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Domain dependent Dirac's delta and derivatives with an application to electromagnetic boundary integral representations

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Abstract

The domain dependent versions of derivatives and Dirac's delta are defined in distributional sense. These operations enable to obtain domain dependent fundamental solutions and global boundary integral representation formulae. A global representation formula is defined everywhere, also on the boundary, and includes the jump relations of the boundary. The use of the domain dependent objects can be interpreted as taking the boundary limit in prior to integrating by parts when deriving the familiar boundary integral equations. As an application, the representation formulae are obtained for the solutions of the Helmholtz equation and the Maxwell equations.

Keywords: Dirac's delta; Fundamental solution; Boundary integral representation; Helmholtz equation; Maxwell equations

1. Introduction

Dirac's delta δ and the integration by parts form the basis in obtaining the boundary integral equations for the solutions of boundary value problems of partial differential equations. Traditionally, the boundary integral equations are obtained so that, first, one derives the representation formula at an observation point x that is strictly in or out of the domain, and then lets the point x tend to the boundary [1,2]. The boundary condition then implies the boundary integral equation. One can also use other boundary integral representations, or ansatz [2], for the solutions. In all cases, the behaviors of the corresponding boundary integral operators have to be studied carefully on the boundary. The operators can be continuous or have jump terms. In this article, we show that the boundary behavior analysis can be done before having any boundary integral operators by defining distributional derivatives and Dirac's delta in a domain dependent way. Then the jump terms follow by integrating by parts the domain dependent derivatives. In this way, one can obtain a global representation formula that is valid also on the boundary.

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What is the fundamental reason for not taking the observation point directly onto the boundary? To understand this, we recall how the representation formula is obtained in the Helmholtz operator case in 3D. Let k be the wave number, $k \geqslant 0$. Consider the Helmholtz operator $-(\Delta + k^2)$ that has a fundamental solution

$$\Phi_x(y) = \frac{1}{4\pi} \frac{e^{ik|x-y|}}{|x-y|}, \tag{1}$$

for which [3]

$$-(\Delta + k^2)\Phi_x = \delta_x. \tag{2}$$

Here δ_x is Dirac's delta at point x. Let $D \subset \mathbb{R}^3$ be an open bounded set with a smooth boundary ∂D and let $x \in \mathbb{R}^3 \setminus \partial D$. The representation formula is obtained by testing (2) with $u\chi_D$, where u is a solution of the homogeneous Helmholtz equation in the domain D and

$$\chi_D(x) = \begin{cases} 1, & x \in D, \\ 0, & x \notin D, \end{cases}$$

is the characteristic function of D. By integrating by parts,

$$\left.\begin{array}{rr} x \in D, & u(x) \\ x \in \mathbb{R}^3 \setminus \overline{D}, & 0 \end{array}\right\} = \text{“}\int_D \delta_x u\, dy\text{”} = \text{“}-\int_D (\Delta + k^2)\Phi_x u\, dy\text{”} = \int_{\partial D} \Phi_x \frac{\partial u}{\partial n}\, dS - \int_{\partial D} \frac{\partial \Phi_x}{\partial n} u\, dS. \tag{3}$$

The integrals marked with the quote (“ ”) symbols are formal, because the integrands are not functions. However, for $x \notin \partial D$, the integral against δ_x can be defined rigorously as a distributional testing

$$\langle \delta_x, \chi_D u\rangle := \langle \delta_x, \psi \chi_D u\rangle + \langle \chi_D u, (1-\psi)\delta_x\rangle, \tag{4}$$

where ψ is a smooth cut off function for which $\psi \equiv 1$ in a neighborhood of x and $\psi \equiv 0$ in a neighborhood of ∂D. Now $\psi\chi_D u$ and $(1-\psi)\delta_x$ are smooth functions so they can be used as test functions. The definition (4) is not possible when $x \in \partial D$ in which case the singular supports of δ_x and χ_D get together.

To be able to do the calculation similar to (3) for $x \in \partial D$ also, we define the domain dependent Dirac's delta δ^D and the domain dependent derivatives ∂_j^D. These can be applied to deduce a global representation formula for which the observation point is allowed to be also on the boundary. We derive the global representation formulae for the solutions of the Helmholtz equation and for the solutions of the extended Maxwell system.

In Section 2 we introduce the domain dependent operators. In Section 3, we apply these to the Helmholtz equation and show that the usual fundamental solution is also the domain dependent fundamental solution. Then the global representation formulae follow immediately. In Section 4, the same is done for the Maxwell equations. The extended Maxwell system [4–6] serves us as the bridge between the Helmholtz side and the Maxwell side. We apply the Helmholtz case results to the extended Maxwell system and then restrict to the Maxwell case. Note that working with the extended Maxwell system allows us to consider also the zero frequency, not only the wave numbers $k > 0$.

