_1|^2 + |a_2|^2 + a_1\overline{a_2}\overline{\lambda} + \overline{a_1}a_2\lambda. \end{aligned}$$

Compare equations (5.10) and (5.11). We have

$$(5.12) \quad \begin{cases} \overline{a_1}a_2 = -\frac{|t|^2}{2}, \\ a_1\overline{a_2} = -\frac{|t|^2}{2}, \\ |a_1|^2 + |a_2|^2 = t_+ + \frac{5}{4}|t|^2. \end{cases}$$

Finally the function x can be written in the form

$$x = \left(\frac{E_1}{D}, \frac{E_1^{\sim 1}}{D}, \frac{D^{\sim 1}}{D} \right),$$

where

$$\begin{cases} x_1(\lambda) = \frac{E_1}{D}(\lambda) = \frac{t(1 - \frac{1}{2}\lambda)}{a_1 + a_2\lambda}, \\ x_2(\lambda) = \frac{E_1^{\sim 1}}{D}(\lambda) = \frac{\overline{t}(\lambda - \frac{1}{2})}{a_1 + a_2\lambda}, \\ x_3(\lambda) = \frac{D^{\sim 1}}{D}(\lambda) = \frac{\overline{a_1}\lambda + \overline{a_2}}{a_1 + a_2\lambda}, \end{cases}$$

where $|a_2| < |a_1|$ and a_1, a_2 are given by solving equations (5.12) as functions of t_+ and t . These formulas give a parametrization of solutions for the above problem.

For example, for the given $t = \sqrt{2}$ and $t_+ = \frac{7}{4}$, the system (5.12) has solutions

$$a_1 = -2i, a_2 = \tfrac{1}{2}i \quad \text{or} \quad a_1 = -2, a_2 = \tfrac{1}{2}.$$

Hence the functions

$$x(\lambda) = \left(\frac{\sqrt{2}(1 - \frac{1}{2}\lambda)}{-2\omega(1 - \frac{1}{4}\lambda)}, \frac{\sqrt{2}(\lambda - \frac{1}{2})}{-2\omega(1 - \frac{1}{4}\lambda)}, \frac{\bar{\omega}(\lambda - \frac{1}{4})}{\omega(1 - \frac{1}{4}\lambda)} \right), \quad \lambda \in \mathbb{D},$$

where $\omega \in \mathbb{T}$, are rational $\overline{\mathbb{E}}$ -inner functions such that $\frac{1}{2}$ is a zero of x_2 and 0 is a royal node of x .

Remark 5.20. Theorem 5.17 shows that there exists a 3-parameter family of rational $\overline{\mathbb{E}}$ -inner functions with given royal nodes and given zeros of x_1 and x_2 . It looks at first sight that the construction in Theorem 5.17 gives us a 4-parameter family of rational $\overline{\mathbb{E}}$ -inner functions with the given data. However, the choice of t_+ , t , D and ω leads to the same x as the choice $1, t/\sqrt{t_+}, D/\sqrt{t_+}$ and ω . Theorem 5.21 tells us that the construction yields all solutions of the problem, and so the family of functions x with the required properties is indeed a 3-parameter family.

Theorem 5.21. *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$ -inner function of degree n such that*

- (1) *the zeros of x_1 are $\alpha_1^1, \dots, \alpha_{k_1}^1 \in \overline{\mathbb{D}}$, repeated according to multiplicity,*
- (2) *the zeros of x_2 are $\alpha_1^2, \dots, \alpha_{k_2}^2 \in \overline{\mathbb{D}}$, repeated according to multiplicity, where $k_1 + k_2 = n$, and*
- (3) *the royal nodes of x are $\sigma_1, \dots, \sigma_n \in \overline{\mathbb{D}}$, repeated according to multiplicity.*

There exists some choice of $t_+ > 0$, $t \in \mathbb{C} \setminus \{0\}$ and $\omega \in \mathbb{T}$ such that the algorithm of Theorem 5.17 with these choices creates the function x .

Proof. By Theorem 4.15, there are polynomials E_1^1, E_2^1 and D^1 such that

- (1) $\deg(E_1^1), \deg(E_2^1)$ and $\deg(D^1) \leq n$,
- (2) $D^1(\lambda) \neq 0$ on $\overline{\mathbb{D}}$,
- (3) $E_2^1(\lambda) = (E_1^1)^{\sim n}(\lambda)$
- (4) $|E_i^1(\lambda)| \leq |D^1(\lambda)|$ on $\overline{\mathbb{D}}, i = 1, 2$ and
- (5) $x_1 = \frac{E_1^1}{D^1}, \quad x_2 = \frac{E_2^1}{D^1}$ and $x_3 = \frac{(D^1)^{\sim n}}{D^1}$ on $\overline{\mathbb{D}}$.

By hypothesis, the zeros of x_1 , repeated according to multiplicity, are $\alpha_1^1, \dots, \alpha_{k_1}^1$, and the zeros of x_2 , repeated according to multiplicity, are $\alpha_1^2, \dots, \alpha_{k_2}^2$ where $k_1 + k_2 = n$.

By Lemma 5.15,

$$E_1^1(\lambda) = t \prod_{j=1}^{k_1} (\lambda - \alpha_j^1) \prod_{j=1}^{k_2} (1 - \bar{\alpha}_j^2 \lambda) \quad \text{for some } t \in \mathbb{C} \setminus \{0\} \text{ and all } \lambda \in \mathbb{D}.$$

By hypothesis, $\sigma_1, \dots, \sigma_n$ are the royal nodes of x . Thus, by Proposition 5.14, for the royal polynomial R^1 of x , there exists $t_+ > 0$ such that

$$R^1(\lambda) = t_+ \prod_{j=1}^n (\lambda - \sigma_j)(1 - \bar{\sigma}_j \lambda).$$

By Proposition 5.3, for $\lambda \in \mathbb{T}$,

$$\lambda^{-n} R^1(\lambda) = |D^1(\lambda)|^2 - |E_1^1(\lambda)|^2.$$

By Theorem 4.15, $D^1(\lambda) \neq 0$ on $\overline{\mathbb{D}}$. Hence, for $\lambda \in \mathbb{T}$

$$\lambda^{-n} R^1(\lambda) + |E_1^1(\lambda)|^2 = |D^1(\lambda)|^2 \neq 0.$$

This implies that $\alpha_1^1, \dots, \alpha_{k_1}^1$ and $\alpha_1^2, \dots, \alpha_{k_2}^2$ which are on \mathbb{T} are distinct from σ_i , $i = 1, \dots, n$. By the construction in Theorem 5.17, for σ_i , $i = 1, \dots, n$ and $\alpha_1^1, \dots, \alpha_{k_1}^1$ and $\alpha_1^2, \dots, \alpha_{k_2}^2$, the rational $\overline{\mathbb{E}}$ -inner function $x = (x_1, x_2, x_3)$ can be defined by

$$\left(\frac{E_1}{D}, \frac{E_1^{\sim n}}{D}, \frac{D^{\sim n}}{D} \right)$$

for a suitable choice of $t_+ > 0$, $t \in \mathbb{C} \setminus \{0\}$ and $\omega \in \mathbb{T}$. Since E_1^1 and R^1 coincide with E_1 and R in the construction of Theorem 5.17 for a suitable choice of $t_+ > 0$ and $t \in \mathbb{C} \setminus \{0\}$, D^1 is a permissible choice for ωD for some choice $\omega \in \mathbb{T}$, as a solution for equation (5.8). Thus the algorithm of Theorem 5.17 creates $x = (x_1, x_2, x_3)$ for the appropriate choices of $t_+ > 0$, $t \in \mathbb{C} \setminus \{0\}$ and $\omega \in \mathbb{T}$. \square

6. CONVEX SUBSETS OF $\overline{\mathbb{E}}$ AND EXTREMALITY

In this section we study convex subsets of $\overline{\mathbb{E}}$. We show that, for a fixed $x_3 \in \overline{\mathbb{D}}$, the subset $\overline{\mathbb{E}} \cap (\mathbb{C}^2 \times \{x_3\})$ is convex. Recall that the distinguished boundaries of the tridisc \mathbb{D}^3 and the ball \mathbb{B}_3 contain no line segments. Thus every inner function in the set of analytic functions $\text{Hol}(\mathbb{D}, \mathbb{D}^3)$ from \mathbb{D} to \mathbb{D}^3 is an extreme point of $\text{Hol}(\mathbb{D}, \mathbb{D}^3)$ and every inner function in the set of analytic functions $\text{Hol}(\mathbb{D}, \mathbb{B}_3)$ from \mathbb{D} to \mathbb{B}_3 is an extreme point of $\text{Hol}(\mathbb{D}, \mathbb{B}_3)$. However, this property contrasts sharply with the situation in the tetrablock. Despite the fact that the set \mathcal{J} of rational tetra-inner functions is not convex, the conventional notion of extreme point of \mathcal{J} is well defined and fruitful. In Theorem 6.19, we prove that for $x \in \mathcal{R}^{n,k}$ with $2k \leq n$, x is not an extreme point. A class of extreme points of the set \mathcal{J} is given in Proposition 6.21.

