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Atkinson's super-linear oscillation theorem for matrix dynamic equations on a time scale

Liuman Ou [1]

Department of Mathematics Sun Yat-Sen (Zhongshan) University, Guangzhou 510275, PR China

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Abstract

By applying the Riccati technique and operator theory, we establish on a time scale $\mathbb{T}$ both oscillation and non-oscillation criteria for Atkinson's super-linear matrix dynamic equation $X^{\Delta^2} + [X^m(t)Q(t)X^{*m}(t)]^\sigma X^\sigma(t) = 0$. These results extend and unify earlier results for the differential and difference equation case.

Keywords: Time scales; Riccati technique; Operator; Oscillation; Non-oscillation

1. Introduction

In 1955, Atkinson [1] proved that the second-order super-linear scalar ordinary differential equation

$$y'' + f(t)y^{2n+1} = 0, \quad t \geqslant 0,$$

E-mail address: ouliuman@163.com.

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with $f(t) > 0$ and continuous for each $t \geqslant 0$, is oscillatory if and only if

$$\int_0^\infty t f(t)\,dt = \infty.$$

In 1982, Kura [2] obtained the same result for the $n \times n$ matrix differential equation

$$Y'' + \left(Y^m Q(t) Y^{*m}\right) Y = 0. \tag{1}$$

Butler and Erbe [3], and Ahlbrandt, Ridenhour and Thompson [4] have done a similar study for (1). In [4] it is shown that when $Q(t)$ is Hermitian, positive definite and continuous for each $t \geqslant 0$, a necessary and sufficient condition for all prepared solution of (1) which extend to infinity to be oscillatory is that

$$\int_0^\infty t\lambda_{\max}\left[Q(t)\right] dt = \infty.$$

Mingarelli [5] has shown that Atkinson's super-linear oscillation theorem is valid for second-order real scalar difference equations. In 1991, Allan Peterson and Jerry Ridenhour [8] used Riccati techniques to establish the necessary and sufficient condition

$$\sum_{t=1}^\infty t\lambda_{\max}\left[Q(t)\right] = \infty$$

for all prepared solutions of Atkinson's super-linear matrix difference equation

$$\Delta^2 Y(t-1) + \left[Y^n(t) Q(t) Y^{*n}(t)\right] Y(t) = 0$$

to be oscillatory.

In the past ten years, theory about calculus on time scales, a unified approach to continuous and discrete calculus, has been studied by many authors [6,7,11,12], etc., and many interesting results have been obtained. For example, see Hilger [6] and Agarwal and Bohner [11]. Motivated by the ideas in [1–5], we shall establish oscillation and non-oscillation criteria for Atkinson's matrix dynamic equation on a time scale.

2. Preliminaries

In this paper we are concerned with Atkinson's super-linear matrix dynamic equation

$$X^{\Delta^2} + \left[X^m(t) Q(t) X^{*m}(t)\right]^\sigma X^\sigma(t) = 0 \tag{2}$$

on a time scale $\mathbb{T}$. Following [6,10], we introduce the following concepts related to the notion of time scale.

Definition. A time scale $\mathbb{T}$ is a closed subset of the set $\mathbb{R}$ of real numbers equipped with the forward jump operator

$$\sigma(t) := \inf\{s > t,\ s \in \mathbb{T}\}$$

and the backward jump operator $\rho(t)$ at t for $t \in \mathbb{T}$ by

$$\rho(t) := \sup\{s < t,\ s \in \mathbb{T}\}.$$

We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. By an interval in $\mathbb{T}$ we mean an interval in $\mathbb{R}$ intersected with $\mathbb{T}$. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ we say t is right-dense, while if $\rho(t) = t$ we say t is left-dense. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous provided f is continuous at right-dense points in $\mathbb{T}$ and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. We use $\mathrm{C}_{rd}\mathbb{T}$ to denote the set of all right-dense continuous function on $\mathbb{T}$. If $\sup \mathbb{T} < \infty$ and $\sup \mathbb{T}$ is left-scattered, we let $\mathbb{T}^k := \mathbb{T} \setminus \{\sup \mathbb{T}\}$. Otherwise, we let $\mathbb{T}^k := \mathbb{T}$. We shall use the notation $\mu(t) := \sigma(t) - t$ which is called the graininess function. Finally, if $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f\big(\sigma(t)\big)$$

for all $t \in \mathbb{T}$, i.e., $f^\sigma = f \circ \sigma$.

Definition. Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}$; then we define $x^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\big|\big[x\big(\sigma(t)\big) - x(s)\big] - x^\Delta(t)\big[\sigma(t) - s\big]\big| \leqslant \varepsilon \big|\sigma(t) - s\big|$$

for all $s \in U$. We call $x^\Delta(t)$ the delta derivative of $x(t)$ at t.

It follows easily that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right-scattered, then

$$x^\Delta(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Definition. We denote by $\mathcal{D}$ the set of all real matrix functions $X(t)$ so that each entry of $X(t)$ is delta differentiable on $\mathbb{T}^k$ and is in $\mathrm{C}_{cd}(\mathbb{T}^{k^2})$. $X^{\delta^2} := X^{\delta\delta}$.

In this paper we always assume that $X(t) \in \mathcal{D}$ and that $a \in \mathbb{T}$. We assume throughout that the coefficient matrix satisfies $Q(t) > 0$, i.e., $Q(t)$ is positive definite, for all $t \in \mathbb{T}^{k^2} := (\mathrm{T}^k)^k$ and $Q(t) = Q^*(t)$, i.e., $Q(t)$ is Hermitian, for all $t \in \mathbb{T}^k$.

Definition. A solution $X(t)$ of (2) is said to be prepared if and only if $X^*(t)X^\Delta(t) = X^{\Delta*}(t)X(t)$, $t \in \mathbb{T}^k$.

Using the formulas $(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\sigma g^\Delta$, one can show that for any solution $X(t)$ of (2), we have

$$X^*(t)X^\Delta(t) - X^{\Delta*}(t)X(t) = K.$$

A solution $X(t)$ is prepared only when K is the zero matrix.

