



## Relation between spectral classes of a self-similar Cantor set

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### Abstract

A self-similar Cantor set is completely decomposed as a class of the lower (upper) distribution sets. We give a relationship between the distribution sets in the distribution class and the subsets in a spectral class generated by the lower (upper) local dimensions of a self-similar measure. In particular, we show that each subset of a spectral class is exactly a distribution set having full measure of a self-similar measure related to the distribution set using the strong law of large numbers. This gives essential information of its Hausdorff and packing dimensions. In fact, the spectral class by the lower (upper) local dimensions of every self-similar measure, except for a singular one, is characterized by the lower or upper distribution class. Finally, we compare our results with those of other authors.

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### 1. Introduction

Many authors [4,5,8–11] have studied multifractal theory including self-similar measures. A spectral class of a self-similar Cantor set, a class of subsets derived from the local dimensions of a self-similar measure on a self-similar Cantor set, has been investigated in [5,6,8] to study its geometrical properties. In [6,8], the Hausdorff and packing dimensions of subsets composing a spectral class were calculated using power equations related to contraction ratios and an associated probability of a self-similar measure. In this paper, we relate a spectral class by the lower (upper) local dimensions of a self-similar measure with the class by the lower or upper distribution sets (cf. [7]). The relationship gives the com-

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parison of a subset in a spectral class with another subset in a different spectral class via a distribution set. Thus all the spectral classes by the self-similar measures and their lower (upper) local dimensions are classified into two classes: the lower distribution class and the upper distribution class. We compute Hausdorff and packing dimensions of the lower (upper) distribution sets in the lower (upper) distribution class finding singularities when calculating packing dimensions of some lower (upper) distribution sets. Using these results with the relationship, we compute the values of dimensions of the subsets composing a spectral class generated by a self-similar measure and its lower (upper) local dimensions. Except for a singular self-similar measure, every self-similar measure gives two spectral classes which are derived from its lower local dimensions and its upper local dimensions. In fact, the set of associated lower (upper) local dimensions of every self-similar measure, except for such a singular one, is some interval of a non-empty interior. In particular, the singular self-similar measure has only one value of its local dimension which is the Hausdorff and packing dimension of the self-similar Cantor set. In the end, we compare our results with those of [6,8].

## 2. Preliminaries

We denote  $F$  a self-similar Cantor set, which is the attractor of the similarities  $f_1(x) = ax$  and  $f_2(x) = bx + (1 - b)$  on  $I = [0, 1]$  with  $a > 0$ ,  $b > 0$  and  $1 - (a + b) > 0$ . Let  $I_{i_1, \dots, i_k} = f_{i_1} \circ \dots \circ f_{i_k}(I)$ , where  $i_j \in \{1, 2\}$  and  $1 \leq j \leq k$ . We note that if  $x \in F$ , then there is  $\sigma \in \{1, 2\}^{\mathbb{N}}$  such that  $\bigcap_{k=1}^{\infty} I_{\sigma|k} = \{x\}$  (here  $\sigma|k = i_1, i_2, \dots, i_k$ , where  $\sigma = i_1, i_2, \dots, i_k, i_{k+1}, \dots$ ). If  $x \in F$  and  $x \in I_{\sigma}$ , where  $\sigma \in \{1, 2\}^k$ ,  $c_k(x)$  denotes  $I_{\sigma}$  and  $|c_k(x)|$  denotes the diameter of  $c_k(x)$  for each  $k = 0, 1, 2, \dots$ . Let  $p \in (0, 1)$  and we denote  $\gamma_p$  a self-similar Borel probability measure on  $F$  satisfying  $\gamma_p(I_1) = p$  (cf. [6]).  $\dim(E)$  denotes the Hausdorff dimension of  $E$  and  $\text{Dim}(E)$  denotes the packing dimension of  $E$  [6]. We note that  $\dim(E) \leq \text{Dim}(E)$  for every set  $E$  [6]. We denote  $n_1(x|k)$  the number of times the digit 1 occurs in the first  $k$  places of  $x = \sigma$  (cf. [7]).