6.1. Convex subsets in the tetrablock.

Definition 6.1. A set Ω in a vector space is convex if for all $z, w \in \Omega$ and all t such that $0 \leq t \leq 1$, the point $tz + (1-t)w$ belongs to Ω .

Proposition 6.2. [1, page 8] $\overline{\mathbb{E}}$ is not convex.

Proof. Take $x = (i, 1, i)$ and $y = (-1, i, -i)$ in $\overline{\mathbb{E}}$. Let us take $t = 1/2$, then the point $w = tx + (1-t)y = \frac{1}{2}(-1+i, 1+i, 0)$. One can show that w is not in $\overline{\mathbb{E}}$. Therefore $\overline{\mathbb{E}}$ is not convex. \square

Let us show that the set $\overline{\mathbb{E}}$ is convex in (x_1, x_2) for a fixed $x_3 \in \overline{\mathbb{D}}$, that is, the set

$$\overline{\mathbb{E}} \cap (\mathbb{C}^2 \times \{x_3\}) = \{x = (x_1, x_2, x_3) \in \mathbb{C}^3 : |x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3| \leq 1 - |x_3|^2\}$$

is convex for every $x_3 \in \overline{\mathbb{D}}$.

Proposition 6.3. The following sets are convex:

- (1) $\overline{\mathbb{E}} \cap (\mathbb{C}^2 \times \{x_3\})$ for any $x_3 \in \overline{\mathbb{D}}$;
- (2) $b\overline{\mathbb{E}} \cap (\mathbb{C}^2 \times \{x_3\})$ for any $x_3 \in \overline{\mathbb{D}}$.

Proof. (1) Let $x, y \in \overline{\mathbb{E}} \cap (\mathbb{C}^2 \times \{x_3\})$, and so, by Theorem 2.5, x and y satisfy the inequalities

$$(6.1) \quad |x_1 - \overline{x_2}x_3| + |x_2 - \overline{x_1}x_3| \leq 1 - |x_3|^2$$

and

$$(6.2) \quad |y_1 - \overline{y_2}x_3| + |y_2 - \overline{y_1}x_3| \leq 1 - |x_3|^2$$

respectively. For all $t \in [0, 1]$,

$$\begin{aligned} w = tx + (1-t)y &= t(x_1, x_2, x_3) + (1-t)(y_1, y_2, x_3) \\ &= (tx_1 + (1-t)y_1, tx_2 + (1-t)y_2, tx_3 + (1-t)x_3) \\ &= (tx_1 + (1-t)y_1, tx_2 + (1-t)y_2, x_3). \end{aligned}$$

Let us check that the point $w \in \mathbb{C}^3$ is in the set $\overline{\mathbb{E}} \cap (\mathbb{C}^2 \times \{x_3\})$. By Theorem 2.5, $w \in \overline{\mathbb{E}}$ if and only if

$$\underbrace{|w_1 - \overline{w}_2 x_3|}_{\text{first term}} + \underbrace{|w_2 - \overline{w}_1 x_3|}_{\text{second term}} \leq 1 - |x_3|^2.$$

Let us consider the first term on the left hand side

$$\begin{aligned} (6.3) \quad |w_1 - \overline{w}_2 x_3| &= |tx_1 + (1-t)y_1 - (t\overline{x}_2 + (1-t)\overline{y}_2)x_3| \\ &= |t(x_1 - \overline{x}_2 x_3) + (1-t)(y_1 - \overline{y}_2 x_3)| \\ &\leq t|x_1 - \overline{x}_2 x_3| + (1-t)|y_1 - \overline{y}_2 x_3|. \end{aligned}$$

For the second term of the left hand side we have

$$\begin{aligned} (6.4) \quad |w_2 - \overline{w}_1 x_3| &= |tx_2 + (1-t)y_2 - (t\overline{x}_1 + (1-t)\overline{y}_1)x_3| \\ &= |t(x_2 - \overline{x}_1 x_3) + (1-t)(y_2 - \overline{y}_1 x_3)| \\ &\leq t|x_2 - \overline{x}_1 x_3| + (1-t)|y_2 - \overline{y}_1 x_3|. \end{aligned}$$

Add inequalities (6.3) and (6.4) we get

$$\begin{aligned} |w_1 - \overline{w}_2 x_3| + |w_2 - \overline{w}_1 x_3| &\leq \\ &t|x_1 - \overline{x}_2 x_3| + (1-t)|y_1 - \overline{y}_2 x_3| + t|x_2 - \overline{x}_1 x_3| + (1-t)|y_2 - \overline{y}_1 x_3|. \end{aligned}$$

Therefore, by inequalities (6.1) and (6.2),

$$\begin{aligned} |w_1 - \overline{w}_2 x_3| + |w_2 - \overline{w}_1 x_3| &\leq t(\underbrace{|x_1 - \overline{x}_2 x_3| + |x_2 - \overline{x}_1 x_3|}_{\text{first part}}) \\ &\quad + (1-t)(\underbrace{|y_1 - \overline{y}_2 x_3| + |y_2 - \overline{y}_1 x_3|}_{\text{second part}}) \\ &\leq t(1 - |x_3|^2) + (1-t)(1 - |x_3|^2) \\ &= 1 - |x_3|^2. \end{aligned}$$

Hence for all $t \in [0, 1]$, $w = tx + (1-t)y \in \overline{\mathbb{E}} \cap (\mathbb{C}^2 \times \{x_3\})$. Therefore $\overline{\mathbb{E}} \cap (\mathbb{C}^2 \times \{x_3\})$ is convex for any fixed $x_3 \in \overline{\mathbb{D}}$.

(2) Let $x_3 \in \overline{\mathbb{D}}$ and $x, y \in b\overline{\mathbb{E}} \cap (\mathbb{C}^2 \times \{x_3\})$, where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, x_3)$. Note that, by Theorem 2.10 (1),

$$(6.5) \quad w \in \mathbb{C}^3 \text{ belongs to } b\overline{\mathbb{E}} \text{ if and only if } w_1 = \overline{w}_2 w_3, |w_2| \leq 1 \text{ and } |w_3| = 1.$$

Thus we have

$$x_1 = \overline{x}_2 x_3, \quad |x_2| \leq 1 \quad \text{and} \quad |x_3| = 1,$$

and

$$y_1 = \overline{y}_2 x_3, \quad |y_2| \leq 1 \quad \text{and} \quad |x_3| = 1.$$

For t such that $0 \leq t \leq 1$, let

$$w = (w_1, w_2, w_3) = tx + (1-t)y = (tx_1 + (1-t)y_1, tx_2 + (1-t)y_2, x_3).$$

To prove the convexity of $b\overline{\mathbb{E}} \cap (\mathbb{C}^2 \times \{x_3\})$, we need to check that, for all t such that $0 \leq t \leq 1$, w lies in $b\overline{\mathbb{E}} \cap (\mathbb{C}^2 \times \{x_3\})$, that is, it satisfies condition (6.5).