Definition. Given a solution $X(t)$ of (2), the Riccati functions $W(t)$ and $V(t)$ defined by

$$W(t) := X^{\Delta}(t)X^{-1}(t) \quad \text{and} \quad V(t) := X^{\Delta}(t)X^{-1}\big(\sigma(t)\big). \tag{3}$$

Then we have the relation

$$I + \mu(t)W(t) = X\big(\sigma(t)\big)X^{-1}(t);$$
$$\big[I + \mu(t)W(t)\big]^{-1} = I - \mu(t)V(t) = X(t)X^{-1}\big(\sigma(t)\big).$$

Let $X(t)$ be a solution of (2). Using

$$\big[A(t)B(t)\big]^{\Delta} = A^{\sigma}(t)B^{\Delta}(t) + A^{\Delta}(t)B(t) = A^{\Delta}(t)B^{\sigma}(t) + A(t)B^{\Delta}(t)$$

and

$$\big[A^{-1}(t)\big]^{\Delta} = -A^{-1}(t)A^{\Delta}(t)A^{-1}\big(\sigma(t)\big) = -A^{-1}\big(\sigma(t)\big)A^{\Delta}(t)A^{-1}(t), \quad \text{for invertible } A,$$

it follows that

$$\begin{aligned}
W^{\Delta}(t) &= \big[X^{\Delta}(t)X^{-1}(t)\big]^{\Delta} = X^{\Delta^2}(t)X^{-1}\big(\sigma(t)\big) + X^{\Delta}(t)\big[X^{-1}(t)\big]^{\Delta} \\
&= X^{\Delta^2}(t)X^{-1}\big(\sigma(t)\big) + X^{\Delta}(t)\big[-X^{-1}\big(\sigma(t)\big)X^{\Delta}(t)X^{-1}(t)\big] \\
&= X^{\Delta^2}(t)X^{-1}\big(\sigma(t)\big) - X^{\Delta}(t)X^{-1}\big(\sigma(t)\big)X^{\Delta}(t)X^{-1}(t) \\
&= -\big[X^{m}(t)Q(t)X^{*m}(t)\big]^{\sigma} - X^{\Delta}(t)X^{-1}\big(\sigma(t)\big)W(t) \\
&= -\big[X^{m}(t)Q(t)X^{*m}(t)\big]^{\sigma} - W(t)\big[I + \mu(t)W(t)\big]^{-1}W(t),
\end{aligned}$$

i.e.,

$$W^{\Delta}(t) + \big[X^{m}(t)Q(t)X^{*m}(t)\big]^{\sigma} + W(t)\big[I + \mu(t)W(t)\big]^{-1}W(t) = 0. \tag{4}$$

We say that (4) is the Riccati equation associated with (2).

3. Main results

Theorem 3.1. *Assume that $X(t)$ is a solution of* (2) *on* T. *Then the following are equivalent*:

(i) *$X(t)$ is a prepared solution*;
(ii) *$X^*(t)X^{\Delta}(t)$ is Hermitian for all $t \in \mathrm{T}^k$*;
(iii) *$X^*(t_0)X^{\Delta}(t_0)$ is Hermitian for some $t_0 \in \mathrm{T}^k$*.

Proof. Assume that $X(t)$ is a solution of (2) on T. Since

$$X^*(t)X^{\Delta}(t) - X^{\Delta *}(t)X(t) = K$$

for $t \in \mathrm{T}^k$, it follows that $X(t)$ is a prepared solution of (2) if and only if $X^*(t)X^{\Delta}(t)$ is Hermitian for all $t \in \mathrm{T}^k$ if and only if $X^*(t_0)X^{\Delta}(t_0)$ is Hermitian for some $t_0 \in \mathrm{T}^k$. $\square$

Lemma 3.2. *Let $X(t)$ be a solution of* (2)*. If $X(t)$ is prepared, then $X^*(\sigma(t))X(t)$ is Hermitian for all $t \in \mathrm{T}^k$. Conversely, if there is $t_0 \in \mathrm{T}^k$ such that $\mu(t_0) > 0$ and $X^*(\sigma(t_0))X(t_0)$ is Hermitian, then $X(t)$ is a prepared solution of* (2)*. Also, if $X(t)$ is a nonsingular prepared solution, then $X(\sigma(t))X^{-1}(t)$, $X(t)X^{-1}(\sigma(t))$, and $W(t)$ and $V(t)$ are Hermitian for all $t \in \mathrm{T}^k$.*

Proof. Let $X(t)$ be a solution of (2). The relation

$$X^*\big(\sigma(t)\big)X(t) = \big(X(t) + \mu(t)X^{\Delta}(t)\big)^*X(t) = X^*(t)X(t) + \mu(t)X^{\Delta^*}(t)X(t)$$

proves the first two statements of this lemma. Now assume that $X(t)$ is a nonsingular prepared solution of (2). Then

$$\begin{aligned} X^*\big(\sigma(t)\big)X(t) &= X^*(t)X(t) + \mu(t)\big(X^{\Delta}(t)\big)^*X(t) = X^*(t)X(t) + \mu(t)X^{*\Delta}(t)X(t) \\ &= X^*(t)\big(X(t) + \mu(t)X^{\Delta}(t)\big) = X^*(t)X^{\sigma}(t), \end{aligned} \tag{5}$$

$$X^*(t)X^{\Delta}(t) = X^{\Delta^*}(t)X(t) \tag{6}$$

by Theorem 3.1 and what we have shown above. Now multiply Eq. (5) on the left by $X^{-1}(t)$ and on the right by $(X^{-1}(t))^*$ to obtain that $X^{\sigma}(t)X^{-1}(t)$ is Hermitian. Next multiply Eq. (5) on the left by $(X^{-1}(\sigma(t)))^*$ and on the right by $X^{-1}(\sigma(t))$ to obtain that $X(t)X^{-1}(\sigma(t))$ is Hermitian. Finally, multiply Eq. (6) $(X^{-1}(t))^*$ from the left and with $X^{-1}(t)$ from the right shows that $W(t)$ is Hermitian. From (3) and $X(t)X^{-1}(\sigma(t))$ being Hermitian we have $V(t)$ is Hermitian. □

Lemma 3.3. *Assume that $X(t)$ is a prepared solution of* (2) *on $\mathbb{T}$. Then the following are equivalent*:

(i) $X^*(\sigma(t))X(t) > 0$ *on* T^k;
(ii) $X(t)$ *is nonsingular and* $X(\sigma(t))X^{-1}(t) > 0$ *on* T^k;
(iii) $X(t)$ *is nonsingular and* $X(t)X^{-1}(\sigma(t))$ *on* T^k.

