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Convergence of derivatives for certain mixed Szasz–Beta operators

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Abstract

In this paper we study the mixed summation–integral type operators having Szasz and Beta basis functions in summation and integration, respectively, we obtain the rate of point-wise convergence, a Voronovskaja type asymptotic formula and an error estimate in simultaneous approximation.

Keywords: Linear positive operators; Summation–integral type operators; Rate of convergence; Asymptotic formula; Error estimate; Simultaneous approximation

1. Introduction

Recently Srivastava and Gupta [6] proposed a general family of summation–integral type operators which include some well-known operators (see, e.g., [3,5]) as special cases. Ispir and Yuksel [4] considered the Bezier variant of the operators studied in [6] and estimated the rate of convergence for bounded variation functions. Several other hybrid

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summation–integral type operators were proposed by Gupta and Gupta [2]. Very recently Finta [1] proposed yet another sequence of linear positive operators. For $f \in C_\gamma[0,\infty) = \{f \in C[0,\infty)\colon |f(t)| \leqslant M(1+t)^\gamma$, for some $M > 0$, $\gamma > 0\}$, the other mixed sequence of summation–integral type operators is defined as

$$S_n(f,x) = \int_0^\infty W_n(x,t) f(t)\,dt = \sum_{v=1}^\infty s_{n,v}(x) \int_0^\infty f(t) b_{n,v}(t)\,dt + s_{n,0}(x) f(0), \qquad (1)$$

where $W_n(x,t) = \sum_{v=1}^\infty s_{n,v}(x) b_{n,v}(t) + s_{n,0}(x)\delta(t)$, $\delta(t)$ being Dirac delta-function and

$$s_{n,v}(x) = e^{-nx}\frac{(nx)^v}{v!}, \qquad b_{n,v}(t) = \frac{1}{B(n+1,v)} \frac{t^{v-1}}{(1+t)^{n+v+1}},$$

are respectively Szasz and Beta basis functions. It is easily verified that the operators (1) are linear positive operators. The behaviour of these operators are very similar to the operators studied by Gupta and Gupta [3], but the approximation properties of the operators S_n are different in comparison to the operators studied in [3]. The main difference is that the operators S_n reproduce not only the constant ones but linear functions also. In the present paper we study some direct results for the operators S_n, we obtain a point-wise rate of convergence, asymptotic formula of Voronovskaja type and an error estimate in simultaneous approximation.

2. Auxiliary results

We need the following lemmas in the sequel.

Lemma 1. *For $m \in N^0 := (0,1,2,3,\ldots)$, if the mth order moment be defined as*

$$U_{n,m}(x) = \sum_{v=0}^\infty s_{n,v}(x)\big(v.n^{-1} - x\big)^m,$$

then $U_{n,0}(x) = 1, U_{n,1}(x) = 0$ and $nU_{n,m+1}(x) = x[U_{n,m}^{(1)}(x) + mU_{n,m-1}(x)]$. Consequently

$$U_{n,m}(x) = O\big(n^{-[(m+1)/2]}\big).$$

Lemma 2. *Let the function $\mu_{n,m}(x), m \in N^0$, be defined as*

$$\mu_{n,m}(x) = \sum_{v=1}^\infty s_{n,v}(x) \int_0^\infty (t-x)^m b_{n,v}(t)\,dt + s_{n,0}(x)(-x)^m.$$

Then

$$\mu_{n,0}(x) = 1, \qquad \mu_{n,1}(x) = 0, \qquad \mu_{n,2}(x) = \frac{x(2+x)}{n-1},$$

also we have the recurrence relation:

$$(n-m)\mu_{n,m+1}(x) = x\big[\mu_{n,m}^{(1)}(x) + m(2+x)\mu_{n,m-1}(x)\big] + m(1+2x)\mu_{n,m}(x).$$

Consequently for each $x \in [0,\infty)$, *we have from this recurrence relation that* $\mu_{n,m}(x) = O(n^{-[(m+1)/2]})$.