Note that

$$\begin{aligned}\overline{w}_2 w_3 &= \overline{(tx_2 + (1-t)y_2)} x_3 \\ &= t\overline{x}_2 x_3 + (1-t)\overline{y}_2 x_3 \\ &= tx_1 + (1-t)y_1 = w_1\end{aligned}$$

and

$$\begin{aligned}|w_2| &= |tx_2 + (1-t)y_2| \\ &\leq t|x_2| + (1-t)|y_2| \\ &\leq t + 1 - t = 1.\end{aligned}$$

Obviously, $|w_3| = |x_3| = 1$. Therefore the set $b\overline{\mathbb{E}} \cap (\mathbb{C}^2 \times \{x_3\})$ is convex for any fixed $x_3 \in \overline{\mathbb{D}}$. \square

Lemma 6.4. *Let $x = (x_1, x_2, x_3)$, $x^1 = (x_1^1, x_2^1, x_3^1)$ and $x^2 = (x_1^2, x_2^2, x_3^2)$ be in $b\overline{\mathbb{E}}$ and satisfy $x = tx^1 + (1-t)x^2$ for some $t \in (0, 1)$. Then $x_3 = x_3^1 = x_3^2$.*

Proof. Since $x, x^1, x^2 \in b\overline{\mathbb{E}}$, by Theorem 2.10,

$$|x_3| = 1, \quad |x_3^1| = 1 \quad \text{and} \quad |x_3^2| = 1.$$

By assumption $x_3 = tx_3^1 + (1-t)x_3^2$. Since every point of \mathbb{T} is an extreme point of $\overline{\mathbb{D}}$, $x_3 = x_3^1 = x_3^2$. \square

6.2. Extremality in the set of $\overline{\mathbb{E}}$ -inner functions. In this section we show that, for a fixed inner function x_3 , the set of rational $\overline{\mathbb{E}}$ -inner functions $x = (x_1, x_2, x_3)$ with third component x_3 is a convex set in \mathcal{J} . We prove that an $\overline{\mathbb{E}}$ -inner function x is not an extreme point of the set \mathcal{J} if the number of the royal nodes of x on \mathbb{T} , counted with multiplicity, is less than or equal to half of the degree of x . In Proposition 6.21 we give a class of extreme rational $\overline{\mathbb{E}}$ -inner functions $x \in \mathcal{R}^{n,k}$ of the set \mathcal{J} for which $2k > n$.

Theorem 6.5. *For a fixed inner function x_3 , the set of $\overline{\mathbb{E}}$ -inner functions (x_1, x_2, x_3) is convex.*

Proof. For the fixed inner function x_3 , let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, x_3)$ be $\overline{\mathbb{E}}$ -inner functions. For $0 \leq t \leq 1$ and $\lambda \in \mathbb{D}$,

$$(tx + (1-t)y)(\lambda) = (tx_1 + (1-t)y_1, tx_2 + (1-t)y_2, x_3)(\lambda).$$

The function

$$w(\lambda) = (w_1, w_2, w_3)(\lambda) = (tx_1 + (1-t)y_1, tx_2 + (1-t)y_2, x_3)(\lambda), \quad \lambda \in \mathbb{D},$$

is analytic on \mathbb{D} and, by Proposition 6.3 (1), $w(\mathbb{D}) \subseteq \overline{\mathbb{E}}$. By Proposition 6.3 (2), since for almost all $\lambda \in \mathbb{T}$, $x(\lambda)$ and $y(\lambda)$ are in $b\overline{\mathbb{E}}$, $w(\lambda)$ has also to be in $b\overline{\mathbb{E}}$. Thus w is an $\overline{\mathbb{E}}$ -inner function. Therefore the set of $\overline{\mathbb{E}}$ -inner functions $x = (x_1, x_2, x_3)$ is convex for any fixed inner function x_3 . \square

Definition 6.6. *A rational $\overline{\mathbb{E}}$ -inner function x is an extreme point of \mathcal{J} if whenever x has a representation of the form $x = tx^1 + (1-t)x^2$ for $t \in (0, 1)$ and x^1, x^2 are rational $\overline{\mathbb{E}}$ -inner functions, $x^1 = x^2$.*

We will show below that \mathcal{J} is not convex, however the notion of extreme points still has the usual sense.

Lemma 6.7. *Let $x = (x_1, x_2, x_3)$, $x^1 = (x_1^1, x_2^1, x_3^1)$ and $x^2 = (x_1^2, x_2^2, x_3^2)$ be rational $\overline{\mathbb{E}}$ -inner functions. If $x = tx^1 + (1-t)x^2$ for some $t \in (0, 1)$ then $x_3 = x_3^1 = x_3^2$.*

Proof. Since $x = tx^1 + (1-t)x^2$, we have

$$(x_1, x_2, x_3) = (tx_1^1, tx_2^1, tx_3^1) + \left((1-t)x_1^2, (1-t)x_2^2, (1-t)x_3^2 \right).$$

Thus $x_3 = tx_3^1 + (1-t)x_3^2$. Hence, for every point $\lambda \in \mathbb{T}$,

$$x_3(\lambda) = tx_3^1(\lambda) + (1-t)x_3^2(\lambda).$$

By assumption, x^1 and x^2 are rational $\overline{\mathbb{E}}$ -inner functions, and so, by Lemma 4.3 (2), x_3^1 and x_3^2 are rational inner functions, that is, for all $\lambda \in \mathbb{T}$,

$$|x_3^1(\lambda)| = 1 \quad \text{and} \quad |x_3^2(\lambda)| = 1.$$

Every point of \mathbb{T} is an extreme point of $\overline{\mathbb{D}}$, and therefore,

$$x_3(\lambda) = x_3^1(\lambda) = x_3^2(\lambda)$$

for all $\lambda \in \mathbb{T}$. Since x^1 and x^2 are rational functions, $x_3 = x_3^1 = x_3^2$. \square

Lemma 6.8. *The set of rational $\overline{\mathbb{E}}$ -inner functions \mathcal{J} is not convex.*

Proof. Suppose that $x^1 = (x_1^1, x_2^1, x_3^1) \in \mathcal{J}$ and $x^2 = (x_1^2, x_2^2, x_3^2) \in \mathcal{J}$ such that $x_3^1 \neq x_3^2$. Then by Lemma 6.7, $x = tx^1 + (1-t)x^2$ is not in \mathcal{J} for all $t \in (0, 1)$. Therefore \mathcal{J} is not convex. \square

For an inner function p of degree n , let \mathcal{R}_p^n be the set of rational $\overline{\mathbb{E}}$ -inner functions with third component p .

Proposition 6.9. *The set \mathcal{R}_p^n is convex for every inner function p of degree n . For any collection S of rational $\overline{\mathbb{E}}$ -inner functions, S is convex if and only if there exists an inner function p of degree n such that S is a convex subset of \mathcal{R}_p^n .*

Proof. It follows from Theorem 6.5 and Lemma 6.7. \square

Proposition 6.10. *Let x be a rational $\overline{\mathbb{E}}$ -inner function of degree n . Then x is a convex combination of at most $2n + 3$ extreme rational $\overline{\mathbb{E}}$ -inner functions of degree at most n .*

Proof. By assumption, $x = (x_1, x_2, x_3)$ is a rational $\overline{\mathbb{E}}$ -inner function of degree n , and so x_3 is an inner function of degree n . Thus $x \in \mathcal{R}_{x_3}^n$. By Remark 4.16, the convex set $\mathcal{R}_{x_3}^n$ is a subset of a $(2n + 2)$ -dimensional real subspace of the rational functions. Therefore, by a theorem of Carathéodory [11, 19], x is a convex combination of at most $2n + 3$ extreme rational $\overline{\mathbb{E}}$ -inner functions of degree at most n . \square

Definition 6.11. *Let f be a real or complex-valued function on a real interval I . We say that f takes a value y to order $m \geq 1$ at a point $t_0 \in I$ if $f \in C^m(I)$, $f(t_0) = y$, $f^{(j)}(t_0) = 0$ for $j = 1, \dots, m-1$ and $f^{(m)}(t_0) \neq 0$. We say that f vanishes to order $m \geq 1$ at a point $t_0 \in I$ if f takes the value 0 to order m at t_0 .*

Lemma 6.12. *Let $f \in C^m(I)$, $f(t_0) = y$ at $t_0 \in I$, and let $y \neq 0$. If f^2 takes the value y^2 to order $m \geq 1$ at t_0 , then f takes the value y to order m at t_0 .*

Proof. Let I be a real interval and let $f \in C^m(I)$. Suppose that f^2 takes the value y^2 to order m at t_0 . Then, by Definition 6.11,

$$(6.6) \quad f^2(t_0) = y^2, [f^2]^{(1)}(t_0) = [f^2]^{(2)}(t_0) = \dots = [f^2]^{(m-1)}(t_0) = 0, [f^2]^{(m)}(t_0) \neq 0.$$

One can check that

$$f(t_0) = y, \quad f^{(j)}(t_0) = 0, \quad \text{for } j = 1, \dots, m-1 \quad \text{and} \quad f^{(m)}(t_0) \neq 0.$$

□

Definition 6.13. A function f is analytic on \mathbb{T} if there exists a function g analytic in a neighbourhood $U_{\mathbb{T}}$ of \mathbb{T} such that $f = g|_{\mathbb{T}}$.