Proof. First note that $X^*(\sigma(t))X(t) > 0$ for $t \in \mathrm{T}^k$ implies that $X(t)$ is nonsingular for $t \in \mathrm{T}^k$. Since $X(t)$ is prepared solution, we have by Lemma 3.2 that

$$X\big(\sigma(t)\big)X^{-1}(t) = \big(X^{-1}(t)\big)^*X^*\big(\sigma(t)\big), \tag{7}$$

$$X(t)X^{-1}\big(\sigma(t)\big) = \big(X^{-1}\big(\sigma(t)\big)\big)^*X^*(t) \tag{8}$$

on T^k. We multiply the right-hand side of (7) on the right by $X(t)X^{-1}(t)$ to obtain the equivalence of (i) and (ii). For equivalence of (i) and (iii), multiply the right-hand side of (8) on the right by $X^{\sigma}(t)X^{-1}(\sigma(t))$. □

Theorem 3.4. *If* (2) *has a prepared solution $X(t)$ such that $X(t)$ is invertible for all $t \in \mathbb{T}$, then $W(t)$ is a Hermitian solution of the matrix Riccati equation* (4) *on T^k. Conversely, if* (4) *has a Hermitian solution $W(t)$ on $\mathbb{T}^k$, then there exists a prepared solution $X(t)$ of* (4) *such that $X(t)$ is invertible for all $t \in \mathbb{T}$ and relation* (3) *holds.*

Proof. From Lemma 3.2 the first conclusion follows. Conversely, let $W(t)$ be a Hermitian solution of (4) on T^k. Let $t_0 \in \mathrm{T}$ and put $X = e_W(\cdot, t_0)$. By Theorem 5.8 in [13], X is defined because $I + \mu W$ is invertible on $\mathbb{T}^k$. Then X is invertible on $\mathbb{T}$ by Theorem 5.21 [13], and we have

$$\begin{aligned}
\big[X^{\Delta}(t)\big]^{\Delta} &= \big(W(t)X(t)\big)^{\Delta} = W^{\Delta}(t)X^{\sigma}(t) + W(t)X^{\Delta}(t) \\
&= -\big[X^m(t)Q(t)X^{*m}(t)\big]^{\sigma} X^{\sigma}(t) - W(t)\big[I + \mu(t)W(t)\big]^{-1} W(t)X^{\sigma}(t) \\
&\quad + W(t)W(t)X(t) \\
&= -\big[X^m(t)Q(t)X^{*m}(t)\big]^{\sigma} X^{\sigma}(t) \\
&\quad + W(t)\big[I + \mu(t)W(t)\big]^{-1}\big[\big(I + \mu(t)W(t)\big)W(t)X(t) - W(t)X^{\sigma}(t)\big] \\
&= -\big[X^m(t)Q(t)X^{*m}(t)\big]^{\sigma} X^{\sigma}(t) \\
&\quad + W(t)\big[I + \mu(t)W(t)\big]^{-1}\big\{W(t)X(t) + \mu(t)W(t)W(t)X(t) \\
&\quad - W(t)\big[X(t) + \mu(t)W(t)X(t)\big]\big\} \\
&= -\big[X^m(t)Q(t)X^{*m}(t)\big]^{\sigma} X^{\sigma}(t)
\end{aligned}$$

on $\mathbb{T}^k$. So that $X(t)$ is a solution of (2) and $X(t)$ is indeed a prepared solution because $X^*(t)X^{\Delta}(t) = W(t)$ is Hermitian. □

From Lemma 3.3 and Theorem 3.4 we have

Theorem 3.5. *Equation* (2) *has a prepared solution* $X(t)$ *on* $\mathbb{T}$ *with* $X^*(\sigma(t))X(t) > 0$ *on* $\mathbb{T}^k$ *if and only if* (4) *has a Hermitian solution* $W(t)$ *on* T^k *satisfying* $I + \mu(t)W(t) > 0$ *for all* $t \in \mathbb{T}^k$.

Definition. Assume $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$. We say that (2) is non-oscillatory on $[a, \infty)$ provided there is a prepared solution $X(t)$ of (2) and a $t_0 \in [a, \infty)$ such that $X^*(\sigma(t))X(t) > 0$ on $[t_0, \infty)$. Otherwise we say (2) is oscillatory on $[a, \infty)$.

We now introduce some notation that we will use in the remainder of this paper. If A is an $n \times n$ Hermitian matrix, let $\lambda_i(A)$ denote the ith eigenvalue of A so that

$$\lambda_{\max}(A) = \lambda_1(A) \geqslant \cdots \geqslant \lambda_n(A) = \lambda_{\min}(A).$$

The trace of a matrix A is denoted by $\operatorname{tr}(A) := \sum_{i=1}^n \lambda_i(A)$. We shall frequently use Weyl's theorem [9, p. 181, Theorem 4.3.1] which says if A and B are Hermitian matrices, then

$$\lambda_i(A) + \lambda_{\max}(B) \geqslant \lambda_i(A+B) \geqslant \lambda_i(A) + \lambda_{\min}(B)$$

and Ostrowski's inequalities [9, pp. 224–225] which give

$$\lambda_i(APA^*) \geqslant \lambda_i(P)\lambda_{\min}(AA^*) \quad \text{and} \quad \lambda_i(APA^*) \geqslant \lambda_i(AA^*)\lambda_{\min}(P).$$

Lemma 3.6. *Suppose* $X(t)$ *is a non-oscillatory prepared solution of* (2). *Then there exists* $t_0 \in \mathrm{T}^k$ *such that* $W(t)$ *and* $V(t)$ *are both positive definite and decreasing for* $t \in [t_0, \infty)$ *with*

$$\lim_{t\to\infty,\ t\in\mathrm{T}} W(t) = \lim_{t\to\infty,\ t\in\mathrm{T}} V(t) = 0. \tag{9}$$

Furthermore, multiplication of $W(t)$ and $V(t)$ is commutative at point where both exist and $W(t)V(t) = V(t)W(t)$ is positive definite for $t \in [t_0, \infty)$.