Remark 1. From Lemma 2, we can easily obtain the following identity:

$$S_n\big(t^i, x\big) = \frac{n^{i-1}.\,(n-i)!}{(n-1)!}x^i + i(i-1)\frac{n^{i-2}.\,(n-i)!}{(n-1)!}x^{i-1} + O\big(n^{-2}\big).$$

Lemma 3. *There exist the polynomials* $Q_{i,j,r}(x)$ *independent of* n *and* v *such that*

$$x^r D^r\big[s_{n,v}(x)\big] = \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i[v-nx]^j Q_{i,j,r}(x)s_{n,v}(x),$$

where $D = \frac{d}{dx}$.

3. Simultaneous approximation

Theorem 1. *Let* $f \in C_\gamma[0,\infty)$, $\gamma > 0$ *and* $f^{(r)}$ *exists at a point* $x \in (0,\infty)$, *then*

$$\lim_{n\to\infty} S_n^{(r)}\big(f(t), x\big) = f^{(r)}(x). \tag{2}$$

Proof. By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!}(t-x)^i + \epsilon(t,x)(t-x)^r,$$

where $\epsilon(t,x) \to 0$ as $t \to x$. Hence

$$\begin{aligned} S_n^{(r)}\big(f(t), x\big) &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \int_0^\infty W_n^{(r)}(t,x)(t-x)^i\,dt \\ &\quad + \int_0^\infty W_n^{(r)}(t,x)\epsilon(t,x)(t-x)^r\,dt \\ &= E_1 + E_2, \quad \text{say}. \end{aligned}$$

First, to estimate E_1, using binomial expansion of $(t-x)^r$, and Lemma 2, we have

$$\begin{aligned} E_1 &= \sum_{i=0}^{r} \frac{f^{(i)}(x)}{i!} \sum_{v=0}^{i} \binom{i}{v}(-x)^{i-v} \int_0^\infty W_n^{(r)}(t,x)t^v\,dt \\ &= \frac{f^{(r)}(x)}{r!} \int_0^\infty W_n^{(r)}(t,x)t^r\,dt = f^{(r)}(x) + o(1), \quad n\to\infty. \end{aligned}$$

Next, using Lemma 3, we obtain

$$
\begin{aligned}
|E_2| &\leqslant \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \frac{|Q_{i,j,r}(x)|}{x^r} \sum_{v=1}^{\infty} |v-nx|^j s_{n,v}(x) \\
&\quad \times \int_0^{\infty} b_{n,v}(t)\big|\epsilon(t,x)\big|(t-x)^r\,dt + (-n)^r e^{-nx}\big|\epsilon(0,x)\big|(-x)^r \\
&= E_3 + E_4.
\end{aligned}
$$

Since $\epsilon(t,x) \to 0$ as $t \to x$ for a given $\epsilon > 0$ there exists a $\delta > 0$ such that $|\epsilon(t,x)| < \epsilon$ whenever $0 < |t-x| < \delta$. Further if $s \geqslant \max\{\gamma, r\}$, where s is any integer, then we can find a constant M_1 such that $|\epsilon(t,x)(t-x)^r| \leqslant M_1|t-x|^s$, for $|t-x| \geqslant \delta$. Thus with $M_2 = \sup_{2i+j\leqslant r} x^{-r}|Q_{i,j,r}(x)|$, we have

$$
\begin{aligned}
E_3 &\leqslant M_2 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \sum_{v=1}^{\infty} s_{n,v}(x)|v-nx|^j \\
&\quad \times \left\{ \epsilon \int_{|t-x|<\delta} b_{n,v}(t)|t-x|^r + \int_{|t-x|\geqslant\delta} b_{n,v}(t) M_1 |t-x|^s\,dt \right\} \\
&= E_5 + E_6.
\end{aligned}
$$

Applying Schwarz inequality for integration and summation, respectively, and using Lemmas 1 and 2, we obtain

$$
\begin{aligned}
E_5 &\leqslant \epsilon M_2 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \sum_{v=1}^{\infty} s_{n,v}(x)|v-nx|^j \left\{ \int_0^{\infty} b_{n,v}(t)\,dt \right\}^{1/2} \\
&\quad \times \left\{ \int_0^{\infty} b_{n,v}(t)(t-x)^{2r}\,dt \right\}^{1/2} \\
&\leqslant \epsilon \cdot M_2 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i O\big(n^{j/2}\big) O\big(n^{-r/2}\big) = \epsilon \cdot O(1).
\end{aligned}
$$

