



Available online at [www.sciencedirect.com](http://www.sciencedirect.com)

SCIENCE @ DIRECT®

J. Math. Anal. Appl. 299 (2004) 392–410

*Journal of*  
MATHEMATICAL  
ANALYSIS AND  
APPLICATIONS

[www.elsevier.com/locate/jmaa](http://www.elsevier.com/locate/jmaa)

# On solutions of neutral nonlocal evolution equations with nondense domain<sup>☆</sup>

Xianlong Fu

*Department of Mathematics, East China Normal University, Shanghai, 200062, PR China*

Received 15 September 2003

Available online 16 September 2004

Submitted by M. Iannelli

---

## Abstract

In this paper, with Sadovskii's and Banach's fixed point theorems applied, we establish some results on the existence of integral solutions, strong solutions, and strict solutions for a class of nondensely defined neutral evolution equations with nonlocal conditions. Also, an example is given in the end to show the applications of the obtained results.

© 2004 Elsevier Inc. All rights reserved.

*Keywords:* Existence of solutions; Nonlocal Cauchy problem; Hille–Yosida condition; Integrated semigroups

---

## 1. Introduction

In this paper we study the existence of solutions for semilinear neutral functional differential evolution equations with nonlocal conditions. More precisely, we consider the following Cauchy problem on a general Banach space  $X$ :

$$\begin{aligned} \frac{d}{dt}[x(t) - F(t, x(h_1(t)))] &= A[x(t) - F(t, x(h_1(t)))] + G(t, x(h_2(t))), \\ 0 \leq t \leq a, \end{aligned}$$

---

<sup>☆</sup> This work is supported by Mathematics Tianyuan Fund (No. A0324624), NNSF of China (No. 10371040), and Shanghai Priority Academic Discipline.

*E-mail address:* [xl fu@math.ecnu.edu.cn](mailto:xl fu@math.ecnu.edu.cn).

$$x(0) + g(x) = x_0 \in X, \quad (1)$$

where the operator  $A$  is not densely defined in  $X$  and generates a integrated semigroup  $\{S(t)\}_{t \geq 0}$ ,  $F$ ,  $G$ ,  $g$ , and  $h_1, h_2$  are given functions to be specified later.

The nonlocal Cauchy problem was considered by Byszewski [5] and the importance of nonlocal conditions in different fields has been discussed in [5] and [8] and the references therein. For example, in [8] the author described the diffusion phenomenon of a small amount of gas in a transparent tube by using the formula

$$g(x) = \sum_{i=0}^p c_i x(t_i), \quad (2)$$

where  $c_i$ ,  $i = 0, 1, \dots, p$ , are given constants and  $0 < t_0 < t_1 < \dots < t_p < a$ . In this case Eq. (2) allows the additional measurement at  $t_i$ ,  $i = 0, 1, \dots, p$ . In the past several years theorems about existence, uniqueness, and stability of differential and functional differential abstract evolution Cauchy problem with nonlocal conditions have been studied by Byszewski and Lakshmikantham [7], Byszewski [6] and the references therein, Balachandran and Chandrasekaran [4], Ntouyas and Tsamiatos [14], Aizicovici and McKibben [1], and by Lin and Liu [13].

When  $F(\cdot, \cdot) = 0$  and  $A$  generating a  $C_0$ -semigroup in Eq. (1), Byszewski has investigated the existence of mild solutions and classical solutions in paper [6] applying Schauder's fixed point theorem. But as pointed out in paper [11], the author employed the following condition as discussing the existence of the classical solutions: the mild solution  $x(\cdot)$  is assumed to satisfy that  $\|x(h_2(t_2)) - x(h_2(t_1))\| \leq L\|x(t_2) - x(t_1)\|$  for some constant  $L > 0$ , which is very difficult to verify since the mild solution  $x(\cdot)$  is unknown. To take away this unsatisfactory condition and extend its results to neutral equations, in [11] the authors have studied the existence of mild solutions and strong solutions for the equations with the form

$$\begin{aligned} \frac{d}{dt}[x(t) + F(t, x(h_1(t)))] + Ax(t) &= G(t, x(h_2(t))), \quad 0 \leq t \leq a, \\ x(0) + g(x) &= x_0 \in X, \end{aligned}$$

where the densely defined operator  $-A$  generates a compact analytic semigroup. The main tools and techniques in [11] are the properties of fractional power and Sadovskii fixed point theorem.

However, as indicated in [16], we sometimes need to deal with the nondensely defined operators, for example, when we look at a one-dimensional heat equation with Dirichlet condition on  $[0, 1]$  and consider  $A = \partial^2/\partial^2 x$  in  $C([0, 1]; R)$ , in order to measure the solution in the sup-norm, we take the domain

$$D(A) = \{x \in C^2([0, 1]; R) : x(0) = x(1) = 0\},$$

then it is not dense in  $C([0, 1]; R)$  (with the sup-norm). See [16] for more examples and remarks concerning the nondensely defined operators.

Our purpose here is to improve the results in paper [11]. The work of this article has two wedges, on one hand, we will extend these results of densely defined evolution equations to nondensely defined evolution equations with nonlocal conditions. On the other hand, we

will obtain better regularity for the solutions than in paper [11]. The obtained results extend and develop the main results reported in [4–11,13,14]. In addition, if we set  $F(\cdot, \cdot) \equiv 0$ ,  $g(\cdot) \equiv 0$ ,  $G(\cdot, \cdot) = f(t)$  and let  $\overline{D(A)} = X$  for Eq. (1) in this article, then the problem we considered here reduces immediately to the classical Cauchy problem discussed in [15, Chapter 4], so our results can also be regarded as an extension of the corresponding results shown in [15, Chapter 4].

This article is organized as follows. First, we introduce some preliminaries about the theory of integrated semigroup in Section 2. The main results of this paper are arranged in Section 3, where the existence and uniqueness of integral solutions are discussed and their regularity are also be investigated. Finally, an example is presented in Section 4 to show the applications of the obtained results.

## 2. Preliminaries

The purpose of this section is to state some preliminaries about the theory of integrated semigroup which are required throughout this paper.

**Definition 2.1** (See [2,3]). Let  $X$  be a Banach space. An integrated semigroup is a family  $\{S(t)\}_{t \geq 0}$  of bounded linear operators  $S(t)$  on  $X$  with the following properties:

- (i)  $S(0) = 0$ ;
- (ii)  $t \rightarrow S(t)$  is strongly continuous;
- (iii)  $S(t)S(s) = \int_0^t [S(s+\tau) - S(\tau)] d\tau$ , for all  $t, s \geq 0$ .