2. Domain-dependent δ^D and ∂_j^D

Let $D \subset \mathbb{R}^3$ be an open, not necessary bounded, set with a C^2-boundary. Let $n = n(x)$ be the unit outer normal of the boundary at point $x \in \partial D$.

The relative solid angle of domain D at point x is

$$\Omega_D(x) = \frac{1}{4\pi} \int_{\partial D_r(x)} \frac{n(y)\cdot(y-x)}{|y-x|^3}\, dS(y), \tag{5}$$

where

$$D_r(x) = D \cap B(x, r) \tag{6}$$

and $B(x,r)$ is the x centered ball with an arbitrary radius $r>0$. It is easy to see that the definition does not depend on r. Because ∂D is C^2, it holds [2]

$$\Omega_D(x)=\begin{cases}1, & x\in D,\\ 1/2, & x\in\partial D,\\ 0, & x\in\mathbb{R}^3\setminus\overline{D}.\end{cases} \tag{7}$$

Similar to the usual Dirac's δ_x, the domain dependent δ_x^D picks the value of the test function at x, but weights the value with the relative solid angle.

Definition 2.1. The domain dependent delta δ_x^D at point $x\in\mathbb{R}^3$ is

$$\langle\delta_x^D,\phi\rangle:=\Omega_D(x)\phi(x),\quad \phi\in C_0^\infty(\mathbb{R}^3). \tag{8}$$

We denote by $L^1_\partial(D)$ the space of functions $u\in L^1_{\mathrm{loc}}(D)$ for which the boundary value $u|^D_{\partial D}\in L^1_{\mathrm{loc}}(\partial D)$ exists (for some L^1 representative) in the sense

$$\lim_{h\to 0}\big\|u|^D_{\partial D}-u(\cdot-hn)\big\|_{L^1_{\mathrm{loc}}(\partial D)}=0. \tag{9}$$

Definition 2.2. Suppose that $u\in L^1_\partial(D)$ with $f=u|^D_{\partial D}$. The domain dependent derivative of function u is

$$\langle\partial_j^D u,\phi\rangle:=\int_{\partial D} n_j f\phi\,dS-\int_D u\partial_j\phi\,dx,\quad \phi\in C_0^\infty(\mathbb{R}^3). \tag{10}$$

The space $L^1_\partial(D)$ was chosen so that terms on the right-hand side of (10) are integrals. Note that if $u_i\to u$ in norm

$$\|u\|_{L^1_{\mathrm{loc}}(D)}+\big\|u|^D_{\partial D}\big\|_{L^1_{\mathrm{loc}}(\partial D)},$$

then

$$\langle\partial_j^D u_i,\phi\rangle\to\langle\partial_j^D u,\phi\rangle$$

for every ϕ.

Both definitions (8) and (10) are valid with less smooth test functions, too. For δ_x^D, it is sufficient that ϕ is continuous at x, and for the domain dependent derivative ∂^D it is enough that

$$\phi\in C_0(\overline{D})\cap C^1(D). \tag{11}$$

For smooth functions, the domain dependent derivative is the restriction of the ordinary derivative to the domain. Namely, if $u\in C^1(\mathbb{R}^3)$, then

$$\langle\partial_j^D u,\phi\rangle=\int_{\partial D} n_j u\phi\,dS-\int_D u\partial_j\phi\,dx=\int_D\partial_j u\phi\,dx=\langle\chi_D\partial_j u,\phi\rangle$$

and so

$$\partial_j^D u=\chi_D\partial_j u.$$

If $\mathrm{supp}(u)\subset D$, then $\partial_j^D u$ is the distribution derivative.

Consider the characteristic function χ_D of D to see how the distribution derivative and the domain dependent derivative differ from each other if the function has discontinuity on the boundary. The distribution derivative for χ_D is [3]

$$\partial_j\chi_D=-n_j\,dS_{\partial D}.$$

However,

$$\langle\partial_j^D\chi_D,\phi\rangle=\int_{\partial D} n_j\phi\,dS-\int_D\partial_j\phi\,dx=0$$

and so

$$\partial_j^D \chi_D = 0.$$

Next, we show that the usual double derivative rules hold when the domain dependent derivative is acting as the second derivative. Let