Again using Schwarz inequality, Lemmas 1 and 2, we get

$$
\begin{aligned}
E_6 &\leqslant M_3 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \sum_{v=1}^{\infty} s_{n,v}(x)|v-nx|^j \int_{|t-x|\geqslant\delta} b_{n,v}(t)|t-x|^s\,dt \\
&\leqslant M_3 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i \left(\sum_{v=1}^{\infty} s_{n,v}(x)(v-nx)^{2j} \right)^{1/2}
\end{aligned}
$$

$$\times\left(\sum_{v=1}^{\infty}s_{n,v}(x)\int_0^{\infty}b_{n,v}(t)(t-x)^{2s}\,dt\right)^{1/2}$$
$$=\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^iO\big(n^{j/2}\big)O\big(n^{-s/2}\big)=O\big(n^{(r-s)/2}\big)=o(1).$$

Thus due to arbitrariness of $\epsilon>0$ it follows that $E_3=o(1)$. Also $E_4\to 0$ as $n\to\infty$ and hence $E_2=o(1)$. Collecting the estimates of E_1 and E_2, we get the required result. □

Theorem 2. *Let $f\in C_\gamma[0,\infty)$, $\gamma>0$ and $f^{(r+2)}$ exists at a point $x\in(0,\infty)$, then*

$$\lim_{n\to\infty}n\big[S_n^{(r)}(f,x)-f^{(r)}(x)\big]$$
$$=\frac{r(r-1)}{2}f^{(r)}(x)+(x+1)rf^{(r+1)}(x)+\big(x^2+x\big)f^{(r+2)}(x).$$

Proof. Using Taylor's expansion of f, we have

$$f(t)=\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}(t-x)^i+\epsilon(t,x)(t-x)^{r+2},$$

where $\epsilon(t,x)\to 0$ as $t\to x$. Applying Lemma 2, we have

$$n\big[S_n^{(r)}(f,x)-f^{(r)}(x)\big]=n\left[\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\int_0^{\infty}W_n^{(r)}(t,x)(t-x)^i\,dt-f^{(r)}(x)\right]$$
$$+n\int_0^{\infty}W_n^{(r)}(t,x)\epsilon(t,x)(t-x)^{r+2}\,dt$$
$$=J_1+J_2,$$

$$J_1=n\sum_{i=0}^{r+2}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\int_0^{\infty}W_n^{(r)}(t,x)t^j\,dt-nf^{(r)}(x)$$
$$=\frac{f^{(r)}(x)}{r!}n\big[B_n^{(r)}\big(t^r,x\big)-(r!)\big]$$
$$+\frac{f^{(r+1)}(x)}{(r+1)!}n\big[(r+1)(-x)B_n^{(r)}\big(t^r,x\big)+B_n^{(r)}\big(t^{r+1},x\big)\big]$$
$$+\frac{f^{(r+2)}(x)}{(r+2)!}n\bigg[\frac{(r+1)(r+2)}{2}x^2B_n^{(r)}\big(t^r,x\big)$$
$$+(r+2)(-x)B_n^{(r)}\big(t^{r+1},x\big)+B_n^{(r+2)}\big(t^{r+2},x\big)\bigg].$$

Using Remark 1 for each $x\in(0,\infty)$, we have

$$\begin{aligned}
J_1 = {} & nf^{(r)}(x)\left[\frac{n^{r-1}(n-r)!}{(n-1)!} - 1\right] + n\frac{f^{(r+1)}(x)}{(r+1)!}\left[(r+1)(-x)(r!)\left\{\frac{n^{r-1}(n-r)!}{(n-1)!}\right\}\right. \\
& \left. + \left\{\frac{n^r(n-r-1)!}{(n-1)!}(r+1)!x + r(r+1)\frac{n^{r-1}(n-r-1)!}{(n-1)!}(r!)\right\}\right] \\
& + n\frac{f^{(r+2)}(x)}{(r+2)!}\left[\frac{(r+2)(r+1)x^2}{2}(r!)\frac{n^r(n-r)!}{(n-1)!}\right. \\
& + (r+2)(-x)\left\{\frac{n^r(n-r-1)!}{(n-1)!}(r+1)!x + r(r+1)\frac{n^{r-1}(n-r-1)!}{(n-1)!}(r!)\right\} \\
& + \left\{\frac{n^{r+1}(n-r-2)!}{(n-1)!}\frac{(r+2)!}{2}x^2 + (r+1)(r+2)\frac{n^r(n-r-2)!}{(n-1)!}(r+1)!x\right\} \\
& \left. + O\big(n^{-2}\big)\right].
\end{aligned}$$