**Definition 2.2** (See [3]). An operator  $A$  is called to be the generator of an integrated semigroup if there exists  $\omega \in \mathbb{R}$  such that  $(\omega, +\infty) \subset \rho(A)$  and there exists a strongly continuous exponentially bounded family  $\{S(t)\}_{t \geq 0}$  of bounded linear operators such that  $S(0) = 0$  and  $(\lambda I - A)^{-1} = \lambda \int_0^\infty e^{-\lambda t} S(t) dt$  exists for all  $\lambda$  with  $\lambda > \omega$ .

**Proposition 2.3** (See [3]). Let  $A$  be the generator of an integrated semigroup  $\{S(t)\}_{t \geq 0}$ . Then for all  $x \in X$  and  $t \geq 0$ ,

$$\int_0^t S(s)x ds \in D(A) \quad \text{and} \quad S(t)x = A \int_0^t S(s)x ds + tx.$$

**Definition 2.4** (See [12]).

- (i) An integrated semigroup  $\{S(t)\}_{t \geq 0}$  is called locally Lipschitz continuous, if, for all  $\tau > 0$ , there exists a constant  $L > 0$  such that

$$\|S(t) - S(s)\| \leq L|t - s|, \quad t, s \in [0, \tau].$$

- (ii) An integrated semigroup  $\{S(t)\}_{t \geq 0}$  is called nondegenerate if  $S(t)x = 0$  for all  $t \geq 0$  implies that  $x = 0$ .

**Definition 2.5.** We say that a linear operator  $A$  satisfies the so-called Hille–Yosida condition, if there exist  $M \geq 0$  and  $w \in \mathbb{R}$  such that  $(w, +\infty) \subset \rho(A)$  and

$$\sup\{(\lambda - w)^n \|R(\lambda, A)^n\|, n \in \mathbb{N}, \lambda > w\} \leq M.$$

**Theorem 2.6** (See [12]). *The following assertions are equivalent:*

- (i)  $A$  is the generator of non-degenerate, locally Lipschitz continuous integrated semigroup;
- (ii)  $A$  satisfies the Hille–Yosida condition.

If  $A$  is the generator of an integrated semigroup  $\{S(t)\}_{t \geq 0}$  which is locally Lipschitz, then from [2],  $S(t)x$  is continuously differentiable if and only if  $x \in \overline{D(A)}$  and  $\{S'(t)\}_{t \geq 0}$  is a  $C_0$ -semigroup on  $\overline{D(A)}$ . Furthermore, let  $A_0$  be the generator of the  $C_0$ -semigroup  $\{S'(t)\}_{t \geq 0}$ , then  $A_0$  is the part of  $A$  on  $\overline{D(A)}$  defined by

$$D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\},$$

$$A_0x = Ax.$$

For the theory of  $C_0$ -semigroup we refer to the book [15].

In the sequel, we give some results for the existence of solutions of the following Cauchy problem:

$$\begin{aligned} \frac{d}{dt}x(t) &= Ax(t) + f(t), \quad t \geq 0, \\ x(0) &= x_0 \in X, \end{aligned} \tag{3}$$

where  $A$  satisfies the Hille–Yosida condition without being densely defined. Let  $\{S(t)\}_{t \geq 0}$  be the integrated semigroup generated by  $A$ , then one has the following theorem.

**Theorem 2.7** (See [2]). *Let  $f : [0, a] \rightarrow X$  is a continuous function. Then for  $x_0 \in \overline{D(A)}$ , there is a unique continuous function  $x : [0, a] \rightarrow X$  such that*

- (i)  $\int_0^t x(s) ds \in D(A)$ ,  $t \in [0, a]$ ;
- (ii)  $x(t) = x_0 + A \int_0^t x(s) ds + \int_0^t f(s) ds$ ,  $t \in [0, a]$ ;
- (iii)  $\|x(t)\| \leq Me^{\omega t}[\|x_0\| + \int_0^t e^{-\omega s} \|f(s)\| ds]$ ,  $t \in [0, a]$ .

**Proposition 2.8** (See [17]). *Let  $A : D(A) \subset X \rightarrow X$  be a linear operator satisfying the Hille–Yosida condition,  $\{S(t)\}_{t \geq 0}$  be the integrated semigroup generated by  $A$  and  $f : [0, t] \rightarrow X$ ,  $T > 0$ , be a Bochner-integrable function. Then the function  $K : [0, T] \rightarrow X$  defined by*

$$K(t) = \int_0^t S(t-s)f(s) ds$$

is continuously differentiable on  $[0, T]$  and satisfies that, for  $\lambda > \omega$  and  $t \in [0, T]$ ,

$$R(\lambda, A)K'(t) = \int_0^t S'(t-s)R(\lambda, A)f(s)ds.$$

This is suggestive to solve Eq. (3) by the following variation of constant formula:

$$x(t) = S'(t)x_0 + \frac{d}{dt} \int_0^t S(t-s)f(s)ds, \quad t \geq 0. \quad (4)$$

### 3. Main results

Throughout this whole section, we will always assume that

( $H_0$ ) The operator  $A$  satisfies the Hille–Yosida condition and generates an integrated semi-group  $\{S(t)\}_{t \geq 0}$  on  $X$ .

#### 3.1. Existence of integral solutions

**Definition 3.1.** A function  $x : [0, a] \rightarrow X$  is said to be an integral solution of Eq. (1) on  $[0, a]$ , if the following conditions hold:

- (i)  $x$  is continuous on  $[0, a]$ ;
- (ii)  $\int_0^t [x(s) - F(s, x(h_1(s)))]ds \in D(A)$  on  $[0, a]$ ;
- (iii)

$$\begin{aligned} x(t) = & x_0 - g(x) - F(0, x(h_1(0))) + F(t, x(h_1(t))) \\ & + A \int_0^t [x(s) - F(s, x(h_1(s)))]ds + \int_0^t G(s, x(h_2(s)))ds, \quad t \geq 0. \end{aligned}$$

**Remark 1.** It is not difficult to prove that, if  $x(\cdot)$  is a integral solution of Eq. (1) on  $[0, a]$ , then for all  $t \in [0, a]$ ,  $x(t) - F(t, x(h_1(t))) \in \overline{D(A)}$ . In particular,  $x(0) - F(0, x(h_1(0))) \in \overline{D(A)}$ .