Proof. Since $X(t)$ is non-oscillatory and prepared, we begin by choosing $t_0 \in \mathrm{T}^k$ so that $I + \mu(t)W(t) > 0$ for $t \in [t_0, \infty)$. Since $Q(t)$ is positive definite, we see from the Riccati equation (4), Ostrowski's inequality, and Weyl's inequality that $W^\Delta(t) < 0$ for $t \in [t_0, \infty)$. Hence by Weyl's inequality, each eigenvalue $\lambda_i[W(t)]$ $(1 \leqslant i \leqslant n)$, is a decreasing function of t for $t \in [t_0, \infty)$. Furthermore, each $\lambda_i[W(t)]$ is bounded below for $t \in [t_0, \infty)$ since $I + \mu(t)W(t) > 0$, so $\lim_{t\to\infty} \lambda_i[W(t)]$ exists for $1 \leqslant i \leqslant n$. Since the eigenvalues of $I + \mu(t)W(t)$ decrease but remain positive, the eigenvalues of $[I + \mu(t)W(t)]^{-1}$ are positive and increasing for $t \in [t_0, \infty)$. From (4) and the eigenvalue inequalities mentioned above we obtain

$$\begin{aligned}\lambda_i\big[-W^\Delta(t)\big] &> \lambda_i\big\{W(t)\big[I + \mu(t)W(t)\big]^{-1}W(t)\big\}\\ &\geqslant \lambda_i\big[W^2(t)\big]\lambda_{\min}\big(\big[I + \mu(t)W(t)\big]^{-1}\big)\\ &\geqslant \big\{\lambda_i\big[W(t)\big]\big\}^2\lambda_{\min}\big(\big[I + \mu(t_0)W(t_0)\big]^{-1}\big)\end{aligned} \tag{10}$$

for $1 \leqslant i \leqslant n$ and $t \in [t_0, \infty)$. Now we claim

$$\lim_{t\to\infty,\ t\in\mathrm{T}} \lambda_i\big[W(t)\big] = 0 \tag{11}$$

holds for $1 \leqslant i \leqslant n$. Suppose not. We choose i_0 with $1 \leqslant i_0 \leqslant n$ such that

$$\lim_{t\to\infty,\ t\in\mathrm{T}} \lambda_{i_0}\big[W(t)\big] = \lambda_0 \neq 0. \tag{12}$$

Combining (10) and (12), we can choose $t_1 \in [t_0, \infty)$ and a positive number δ such that

$$\lambda_{i_0}\big[-W^\Delta(t)\big] > \delta \quad \text{for } t \in [t_1, \infty). \tag{13}$$

But

$$\begin{aligned}\lambda_{\max}\big[-W(t) + W(t_1)\big] &= \lambda_{\max}\left[\int_{t_1}^{t} -W^\Delta(\tau)\Delta\tau\right] \geqslant \frac{1}{n}\operatorname{tr}\left[\int_{t_1}^{t} -W^\Delta(\tau)\Delta\tau\right]\\ &= \frac{1}{n}\int_{t_1}^{t} \operatorname{tr}\big[-W^\Delta(\tau)\big]\Delta\tau \geqslant \frac{1}{n}\int_{t_1}^{t} \lambda_{i_0}\big[-W^\Delta(\tau)\big]\Delta\tau.\end{aligned}$$

By (13), this implies that $\lambda_{\max}[-W(t) + W(t_1)] \to \infty$ and $\lambda_{\max}[W(t)] \to -\infty$ as $t \to \infty$. This contradicts the fact the eigenvalues of $W(t)$ are bounded below for $t \in [t_0, \infty)$ and proves that (11) holds. Consequently, $\lim_{t\to\infty,\ t\in\mathrm{T}} W(t) = 0$. Therefore, $W(t)$ is positive for $t \in [t_0, \infty)$. From (3), $\mu(t)V(t) = I - [I + \mu(t)W(t)]^{-1}$, so the eigenvalues of $V(t)$ are also positive and decreasing for $t \in [t_0, \infty)$ with $\lim_{t\to\infty} \lambda_i[V(t)] = 0$ for $1 \leqslant i \leqslant n$ showing that (9) holds.

Finally, from (3), we see that

$$\big[I + \mu(t)W(t)\big]\big[I - \mu(t)V(t)\big] = I = \big[I - \mu(t)V(t)\big]\big[I + \mu(t)W(t)\big],$$

from which it follows that

$$\mu(t)W(t)V(t) = \mu(t)V(t)W(t) = W(t) - V(t)$$

at all $t \in \mathrm{T}^k$ where both $W(t)$ and $V(t)$ exist. Since

$$\begin{aligned} V(t)\big[I + \mu(t)W(t)\big]V(t) &= \mu(t)V(t)W(t)V(t) + V^2(t) \\ &= \big[W(t) - V(t)\big]V(t) + V^2(t) = W(t)V(t), \end{aligned}$$

we see that $W(t)V(t) = V(t)W(t)$ is positive for $t \in [t_0, \infty)$ completing the proof of the lemma. $\square$

Theorem 3.7. *Suppose $Q(t)$ is Hermitian and positive definite for all $t \in \mathbb{T}^k$. Then* (2) *is oscillatory if and only if*

$$\int_a^\infty t\lambda_{\max}\big[Q(t)\big]\Delta t = \infty \tag{14}$$

holds.