Lemma 6.14. Let $\tau = e^{it_0}$ and let $f(t) = (e^{it} - \tau)^{2v} G(e^{it})$ in a neighbourhood of t_0 where $G(z)$ is analytic on \mathbb{T} and $G(\tau) \neq 0$. Then

$$(6.7) \quad f^{(j)}(t_0) = 0 \quad \text{for } j = 0, 1, \dots, 2v-1 \quad \text{and} \quad f^{(2v)}(t_0) \neq 0.$$

Proof. Since G is analytic on \mathbb{T} , by Definition 6.13, there exists $U_{\mathbb{T}}$ a neighbourhood of \mathbb{T} and there exists \tilde{G} analytic on $U_{\mathbb{T}}$ such that $G = \tilde{G}|_{\mathbb{T}}$. Let $z = e^{it}$, $\varphi(z) = (z - \tau)^{2v} G(z)$ and $\tilde{\varphi}(z) = (z - \tau)^{2v} \tilde{G}(z)$. Define $\gamma(\tau, r)$ to be an anticlockwise circle centred at τ with radius r

$$\gamma(\tau, r) = \{z \in \mathbb{C} : |z - \tau| = r\},$$

where r is taken sufficiently small that $\gamma \subset U_{\mathbb{T}}$. Hence the function $\tilde{\varphi}$ is analytic inside the curve γ . Therefore, by Cauchy's integral formula,

$$\begin{aligned} \tilde{\varphi}^{(j)}(\tau) &= \frac{j!}{2\pi i} \int_{\gamma} \frac{\tilde{\varphi}(z)}{(z - \tau)^{j+1}} dz, \\ &= \frac{j!}{2\pi i} \int_{\gamma} \frac{(z - \tau)^{2v} \tilde{G}(z)}{(z - \tau)^{j+1}} dz \\ (6.8) \quad &= \frac{j!}{2\pi i} \int_{\gamma} (z - \tau)^{2v-j-1} \tilde{G}(z) dz. \end{aligned}$$

For j such that $0 \leq j \leq 2v-1$, the function $(z - \tau)^{2v-j-1} \tilde{G}(z)$ is analytic on $U_{\mathbb{T}}$. Therefore, by Cauchy's Theorem,

$$(6.9) \quad \tilde{\varphi}^{(j)}(\tau) = \frac{j!}{2\pi i} \int_{\gamma} (z - \tau)^{2v-j-1} \tilde{G}(z) dz = 0.$$

If $j = 2v$, then equation (6.8) becomes

$$\tilde{\varphi}^{(2v)}(\tau) = \frac{(2v)!}{2\pi i} \int_{\gamma} \frac{\tilde{G}(z)}{(z - \tau)} dz.$$

By Cauchy's integral formula,

$$(6.10) \quad \tilde{\varphi}^{(2v)}(\tau) = \frac{(2v)!}{2\pi i} \int_{\gamma} \frac{\tilde{G}(z)}{(z - \tau)} dz = (2v)! G(\tau) \neq 0.$$

Hence $\varphi^{(2v)}(\tau) \neq 0$ because $\tilde{\varphi}$ agrees with φ on \mathbb{T} . Note that,

$$f(t) = (e^{it} - e^{it_0})^{(2v)} G(e^{it}) = \varphi(e^{it}).$$

By the chain rule,

$$\begin{aligned} \frac{df}{dt} &= \frac{d\varphi}{dz} \frac{dz}{dt} \\ \frac{d^2 f}{dt^2} &= \frac{d^2 \varphi}{dz^2} \left(\frac{dz}{dt} \right)^2 + \frac{d\varphi}{dz} \frac{d^2 z}{dt^2} \\ \dots &= \dots \\ \frac{d^{2v-1} f}{dt^{2v-1}} &= \frac{d^{2v-1} \varphi}{dz^{2v-1}} \left(\frac{dz}{dt} \right)^{2v-1} + \dots + \frac{d\varphi}{dz} \frac{d^{2v-1} z}{dt^{2v-1}}. \end{aligned}$$

By equation (6.9) and since $\tilde{\varphi}$ and φ agree on \mathbb{T} ,

$$\frac{d^j \tilde{\varphi}}{dz^j}(\tau) = 0, \quad \text{for } j = 1, \dots, 2v - 1,$$

and so,

$$\frac{d^j \varphi}{dz^j}(\tau) = 0, \quad \text{for } j = 1, \dots, 2v - 1.$$

Therefore, $f^{(j)}(t_0) = 0$ for $j = 1, \dots, 2v - 1$. For the $(2v)$ th derivative of f , we have

$$\frac{d^{2v} f}{dt^{2v}} = \frac{d^{2v} \varphi}{dz^{2v}} \left(\frac{dz}{dt} \right)^{2v} + \dots + \frac{d\varphi}{dz} \frac{d^{2v} z}{dt^{2v}}.$$

By equations (6.9) and (6.10),

$$\frac{d^j \varphi}{dz^j}(\tau) = 0, \quad \text{for } j = 1, \dots, 2v - 1 \quad \text{and} \quad \frac{d^{2v} \varphi}{dz^{2v}}(\tau) \neq 0.$$

$$\text{Hence } f^{(2v)}(t_0) = \frac{d^{2v} \varphi}{dz^{2v}}(\tau) \left(\frac{dz}{dt} \right)^{2v} (t_0) \neq 0.$$

Therefore $f^{(j)}(t_0) = 0$ for $j = 0, \dots, 2v - 1$ and $f^{(2v)}(t_0) \neq 0$. \square

Lemma 6.15. *Let $x = (x_1, x_2, x_3)$ be a rational $\overline{\mathbb{E}}$ -inner function. For $\tau \in \mathbb{T}$,*

(1) $|x_1(\tau)| = 1 \Leftrightarrow \tau$ is a royal node of x ;

(2) $|x_2(\tau)| = 1 \Leftrightarrow \tau$ is a royal node of x .

Moreover,

(3) $\tau = e^{it_0}$ is a royal node of x of multiplicity v if and only if $|x_1(e^{it})| = 1$ to order $2v$ at $t = t_0$;

(4) $\tau = e^{it_0}$ is a royal node of x of multiplicity v if and only if $|x_2(e^{it})| = 1$ to order $2v$ at $t = t_0$.

Proof. (1) If $\tau = e^{it_0}$ is a royal node of x of multiplicity v , by Definition 5.8,

$$(6.11) \quad (x_3 - x_1 x_2)(\lambda) = (\lambda - \tau)^{2v} F(\lambda),$$

where F is a rational function, analytic on \mathbb{T} and $F(\tau) \neq 0$. By Lemma 2.11, since x is an $\overline{\mathbb{E}}$ -inner function, $x_2 = \overline{x_1} x_3$ on \mathbb{T} . Therefore, for $\lambda \in \mathbb{T}$,

$$\begin{aligned} (6.12) \quad (x_3 - x_1 x_2)(\lambda) &= (x_3 - x_1 \overline{x_1} x_3)(\lambda) \\ &= x_3(\lambda) - x_3(\lambda) |x_1(\lambda)|^2 \\ &= x_3(\lambda) (1 - |x_1(\lambda)|^2). \end{aligned}$$

Therefore, for any $\lambda \in \mathbb{T}$,

$$|x_1(\lambda)| = 1 \quad \Longleftrightarrow \quad (x_3 - x_1 x_2)(\lambda) = 0,$$

that is, if and only if λ is a royal node of x . Hence $\tau \in \mathbb{T}$ is a royal node of x if and only if $|x_1(\tau)| = 1$.

(2) Since x is rational $\overline{\mathbb{E}}$ -inner function, by Theorem 2.10, $x_1 = \overline{x_2} x_3$ on \mathbb{T} . The rest of the proof is similar to the above proof of (1).