From Theorem 2.7 and (4) we know that  $x : [0, a] \rightarrow X$  is an integral solution of Eq. (1) if and only if  $x$  solves the following equation:

$$\begin{aligned} x(t) = & S'(t)[x_0 - g(x) - F(0, x(h_1(0)))] + F(t, x(h_1(t))) \\ & + \frac{d}{dt} \int_0^t S(t-s)G(s, x(h_2(s)))ds, \quad 0 \leq t \leq a. \end{aligned} \quad (5)$$

For Eq. (1) we assume that

(H<sub>1</sub>) The function  $F : [0, a] \times X \rightarrow X$  is Lipschitz continuous, that is, there exists a constant  $L > 0$  such that

$$\|F(t_1, x_1) - F(t_2, x_2)\| \leq L(|t_1 - t_2| + \|x_1 - x_2\|), \quad (6)$$

for every  $0 \leq t_1, t_2 \leq a, x_1, x_2 \in X$ . And there is an  $L_1 > 0$  such that

$$\|F(t, x)\| \leq L_1(\|x\| + 1),$$

for any  $0 \leq t \leq a$  and  $x \in X$ .

(H<sub>2</sub>) The function  $G : [0, a] \times X \rightarrow X$  satisfies the following conditions:

- (i) for each  $t \in [0, a]$ , the function  $G(t, \cdot) : X \rightarrow X$  is continuous and for each  $x \in X$  the function  $G(\cdot, x) : [0, a] \rightarrow X$  is strongly measurable;
- (ii) for each positive number  $k \in \mathbb{N}$ , there is a positive function  $g_k \in L^1([0, a])$  such that

$$\sup_{\|x\| \leq k} \|G(t, x)\| \leq g_k(t) \quad \text{and} \quad \liminf_{k \rightarrow \infty} \frac{1}{k} \int_0^a g_k(s) ds = \gamma < \infty.$$

(H<sub>3</sub>)  $h_i \in C([0, a]; [0, a]), i = 1, 2$ .  $g \in C(E; \overline{D(A)})$ , here and hereafter  $E = C([0, a]; X)$ , and  $g$  satisfies that

- (i) there exist positive constants  $L_2$  and  $L'_2$  such that  $\|g(u)\| \leq L_2\|u\| + L'_2$  for all  $u \in E$ ;
- (ii)  $g$  is a completely continuous map.

(H<sub>4</sub>) The semigroup  $\{S'(t)\}_{t \geq 0}$  is compact on  $(\overline{D(A)}, \|\cdot\|)$ , and there is a constant  $M' \geq 1$  such that

$$\|S'(t)\| \leq M'$$

for all  $t \in [0, a]$ .

**Theorem 3.2.** Let  $x_0 \in \overline{D(A)}$ . If the assumptions (H<sub>1</sub>)–(H<sub>4</sub>) are satisfied together with  $(M' + 1)L < 1$ , then the nonlocal Cauchy problem (1) has an integral solution provided that

$$(L_2 + \gamma M)M' + (1 + M')L_1 < 1. \quad (7)$$

**Proof.** From (5), we consider the operator  $P$  defined on  $E = C([0, a]; X)$  by the formula

$$\begin{aligned} (Px)(t) &= S'(t)[x_0 - g(x) - F(0, x(h_1(0)))] + F(t, x(h_1(t))) \\ &\quad + \frac{d}{dt} \int_0^t S(t-s)G(s, x(h_2(s))) ds, \quad 0 \leq t \leq a. \end{aligned} \quad (8)$$

For each positive integer  $k$ , let

$$B_k = \{x \in E : \|x(t)\| \leq k, 0 \leq t \leq a\}.$$

Then for each  $k$ ,  $B_k$  is clearly a bounded closed convex set in  $E$ . We claim that there exists a positive integer  $k$  such that  $PB_k \subseteq B_k$ . If it is not true, then for each positive integer  $k$ ,

there is a function  $x_k(\cdot) \in B_k$ , but  $Px_k \notin B_k$ , that is  $\|Px_k(t)\| > k$  for some  $t(k) \in [0, a]$ , where  $t(k)$  denotes  $t$  is dependent on  $k$ . However, on the other hand, we have

$$\begin{aligned} k &< \|(Px_k)(t)\| \\ &= \left\| S'(t)[x_0 - g(x_k) + F(0, x(h_1(0)))] - F(t, x(h_1(t))) \right. \\ &\quad \left. + \frac{d}{dt} \int_0^t S(t-s)G(s, x(h_2(s))) ds \right\| \\ &\leq \|S'(t)[x_0 - g(x_k) + F(0, x(h_1(0)))]\| + \|F(t, x(h_1(t)))\| \\ &\quad + \left\| \frac{d}{dt} \int_0^t S(t-s)G(s, x(h_2(s))) ds \right\| \\ &\leq M'[\|x_0\| + L_2k + L_2' + L_1(k+1)] + L_1(k+1) \\ &\quad + \left\| \frac{d}{dt} \int_0^t S(t-s)G(s, x(h_2(s))) ds \right\|. \end{aligned}$$

From Proposition 2.8, we get

$$\begin{aligned} &\left\| \lambda R(\lambda, A) \frac{d}{dt} \int_0^t S(t-s)G(s, x(h_2(s))) ds \right\| \\ &= \left\| \int_0^t S'(t-s)\lambda R(\lambda, A)G(s, x(h_2(s))) ds \right\| \\ &\leq \frac{\lambda}{\lambda - \omega} MM' \int_0^a g_k(s) ds. \end{aligned}$$

Letting  $\lambda \rightarrow +\infty$ , we obtain that

$$\left\| \frac{d}{dt} \int_0^t S(t-s)G(s, x(h_2(s))) ds \right\| \leq MM' \int_0^a g_k(s) ds.$$

So,

$$k < \|(Px_k)(t)\| \leq M'(\|x_0\| + L_2k + L_2') + (1 + M')L_1(k+1) + MM' \int_0^a g_k(s) ds.$$

Dividing on both sides by  $k$  and taking the lower limit as  $k \rightarrow +\infty$ , we get

$$(L_2 + \gamma M)M' + (1 + M')L_1 \geq 1.$$

This contradicts (7). Hence for some positive integer  $k$ ,  $PB_k \subseteq B_k$ .

Next we will show that the operator  $P$  has a fixed point on  $B_k$ , which implies Eq. (1) has an integral solution. To this end, we decompose  $P$  as  $P = P_1 + P_2$ , where the operators  $P_1, P_2$  are defined on  $B_k$  respectively by

$$(P_1x)(t) = S'(t)F(0, x(h_1(0))) - F(t, x(h_1(t)))$$

and

$$(P_2x)(t) = S'(t)[x_0 - g(x)] + \frac{d}{dt} \int_0^t S(t-s)G(s, x(h_2(s)))ds,$$

for  $0 \leq t \leq a$ , and we will verify that  $P_1$  is a contraction while  $P_2$  is a compact operator.