Proof. First, we assume $m = 1$ in (2), the general case will be treated later. Suppose (14) holds but (2) has a non-oscillatory prepared solution $X(t)$. Applying Lemma 3.6, we choose $t_0 \in \mathbb{T}^k$ so that $X(t)$ is invertible and matrices $W(t), V(t)$ and $W(t)V(t) = V(t)W(t)$ are all positive definite for $t \in [t_0, \infty)$. Then

$$\begin{aligned} \big[X^{-1}(t)X^{*-1}(t)\big]^\Delta &= \big[X^{-1}(t)\big]^\Delta X^{*-1}\big(\sigma(t)\big) + X^{-1}(t)\big[X^{*-1}(t)\big]^\Delta \\ &= -X^{-1}\big(\sigma(t)\big)X^\Delta(t)X^{-1}(t)X^{*-1}\big(\sigma(t)\big) \\ &\quad - X^{-1}(t)X^{*-1}\big(\sigma(t)\big)X^{*\Delta}(t)X^{*-1}(t) \\ &= -X^{-1}\big(\sigma(t)\big)W(t)X^{*-1}\big(\sigma(t)\big) - X^{-1}(t)V(t)X^{*-1}(t). \end{aligned} \tag{15}$$

Using the product rule for Δ-derivatives we have

$$\begin{aligned} &\big[A(t)B(t)C(t)D(t)E(t)\big]^\Delta \\ &\quad = A^\sigma(t)B^\sigma(t)C^\sigma(t)D^\sigma(t)E^\Delta(t) + A^\sigma(t)B^\Delta(t)C^\sigma(t)D^\sigma(t)E(t) \\ &\qquad + A^\sigma(t)B(t)C^\Delta(t)D^\sigma(t)E(t) + A^\sigma(t)B(t)C(t)D^\Delta(t)E(t) \\ &\qquad + A^\Delta(t)B(t)C(t)D(t)E(t), \end{aligned}$$

and we obtain

$$\begin{aligned} &\big[tX^{-1}\big(\sigma(t)\big)X^\Delta(t)X^{-1}(t)X^{*-1}\big(\sigma(t)\big)\big]^\Delta \\ &\quad = \sigma(t)X^{-1}\big(\sigma^2(t)\big)X^\Delta\big(\sigma(t)\big)X^{-1}\big(\sigma(t)\big)\big[X^{*-1}\big(\sigma(t)\big)\big]^\Delta \\ &\qquad + \sigma(t)\big[X^{-1}\big(\sigma(t)\big)\big]^\Delta X^\Delta\big(\sigma(t)\big)X^{-1}\big(\sigma(t)\big)X^{*-1}\big(\sigma(t)\big) \end{aligned}$$

$$
\begin{aligned}
&+\sigma(t)X^{-1}\big(\sigma(t)\big)\big[X^{\Delta}(t)\big]^{\Delta}X^{-1}\big(\sigma(t)\big)X^{*-1}\big(\sigma(t)\big)\\
&+\sigma(t)X^{-1}\big(\sigma(t)\big)X^{\Delta}(t)\big[X^{-1}(t)\big]^{\Delta}X^{*-1}\big(\sigma(t)\big)\\
&+X^{-1}\big(\sigma(t)\big)X^{\Delta}(t)X^{-1}(t)X^{*-1}\big(\sigma(t)\big)\\
={}&-\sigma(t)X^{-1}\big(\sigma^2(t)\big)W^2\big(\sigma(t)\big)X^{*-1}\big(\sigma^2(t)\big)\\
&-\sigma(t)X^{-1}\big(\sigma(t)\big)V\big(\sigma(t)\big)W\big(\sigma(t)\big)X^{*-1}\big(\sigma(t)\big)-\sigma(t)Q\big(\sigma(t)\big)\\
&-\sigma(t)X^{-1}\big(\sigma(t)\big)V(t)W(t)X^{*-1}\big(\sigma(t)\big)+X^{-1}\big(\sigma(t)\big)W(t)X^{*-1}\big(\sigma(t)\big). \qquad (16)
\end{aligned}
$$

Applying the product rule gives

$$
\begin{aligned}
&\big[A(t)B(t)C(t)D(t)E(t)\big]^{\Delta}\\
&\quad=A^{\sigma}(t)B^{\sigma}(t)C^{\sigma}(t)D^{\Delta}(t)E^{\sigma}(t)+A^{\sigma}(t)B^{\sigma}(t)C^{\Delta}(t)D(t)E^{\sigma}(t)\\
&\qquad+A^{\sigma}(t)B^{\Delta}(t)C(t)D(t)E^{\sigma}(t)+A^{\sigma}(t)B(t)C(t)D(t)E^{\Delta}(t)\\
&\qquad+A^{\Delta}(t)B(t)C(t)D(t)E(t),
\end{aligned}
$$

and we find that

$$
\begin{aligned}
&\big[tX^{-1}(t)X^{\Delta}(t)X^{-1}\big(\sigma(t)\big)X^{*-1}(t)\big]^{\Delta}\\
&\quad=-\sigma(t)X^{-1}\big(\sigma(t)\big)W\big(\sigma(t)\big)V\big(\sigma(t)\big)X^{*-1}\big(\sigma(t)\big)-\sigma(t)Q\big(\sigma(t)\big)\\
&\qquad-\sigma(t)X^{-1}\big(\sigma(t)\big)W(t)V(t)X^{*-1}\big(\sigma(t)\big)-\sigma(t)X^{-1}(t)V^2(t)X^{*-1}(t)\\
&\qquad+X^{-1}(t)V(t)X^{*-1}(t). \qquad (17)
\end{aligned}
$$

Combining (15)–(17), we have

$$
\begin{aligned}
&\big[X^{-1}(t)X^{*-1}(t)\big]^{\Delta}\\
&\quad=-X^{-1}\big(\sigma(t)\big)W(t)X^{*-1}\big(\sigma(t)\big)-X^{-1}(t)V(t)X^{*-1}(t)\\
&\quad=-\big[tX^{-1}\big(\sigma(t)\big)W(t)X^{*-1}\big(\sigma(t)\big)\big]^{\Delta}-\big[tX^{-1}(t)V(t)X^{*-1}(t)\big]^{\Delta}\\
&\qquad-2\sigma(t)Q\big(\sigma(t)\big)-\sigma(t)H(t), \qquad (18)
\end{aligned}
$$

where

$$
\begin{aligned}
H(t)={}&X^{-1}\big(\sigma^2(t)\big)W^2\big(\sigma(t)\big)X^{*-1}\big(\sigma^2(t)\big)+X^{-1}(t)V^2(t)X^{*-1}(t)\\
&+X^{-1}\big(\sigma(t)\big)\big[2V\big(\sigma(t)\big)W\big(\sigma(t)\big)+2V(t)W(t)\big]X^{*-1}\big(\sigma(t)\big).
\end{aligned}
$$