(3) Suppose that $\tau = e^{it_0}$ is a royal node of x of multiplicity $v \geq 1$. Then on combining equations (6.11) and (6.12), we have, for all $t \in \mathbb{R}$,

$$x_3(e^{it})(1 - |x_1(e^{it})|^2) = (e^{it} - \tau)^{2v} F(e^{it}).$$

This gives

$$1 - |x_1(e^{it})|^2 = (e^{it} - \tau)^{2v} \frac{F(e^{it})}{x_3(e^{it})}.$$

The rational function $G = \frac{F}{x_3}$ is analytic on \mathbb{T} and is not equal to zero at $\tau = e^{it_0}$. Thus we have

$$1 - |x_1(e^{it})|^2 = (e^{it} - \tau)^{2v} G(e^{it}).$$

Since x is rational and $|x_1(e^{it_0})| = 1$, the function $f(t) = 1 - |x_1(e^{it})|^2$ is C^∞ on a neighbourhood of t_0 . By Lemma 6.14,

$$f^{(j)}(t_0) = 0 \quad \text{for } j = 0, 1, \dots, 2v - 1 \quad \text{and} \quad f^{(2v)}(t_0) \neq 0.$$

Therefore f takes the value 0 to order $2v$ at t_0 , which implies, by Lemma 6.12, $|x_1(e^{it})| = 1$ to order $2v$ at t_0 .

(4) The proof of this statement follows from (2) and is similar to the above proof of (3). \square

For an inner function p of degree n and $k = 0, 1, \dots, n$, let

$$(6.13) \quad \mathcal{R}_p^{n,k} = \{(x_1, x_2, x_3) \in \mathcal{R}^{n,k} : x_3 = p\}.$$

Lemma 6.16. *Let $x = (x_1, x_2, x_3) \in \mathcal{R}_{x_3}^{n,k}$ and let $\tau_1, \tau_2, \dots, \tau_k \in \mathbb{T}$ be royal nodes of x . Suppose $x = tx^1 + (1-t)x^2$ for some t such that $0 < t < 1$, where $x^1 = (x_1^1, x_2^1, x_3^1)$ and $x^2 = (x_1^2, x_2^2, x_3^2)$ are rational $\overline{\mathbb{E}}$ -inner functions. Then $x_3 = x_3^1 = x_3^2$,*

$$x_1^i(\tau_j) = x_1(\tau_j) \quad \text{for } j = 1, \dots, k \text{ and } i = 1, 2,$$

and

$$x_2^i(\tau_j) = x_2(\tau_j) \quad \text{for } j = 1, \dots, k \text{ and } i = 1, 2.$$

Moreover, $\tau_1, \tau_2, \dots, \tau_k \in \mathbb{T}$ are royal nodes of x^1 and x^2 .

Proof. By Lemma 6.7, $x_3 = x_3^1 = x_3^2$. By Lemma 6.15, $|x_1(\tau_j)| = 1$ and $|x_2(\tau_j)| = 1$ at each royal node $\tau_j \in \mathbb{T}$. By assumption,

$$x_1(\tau_j) = tx_1^1(\tau_j) + (1-t)x_1^2(\tau_j)$$

for t such that $0 < t < 1$ and $|x_1^i(\tau_j)| \leq 1$ for $j = 1, \dots, k$ and $i = 1, 2$. Similarly,

$$x_2(\tau_j) = tx_2^1(\tau_j) + (1-t)x_2^2(\tau_j)$$

for t such that $0 < t < 1$ and $|x_2^i(\tau_j)| \leq 1$ for $j = 1, \dots, k$ and $i = 1, 2$. Every point on the circle \mathbb{T} is an extreme point of \mathbb{D} , and so $x_1^1(\tau_j) = x_1(\tau_j)$ and $x_2^1(\tau_j) = x_2(\tau_j)$ for $j = 1, \dots, k$ and $i = 1, 2$. Therefore $|x_1^i(\tau_j)| = 1$, $|x_2^i(\tau_j)| = 1$ for $j = 1, \dots, k$ and $i = 1, 2$. By Lemma 6.15, $\tau_1, \tau_2, \dots, \tau_k \in \mathbb{T}$ are royal nodes of x^1 and x^2 . \square

Lemma 6.17. *Let $n \geq 1$. Any $x = (x_1, x_2, x_3) \in \mathcal{R}^{n,0}$ is not an extreme point of \mathcal{J} .*

Proof. Since x has no royal nodes on \mathbb{T} , by Lemma 6.15, for all $\lambda \in \mathbb{T}$,

$$|x_1(\lambda)| < 1 \quad \text{and} \quad |x_2(\lambda)| < 1.$$

Since \mathbb{T} is compact, the supremum of x_1 and x_2 is attained on \mathbb{T} , that is, there exist $\lambda_1, \lambda_2 \in \mathbb{T}$ such that

$$(6.14) \quad \sup_{\lambda \in \mathbb{T}} |x_1(\lambda)| = |x_1(\lambda_1)| < 1 \quad \text{and} \quad \sup_{\lambda \in \mathbb{T}} |x_2(\lambda)| = |x_2(\lambda_2)| < 1.$$

Choose $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$(6.15) \quad |x_1(\lambda_1)|(1 + \varepsilon_1) < 1 \quad \text{and} \quad |x_2(\lambda_2)|(1 + \varepsilon_2) < 1.$$

Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. If $x_1(\lambda_1) = 0$, then

$$x_1(\lambda) = 0 \quad \text{for all } \lambda \in \mathbb{T}.$$

Likewise, if $x_2(\lambda_2) = 0$ then

$$x_2(\lambda) = 0, \quad \text{for all } \lambda \in \mathbb{T}.$$

Define x^1 and x^2 to be

$$x^1 = ((1 + \varepsilon)x_1, (1 + \varepsilon)x_2, x_3) \quad \text{and} \quad x^2 = ((1 - \varepsilon)x_1, (1 - \varepsilon)x_2, x_3).$$

Since $x = (x_1, x_2, x_3)$ is a rational $\overline{\mathbb{E}}$ -inner function, for almost all $\lambda \in \mathbb{T}$,

$$(6.16) \quad x_1(\lambda) = \overline{x}_2(\lambda)x_3(\lambda), \quad |x_2(\lambda)| \leq 1 \quad \text{and} \quad |x_3(\lambda)| = 1.$$

Let us check that x^1 and x^2 are rational $\overline{\mathbb{E}}$ -inner functions. By Theorem 2.10 (1), this will follow if we show that

$$(1 + \varepsilon)x_1(\lambda) = (1 + \varepsilon)\overline{x}_2(\lambda)x_3(\lambda), \quad (1 + \varepsilon)|x_2(\lambda)| \leq 1 \quad \text{and} \quad |x_3(\lambda)| = 1,$$

and $x^1(\mathbb{D}) \subset \mathbb{E}$. By equations (6.16), we have to show only that

$$(1 + \varepsilon)|x_2(\lambda)| \leq 1 \quad \text{on } \mathbb{T}.$$

This statement follows from inequalities (6.14) and (6.15). Thus $x^1(\mathbb{T}) \subset b\overline{\mathbb{E}}$. By Theorem 2.10 (2), for almost all $\lambda \in \mathbb{T}$,

$$x^1(\lambda) \in b\overline{\mathbb{E}} \Leftrightarrow \Psi(., x^1(\lambda)) \text{ is an automorphism of } \mathbb{D}.$$

By the maximum principle, for all $\lambda \in \mathbb{D}$, $\|\Psi(., x^1(\lambda))\|_{H^\infty} < 1$. Therefore, by Theorem 2.4, for all $\lambda \in \mathbb{D}$, $x^1(\lambda) \in \mathbb{E}$. This completes the proof that x^1 is a rational $\overline{\mathbb{E}}$ -inner function.

In a similar way we can show that x^2 is a rational $\overline{\mathbb{E}}$ -inner function. Moreover, by Lemma 6.15, x^1, x^2 have no royal nodes on \mathbb{T} and therefore $x^1, x^2 \in \mathcal{R}^{n,0}$. However $x = \frac{1}{2}x^1 + \frac{1}{2}x^2$, which implies that x cannot be an extreme point of \mathcal{J} since $x^1 \neq x^2$. \square

Proposition 6.18. *Let $x = (x_1, x_2, x_3)$ be superficial and $x = tx^1 + (1 - t)x^2$ for some $0 < t < 1$, where $x^1 = (x_1^1, x_2^1, x_3^1)$ and $x^2 = (x_1^2, x_2^2, x_3^2)$ are rational $\overline{\mathbb{E}}$ -inner functions. Then x^1 and x^2 are superficial and $x_3 = x_3^1 = x_3^2$.*

Proof. By Lemma 6.7, $x_3 = x_3^1 = x_3^2$. Suppose, for a contradiction, x^1 is not superficial. It means there exists $\lambda_0 \in \mathbb{D}$ such that $x^1(\lambda_0) \in \mathbb{E}$. We will show that in this case $x(\lambda_0) \in \mathbb{E}$, and so x is not superficial.