It is easy to see from the conditions that  $P_1$  is a contraction. To prove that  $P_2$  is compact, firstly we prove that  $P_2$  is continuous on  $B_k$ . Let  $\{x_n\} \subseteq B_k$  with  $x_n \rightarrow x$  in  $B_k$ , then by  $H_2(i)$ , we have

$$G(s, x_n(h_2(s))) \rightarrow G(s, x(h_2(s))), \quad n \rightarrow +\infty.$$

Moreover

$$\|G(s, x_n(h_2(s))) - G(s, x(h_2(s)))\| \leq 2g_k(s),$$

and

$$\begin{aligned} & \left\| \lambda R(\lambda, A) \frac{d}{dt} \int_0^t S(t-s)[G(s, x_n(h_2(s))) - G(s, x(h_2(s)))]ds \right\| \\ &= \left\| \int_0^t S'(t-s)\lambda R(\lambda, A)[G(s, x_n(h_2(s))) - G(s, x(h_2(s)))]ds \right\| \\ &\leq \frac{\lambda}{\lambda - \omega} MM' \int_0^a \|G(s, x_n(h_2(s))) - G(s, x(h_2(s)))\| ds. \end{aligned}$$

Letting  $\lambda \rightarrow +\infty$ , we obtain that

$$\begin{aligned} & \left\| \frac{d}{dt} \int_0^t S(t-s)[G(s, x_n(h_2(s))) - G(s, x(h_2(s)))]ds \right\| \\ &\leq MM' \int_0^a \|G(s, x_n(h_2(s))) - G(s, x(h_2(s)))\| ds. \end{aligned}$$

By the dominated convergence theorem, we have

$$\|P_2x_n - P_2x\| = \sup_{0 \leq t \leq a} \left\| S'(t)[g(x) - g(x_n)] \right\|$$

$$\begin{aligned}
& + \frac{d}{dt} \int_0^t S(t-s) [G(s, x_n(h_2(s))) - G(s, x(h_2(s)))] ds \Big\| \\
& \rightarrow 0, \quad \text{as } n \rightarrow +\infty,
\end{aligned}$$

i.e.,  $P_2$  is continuous.

Next we prove the family  $\{P_2x: x \in B_k\}$  is a equicontinuous family of functions. To see this we fix  $t_1 > 0$  and let  $t_2 > t_1$  and  $\varepsilon > 0$  enough small. Then

$$\begin{aligned}
& \|\lambda R(\lambda, A)[(P_2x)(t_2) - (P_2x)(t_1)]\| \\
& = \left\| \lambda R(\lambda, A)[S'(t_2) - S'(t_1)](x_0 - g(x)) \right. \\
& \quad + \int_0^{t_1-\varepsilon} [S'(t_2-s) - S'(t_1-s)] \lambda R(\lambda, A) G(s, x(h_2(s))) ds \\
& \quad + \int_{t_1-\varepsilon}^{t_1} [S'(t_2-s) - S'(t_1-s)] \lambda R(\lambda, A) G(s, x(h_2(s))) ds \\
& \quad \left. + \int_{t_1}^{t_2} S'(t_2-s) \lambda R(\lambda, A) G(s, x(h_2(s))) ds \right\| \\
& \leq \|\lambda R(\lambda, A)[S'(t_2) - S'(t_1)](x_0 - g(x))\| \\
& \quad + \frac{\lambda M}{\lambda - \omega} \int_0^{t_1-\varepsilon} \|S'(t_2-s) - S'(t_1-s)\| \|G(s, x(h_2(s)))\| ds \\
& \quad + \frac{\lambda M}{\lambda - \omega} \int_{t_1-\varepsilon}^{t_1} \|S'(t_2-s) - S'(t_1-s)\| \|G(s, x(h_2(s)))\| ds \\
& \quad + \frac{\lambda M}{\lambda - \omega} \int_{t_1}^{t_2} \|S'(t_2-s)\| \|G(s, x(h_2(s)))\| ds.
\end{aligned}$$

Noting that  $\|G(s, x(h_2(s)))\| \leq g_k(s)$  and  $g_k(s) \in L^1$  and using the same limitation method as above, we see that  $\|(P_2x)(t_2) - (P_2x)(t_1)\|$  tends to zero independently of  $x \in B_k$  as  $t_2 - t_1 \rightarrow 0$  since the compactness of  $S'(t)(t > 0)$  implies the continuity of  $S'(t)(t > 0)$  in  $t$  in the uniform operators topology. Similarly, using the compactness of the set  $g(B_k)$  we can prove that the functions  $P_2x, x \in B_k$  are equicontinuous at  $t = 0$ . Hence  $P_2$  maps  $B_k$  into an equicontinuous family of functions.

It remains to prove that  $V(t) = \{(P_2x)(t): x \in B_k\}$  is relatively compact in  $X$ . Obviously, by condition  $(H_3)$ ,  $V(0)$  is relatively compact in  $X$ . Let  $0 < t \leq a$  be fixed and  $0 < \varepsilon < t$ . For  $z \in B_k$ , let

$$(P_{2,\varepsilon}x)(t) = S'(t)[x_0 - g(x)] + \frac{d}{dt} \int_0^{t-\varepsilon} S(t-s)G(s, x(h_2(s))) ds,$$

then

$$\begin{aligned} \lambda R(\lambda, A)(P_{2,\varepsilon}x)(t) &= \lambda R(\lambda, A)S'(t)[x_0 - g(x)] \\ &\quad + S'(\varepsilon) \int_0^{t-\varepsilon} S'(t-\varepsilon-s)\lambda R(\lambda, A)G(s, x(h_2(s))) ds. \end{aligned}$$

Then from the compactness of  $S'(\varepsilon)$  ( $\varepsilon > 0$ ), we obtain that  $V_\varepsilon(t) = \{(P_{2,\varepsilon}x)(t): x \in B_k\}$  is relatively compact in  $X$  for every  $\varepsilon$ ,  $0 < \varepsilon < t$ . Moreover, for every  $x \in B_k$ , we have

$$\begin{aligned} \|\lambda R(\lambda, A)[(P_2x)(t) - (P_{2,\varepsilon}x)(t)]\| &\leq \int_{t-\varepsilon}^t \|S'(t-s)\lambda R(\lambda, A)G(s, x(h_2(s)))\| ds \\ &\leq \frac{\lambda}{\lambda - \omega} \int_{t-\varepsilon}^t MM'g_k(s) ds. \end{aligned}$$

Therefore, letting  $\lambda \rightarrow +\infty$  and  $\varepsilon \rightarrow 0$ , we see that, there are relatively compact sets arbitrarily close to the set  $V(t)$ , hence the set  $V(t)$  is also relatively compact in  $X$ .