For  $r \in [0, 1]$ , we define the lower (upper) distribution set  $\underline{F}(r)$  ( $\overline{F}(r)$ ) containing the digit 1 in proportion  $r$  by

$$\underline{F}(r) = \left\{ x \in F : \liminf_{k \rightarrow \infty} \frac{n_1(x|k)}{k} = r \right\},$$

$$\overline{F}(r) = \left\{ x \in F : \limsup_{k \rightarrow \infty} \frac{n_1(x|k)}{k} = r \right\}.$$

We call  $\{\underline{F}(r) : 0 \leq r \leq 1\}$  the lower distribution class and  $\{\overline{F}(r) : 0 \leq r \leq 1\}$  the upper distribution class. We write  $\underline{E}_{\alpha}^{(p)}$  ( $\overline{E}_{\alpha}^{(p)}$ ) for the set of points at which the lower (upper) local dimension of  $\gamma_p$  on  $F$  is exactly  $\alpha$ , so that

$$\underline{E}_{\alpha}^{(p)} = \left\{ x : \liminf_{r \rightarrow 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha \right\},$$

$$\overline{E}_{\alpha}^{(p)} = \left\{ x : \limsup_{r \rightarrow 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \alpha \right\}.$$

We call  $\{\underline{E}_\alpha^{(p)} (\neq \emptyset): \alpha \in \mathbb{R}\}$  the spectral class generated by the lower local dimensions of a self-similar measure  $\gamma_p$  and  $\{\bar{E}_\alpha^{(p)} (\neq \emptyset): \alpha \in \mathbb{R}\}$  the spectral class generated by the upper local dimensions of a self-similar measure  $\gamma_p$ . We call  $\alpha$  satisfying  $\underline{E}_\alpha^{(p)} (\neq \emptyset)$  ( $\bar{E}_\alpha^{(p)} (\neq \emptyset)$ ) an associated lower (upper) local dimension of  $\gamma_p$ .

In this paper, we assume that  $0 \log 0 = 0$  for convenience.

### 3. Main results

**Lemma 1.** Let  $p \in (0, 1)$  and consider a self-similar measure  $\gamma_p$  on  $F$  and let  $r \in [0, 1]$  and

$$g(r, p) = \frac{r \log p + (1-r) \log(1-p)}{r \log a + (1-r) \log b}.$$

Then for a real number  $s$  satisfying  $a^s + b^s = 1$ ,

(1) for  $0 < p < a^s$

$$\liminf_{k \rightarrow \infty} \frac{n_1(x|k)}{k} = r \iff \liminf_{k \rightarrow \infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = g(r, p),$$

(2) for  $a^s < p < 1$

$$\liminf_{k \rightarrow \infty} \frac{n_1(x|k)}{k} = r \iff \limsup_{k \rightarrow \infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = g(r, p),$$

(3) for  $0 < p < a^s$

$$\limsup_{k \rightarrow \infty} \frac{n_1(x|k)}{k} = r \iff \limsup_{k \rightarrow \infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = g(r, p),$$

(4) for  $a^s < p < 1$

$$\limsup_{k \rightarrow \infty} \frac{n_1(x|k)}{k} = r \iff \liminf_{k \rightarrow \infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = g(r, p).$$

**Proof.** For (2), we note that the function

$$f(q) = \frac{q \log p + (1-q) \log(1-p)}{q \log a + (1-q) \log b}$$

is a strictly decreasing function where  $0 \leq q \leq 1$  under the assumption that

$$a^s < p < 1 \iff \frac{\log p}{\log a} < \frac{\log(1-p)}{\log b}.$$

Assume that  $\liminf_{k \rightarrow \infty} n_1(x|k)/k = r$ . Consider a convergent subsequence  $\{n_1(x|k_n)/k_n\}$  of  $\{n_1(x|k)/k\}$  whose limit is  $t$ . Clearly  $t \geq r$ . Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\log \gamma_p(c_{k_n}(x))}{\log |c_{k_n}(x)|} &= \lim_{n \rightarrow \infty} \frac{n_1(x|k_n) \log p + (k_n - n_1(x|k_n)) \log(1-p)}{n_1(x|k_n) \log a + (k_n - n_1(x|k_n)) \log b} \\ &= \frac{t \log p + (1-t) \log(1-p)}{t \log a + (1-t) \log b}. \end{aligned}$$