By Theorem 2.4, it is enough to prove that

$$(6.17) \quad |x_1(\lambda_0) - \overline{x}_2(\lambda_0)x_3(\lambda_0)| + |x_2(\lambda_0) - \overline{x}_1(\lambda_0)x_3(\lambda_0)| < 1 - |x_3(\lambda_0)|^2.$$

Since $x^1(\lambda_0) \in \mathbb{E}$ and x^2 is a rational $\overline{\mathbb{E}}$ -inner function, this implies that

$$(6.18) \quad |x_1^1(\lambda_0) - \overline{x}_2^1(\lambda_0)x_3^1(\lambda_0)| + |x_2^1(\lambda_0) - \overline{x}_1^1(\lambda_0)x_3^1(\lambda_0)| < 1 - |x_3^1(\lambda_0)|^2$$

and

$$(6.19) \quad |x_1^2(\lambda_0) - \overline{x}_2^2(\lambda_0)x_3^2(\lambda_0)| + |x_2^2(\lambda_0) - \overline{x}_1^2(\lambda_0)x_3^2(\lambda_0)| \leq 1 - |x_3^2(\lambda_0)|^2.$$

Let us begin with the first term on the left hand side of inequality (6.17).

$$\begin{aligned}
 & |x_1(\lambda_0) - \bar{x}_2(\lambda_0)x_3(\lambda_0)| \\
 &= |tx_1^1(\lambda_0) + (1-t)x_1^2(\lambda_0) - (t\bar{x}_2^1(\lambda_0) + (1-t)\bar{x}_2^2(\lambda_0))x_3(\lambda_0)| \\
 (6.20) \quad &\leq |t(x_1^1(\lambda_0) - \bar{x}_2^1(\lambda_0)x_3(\lambda_0))| + |(1-t)(x_1^2(\lambda_0) - \bar{x}_2^2(\lambda_0)x_3(\lambda_0))|.
 \end{aligned}$$

The second term on inequality (6.17)

$$\begin{aligned}
 & |x_2(\lambda_0) - \bar{x}_1(\lambda_0)x_3(\lambda_0)| \\
 &= |tx_2^1(\lambda_0) + (1-t)x_2^2(\lambda_0) - (t\bar{x}_1^1(\lambda_0) + (1-t)\bar{x}_1^2(\lambda_0))x_3(\lambda_0)| \\
 (6.21) \quad &\leq |t(x_2^1(\lambda_0) - \bar{x}_1^1(\lambda_0)x_3(\lambda_0))| + |(1-t)(x_2^2(\lambda_0) - \bar{x}_1^2(\lambda_0)x_3(\lambda_0))|.
 \end{aligned}$$

Add inequalities (6.20) and (6.21), and use inequalities (6.18) and (6.19) to obtain

$$\begin{aligned}
 & |x_1(\lambda_0) - \bar{x}_2(\lambda_0)x_3(\lambda_0)| + |x_2(\lambda_0) - \bar{x}_1(\lambda_0)x_3(\lambda_0)| \\
 &\leq t\left(|x_1^1(\lambda_0) - \bar{x}_2^1(\lambda_0)x_3(\lambda_0)| + |x_2^1(\lambda_0) - \bar{x}_1^1(\lambda_0)x_3(\lambda_0)|\right) \\
 &\quad + (1-t)\left(|x_1^2(\lambda_0) - \bar{x}_2^2(\lambda_0)x_3(\lambda_0)| + |x_2^2(\lambda_0) - \bar{x}_1^2(\lambda_0)x_3(\lambda_0)|\right) \\
 (6.22) \quad &< t(1 - |x_3(\lambda_0)|^2) + (1-t)(1 - |x_3(\lambda_0)|^2) = 1 - |x_3(\lambda_0)|^2.
 \end{aligned}$$

□

Theorem 6.19. *Let $x \in \mathcal{R}^{n,k}$. If $2k \leq n$, then x is not an extreme point of the set \mathcal{J} of rational \mathbb{E} -inner functions.*

Proof. Let $x \in \mathcal{R}^{n,k}$. By Definition 5.8, x has n royal nodes in \mathbb{D} and k royal nodes that lie in \mathbb{T} . By Theorem 4.15, there exist polynomials E_1 , E_2 and D of degree at most n such that

$$x = \left(\frac{E_1}{D}, \frac{E_1^{\sim n}}{D}, \frac{D^{\sim n}}{D} \right),$$

where, for all $\lambda \in \mathbb{D}$, $D(\lambda) \neq 0$ and $E_2(\lambda) = E_1^{\sim n}(\lambda)$. Let $\tau_1, \dots, \tau_k \in \mathbb{T}$ and $\alpha_{k+1}, \dots, \alpha_n \in \mathbb{D}$ be the royal nodes of x in \mathbb{D} repeated according to multiplicity. By Proposition 5.14, the royal polynomial of x is

$$R = r \prod_{j=1}^k Q_{\tau_j} \prod_{j=k+1}^n Q_{\alpha_j},$$

for some $r > 0$. Thus for all $\lambda \in \mathbb{T}$,

$$\begin{aligned}
 \lambda^{-n}R(\lambda) &= r\lambda^{-n} \left\{ \prod_{j=1}^k (\lambda - \tau_j)(1 - \bar{\tau}_j\lambda) \prod_{j=k+1}^n (\lambda - \alpha_j)(1 - \bar{\alpha}_j\lambda) \right\} \\
 (6.23) \quad &= r \prod_{j=1}^k |\lambda - \tau_j|^2 \prod_{j=k+1}^n |\lambda - \alpha_j|^2.
 \end{aligned}$$

By Proposition 5.3 and equation (6.23), for all $\lambda \in \mathbb{T}$,

$$(6.24) \quad |D(\lambda)|^2 - |E_1(\lambda)|^2 = \lambda^{-n}R(\lambda) = r \prod_{j=1}^k |\lambda - \tau_j|^2 \prod_{j=k+1}^n |\lambda - \alpha_j|^2.$$

Assume first that n is even and write $n = 2m$. This implies that $k \leq m$. Define a polynomial g by

$$g(\lambda) = \bar{\tau}_1 \dots \bar{\tau}_k \lambda^{m-k} \prod_{j=1}^k (\lambda - \tau_j)^2.$$

Clearly, the polynomial g has degree $m + k \leq n$. Moreover, g is n -symmetric since

$$\begin{aligned} g^{\sim n}(\lambda) &= \lambda^n \overline{g\left(\frac{1}{\lambda}\right)} = \lambda^n \left\{ \overline{\bar{\tau}_1 \dots \bar{\tau}_k \frac{1}{(\bar{\lambda})^{m-k}} \prod_{j=1}^k \left(\frac{1}{\bar{\lambda}} - \tau_j\right)^2} \right\} \\ &= \lambda^{2m} \left\{ \tau_1 \dots \tau_k \frac{1}{\lambda^{m-k}} \prod_{j=1}^k \left(\frac{1}{\lambda} - \bar{\tau}_j\right)^2 \right\} \\ &= \bar{\tau}_1 \dots \bar{\tau}_k \lambda^{m-k} \prod_{j=1}^k (\lambda - \tau_j)^2 = g(\lambda). \end{aligned}$$

Let

$$E_1^t = E_1 + tg \quad \text{and} \quad E_2^t = E_1^{\sim n} + tg \quad \text{for } t \in \mathbb{R}.$$

The polynomial E_1^t has degree at most n . We also have, for all $\lambda \in \overline{\mathbb{D}}$,

$$\begin{aligned} (E_2^t)^{\sim n}(\lambda) &= (E_1^{\sim n} + tg)^{\sim n}(\lambda) \\ (6.25) \quad &= (E_1^{\sim n})^{\sim n}(\lambda) + (tg)^{\sim n}(\lambda) = (E_1^t + tg)(\lambda) = E_1^t(\lambda). \end{aligned}$$

Note that, on \mathbb{T} ,

$$\begin{aligned} |D|^2 - |E_1^t|^2 &= |D|^2 - |E_1 + tg|^2 \\ &= |D|^2 - (E_1 + tg)\overline{(E_1 + tg)} \\ (6.26) \quad &= |D|^2 - |E_1|^2 - t^2|g|^2 - 2\operatorname{Re}(tg\bar{E}_1). \end{aligned}$$