Integrating both side of (18) from t_0 to t yields

$$
\begin{aligned}
&\int_{t_0}^{t}\big[X^{-1}(\tau)X^{*-1}(\tau)\big]^{\Delta}\Delta\tau\\
&\quad=-\int_{t_0}^{t}\big[\tau X^{-1}\big(\sigma(\tau)\big)W(\tau)X^{*-1}\big(\sigma(\tau)\big)\big]^{\Delta}\Delta\tau-2\int_{t_0}^{t}\sigma(\tau)Q\big(\sigma(\tau)\big)\Delta\tau
\end{aligned}
$$

$$- \int_{t_0}^{t} \left[\tau X^{-1}(\tau) V(\tau) X^{*-1}(\tau)\right]^{\Delta} \Delta\tau - \int_{t_0}^{t} \sigma(\tau) H(\tau) \Delta\tau,$$

and hence

$$\begin{aligned} 2\int_{t_0}^{t} \sigma(\tau) Q\big(\sigma(\tau)\big) \Delta\tau = {} & - \int_{t_0}^{t} \sigma(\tau) H(\tau) \Delta\tau - X^{-1}(t) X^{*-1}(t) \\ & - t X^{-1}\big(\sigma(t)\big) W(t) X^{*-1}\big(\sigma(t)\big) \\ & - t X^{-1}(t) V(t) X^{*-1}(t) + C, \end{aligned} \tag{19}$$

where C is a constant Hermitian matrix.

Now all the terms except C on the right-hand side of (19) are negative definite for all $t \in [t_0, \infty)$, and consequently there is a real constant M_1 such that

$$\lambda_{\max}\left[\int_{t_0}^{t} \sigma(\tau) Q\big(\sigma(\tau)\big) \Delta\tau\right] \leqslant M_1 \quad \text{for } t \in \big[\sigma(t_0), \infty\big).$$

By Weyl's inequality, there is another constant M_2 so that

$$\lambda_{\max}\left[\int_{a}^{t} \sigma(\tau) Q\big(\sigma(\tau)\big) \Delta\tau\right] \leqslant M_2 \quad \text{for } t \in \big[\sigma(t_0), \infty\big). \tag{20}$$

However,

$$\begin{aligned} \lambda_{\max}\left[\int_{a}^{t} \sigma(\tau) Q\big(\sigma(\tau)\big) \Delta\tau\right] & \geqslant \frac{1}{n} \operatorname{tr}\left[\int_{a}^{t} \sigma(\tau) Q\big(\sigma(\tau)\big) \Delta\tau\right] \\ & = \frac{1}{n} \int_{a}^{t} \operatorname{tr}\big[\sigma(\tau) Q\big(\sigma(\tau)\big)\big] \Delta\tau \\ & \geqslant \frac{1}{n} \int_{a}^{t} \lambda_{\max}\big[\sigma(\tau) Q\big(\sigma(\tau)\big)\big] \Delta\tau. \end{aligned}$$

By (14), $\int_a^t \lambda_{\max}[\sigma(\tau) Q(\sigma(\tau))] \Delta\tau \to \infty$ as $t \to \infty$ contradicting (20). This proves that (14) is a sufficient condition for (2) to be oscillatory in the case $m = 1$.

Next, we will prove the general case. Suppose (14) holds but there is a positive integer m such that (2) has a prepared non-oscillatory solution $X_0(t)$. Since $X_0(t)$ is non-oscillatory, we choose $t_0 \in [a, \infty)$ such that $X_0(t)$ is invertible for $t \in [t_0, \infty)$ and $W_0(t)$ and $V_0(t)$ are positive definite for $t \in [t_0, \infty)$. Set

$$Q_0(t) = X_0^{m-1}\big(\sigma(t)\big) Q\big(\sigma(t)\big) X_0^{*m-1}\big(\sigma(t)\big), \quad t \in [t_0, \infty).$$

It is not difficult to verify that $Q_0(t)$ is Hermitian and positive definite for $t \in [t_0, \infty)$ and $X_0(t)$ is also a non-oscillatory prepared solution of

$$X^{\Delta^2} + X\big(\sigma(t)\big)Q_0(t)X^*\big(\sigma(t)\big)X\big(\sigma(t)\big) = 0, \quad t \in [t_0, \infty). \tag{21}$$

So $X_0^*(t)X_0(t)$ and $X_0(t)X_0^*(t)$ have the same eigenvalues; furthermore, we have

$$\begin{aligned}\big[X_0^*(t)X_0(t)\big]^\Delta &= X_0^*\big(\sigma(t)\big)X_0^\Delta(t) + X_0^{*\Delta}(t)X_0(t)\\ &= X_0^*\big(\sigma(t)\big)V_0(t)X_0\big(\sigma(t)\big) + X_0^*(t)W_0(t)X_0(t).\end{aligned}$$

It follows that $[X_0^*(t)X_0(t)]^\Delta > 0$ for $t \in [t_0, \infty)$, so the eigenvalues of $X_0(t)X_0^*(t)$ are increasing. Hence we can choose a positive real number δ so that $\lambda_{\min}[X_0(t)X_0^*(t)] > \delta$ for $t \in [t_0, \infty)$. By Ostrowski's inequality

$$\lambda_{\max}\big[Q_0(t)\big] \geqslant \lambda_{\max}\big[Q\big(\sigma(t)\big)\big]\delta^{m-1} \quad \text{for } t \in [t_0, \infty).$$

Hence $Q_0(t)$ is Hermitian and positive definite for $t \in [t_0, \infty)$ with

$$\int_a^\infty t\lambda_{\max}\big[Q_0(t)\big]\Delta t = \infty.$$

So $X_0(t)$ is oscillatory solution of (20), even though $Q_0(t)$ may only be positive semi-definite rather than positive definite for $t \in [a, t_0)$, it is clear from the first part of the proof that (21) is oscillatory. Since $X_0(t)$ is a non-oscillatory solution, we get a contradiction. This completes the proof that (2) is oscillatory if (14) holds.