Since  $f(q)$  is strictly decreasing,

$$\limsup_{k \rightarrow \infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = \frac{r \log p + (1-r) \log(1-p)}{r \log a + (1-r) \log b}.$$

For the converse, assume that  $\liminf_{k \rightarrow \infty} n_1(x|k)/k = r'$ , where  $r' \neq r$ . Similarly we get

$$\limsup_{k \rightarrow \infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = \frac{r' \log p + (1-r') \log(1-p)}{r' \log a + (1-r') \log b}.$$

(2) follows since  $f(q)$  is strictly decreasing. The dual arguments give (1), (3), and (4).  $\square$

**Theorem 2.** Let  $s$  be the unique real number satisfying  $a^s + b^s = 1$  and let  $r \in [0, 1]$ . Then

- (1)  $\underline{F}(r) = \underline{E}_{g(r,p)}^{(p)}$  if  $0 < p < a^s$ ,
- (2)  $\underline{F}(r) = \overline{E}_{g(r,p)}^{(p)}$  if  $a^s < p < 1$ ,
- (3)  $\overline{F}(r) = \overline{E}_{g(r,p)}^{(p)}$  if  $0 < p < a^s$ ,
- (4)  $\overline{F}(r) = \underline{E}_{g(r,p)}^{(p)}$  if  $a^s < p < 1$ .

**Proof.** We note that

$$\liminf_{r \rightarrow 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \liminf_{k \rightarrow \infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|}$$

and

$$\limsup_{r \rightarrow 0} \frac{\log \gamma_p(B_r(x))}{\log r} = \limsup_{k \rightarrow \infty} \frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} \quad [3,6].$$

It is immediate from the above lemma.  $\square$

**Corollary 3.** Let  $s$  be the unique real number satisfying  $a^s + b^s = 1$  and let  $p \in (0, 1)$  and  $\alpha \in \mathbb{R}$ . For the solution  $r$  of the equation  $\alpha = g(r, p)$ ,

- (1) if  $0 < p < a^s$  and  $\alpha \in [\log(1-p)/\log b, \log p/\log a]$ , then  $\underline{E}_\alpha^{(p)} = \underline{F}(r) = \underline{E}_{\alpha'}^{(p')} = \overline{E}_{\alpha''}^{(p'')}$  for  $0 < p' < a^s$  and  $\alpha' \in [\log(1-p')/\log b, \log p'/\log a]$  with  $\alpha' = g(r, p')$  and for  $a^s < p'' < 1$  and  $\alpha'' \in [\log p''/\log a, \log(1-p'')/\log b]$  with  $\alpha'' = g(r, p'')$ ,
- (2) if  $a^s < p < 1$  and  $\alpha \in [\log p/\log a, \log(1-p)/\log b]$ , then  $\underline{E}_\alpha^{(p)} = \overline{F}(r) = \underline{E}_{\alpha'}^{(p')} = \overline{E}_{\alpha''}^{(p'')}$  for  $a^s < p' < 1$  and  $\alpha' \in [\log p'/\log a, \log(1-p')/\log b]$  with  $\alpha' = g(r, p')$  and for  $0 < p'' < a^s$  and  $\alpha'' \in [\log(1-p'')/\log b, \log p''/\log a]$  with  $\alpha'' = g(r, p'')$ ,
- (3) if  $0 < p < a^s$  and  $\alpha \in [\log(1-p)/\log b, \log p/\log a]$ , then  $\overline{E}_\alpha^{(p)} = \overline{F}(r) = \overline{E}_{\alpha'}^{(p')} = \underline{E}_{\alpha''}^{(p'')}$  for  $0 < p' < a^s$  and  $\alpha' \in [\log(1-p')/\log b, \log p'/\log a]$  with  $\alpha' = g(r, p')$  and for  $a^s < p'' < 1$  and  $\alpha'' \in [\log p''/\log a, \log(1-p'')/\log b]$  with  $\alpha'' = g(r, p'')$ ,
- (4) if  $a^s < p < 1$  and  $\alpha \in [\log p/\log a, \log(1-p)/\log b]$ , then  $\overline{E}_\alpha^{(p)} = \underline{F}(r) = \overline{E}_{\alpha'}^{(p')} = \underline{E}_{\alpha''}^{(p'')}$  for  $a^s < p' < 1$  and  $\alpha' \in [\log p'/\log a, \log(1-p')/\log b]$  with  $\alpha' = g(r, p')$  and for  $0 < p'' < a^s$  and  $\alpha'' \in [\log(1-p'')/\log b, \log p''/\log a]$  with  $\alpha'' = g(r, p'')$ .