In order to complete the proof of the theorem it is sufficient to show that $J_2 \to 0$ as $n \to \infty$, which can easily be proved along the lines of the proof of Theorem 1 and by using Lemmas 1–3. $\square$

Remark 2. In particular if $r = 0$, we obtain the following conclusion of the above asymptotic formula in ordinary approximation:

$$\lim_{n\to\infty} n\big[S_n(f,x) - f(x)\big] = \big(x^2 + x\big)f^{(2)}(x).$$

Theorem 3. *Let $f \in C_\gamma[0,\infty)$ and $r \leqslant m \leqslant (r+2)$. If $f^{(m)}$ exists and is continuous on $(a-\eta, b+\eta)$, then for n sufficiently large*

$$\big\|S_n^{(r)}(f,x) - f^{(r)}\big\| \leqslant M_4 n^{-1}\sum_{i=r}^{m}\big\|f^{(i)}\big\| + M_5 n^{-1/2}\omega\big(f^{(r+1)}, n^{-1/2}\big) + O\big(n^{-2}\big),$$

where the constants M_4 and M_5 are independent of f and n, $\omega(f,\delta)$ is the modulus of continuity of f on $(a-\eta, b+\eta)$ and $\|\cdot\|$ denotes the sup-norm on the interval $[a,b]$.

Proof. By Taylor's expansion of f, we have

$$f(t) = \sum_{i=0}^{m}(t-x)^i\frac{f^{(i)}(x)}{i!} + (t-x)^m\zeta(t)\frac{f^{(m)}(\xi) - f^{(m)}(x)}{m!} + h(t,x)\big(1-\zeta(t)\big),$$

where ξ lies between t and x and $\zeta(t)$ is the characteristic function on the interval $(a-\eta, b+\eta)$. For $t \in (a-\eta, b+\eta)$, $x \in [a,b]$, we have

$$f(t) = \sum_{i=0}^{m}(t-x)^i\frac{f^{(i)}(x)}{i!} + (t-x)^i\frac{f^{(m)}(\xi) - f^{(m)}(x)}{m!}.$$

For $t \in [0,\infty) \setminus (a-\eta, b+\eta)$, we define

$$h(t,x) = f(t) - \sum_{i=0}^{m}(t-x)^i\frac{f^{(i)}(x)}{i!}.$$

Thus

$$
\begin{aligned}
S_n^{(r)}(f,x)-f^{(r)}(x) &= \left\{\sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}\int_0^\infty W_n^{(r)}(t,x)(t-x)^i\,dt - f^{(r)}(x)\right\}\\
&\quad + \left\{\int_0^\infty W_n^{(r)}(t,x)\frac{f^{(m)}(\xi)-f^{(m)}(x)}{m!}(t-x)^m\zeta(t)\,dt\right\}\\
&\quad + \left\{\int_0^\infty W_n^{(r)}(t,x)h(t,x)\bigl(1-\zeta(t)\bigr)\,dt\right\}\\
&= K_1+K_2+K_3.
\end{aligned}
$$

Using Remark 1, we obtain

$$
\begin{aligned}
K_1 &= \sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\int_0^\infty W_n^{(r)}(t,x)t^j\,dt - f^{(r)}(x)\\
&= \sum_{i=0}^{m}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}\frac{\partial^r}{\partial x^r}\\
&\quad\times\left[\frac{n^{j-1}(n-j)!}{(n-1)!}x^j + j(j-1)\frac{n^{j-2}(n-j)!}{(n-1)!}x^{j-1}+O\bigl(n^{-2}\bigr)\right] - f^{(r)}(x).
\end{aligned}
$$