Let $\|E_1\|_\infty$ denote the supremum of $|E_1|$ on \mathbb{T} . Then, for all $\lambda \in \mathbb{T}$,

$$\begin{aligned} \operatorname{Re}(tg(\lambda)\bar{E}_1(\lambda)) \leq |tg(\lambda)E_1(\lambda)| &= |tE_1(\lambda)| \left| \bar{\tau}_1 \dots \bar{\tau}_k \lambda^{m-k} \prod_{j=1}^k (\lambda - \tau_j)^2 \right| \\ &= |tE_1(\lambda)| \prod_{j=1}^k |\lambda - \tau_j|^2 \\ (6.27) \quad &\leq |t| \|E_1\|_\infty \prod_{j=1}^k |\lambda - \tau_j|^2. \end{aligned}$$

Note that, for all $\lambda \in \mathbb{T}$,

$$|g(\lambda)|^2 = \left| \bar{\tau}_1 \dots \bar{\tau}_k \lambda^{m-k} \prod_{j=1}^k (\lambda - \tau_j)^2 \right|^2 = \left| \prod_{j=1}^k (\lambda - \tau_j)^2 \right|^2.$$

Combine equations (6.24) and (6.26) and inequality (6.27), for all $\lambda \in \mathbb{T}$, to get

$$\begin{aligned}
& |D(\lambda)|^2 - |E_1^t(\lambda)|^2 \\
&= |D(\lambda)|^2 - |E_1(\lambda)|^2 - |t|^2 |g(\lambda)|^2 - 2\operatorname{Re}(tg(\lambda)\overline{E_1}(\lambda)), \quad (\text{by equation (6.26)}) \\
&= r \prod_{j=1}^k |\lambda - \tau_j|^2 \prod_{j=k+1}^n |\lambda - \alpha_j|^2 - |t|^2 |g(\lambda)|^2 - 2\operatorname{Re}(tg(\lambda)\overline{E_1}(\lambda)), \quad (\text{by equation (6.24)}) \\
&\geq r \prod_{j=1}^k |\lambda - \tau_j|^2 \prod_{j=k+1}^n |\lambda - \alpha_j|^2 - |t|^2 |g(\lambda)|^2 - 2|t| \|E_1\|_\infty \prod_{j=1}^k |\lambda - \tau_j|^2, \quad (\text{by inequality (6.27)}) \\
&= \prod_{j=1}^k |\lambda - \tau_j|^2 r \prod_{j=k+1}^n |Q_{\alpha_j}(\lambda)| - |t|^2 \left| \prod_{j=1}^k |\lambda - \tau_j|^2 \right|^2 - 2|t| \|E_1\|_\infty \prod_{j=1}^k |\lambda - \tau_j|^2 \\
&\geq \prod_{j=1}^k |\lambda - \tau_j|^2 \left\{ rM - \left(|t|^2 \prod_{j=1}^k |\lambda - \tau_j|^2 + 2|t| \|E_1\|_\infty \right) \right\} \\
&\geq \prod_{j=1}^k |\lambda - \tau_j|^2 \left\{ rM - |t| (|t| \|g\|_\infty + 2\|E_1\|_\infty) \right\},
\end{aligned}$$

where $M = \inf_{\lambda \in \mathbb{T}} \prod_{j=k+1}^n |Q_{\alpha_j}(\lambda)| > 0$.

Let us show that, for $|t|$ sufficiently small, $|D(\lambda)|^2 - |E_1^t(\lambda)|^2 \geq 0$ on \mathbb{T} . It suffices to find $|t|$ such that

$$rM - |t| (|t| \|g\|_\infty + 2\|E_1\|_\infty) > 0,$$

or equivalently,

$$|t| \left(|t| + 2 \frac{\|E_1\|_\infty}{\|g\|_\infty} \right) - \frac{rM}{\|g\|_\infty} < 0.$$

If we take $|t| \leq \min \left\{ \frac{2\|E_1\|_\infty}{\|g\|_\infty}, \frac{rM}{8\|E_1\|_\infty} \right\}$, then

$$\begin{aligned}
|t| \left(|t| + 2 \frac{\|E_1\|_\infty}{\|g\|_\infty} \right) - \frac{rM}{\|g\|_\infty} &\leq |t| \left(2 \frac{\|E_1\|_\infty}{\|g\|_\infty} + 2 \frac{\|E_1\|_\infty}{\|g\|_\infty} \right) - \frac{rM}{\|g\|_\infty} \\
&\leq \frac{rM}{8\|E_1\|_\infty} \left(4 \frac{\|E_1\|_\infty}{\|g\|_\infty} \right) - \frac{rM}{\|g\|_\infty} \\
(6.28) \quad &= \frac{rM}{2\|g\|_\infty} - \frac{rM}{\|g\|_\infty} = -\frac{rM}{2\|g\|_\infty} < 0.
\end{aligned}$$

Therefore

$$|D|^2 - |E_1^t|^2 \geq 0 \quad \text{on } \mathbb{T},$$

and, by Theorem 5.17, the functions

$$x_{+t} = \left(\frac{E_1^{+t}}{D}, \frac{E_2^{+t}}{D}, \frac{D^{\sim n}}{D} \right) \quad \text{and} \quad x_{-t} = \left(\frac{E_1^{-t}}{D}, \frac{E_2^{-t}}{D}, \frac{D^{\sim n}}{D} \right)$$

are rational $\overline{\mathbb{E}}$ -inner functions. Obviously,

$$\begin{aligned} \frac{1}{2}x_{+t} + \frac{1}{2}x_{-t} &= \left(\frac{E_1^{+t} + E_1^{-t}}{2D}, \frac{E_2^{+t} + E_2^{-t}}{2D}, \frac{D^{\sim n}}{D} \right) \\ &= \left(\frac{E_1 + tg + E_1 - tg}{2D}, \frac{E_1^{\sim n} + tg + E_1^{\sim n} - tg}{2D}, \frac{D^{\sim n}}{D} \right) \\ &= \left(\frac{E_1}{D}, \frac{E_1^{\sim n}}{D}, \frac{D^{\sim n}}{D} \right) = x. \end{aligned}$$

Hence x is not an extreme point of \mathcal{J} .

If n is odd, assume $n = 2m + 1$. This case requires a slight modification. By assumption, $2k \leq n$ thus $2k \leq 2m + 1$. This implies that $k \leq m$. Choose $\omega \in \mathbb{T}$ such that

$$\omega^2 = -\bar{\tau}_1 \prod_{j=1}^k \bar{\tau}_j^2.$$

Let

$$g(\lambda) = \omega \lambda^{m-k} (\lambda - \tau_1) \prod_{j=1}^k (\lambda - \tau_j)^2, \quad \lambda \in \mathbb{C}.$$

Clearly, the polynomial g has degree $m+k+1 \leq n$. Let us check that the g is n -symmetric

$$\begin{aligned} g^{\sim n}(\lambda) &= \lambda^n \overline{g(1/\bar{\lambda})} = \lambda^n \overline{\left(\omega \frac{1}{\lambda^{m-k}} \left(\frac{1}{\lambda} - \tau_1 \right) \prod_{j=1}^k \left(\frac{1}{\lambda} - \tau_j \right)^2 \right)} \\ &= \lambda^n \left(\bar{\omega} \frac{1}{\lambda^{m-k}} \left(\frac{1}{\lambda} - \bar{\tau}_1 \right) \prod_{j=1}^k \left(\frac{1}{\lambda} - \bar{\tau}_j \right)^2 \right) \\ &= \bar{\omega} \lambda^{m-k} \bar{\tau}_1 (\tau_1 - \lambda) \prod_{j=1}^k \bar{\tau}_j^2 (\tau_j - \lambda)^2 \\ &= \bar{\omega} \left(-\bar{\tau}_1 \prod_{j=1}^k \bar{\tau}_j^2 \right) \lambda^{m-k} (\lambda - \tau_1) \prod_{j=1}^k (\lambda - \tau_j)^2 \\ &= \bar{\omega} \omega^2 \lambda^{m-k} (\lambda - \tau_1) \prod_{j=1}^k (\lambda - \tau_j)^2 \\ &= \omega \lambda^{m-k} (\lambda - \tau_1) \prod_{j=1}^k (\lambda - \tau_j)^2 = g(\lambda). \end{aligned}$$

As in the even case, define the polynomials on $\overline{\mathbb{D}}$

$$E_1^t = E_1 + tg \quad \text{and} \quad E_2^t = E_1^{\sim n} + tg \quad \text{for } t \in \mathbb{R}.$$