**Proof.** It is immediate from the above theorem.  $\square$

**Remark 1.** For  $p = a^s$  where  $a^s + b^s = 1$ ,  $\underline{E}_\alpha^{(p)} = F = \overline{E}_\alpha^{(p)}$  for  $\alpha = s$  since

$$\frac{\log \gamma_p(c_k(x))}{\log |c_k(x)|} = s$$

for each  $k \in \mathbb{N}$  and we note that

$$\left[ \frac{\log p}{\log a}, \frac{\log(1-p)}{\log b} \right] = \left[ \frac{\log(1-p)}{\log b}, \frac{\log p}{\log a} \right] = \{s\}.$$

We note that for every  $r \in [0, 1]$  we can relate  $\underline{F}(r)$  and  $\overline{F}(r)$  with  $\underline{E}_\alpha^{(p)}$  and  $\overline{E}_\alpha^{(p)}$  for some  $p \neq a^s$  and some  $\alpha$ . In particular, for  $0 < p < a^s$  and for  $a^s < p'' < 1$  since  $g(r, p)$  is continuous for  $r \in [0, 1]$  and  $g(0, p) - g(0, p'') < 0 < g(1, p) - g(1, p'')$ , there exists  $0 < r < 1$  such that  $g(r, p) = g(r, p'')$  by the intermediate value theorem. Let this value  $g(r, p)$  be  $\alpha$ . Then  $\underline{E}_\alpha^{(p)} = \underline{F}(r) = \overline{E}_\alpha^{(p'')}$ . We note that there exists unique such  $r$  for such  $p$  and  $p''$  since  $g(r, p)$  is strictly increasing for  $r \in [0, 1]$  and  $g(r, p'')$  is strictly decreasing for  $r \in [0, 1]$ . It is just for the case (1) in the above corollary. Duality holds for (2)–(4). From the above corollary we also note that  $E_\alpha^{(p)} = F(r)$  for  $p \neq a^s$ , where  $E_\alpha^{(p)} = \underline{E}_\alpha^{(p)} \cap \overline{E}_\alpha^{(p)}$  and  $F(r) = \underline{F}(r) \cap \overline{F}(r)$ , whereas  $F(a^s) \subset E_s^{(a^s)} = F$  and  $F(a^s) \neq E_s^{(a^s)}$ .

**Corollary 4.** Let  $s$  be the unique real number satisfying  $a^s + b^s = 1$  and let

$$\delta(p) = \frac{p \log p + (1-p) \log(1-p)}{p \log a + (1-p) \log b}.$$

Then

- (1)  $\underline{F}(p) = \underline{E}_{\delta(p)}^{(p)}$  if  $0 < p < a^s$ ,
- (2)  $\underline{F}(p) = \overline{E}_{\delta(p)}^{(p)}$  if  $a^s < p < 1$ ,
- (3)  $\overline{F}(p) = \overline{E}_{\delta(p)}^{(p)}$  if  $0 < p < a^s$ ,
- (4)  $\overline{F}(p) = \underline{E}_{\delta(p)}^{(p)}$  if  $a^s < p < 1$ .

**Proof.** It is immediate from Theorem 2 with  $r = p$ .  $\square$

**Corollary 5.** Let  $s$  be the unique real number satisfying  $a^s + b^s = 1$  and let

$$\delta(p) = \frac{p \log p + (1-p) \log(1-p)}{p \log a + (1-p) \log b}.$$

Then

- (1)  $\dim(\underline{F}(p)) = \dim(\overline{F}(p)) = \delta(p)$  and  $\text{Dim}(\overline{F}(p)) = \delta(p)$  if  $0 < p < a^s$ ,
- (2)  $\dim(\underline{F}(p)) = \dim(\overline{F}(p)) = \delta(p)$  and  $\text{Dim}(\underline{F}(p)) = \delta(p)$  if  $a^s < p < 1$ ,
- (3)  $\dim(\underline{F}(a^s)) = \dim(\overline{F}(a^s)) = s$  and  $\text{Dim}(\underline{F}(a^s)) = \text{Dim}(\overline{F}(a^s)) = s$ .