Thus, by Arzela–Ascoli theorem  $P_2$  is a compact operator. These arguments enable us to conclude that  $P = P_1 + P_2$  is a condensing map on  $B_k$ , and by the fixed point theorem of Sadovskii there exists a fixed point  $x(\cdot)$  for  $P$  on  $B_k$ . Therefore the nonlocal Cauchy problem (1) has a integral solution, and the proof is completed.  $\square$

To obtain the uniqueness of integral solutions for Eq. (1), we impose the following conditions on it:

$(H_5)$  The functions  $F, G: [0, a] \times X \rightarrow X$  are both continuous and Lipschitz continuous in the second variable, that is, there exist constants  $L_3 > 0$  and  $L_4 > 0$  such that

$$\|F(t, x_1) - F(t, x_2)\| \leq L_3\|x_1 - x_2\| \quad \text{and} \quad (9)$$

$$\|G(t, x_1) - G(t, x_2)\| \leq L_4\|x_1 - x_2\|, \quad (10)$$

for every  $0 \leq t \leq a$ ,  $x_1, x_2 \in X$ .

$(H_6)$  The function  $g: E = C([0, a]; X) \rightarrow \overline{D(A)}$  is also Lipschitz continuous, i.e., there is a constant  $L_5 > 0$  such that

$$\|g(u_1) - g(u_2)\| \leq L_5\|u_1 - u_2\|, \quad (11)$$

for any  $u_1, u_2 \in E$ .

$(H_7)$   $h_i \in C([0, a]; [0, a])$ ,  $i = 1, 2$ .

**Theorem 3.3.** Let  $x_0 \in \overline{D(A)}$ . If the assumptions  $(H_5)$ – $(H_7)$  are satisfied, then the nonlocal Cauchy problem (1) has a unique integral solution provided that

$$L_0 =: M'L_5 + (M' + 1)L_3 + aMM'L_4 < 1. \quad (12)$$

**Proof.** We consider the operator  $P$  defined by (8), we will show that  $P$  is a contraction on  $E$ .

In fact, let  $x_1, x_2 \in E$ , then for each  $t \in [0, a]$  and by condition (9), (10), and (11), we have that

$$\begin{aligned} & \| (Px_1)(t) - (Px_2)(t) \| \\ & \leq \| S'(t)[g(x_1) - g(x_2)] \| + \| S'(t)[F(0, x_1(h_1(0))) - F(0, x_2(h_1(0)))] \| \\ & \quad + \| F(t, x_1(h_1(t))) - F(t, x_2(h_1(t))) \| \\ & \quad + M \left\| \int_0^t S'(t-s)[G(s, x_1(h_2(s))) - G(s, x_2(h_2(s)))] ds \right\| \\ & \leq M'L_5 \sup_{0 \leq s \leq a} \|x_1(s) - x_2(s)\| + (M' + 1)L_3 \sup_{0 \leq s \leq a} \|x_1(s) - x_2(s)\| \\ & \quad + aMM'L_4 \sup_{0 \leq s \leq a} \|x_1(s) - x_2(s)\| \\ & = [M'L_5 + (M' + 1)L_3 + aMM'L_4] \sup_{0 \leq s \leq a} \|x_1(s) - x_2(s)\| \\ & = L_0 \sup_{0 \leq s \leq a} \|x_1(s) - x_2(s)\|. \end{aligned}$$

Thus

$$\|Px_1 - Px_2\| \leq L_0 \|x_1 - x_2\|.$$

So  $P_1$  is a contraction since  $L_0 < 1$  by assumption (12).

Thus, by Banach fixed point theorem we conclude that there exists a unique fixed point  $x(\cdot)$  for  $P$  on  $E$ , therefore the nonlocal Cauchy problem (1) has a unique integral solution. The proof is completed.  $\square$

### 3.2. Existence of strong solutions

In the following two subsections we are going to discuss the regularity of integral solutions for Eq. (1), we will provide conditions which allow the differentiation of the integral solutions obtained above, i.e., these derivatives are shown to satisfy differential equation of the form (1). In this section, we will get the a.e. differentiation of the integral solutions with  $X$  reflexive. Then we will obtain the better regularity in Section 3.3 by imposing some stronger conditions on functions  $F$ ,  $G$ , and  $h_i$  (without the reflexivity of  $X$ ). Firstly we give the definition of strong solution.

**Definition 3.4.** A function  $x(\cdot) : [0, a] \rightarrow X$  is said to be a strong solution of the nonlocal Cauchy problem (1), if

- (1)  $x(\cdot) - F(\cdot, x(h_1(\cdot)))$  is continuous on  $[0, a]$  and differentiable a.e. on  $[0, a]$ ,  $x' \in L^1([0, a]; X)$ ;  
 (2)  $x$  satisfies

$$\frac{d}{dt} [x(t) + F(t, x(h_1(t)))] = A[x(t) + F(t, x(h_1(t)))] + G(t, x(h_2(t))),$$

a.e. on  $(0, a)$  and

$$x(0) + g(x) = x_0.$$

**Remark 2.** By the standard arguments we can easily show that, if  $x(\cdot) : [0, a] \rightarrow X$  is an integral solution of Eq. (1) such that  $x(t) - F(t, x(h_1(t)))$  is continuous on  $[0, a]$  and differentiable a.e. on  $[0, a]$ , then  $x(t) - F(t, x(h_1(t))) \in D(A)$  a.e. on  $[0, a]$ . It also occurs on whole  $[0, a]$  when  $x(t) - F(t, x(h_1(t))) \in C^1([0, a]; X)$ .

**Theorem 3.5.** Let  $X$  be a reflexive Banach space. Suppose that conditions  $(H_1)$ ,  $(H_3)$ , and  $(H_4)$  are satisfied with  $F([0, a] \times X) \subset D(A)$  and  $(M' + 1)L < 1$ , and the function  $AF(0, \cdot) : X \rightarrow \overline{D(A)}$  maps bounded sets into bounded sets. Additionally the following conditions hold:

$(H_8)$   $G(\cdot, \cdot)$  is Lipschitz continuous, i.e., there exists a constant  $L_6 > 0$ , such that

$$\|G(t_1, x_1) - G(t_2, x_2)\| \leq L_6[|t_1 - t_2| + \|x_1 - x_2\|]$$

for any  $(t_1, x_1), (t_2, x_2) \in [0, a] \times X$ . Moreover, there is an  $L_7 > 0$  such that

$$\|G(t, x)\| \leq L_7(\|x\| + 1)$$

for any  $(t, x) \in [0, a] \times X$ .

$(H_9)$  There are constants  $l_1, l_2 > 0$ , such that

$$\|h_i(t_1) - h_i(t_2)\| \leq l_i|t_1 - t_2|,$$

for  $t_1, t_2 \in [0, a]$ ,  $i = 1, 2$ ;

$(H_{10})$   $x_0 - g(x) - F(0, x(h_1(0))) \in D(A_0)$ .