As in equation (6.25), for all $\lambda \in \overline{\mathbb{D}}$, $E_1^t(\lambda) = (E_2^t)^{\sim n}(\lambda)$ and as in equation (6.26), for all $\lambda \in \mathbb{T}$,

$$(6.29) \quad |D(\lambda)|^2 - |E_1^t(\lambda)|^2 = |D(\lambda)|^2 - |E_1(\lambda)|^2 - t^2 |g(\lambda)|^2 - 2\operatorname{Re}(tg(\lambda) \overline{E_1(\lambda)}).$$

For all $\lambda \in \mathbb{T}$,

$$\begin{aligned} \operatorname{Re}(tg\overline{E_1(\lambda)}) &\leq |tgE_1(\lambda)| = |tE_1(\lambda)| \left| \omega \lambda^{m-k} (\lambda - \tau_1) \prod_{j=1}^k (\lambda - \tau_j)^2 \right| \\ &\leq |t| \|E_1\|_\infty |\lambda - \tau_1| \prod_{j=1}^k |\lambda - \tau_j|^2. \end{aligned}$$

Combine equations (6.24), (6.29) and inequality (6.30) to obtain, for all $\lambda \in \mathbb{T}$,

$$\begin{aligned} &|D(\lambda)|^2 - |E_1^t(\lambda)|^2 \\ &= |D(\lambda)|^2 - |E_1(\lambda)|^2 - |t|^2 |g(\lambda)|^2 - 2\operatorname{Re}(tg(\lambda)\overline{E_1(\lambda)}) \\ &= r \prod_{j=1}^k |\lambda - \tau_j|^2 \prod_{j=k+1}^n |\lambda - \alpha_j|^2 - |t|^2 |g(\lambda)|^2 - 2\operatorname{Re}(tg(\lambda)\overline{E_1(\lambda)}), \text{ by equation (6.24)} \\ &\geq r \prod_{j=1}^k |\lambda - \tau_j|^2 \prod_{j=k+1}^n |\lambda - \alpha_j|^2 - |t|^2 |g(\lambda)|^2 - 2|t| \|E_1\|_\infty |\lambda - \tau_1| \prod_{j=1}^k |\lambda - \tau_j|^2, \text{ by inequality (6.30)} \\ &= \prod_{j=1}^k |\lambda - \tau_j|^2 r \prod_{j=k+1}^n |Q_{\alpha_j}(\lambda)| - |t|^2 |\lambda - \tau_1|^2 \prod_{j=1}^k |\lambda - \tau_j|^4 - 2|t| \|E_1\|_\infty |\lambda - \tau_1| \prod_{j=1}^k |\lambda - \tau_j|^2 \\ &\geq \prod_{j=1}^k |\lambda - \tau_j|^2 \left\{ rM - \underbrace{|\lambda - \tau_1|}_{\leq 2} \left(\underbrace{|t|^2 |\lambda - \tau_1| \prod_{j=1}^k |\lambda - \tau_j|^2}_{\leq |t|^2 \|g\|_\infty} + 2|t| \|E_1\|_\infty \right) \right\} \\ &\geq \prod_{j=1}^k |\lambda - \tau_j|^2 \left\{ rM - 2(|t|^2 \|g\|_\infty + 2|t| \|E_1\|_\infty) \right\}, \end{aligned}$$

where $M = \inf_{\mathbb{T}} \prod |Q_{\alpha_j}| > 0$. By similar arguments to those in equations (6.28), one can find $|t|$ such that

$$rM - 2(|t|^2 \|g\|_\infty + 2|t| \|E_1\|_\infty) > 0.$$

Therefore,

$$|D|^2 - |E_1^t|^2 \geq 0, \quad \text{on } \mathbb{T}.$$

Hence, by Theorem 5.17, the functions

$$x_{\pm t} = \left(\frac{E_1^{\pm t}}{D}, \frac{(E_1^{\sim n})^{\pm t}}{D}, \frac{D^{\sim n}}{D} \right)$$

are rational $\overline{\mathbb{E}}$ -inner functions. One can check that $x = \frac{1}{2}x_{+t} + \frac{1}{2}x_{-t}$ and therefore x is not an extreme point of \mathcal{J} . \square

Theorem 6.20. [6, Theorem 5.13] *A rational Γ -inner function $h \in \mathcal{R}_\Gamma^{n,k}$ is extreme in the set of rational Γ -inner functions if and only if $2k > n$.*

Proposition 6.21. *Let $x = (x_1, x_2, x_3) \in \mathcal{R}^{n,k}$ be a rational $\overline{\mathbb{E}}$ -inner function such that $x_1 = x_2$ and $2k > n$. Then x is an extreme point of the set \mathcal{J} of rational $\overline{\mathbb{E}}$ -inner functions.*

Proof. By Lemma 4.8 (1), the function $h = (s, p) = (2x_1, x_3)$ is Γ -inner. By Theorem 4.15, there are polynomials E_1, E_2, D such that $x = (\frac{E_1}{D}, \frac{E_2}{D}, \frac{D^{\sim n}}{D})$. Here, since $x_1 = x_2$, necessarily $E_1 = E_2$. By Definition 3.7, the royal polynomial R_h of h is

$$\begin{aligned} R_h(\lambda) &= D^2(\lambda) \left(4x_3 - 4x_1^2 \right)(\lambda) \\ &= D^2(\lambda) \left(4 \frac{D^{\sim n}}{D} - 4 \frac{E_1^2}{D^2} \right)(\lambda) \\ &= 4(DD^{\sim n} - E_1^2)(\lambda) = 4R_x(\lambda). \end{aligned}$$

It is clear that if $x \in \mathcal{R}^{n,k}$, then h has degree n and k royal nodes on \mathbb{T} , counted with multiplicities, such that $2k > n$. Thus, by Theorem 6.20, h is an extreme point of the set of rational Γ -inner functions. That is, if $h^1 = (s^1, p^1)$ and $h^2 = (s^2, p^2)$ are Γ -inner functions such that

$$h = th^1 + (1-t)h^2 \quad \text{for some } t \in (0, 1),$$

then $h = h^1 = h^2$. Note that, in this case, we have

$$(6.30) \quad \begin{cases} s = ts^1 + (1-t)s^2 & \Rightarrow s = s^1 = s^2 \\ p = tp^1 + (1-t)p^2 & \Rightarrow p = p^1 = p^2. \end{cases}$$

Suppose

$$x = tx^1 + (1-t)x^2, \quad \text{for some } t \in (0, 1),$$

and for rational \mathbb{E} -inner functions $x^1 = (x_1^1, x_2^1, x_3^1)$ and $x^2 = (x_1^2, x_2^2, x_3^2)$. This implies that

$$\begin{cases} x_1 = tx_1^1 + (1-t)x_1^2 \\ x_2 = tx_2^1 + (1-t)x_2^2 \\ x_3 = p = tx_3^1 + (1-t)x_3^2. \end{cases}$$

Recall that $(s, p) = (2x_1, x_3)$, hence

$$(6.31) \quad \begin{cases} s = 2tx_1^1 + 2(1-t)x_1^2 \\ s = 2tx_2^1 + 2(1-t)x_2^2 \\ p = tx_3^1 + (1-t)x_3^2. \end{cases}$$

Therefore

$$(s, p) = t(2x_1^1, x_3^1) + (1-t)(2x_1^2, x_3^2)$$

and

$$(s, p) = t(2x_2^1, x_3^1) + (1-t)(2x_2^2, x_3^2).$$

Since h is an extreme rational Γ -inner function, we have

$$\begin{cases} 2x_1^1 = 2x_1^2 = s \\ 2x_2^1 = 2x_2^2 = s \\ x_3^1 = x_3^2 = p. \end{cases}$$

Therefore $x = x^1 = x^2$. Hence x is extreme in the set \mathcal{J} . \square

Remark 6.22. It is not clear that a rational \mathbb{E} -inner function $x = (x_1, x_2, x_3) \in \mathcal{R}^{n,k}$ such that $2k > n$ and $x_1 \neq x_2$, is an extreme point of the set \mathcal{J} of rational \mathbb{E} -inner functions. Could we claim that in Lemma 6.16, if $\tau_i \in \mathbb{T}$ is a royal node of x of multiplicity ν , then τ_i is a royal node of x^1 and x^2 of the same multiplicity ν ? Here

$x = tx^1 + (1 - t)x^2$ for some t such that $0 < t < 1$. If so then x is an extreme point of \mathcal{J} if and only if $2k > n$.

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