**Proof.** From the strong law of large numbers we obtain  $\gamma_p(F(p)) = 1$  where  $F(p) = \underline{E}(p) \cap \overline{F}(p)$ . Applying [6, Proposition 2.3] to above corollary, we easily get  $\dim(\underline{E}(p)) = \delta(p)$  and  $\text{Dim}(\overline{F}(p)) = \delta(p)$  if  $0 < p < a^s$ . We also note that  $\delta(p) \leq \dim(F(p)) \leq \dim(\overline{F}(p)) \leq \text{Dim}(\overline{F}(p)) \leq \delta(p)$  if  $0 < p < a^s$ . Similarly (2) holds. For (3),  $F(a^s) \subset \underline{E}(a^s)$  and  $F(a^s) \subset E_s^{(a^s)} = F$ , where  $E_s^{(a^s)} = \underline{E}_s^{(a^s)} \cap \overline{E}_s^{(a^s)}$  and  $\gamma_{a^s}(F(a^s)) = 1$  by the strong law of large numbers. Similarly it holds for  $\overline{F}(a^s)$ .  $\square$

**Theorem 6.**  $\dim(\underline{F}(0)) = 0$ ,  $\text{Dim}(\overline{F}(0)) = 0$ ,  $\text{Dim}(\underline{F}(1)) = 0$  and  $\dim(\overline{F}(1)) = 0$ .

**Proof.** From (1) of Theorem 2,  $\dim(\underline{F}(0)) \leq \log(1-p)/\log b$  for every  $0 < p < a^s$  where  $a^s + b^s = 1$ . Hence  $\dim(\underline{F}(0)) = 0$ . From (3) of Theorem 2,  $\text{Dim}(\overline{F}(0)) \leq \log(1-p)/\log b$  for every  $0 < p < a^s$ , so  $\text{Dim}(\overline{F}(0)) = 0$ . From (2) of Theorem 2,  $\text{Dim}(\underline{F}(1)) \leq \log p/\log a$  for every  $a^s < p < 1$ , so  $\text{Dim}(\underline{F}(1)) = 0$ . From (4) of Theorem 2, using the same arguments we get  $\dim(\overline{F}(1)) = 0$ .  $\square$

**Remark 2.** Since  $\dim(E) \leq \text{Dim}(E)$  for every set  $E$ , we have that  $\dim(\overline{F}(0)) = \dim(\underline{F}(1)) = 0$ . Unlike the Hausdorff dimension case, we do not have the informations of packing dimension about  $\underline{F}(0)$  and  $\overline{F}(1)$ .

**Corollary 7.** Let  $p \in (0, 1)$  and  $p \neq a^s$  where  $a^s + b^s = 1$ . Let  $\alpha$  be in

$$\left[ \frac{\log p}{\log a}, \frac{\log(1-p)}{\log b} \right] \quad \text{or} \quad \left[ \frac{\log(1-p)}{\log b}, \frac{\log p}{\log a} \right].$$

For the solution  $r = r(\alpha)$  of the equation

$$\alpha = \frac{r \log p + (1-r) \log(1-p)}{r \log a + (1-r) \log b} \quad \text{and} \quad \delta(r) = \frac{r \log r + (1-r) \log(1-r)}{r \log a + (1-r) \log b},$$