$(H_{11})$  There hold the inequalities

$$(L_2 + L_7M)M' + (1 + M')L_1 < 1$$

and

$$LMl_1 + aMM'L_3l_2 < 1. \quad (13)$$

Then the nonlocal Cauchy problem (1) has a strong solution on  $[0, a]$ .

**Proof.** Let  $P$  be the operator defined in the proof of Theorem 3.2. Consider the set

$$B = \{x \in E : \|x\| \leq k, \|x(t) - x(s)\| \leq L^*|t - s|, t, s \in [0, a]\}$$

for some positive constants  $k$  and  $L$  large enough. It is clear that  $B$  is a convex, closed, and non-empty set. We shall prove that  $P$  has a fixed point on  $B$ . Obviously, from the proof of Theorem 3.2 we see that it is sufficient to show that for any  $x \in B$ ,

$$\|(Px)(t_2) - (Px)(t_1)\| \leq L^*|t_2 - t_1|, \quad t_2, t_1 \in [0, a].$$

In fact,

$$\begin{aligned} & \|\lambda R(\lambda, A)[(Px)(t_2) - (Px)(t_1)]\| \\ & \leq \|\lambda R(\lambda, A)[S'(t_2) - S'(t_1)][x_0 + F(0, x(h_1(0))) - g(x)]\| \\ & \quad + \|\lambda R(\lambda, A)[F(t_2, x(h_1(t_2))) - F(t_1, x(h_1(t_1)))]\| \\ & \quad + \left\| \int_0^{t_2} S'(t_2 - s) \lambda R(\lambda, A) G(s, x(h_2(s))) ds \right. \\ & \quad \left. - \int_0^{t_1} S'(t_1 - s) \lambda R(\lambda, A) G(s, x(h_2(s))) ds \right\| \\ & = \|\lambda R(\lambda, A)[S'(t_2) - S'(t_1)][x_0 + F(0, x(h_1(0))) - g(x)]\| \\ & \quad + \|\lambda R(\lambda, A)[F(t_2, x(h_1(t_2))) - F(t_1, x(h_1(t_1)))]\| \\ & \quad + \left\| \int_0^{t_1} S'(t_1 - s) \lambda R(\lambda, A) \right. \\ & \quad \times [G(s + t_2 - t_1, x(h_2(s + t_2 - t_1))) - G(s, x(h_2(s)))] ds \\ & \quad \left. + \int_0^{t_2 - t_1} S'(t_2 - s) \lambda R(\lambda, A) G(s, x(h_2(s))) ds \right\|. \end{aligned}$$

Then from conditions  $(H_1)$ ,  $(H_8)$ ,  $(H_9)$  and the complete continuity of  $g$  and let  $\lambda \rightarrow +\infty$ , it yields that

$$\begin{aligned} \|(Px)(t_2) - (Px)(t_1)\| & \leq MM' \|A_0[x_0 + F(0, x(h_1(0))) - g(x)]\| |t_2 - t_1| \\ & \quad + ML[|t_2 - t_1| + l_1 L^* |t_2 - t_1|] \\ & \quad + aMM' L_3[|t_2 - t_1| + l_2 L^* |t_2 - t_1|] \\ & \quad + L_4 MM'(k + 1) |t_2 - t_1| \\ & \leq [C^* + (LL_1 + aM'L_3 l_2) ML^*] |t_2 - t_1|, \end{aligned}$$

where  $A_0$  is the generator of  $C_0$ -semigroup  $\{S'(t)\}_{t \geq 0}$ ,  $C^*$  is a constant independent of  $L^*$  and  $x \in B$ . So it follows from (13) that  $\|(Px)(t_2) - (Px)(t_1)\| \leq L^* |t_2 - t_1|$ ,  $t_2, t_1 \in [0, a]$ , as long as  $L^*$  is large enough. Thus,  $P$  has a fixed point  $x$  which is an integral solution of Eq. (1). Since  $x$  is Lipschitz continuous on  $[0, a]$  and the space  $X$  is reflexive, we see that  $x(\cdot)$  is a.e. differentiable on  $[0, a]$  and that  $x'(\cdot) \in L^1([0, a]; X)$ . From Remark 2 we know that  $x(t) - F(t, x(h_1(t))) \in D(A)$  and so  $x$  satisfies Eq. (1) a.e.. Therefore,  $x(\cdot)$  is also a strong solution of the nonlocal Cauchy problem (1). Thus the proof is completed.  $\square$

### 3.3. Existence of strict solutions

**Definition 3.6.** A function  $x(\cdot) : [0, a] \rightarrow X$  is said to be a strict solution of the nonlocal Cauchy problem (1), if

- (1)  $x(\cdot) - F(\cdot, x(h_1(\cdot))) \in C^1([0, a]; X)$ ;
- (2)  $x$  satisfies

$$\frac{d}{dt}[x(t) - F(t, x(h_1(t)))] = A[x(t) - F(t, x(h_1(t)))] + G(t, x(h_2(t)))$$

on  $[0, a]$ ; and

- (3)  $x(0) + g(x) = x_0$ .

**Theorem 3.7.** Suppose that  $x(\cdot)$  is the unique integral solution of Eq. (1) obtained by Theorem 3.3, and additionally the following conditions hold:

- (H<sub>12</sub>) Functions  $F, G \in C^1([0, a] \times X; X)$ , and the derivative mappings  $D_1F(\cdot, \cdot)$ ,  $D_2F(\cdot, \cdot)$ ,  $D_1G(\cdot, \cdot)$ ,  $D_2G(\cdot, \cdot)$  all satisfy the Lipschitz condition in the second variable.
- (H<sub>13</sub>) Functions  $h_1(\cdot)$  and  $h_2(\cdot)$  are continuous differentiable on  $[0, a]$  with  $|h'_i(t)| \leq 1$  and  $h_i(t) \leq t$  for  $t \in [0, a]$  ( $i = 1, 2$ ), and so  $h_i(0) = 0$ .
- (H<sub>14</sub>)  $x_0 - g(x) - F(0, x(0)) \in D(A_0)$ .

Then  $x(\cdot)$  is also a strict solution of the nonlocal Cauchy problem (1).