Now we prove that (14) is a necessary condition if (14) is to be oscillatory. Suppose that

$$\int_a^\infty t\lambda_{\max}\big[Q(t)\big]\Delta t < \infty.$$

We need to show that there is at least one non-oscillatory prepared solution $X(t)$ of (14). Here we recall some facts from [9] that will be used in what follows. Let M_n denote the set of $n \times n$ complex matrices, $|x|$ denote the modulus of the complex number x, and let A_{ij} denote the entry in the ith row and jth column of a matrix A. Let $\|\cdot\|_\infty$, $\|\cdot\|_1$ and $\|\cdot\|_2$ be the matrix norms on M_n induced by the l_∞, l_1, and l_2, respectively. Then $\|\cdot\|_\infty$ is the maximum row sum norm, $\|\cdot\|_1$ is the maximum column sum norm, and $\|\cdot\|_2$ is the spectral norm with $\|A\|_2 = [\lambda_{\max}(AA^*)]^{1/2}$ for $A \in \mathrm{M}_n$. $\|H\|_2 = \lambda_{\max}(H)$ when H is Hermitian and positive semi-definite. The relations

$$\|A\|_\infty \leqslant \sqrt{n}\|A\|_2, \qquad \|A\|_1 \leqslant \sqrt{n}\|A\|_2, \qquad \|A\|_1 \leqslant n\|A\|_\infty$$

hold for all $A \in \mathrm{M}_n$.

Let $t_0 \in \mathrm{T}$ be a fixed point. We define an $n \times n$ complex matrix-valued function $X(t)$ for $t \in [t_0, \infty)$ by

$$X(t) = I - \int_{\sigma(t)}^\infty (s-t)\big[X^m(s)Q(s)X^{*m}(s)\big]X(s)\Delta s \tag{22}$$

which satisfies (2) for $t \in [a, \infty)$. In the following discussion we use operator theory to show that $X(t)$ is a prepared non-oscillatory solution of (2). From the assumption, we choose $t_0 \in [a, \infty)$ so large that

$$\int_{\sigma(t_0)}^{\infty} s\lambda_{\max}\big[Q(s)\big]\Delta s < n^{-1/2}2^{-2m}\left(\frac{3m}{2}+1\right)^{-1}. \tag{23}$$

Let H_{t_0} denote the set of all $n \times n$ complex matrix-valued functions $Z(t)$ defined for $t \in [t_0, \infty)$ and such that $\lim_{t\to\infty} Z(t)$ exists as a finite matrix. For $Z \in \mathrm{H}_{t_0}$, let

$$\|Z\| = \sup_{t\in[t_0,\infty)} \big\|Z(t)\big\|_\infty.$$

H_{t_0} equipped with this norm is a Banach space. Let $\mathcal{A} = \{Z \in \mathrm{H}_{t_0}\colon \|Z - I\| \leqslant 1\}$. Then $\mathcal{A}$ is a nonempty closed subset of H_{t_0}. Define TX by

$$TX(t) = I - \int_{\sigma(t)}^{\infty} (s-t)\big[X^m(s)Q(s)X^{*m}(s)\big]X(s)\Delta s \quad \text{for } X \in \mathcal{A}. \tag{24}$$

Then for $X \in \mathcal{A}$, and $s \geqslant t$,

$$\begin{aligned}
&\big|\big[(s-t)X^m(s)Q(s)X^{*m}(s)X(s)\big]_{ij}\big| \\
&\quad\leqslant \big\|(s-t)X^m(s)Q(s)X^{*m}(s)X(s)\big\|_\infty \\
&\quad\leqslant s\big\|X(s)\big\|_\infty^{m+1}\big\|X^m(s)\big\|_\infty\big\|Q(s)\big\|_\infty \leqslant s\|X\|^{m+1}\big\|X^m(s)\big\|_\infty\sqrt{n}\,\big\|Q(s)\big\|_2 \\
&\quad\leqslant 2^{2m+1}n^{1/2}s\lambda_{\max}\big[Q(s)\big].
\end{aligned} \tag{25}$$

From (25), we see that the integral on the right-hand side of (24) is convergent as $t \to \infty$. Moreover, from (23) and (24) we see that, for $t \in [t_0, \infty)$,

$$\big\|TX(t) - I\big\|_\infty \leqslant 2^{2m+1}n^{1/2}\int_{\sigma(t_0)}^{\infty} s\lambda_{\max}\big[Q(s)\big]\Delta s < 1,$$

so $\|TX - I\| \leqslant 1$, that is, $TX(t) \in \mathcal{A}$. Thus T is a mapping from $\mathcal{A}$ into $\mathcal{A}$. For X and Y both in $\mathcal{A}$, we have

$$\begin{aligned}
&\big|\big[TX(t) - TY(t)\big]_{ij}\big| \\
&\quad\leqslant \int_{\sigma(t_0)}^{\infty} s\big\|X^m(s)Q(s)X^{*m}(s)X(s) - Y^m(s)Q(s)Y^{*m}(s)Y(s)\big\|_\infty\Delta s.
\end{aligned} \tag{26}$$

Shortening the notation in a self-evident way,

$$\begin{aligned}
&\|X^mQX^{*m}X - Y^mQY^{*m}Y\|_\infty \\
&\quad\leqslant \|X^mQX^{*m}X - X^mQX^{*m}Y\|_\infty + \|X^mQX^{*m}Y - X^mQY^{*m}Y\|_\infty \\
&\qquad + \|X^mQY^{*m}Y - Y^mQY^{*m}Y\|_\infty
\end{aligned}$$

$$\begin{aligned}&\leqslant \|X\|_\infty^m \sqrt{n}\,\lambda_{\max}[Q]\|X^m\|_\infty\|X-Y\|_\infty + \|X^m\|_\infty\sqrt{n}\,\lambda_{\max}[Q]\|X^m \\ &\quad - Y^m\|_\infty\|Y\|_\infty + \|X^m - Y^m\|_\infty\sqrt{n}\,\lambda_{\max}[Q]\|Y\|_\infty^{m+1}\end{aligned} \tag{27}$$

and

$$\begin{aligned}&\|X^m - Y^m\|_\infty \\ &\quad\leqslant \left\|X^m - X^{m-1}Y\right\|_\infty + \left\|X^{m-1}Y - X^{m-2}Y^2\right\|_\infty + \cdots + \left\|XY^{m-1} - Y^m\right\|_\infty \\ &\quad\leqslant \left\|X^{m-1}(X-Y)\right\|_\infty + \left\|X^{m-2}(X-Y)Y\right\|_\infty + \cdots + \left\|(X-Y)X^{m-1}\right\|_\infty \\ &\quad\leqslant m2^{m-1}\|X-Y\|_\infty.\end{aligned} \tag{28}$$