- (1)  $\dim(\underline{E}_\alpha^{(p)}) = \dim(\overline{E}_\alpha^{(p)}) = \dim(E_\alpha^{(p)}) = \dim(F(r)) = \delta(r)$ ,
- (2)  $\text{Dim}(E_\alpha^{(p)}) = \text{Dim}(F(r)) = \delta(r)$ ,
- (3)  $\text{Dim}(\underline{E}_\alpha^{(p)}) = \delta(r)$  if  $0 < p < a^s$  with  $a^s \leq r \leq 1$  or  $a^s < p < 1$  with  $0 \leq r \leq a^s$ ,
- (4)  $\text{Dim}(\overline{E}_\alpha^{(p)}) = \delta(r)$  if  $0 < p < a^s$  with  $0 \leq r \leq a^s$  or  $a^s < p < 1$  with  $a^s \leq r \leq 1$ ,
- (5)  $\dim(\underline{E}_s^{(a^s)}) = \dim(\overline{E}_s^{(a^s)}) = \text{Dim}(\underline{E}_s^{(a^s)}) = \text{Dim}(\overline{E}_s^{(a^s)}) = s$ .

**Proof.** For  $p (\neq a^s) \in (0, 1)$  and  $\alpha$  in the assumption, we note that  $E_\alpha^{(p)} = \underline{E}_\alpha^{(p)} \cap \overline{E}_\alpha^{(p)} = \underline{F}(r) \cap \overline{F}(r) = F(r)$  where  $r = r(\alpha)$  of the equation

$$\alpha = \frac{r \log p + (1-r) \log(1-p)}{r \log a + (1-r) \log b}.$$

From Corollary 5 and Theorem 6,  $\delta(r) = \dim(F(r)) = \dim(\underline{F}(r)) = \dim(\overline{F}(r))$  for every  $r \in [0, 1]$ . Since from Theorem 2,  $\overline{E}_\alpha^{(p)}$  is  $\underline{F}(r)$  or  $\overline{F}(r)$  and  $\underline{E}_\alpha^{(p)}$  is  $\underline{F}(r)$  or  $\overline{F}(r)$ , where  $r = r(\alpha)$  of the equation

$$\alpha = \frac{r \log p + (1-r) \log(1-p)}{r \log a + (1-r) \log b},$$

so (1) follows. Similarly (2)–(5) follow from Theorem 2, Corollary 7 and Theorem 6.  $\square$

**Remark 3.** We find the Hausdorff and packing dimensions of  $\underline{F}(r)$ ,  $\overline{F}(r)$ ,  $F(r)$ ,  $\underline{E}_\alpha^{(p)}$ ,  $\overline{E}_\alpha^{(p)}$  and  $E_\alpha^{(p)}$  where  $r \in [0, 1]$  and  $p \in (0, 1)$  and

$$\alpha \in \left[ \frac{\log p}{\log a}, \frac{\log(1-p)}{\log b} \right] \text{ or } \left[ \frac{\log(1-p)}{\log b}, \frac{\log p}{\log a} \right]$$

except for the informations of upper bound of packing dimension about  $\underline{F}(r)$  where  $0 \leq r < a^s$  with  $a^s + b^s = 1$  and  $\overline{F}(r)$  where  $a^s < r \leq 1$  and  $\underline{E}_\alpha^{(p)}$  where  $0 < p < a^s$  with  $0 \leq r(\alpha) < a^s$  or  $a^s < p < 1$  with  $a^s < r(\alpha) \leq 1$  and  $\overline{E}_\alpha^{(p)}$  where  $0 < p < a^s$  with  $a^s < r(\alpha) \leq 1$  or  $a^s < p < 1$  with  $0 \leq r(\alpha) < a^s$  (cf. [6]).

**Remark 4.** We compare our results with those of [6,8]. We note that, for every  $p \in (0, 1)$  with  $p \neq a^s$  where  $a^s + b^s = 1$ ,  $F(p) = E_{\delta(p)}^{(p)}$  where