**Proof.** For the integral solution  $x(\cdot)$ , we consider the following equation:

$$\begin{aligned} y(t) = & S'(t) \{ A[x_0 - g(x) - F(0, x(0))] + G(0, x(0)) \} \\ & + D_1F(t, x(h_1(t))) + D_2F(t, x(h_1(t)))y(h_1(t))h'_1(t) \\ & + \frac{d}{dt} \int_0^t S(t-s) [D_1G(s, x(h_2(s))) \\ & + D_2G(s, x(h_2(s)))y(h_2(s))h'_2(s)] ds. \end{aligned} \quad (14)$$

We now prove that Eq. (14) has a solution on  $[0, a]$ . Let  $P_1 : E \rightarrow E$  be the operator defined by

$$\begin{aligned} (P_1y)(t) = & S'(t) \{ A[x_0 - g(x) - F(0, x(0))] + G(0, x(0)) \} \\ & + D_1F(t, x(h_1(t))) + D_2F(t, x(h_1(t)))y(h_1(t))h'_1(t) \\ & + \frac{d}{dt} \int_0^t S(t-s) [D_1G(s, x(h_2(s))) \\ & + D_2G(s, x(h_2(s)))y(h_2(s))h'_2(s)] ds, \end{aligned}$$

and let  $y_1, y_2 \in E$ , then for each  $t \in [0, a]$ , we have that

$$\begin{aligned}
& \| (Py_1)(t) - (Py_2)(t) \| \\
& \leq \| D_2 F(t, x(h_1(t))) \| |h'_1(t)| \sup_{0 \leq s \leq a} \| y_1(s) - y_2(s) \| \\
& \quad + aMM' \| D_2 G(t, x(h_2(t))) \| |h'_2(t)| \sup_{0 \leq s \leq a} \| y_1(s) - y_2(s) \| \\
& = \left[ \| D_2 F(t, x(h_1(t))) \| |h'_1(t)| + aMM' \sup_{0 \leq s \leq a} \| D_2 G(s, x(h_2(s))) \| |h'_2(s)| \right] \\
& \quad \times \sup_{0 \leq s \leq a} \| y_1(s) - y_2(s) \|.
\end{aligned}$$

Evidently, (9), (10), (12), and condition  $(H_{13})$  imply that

$$\sup_{0 \leq s \leq a} \| D_2 F(s, x(h_1(s))) \| |h'_1(s)| + aMM' \sup_{0 \leq s \leq a} \| D_2 G(s, x(h_2(s))) \| |h'_2(s)| < 1.$$

So  $P_1$  has a fixed point and Eq. (14) has a solution. Suppose that  $y(t)$  is a solution of Eq. (14). Let

$$z(t) = x(0) + \int_0^t y(s) ds,$$

for  $t \in [0, a]$ . We will prove that  $x(\cdot) = z(\cdot)$  on  $[0, a]$ . In fact, we have that

$$\begin{aligned}
z(t) &= x(0) + \int_0^t S'(s)A[x_0 - g(x) - F(0, x(0))] ds + \int_0^t S'(s)G(0, x(0)) ds \\
&\quad + \int_0^t [D_1 F(s, x(h_1(s))) + D_2 F(s, x(h_1(s)))y(h_1(s))h'_1(s)] ds \\
&\quad + \int_0^t \frac{d}{ds} \int_0^s S(s-\tau) \\
&\quad \times [D_1 G(\tau, x(h_2(\tau))) + D_2 G(\tau, x(h_2(\tau)))y(h_2(\tau))h'_2(\tau)] d\tau ds \\
&= x(0) + S(t)A[x_0 - g(x) - F(0, x(0))] + S(t)G(0, x(0)) \\
&\quad + \int_0^t [D_1 F(s, x(h_1(s))) + D_2 F(s, x(h_1(s)))y(h_1(s))h'_1(s)] ds \\
&\quad + \int_0^t S(t-s)[D_1 G(s, x(h_2(s))) + D_2 G(s, x(h_2(s)))y(h_2(s))h'_2(s)] ds.
\end{aligned}$$

Since

$$\begin{aligned}
S(t)A[x_0 - g(x) - F(0, x(0))] &= S'(t)[x_0 - g(x) - F(0, x(0))] \\
&\quad - [x_0 - g(x) - F(0, x(0))]
\end{aligned}$$

and

$$\begin{aligned} F(t, z(h_1(t))) &= \int_0^t \frac{d}{ds} F(s, z(h_1(s))) ds + F(0, z(0)) \\ &= \int_0^t [D_1 F(s, z(h_1(s))) + D_2 F(s, z(h_1(s))) y(h_1(s)) h_1'(s)] ds \\ &\quad + F(0, z(0)), \end{aligned}$$

we get that (noting that  $x(0) = z(0)$ )

$$\begin{aligned} z(t) - F(t, z(h_1(t))) &= S'(t)[x_0 - g(x) - F(0, x(0))] + S(t)G(0, x(0)) \\ &\quad + \int_0^t [D_1 F(s, x(h_1(s))) - D_1 F(s, z(h_1(s)))] ds \\ &\quad + \int_0^t [D_2 F(s, x(h_1(s))) - D_2 F(s, z(h_1(s)))] y(h_1(s)) h_1'(s) ds \\ &\quad + \int_0^t S(t-s)[D_1 G(s, x(h_2(s))) + D_2 G(s, x(h_2(s))) y(h_2(s)) h_2'(s)] ds. \quad (15) \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{d}{dt} \int_0^t S(t-s)G(s, z(h_2(s))) ds &= \frac{d}{dt} \int_0^t S(s)G(t-s, z(h_2(t-s))) ds \\ &= S(t)G(0, z(0)) \\ &\quad + \int_0^t S(t-s)[D_1 G(s, z(h_2(s))) + D_2 G(s, z(h_2(s))) y(h_2(s)) h_2'(s)] ds. \quad (16) \end{aligned}$$

It follows from (15) and (16) that

$$\begin{aligned} z(t) - x(t) - [F(t, z(h_1(t))) - F(t, x(h_1(t)))] &= \frac{d}{dt} \int_0^t S(t-s)[G(s, z(h_2(s))) - G(s, x(h_2(s)))] ds \end{aligned}$$

$$\begin{aligned}
& + \int_0^t [D_1 F(s, x(h_1(s))) - D_1 F(s, z(h_1(s)))] ds \\
& + \int_0^t [D_2 F(s, x(h_1(s))) - D_2 F(s, z(h_1(s)))] y(h_1(s)) h_1'(s) ds \\
& + \int_0^t S(t-s) [D_1 G(s, x(h_2(s))) - D_1 G(s, z(h_2(s)))] ds \\
& + \int_0^t S(t-s) [D_2 G(s, x(h_2(s))) - D_2 G(s, z(h_2(s)))] y(h_2(s)) h_2'(s) ds.
\end{aligned}$$

Then from conditions  $(H_{12})$  and  $(H_{13})$  we obtain that

$$\sup_{0 \leq s \leq t} \|x(s) - z(s)\| \leq L_3 \sup_{0 \leq s \leq t} \|x(s) - z(s)\| + C \int_0^t \sup_{0 \leq \tau \leq s} \|x(\tau) - z(\tau)\| ds,$$

where  $C$  is a constant. Since (12) implies that  $L_3 < 1$ , from which we get

$$\sup_{0 \leq s \leq t} \|x(s) - z(s)\| \leq \frac{C}{1 - L_3} \int_0^t \sup_{0 \leq \tau \leq s} \|x(\tau) - z(\tau)\| ds.$$