Combining (27) and (28) yields

$$\|X^m QX^{*m}X - Y^m QY^{*m}Y\|_\infty \leqslant \left(\frac{3m}{2}+1\right)2^{2m}n^{1/2}\lambda_{\max}[Q]\|X-Y\|_\infty. \tag{29}$$

From (26) and (29) we find

$$\|TX - TY\| \leqslant \left[\left(\frac{3m}{2}+1\right)2^{2m}n^{1/2}\int_{\sigma(t_0)}^{\infty} s\lambda_{\max}\bigl(Q(s)\bigr)\Delta s\right]\|X-Y\|.$$

Therefore, from (23) it follows that $T:\mathcal{A}\to\mathcal{A}$ is a contraction mapping. Consequently, there is a solution $X(t)$ of (24) which is also a solution of (2) for $t\in[t_0,\infty)$. Extending this solution backward to $t=a$, we obtain a solution satisfying (2) for $t\in[a,\infty)$. Since

$$\lim_{t\to\infty} X(t) = I \quad \text{and} \quad \lim_{t\to\infty} X^{\Delta}(t) = 0,$$

it follows that $X(t)$ is a prepared solution of (2). Finally,

$$\lim_{t\to\infty} W(t) = \lim_{t\to\infty} X^{\Delta}(t)X^{-1}(t) = 0.$$

So $\lim_{t\to\infty}[W(t)+I] = I$ making $X(t)$ a non-oscillatory solution of (2). This completes the proof of Theorem 3.7. □

4. Examples

The following examples illustrate the applications of our oscillation criteria.

Example 1. Consider the second-order matrix system (2) on time scale $\mathbb{T} = h\mathbb{Z} = \{hk\colon k\in\mathbb{Z}\}$ $(h>0)$; where

$$Q(t) = \begin{pmatrix} t+2 & 0 \\ 0 & 3 \end{pmatrix}.$$

For $t\in\mathbb{T}$; $\sigma(t) = t+h$; $\mu(t) = h$. Let $t_0\in\mathbb{T}$ be given and pick $k_0\in\mathbb{Z}$ so that $a := hk_0 > t_0$. For $t\in\mathbb{T}$; consider

$$\int_a^\infty (t+h)(t+h+2)\Delta t = \sum_{j=k_0}^{\infty}(hj+h)(hj+h+2)$$
$$= h^2 \sum_{j=k_0}^{\infty}\left[(j+1)^2 + \frac{2}{h}(j+h)\right] = \infty.$$

Hence from Theorem 3.7, we get that this equation is oscillatory on $\mathbb{T}$.

Example 2. It will show that the q-difference equation

$$X^{\Delta^2}(t) + \left[X^m(t)q(q-1)t^2X^{*m}(t)\right]^\sigma X\big(\sigma(t)\big) = 0$$

is oscillatory on $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$ is a constant. In fact, $t_0 \in [1;\infty)$ is given and dick $k_0 \in \mathbb{N}$ so that $a := q^{k_0} > t_0$. For $t \in \mathbb{T}$; let

$$\bar{Q}(t) = \sigma(t)\lambda_{\max}\left[Q\big(\sigma(t)\big)\right] = qt\frac{1}{q(q-1)(qt)^2} = \frac{1}{q^2(q-1)t}$$

and

$$\int_a^{n'} \bar{Q}(t)\Delta t = \int_{q^{k_0}}^{q^n} \bar{Q}(t)\Delta t = \sum_{j=k_0}^{n}\bar{Q}\big(q^j\big)\mu\big(q^j\big) = \sum_{j=k_0}^{n}\frac{1}{(q-1)q^{j+2}}(q-1)q^j$$
$$= \sum_{j=k_0}^{n}\frac{1}{q^2} = \frac{1}{2q^2}(n+k_0)(n-k_0+1) = \frac{n^2+n-k_0^2+k}{2q^2} = \infty.$$

Since $\lim_{n\to\infty}\frac{n^2+n-k_0^2+k}{2q^2} = \infty$. That is $\int_a^\infty \sigma(t)\lambda_{\max}[Q(\sigma(t))]\Delta t = \infty$.

Remark. When $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, argument in this paper is just as [4,8], respectively.

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References

[1] F.V. Atkinson, On second-order non-linear oscillation, Pacific J. Math. 5 (1955) 643–647.
[2] T. Kura, A matrix analogue of Atkinson's oscillation theorem, Funkcial Ekval. 25 (1982) 223–226.
[3] G.J. Butler, L.H. Erbe, Oscillation theorems for second order differential systems with functionally commutative matrix coefficients, Funkcial Ekval. 28 (1985) 47–55.
[4] C.D. Ahlbrandt, J. Ridenhour, R.C. Thompson, Oscillation of superlinear matrix differential equation, Proc. Amer. Math. Soc. 105 (1989) 141–148.
[5] A.B. Mingarelli, Volterra–Stieltjes integral equations and generalized ordinary differential expressions, Lecture Notes in Math., vol. 989, Spinger-Verlag, Berlin, 1983.

[6] S. Hilger, Analysis on measure chain-a unified approach to continuous and discrete calculus, Results Math. 183 (1990) 18–56.
[7] L. Erbe, A. Peterson, Oscillation criteria for second-order matrix dynamic equations on a time scale, J. Comput. Appl. Math. 141 (2002) 169–186.
[8] A. Peterson, J. Ridenhour, Atkinson's superlinear oscillation theorem for matrix difference equations, SIAM. J. Math. Anal. 22 (1991) 774–784.
[9] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge Univ. Press, Cambridge, 1991.
[10] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, Birkhäuser, Boston, 2001.
[11] R. Agarwal, M. Bohner, Quadratic functional for second order matrix equation on time scales, Nonlinear Anal. 33 (1998) 675–692.
[12] R. Agarwal, M. Bohner, P. Wong, Sturm–Liouville eigenvalue problems on time scales, Appl. Math. Comput. 99 (1999) 153–166.
[13] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction and Applications, Birkhäuser, Boston, 2001.