$$\delta(p) = \frac{p \log p + (1-p) \log(1-p)}{p \log a + (1-p) \log b},$$

from which with the strong law of large numbers we see that  $\gamma_p(E_{\delta(p)}^{(p)}) = 1$ . Now let  $\alpha$  be in  $[\log p / \log a, \log(1-p) / \log b]$  or  $[\log(1-p) / \log b, \log p / \log a]$ . Let  $\beta = \beta(q)$  as the positive number satisfying  $p^q a^{\beta(q)} + (1-p)^q b^{\beta(q)} = 1$ . We also note that  $E_\alpha^{(p)} = E_{q\alpha + \beta(q)}^{(p^q a^{\beta(q)})}$  for every  $(q, \beta(q))$ . For  $(q, \beta(q))$  satisfying the derivative  $\beta'(q) = -\alpha$  of  $\beta$  at  $q$ ,  $\gamma_{p^q a^{\beta(q)}}(E_{q\alpha + \beta(q)}^{(p^q a^{\beta(q)})}) = 1$  since  $\delta(p^q a^{\beta(q)}) = q\alpha + \beta(q)$ . Hence  $\dim(E_\alpha^{(p)}) = \text{Dim}(E_\alpha^{(p)}) = q\alpha + \beta(q)$  where  $p^q a^{\beta(q)} + (1-p)^q b^{\beta(q)} = 1$  and  $\beta'(q) = -\alpha$ . We note that above

$$q\alpha + \beta(q) = \delta(r) = \frac{r \log r + (1-r) \log(1-r)}{r \log a + (1-r) \log b},$$

where

$$\alpha = \frac{r \log p + (1-r) \log(1-p)}{r \log a + (1-r) \log b}$$

in the above corollary.

**Remark 5.** A self-similar Cantor set  $F$  is completely decomposed into classes by the lower and upper distribution sets as  $F = \bigcup_{0 \leq r \leq 1} \underline{F}(r)$  and  $F = \bigcup_{0 \leq r \leq 1} \overline{F}(r)$ . Let  $\{\underline{F}(r): 0 \leq r \leq 1\} = S_1$  and  $\{\overline{F}(r): 0 \leq r \leq 1\} = S_2$ . Then  $F$  is partitioned into  $S_1$  by the equivalence relation that  $x$  is equivalent to  $y \Leftrightarrow x, y \in \underline{F}(r)$  for some  $r \in [0, 1]$ . Similarly,  $F$  is partitioned into  $S_2$  by the equivalence relation that  $x$  is equivalent to  $y \Leftrightarrow x, y \in \overline{F}(r)$  for some  $r \in [0, 1]$ . Fix  $p (\neq a^s) \in (0, 1)$  where  $a^s + b^s = 1$ . Since  $g(r, p)$  is strictly monotone for  $r \in [0, 1]$  from Remark 1, the function  $g(r, p)$  of  $r$  is a one-to-one correspondence between  $[0, 1]$  and  $[\log(1-p) / \log b, \log p / \log a]$  or  $[\log p / \log a, \log(1-p) / \log b]$ . So from Remark 1 we can easily see that the two classes  $S_1, S_2$  characterize the spectral classes by

self-similar measures and their lower and upper local dimensions. That is,  $F$  is completely decomposed into the spectral classes by the lower and upper local dimensions of a self-similar measure  $\gamma_p$  as

$$F = \bigcup_{\alpha \in [\frac{\log(1-p)}{\log b}, \frac{\log p}{\log a}]} \underline{E}_\alpha^{(p)} = \bigcup S_1 \quad \text{if } 0 < p < a^s,$$

$$F = \bigcup_{\alpha \in [\frac{\log p}{\log a}, \frac{\log(1-p)}{\log b}]} \underline{E}_\alpha^{(p)} = \bigcup S_2 \quad \text{if } a^s < p < 1,$$

$$F = \bigcup_{\alpha \in [\frac{\log(1-p)}{\log b}, \frac{\log p}{\log a}]} \overline{E}_\alpha^{(p)} = \bigcup S_2 \quad \text{if } 0 < p < a^s,$$

$$F = \bigcup_{\alpha \in [\frac{\log p}{\log a}, \frac{\log(1-p)}{\log b}]} \overline{E}_\alpha^{(p)} = \bigcup S_1 \quad \text{if } a^s < p < 1.$$

We note that all the spectral classes by the lower or upper local dimensions of all self-similar measures, except for the singular self-similar measure having the exact dimension of the Hausdorff and packing dimension of the self-similar Cantor set, are just classes  $S_1$  or  $S_2$ .

**Remark 6.** We hope that these methods for finding Hausdorff and packing dimensions and relationships of the subsets in the spectral classes will be applied to a perturbed Cantor set or a deranged Cantor set [1,2].

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