By Gronwall inequality we infer that  $x(\cdot) = z(\cdot)$  on  $[0, a]$ . Therefore we conclude that  $x \in C^1([0, a]; X)$  since  $z$  has obviously this property. In addition, Remark 2 shows that  $x(t) - F(t, x(h_1(t))) \in D(A)$  and satisfies Eq. (1). Hence  $x$  is a strict solution of Eq. (1) and this ends the proof.  $\square$

#### 4. An example

To illustrate the applications of the obtained results of this paper, we study the following example in this section:

$$\begin{aligned}
& \frac{\partial}{\partial t} [z(t, x) - f(t, z(\sin t, x))] = \frac{\partial^2}{\partial x^2} [z(t, x) - f(t, z(\sin t, x))] + g(t, z(\sin t, x)), \\
& 0 \leq t \leq 1, 0 \leq x \leq \pi, \\
& z(t, 0) = z(t, \pi) = 0, \\
& z(0, x) + \sum_{i=0}^p \int_0^\pi c_i(x, y) z(t_i, y) dy = z_0(x), \quad 0 \leq x \leq \pi,
\end{aligned} \tag{17}$$

where  $p \in \mathbf{N}^+$ ,  $0 < t_0 < t_1 < \cdots < t_p < 1$ ,  $z_0(x) \in X = C([0, \pi])$  and  $z_0(0) = z_0(\pi) = 0$ .  $c_i(\cdot, \cdot)$  are continuous functions with  $c_i(0, y) = c_i(\pi, y) = 0$ .

Let  $A$  be the operator defined by

$$Af = -f''$$

with the domain

$$D(A) = \{f(\cdot) \in X: f'' \in X, f(0) = f(\pi) = 0\}.$$

We have  $\overline{D(A)} = \{f(\cdot) \in X: f(0) = f(\pi) = 0\} \neq X$  and

$$\rho(A) = (0, +\infty), \quad \|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}, \quad \text{for } \lambda > 0.$$

This implies that  $A$  satisfies the Hille–Yosida condition on  $X$ .

It is well known that  $A$  generates a compact  $C_0$ -semigroup  $T_0(t)_{t \geq 0}$  on  $\overline{D(A)}$  such that  $\|T_0(t)\| \leq e^{-t}$  for  $t \geq 0$ .

We assume that

( $H_{15}$ ) The functions  $f$  and  $g$  are both continuous and Lipschitz continuous in the second variable, that is, there are  $k_1 > 0$  and  $k_2 > 0$  such that

$$|f(t, x) - f(t, y)| < k_1|x - y|, \quad |g(t, x) - g(t, y)| < k_2|x - y|.$$

Define respectively  $F, G: [0, 1] \times X \rightarrow X$ , and  $g: E \rightarrow \overline{D(A)}$  by

$$F(t, z)(x) = f(t, z(x)), \quad G(t, z)(x) = g(t, z(x)),$$

and

$$g(w(t)) = \sum_{i=0}^p K_i w(t_i), \quad w \in E,$$

where  $K_i(z)(x) = \int_0^\pi c_i(x, y)z(y)dy$ . Let  $h_1(t) = h_2(t) = \sin t$ .

Then from Theorem 3.3 we know that, if

$$c + 2k_1 + \pi k_2 < 1,$$

where  $c = \max\{\|K_i\|, i = 0, 1, \dots, p\}$ , then System (17) admits a unique integral solution.

Furthermore, if  $c_i \in C^2$ ,  $c''_{i_{xx}}(0, y) = c''_{i_{xx}}(\pi, y) = 0$ ,  $z_0(x) - f(0, z(0, x)) \in D(A_0)$  and

( $H_{16}$ ) Functions  $f, g \in C^1$ , and the derivative mappings  $D_1 f(\cdot, \cdot)$ ,  $D_2 f(\cdot, \cdot)$ ,  $D_1 g(\cdot, \cdot)$ ,  $D_2 g(\cdot, \cdot)$  all satisfy the Lipschitz condition in the second variable.

Then by Theorem 3.7 the integral solution is also a strict solution of System (17).

## Acknowledgment

I thank the referees very much for the valuable revision suggestions.

## References

- [1] S. Aizicovici, M. McKibben, Existence results for a class of abstract nonlocal Cauchy problems, *Nonlinear Anal.* 39 (2000) 649–668.
- [2] W. Arendt, Vector valued Laplace transforms and Cauchy problems, *Israel J. Math.* 59 (1987) 327–352.
- [3] W. Arendt, Resolvent positive operators and integrated semigroup, *Proc. London Math. Soc.* 54 (1987) 321–349.
- [4] K. Balachandran, M. Chandrasekaran, Existence of solutions of a delay differential equation with nonlocal condition, *Indian J. Pure Appl. Math.* 27 (1996) 443–449.
- [5] L. Byszewski, Theorems about existence and uniqueness of a solution of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.* 162 (1991) 496–505.
- [6] L. Byszewski, Existence of solutions of semilinear functional-differential evolution nonlocal problem, *Nonlinear Anal.* 34 (1998) 65–72.
- [7] I. Byszewski, V. Lakshmikantham, Theorem about existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Appl. Anal.* 40 (1990) 11–19.
- [8] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, *J. Math. Anal. Appl.* 179 (1993) 630–637.
- [9] J.P. Dauer, K. Balachandran, Existence of solutions of nonlinear neutral integrodifferential equations in Banach space, *J. Math. Anal. Appl.* 251 (2000) 93–105.
- [10] K. Ezzinbi, J.H. Liu, Nondensely defined evolution equations with nonlocal conditions, *Math. Comput. Modelling* 36 (2002) 1027–1038.
- [11] X. Fu, K. Ezzinbi, Existence of solutions for neutral functional evolution equations with nonlocal conditions, *Nonlinear Anal.* 54 (2003) 215–227.
- [12] H. Kellermann, M. Hieber, Integrated semigroup, *J. Funct. Anal.* 84 (1989) 160–180.
- [13] Y. Lin, H. Liu, Semilinear integrodifferential equations with nonlocal Cauchy problem, *Nonlinear Anal.* 26 (1996) 1023–1033.
- [14] S.K. Ntouyas, P.Ch. Tsamatos, Global existence for semilinear evolution equations with nonlocal conditions, *J. Math. Anal. Appl.* 210 (1997) 679–687.
- [15] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.
- [16] G. Da Prato, E. Sinestrari, Differential operators with non-dense domain, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 14 (1987) 285–344.
- [17] H. Thieme, Integrated semigroup and integral solutions to abstract Cauchy problems, *J. Math. Anal. Appl.* 152 (1990) 416–447.