Hence

$$
\|K_1\| \leqslant M_4 n^{-1}\sum_{i=r}^{m}\bigl\|f^{(i)}\bigr\| + O\bigl(n^{-2}\bigr),
$$

uniformly in $x\in[a,b]$. Next

$$
\begin{aligned}
|K_2| &\leqslant \int_0^\infty W_n^{(r)}(t,x)\frac{|f^{(m)}(\xi)-f^{(m)}(x)|}{m!}|t-x|^m\zeta(t)\,dt\\
&\leqslant \frac{\omega(f^{(m)},\delta)}{m!}\int_0^\infty \bigl|W_n^{(r)}(t,x)\bigr|\left(1+\frac{|t-x|}{\delta}\right)|t-x|^m\,dt.
\end{aligned}
$$

Next, we shall show that for $q=0,1,2,\ldots$

$$
\sum_{v=1}^{\infty} s_{n,v}(x)|v-nx|^j\int_0^\infty b_{n,v}(t)|t-x|^q\,dt = O\bigl(n^{(j-q)/2}\bigr).
$$

Now by using Lemmas 1 and 2, we have

$$\begin{aligned}
&\sum_{v=1}^{\infty} s_{n,v}(x)|v-nx|^{j} \int_{0}^{\infty} b_{n,v}(t)|t-x|^{q}\,dt \\
&\quad \leqslant \left(\sum_{v=1}^{\infty} s_{n,v}(x)(v-nx)^{2j}\right)^{1/2} \left(\sum_{v=1}^{\infty} p_{n,v}(x) \int_{0}^{\infty} b_{n,v}(t)(t-x)^{2q}\,dt\right)^{1/2} \\
&\quad = O\big(n^{j/2}\big) O\big(n^{-q/2}\big) = O\big(n^{(j-q)/2}\big),
\end{aligned}$$

uniformly in x. Thus by Lemma 3, we obtain

$$\begin{aligned}
&\sum_{v=1}^{\infty} \big|s_{n,v}^{(r)}(x)\big| \int_{0}^{\infty} b_{n,v}(t)|t-x|^{q}\,dt \\
&\quad \leqslant M_6 \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^{i} \left[\sum_{v=1}^{\infty} s_{n,v}(x)|v-nx|^{j} \int_{0}^{\infty} b_{n,v}(t)|t-x|^{q}\,dt\right] = O\big(n^{(r-q)/2}\big),
\end{aligned}$$

uniformly in x, where $M_6 = \sup_{2i+j\leqslant r,\, i,j\geqslant 0} \sup_{x\in[a,b]} |Q_{i,j,r}(x)|x^{-r}$. Choosing $\delta = n^{-1/2}$, we get for any $s > 0$

$$\begin{aligned}
\|K_2\| &\leqslant \frac{\omega(f^{(m)}, n^{-1/2})}{m!} \big[O\big(n^{(r-m)/2}\big) + n^{1/2} O\big(n^{(r-m-1)/2}\big) + O\big(n^{-s}\big)\big] \\
&\leqslant M_5 \omega\big(f^{(m)}, n^{-1/2}\big) n^{-(m-r)/2}.
\end{aligned}$$

Since $t \in [0,\infty) \setminus (a-\eta, b+\eta)$, we can choose a $\delta > 0$ in such a way that $|t-x| \geqslant \delta$ for all $x \in [a,b]$. Applying Lemma 3, we obtain

$$\begin{aligned}
\|K_3\| &\leqslant \sum_{v=1}^{\infty} \sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^{i} \frac{|Q_{i,j,r}(x)|}{x^{r}} |v-nx|^{j} s_{n,v}(x) \int_{|t-x|\geqslant\delta} b_{n,v}(t)\big|h(t,x)\big|\,dt \\
&\quad + (-n)^{r} e^{-nx} \big|h(0,x)\big| \\
&= E_3 + E_4.
\end{aligned}$$

If β is any integer greater than equal to $\{\gamma, m\}$, then we can find a constant M_7 such that $|h(t,x)| \leqslant M_7|t-x|^{\beta}$ for $|t-x| \geqslant \delta$. Now applying Lemmas 1 and 2, it is easily verified that $K_3 = O(n^{-q})$ for any $q > 0$ uniformly on $[a,b]$. Combining the estimates of K_1, K_2 and K_3, we get the required result. □

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