$$\langle f, g\rangle = \sum_j \langle f_j, g_j\rangle \tag{12}$$

for vectors $f = (f_j)$ and $g = (g_j)$. Define

$$\nabla^D = \begin{pmatrix} \partial_1^D \\ \partial_2^D \\ \partial_3^D \end{pmatrix}$$

and

$$\Delta^D = \nabla^D \cdot \nabla. \tag{13}$$

Note that for the vector valued first order linear constant coefficient differential operator

$$P(\nabla) = \sum_{j=1}^{3} P^j \partial_j,$$

the formula (10) takes form

$$\big\langle P\big(\nabla^D\big)U, \phi\big\rangle = \int_{\partial D} \big(P(n)U\big)^T \phi\, dS - \int_D U^T P(\nabla)^T \phi\, dy, \tag{14}$$

where ϕ is a test vector field. In (14), it is sufficient that the components of $P(n)U|_{\partial D}^D$ are in $L^1_{\mathrm{loc}}(\partial D)$.

Lemma 2.3.

$$\nabla^D \times (\nabla\times) = \nabla^D \cdot \nabla - \Delta^D, \tag{15}$$

$$\nabla^D \cdot \nabla\times = 0, \tag{16}$$

$$\nabla^D \times \nabla = 0. \tag{17}$$

Proof. Let u be a vector field with $\partial_j u_i \in L^1_\partial(D)$, $i, j = 1, 2, 3$. Let ϕ be a test field, $\phi \in C_0^\infty(\mathbb{R}^3)^3$. The idea is to move the domain dependent derivatives to the test function and use the corresponding properties of usual derivatives.

For (15), compute

$$\begin{aligned}
&\big\langle \nabla^D \times (\nabla \times u) - \nabla^D \nabla \cdot u + \Delta^D u, \phi\big\rangle \\
&\quad = \int_{\partial D} \big[n \times (\nabla \times u) \cdot \phi - (\nabla \cdot u)(n \cdot \phi) + (n \cdot \nabla)u \cdot \phi\big]\, dS \\
&\qquad + \int_{\partial D} \big[(n \times u) \cdot (\nabla \times \phi) + (n \cdot u)\nabla \cdot \phi - u \cdot (n \cdot \nabla)\phi\big]\, dS + \int_D u \cdot \big[(\nabla\times)^2\phi - \nabla\nabla \cdot \phi + \Delta\phi\big]\, dx \\
&\quad = \int_{\partial D} \mathrm{Div}\big(n \times (\phi \times u)\big)\, dS + \int_{\partial D} \big[u \cdot (\phi \cdot \nabla)n - \phi \cdot (u \cdot \nabla)n\big]\, dS.
\end{aligned}$$

The first term is

$$\int_{\partial D} \mathrm{Div}\big(n \times (\phi \times u)\big)\, dS = -\int_{\partial D} \big(n \times (\phi \times u)\big) \cdot \nabla 1\, dS = 0.$$

The second term vanishes because $n = \nabla\psi$ for some function ψ and so

$$\phi \cdot (u \cdot \nabla)n = \sum_{i,j} \phi_j u_i \partial_j \partial_i \psi = u \cdot (\phi \cdot \nabla)n.$$

This proves (15).

Eq. (16) follows from

$$\begin{aligned}\langle \nabla^D \cdot \nabla \times u, \phi \rangle &= \int_{\partial D} n \cdot (\nabla \times u)\phi \, dS - \int_D (\nabla \times u) \cdot \nabla\phi \, dS \\ &= -\int_{\partial D} \mathrm{Div}(n \times u)\phi \, dS - \int_{\partial D} (n \times u) \cdot \nabla\phi \, dS - \int_D u \cdot \nabla \times \nabla\phi \, dx \\ &= 0,\end{aligned}$$

because

$$\int_{\partial D} \mathrm{Div}(n \times u)\phi \, dS = -\int_{\partial D} (n \times u) \cdot \nabla\phi \, dS,$$

and $\nabla \times \nabla\phi = 0$.

Similarly, for (17),

$$\begin{aligned}\langle \nabla^D \times \nabla\psi, \phi \rangle &= \int_{\partial D} (n \times \nabla\psi) \cdot \phi \, dS + \int_D \nabla\psi \cdot (\nabla \times \phi) \, dx \\ &= -\int_{\partial D} \nabla\psi \cdot (n \times \phi) \, dS + \int_{\partial D} \psi n \cdot (\nabla \times \phi) \, dS - \int_D \psi \nabla \cdot (\nabla \times \phi) \, dx \\ &= \int_{\partial D} \psi \, \mathrm{Div}(n \times \phi) \, dS - \int_D \psi \, \mathrm{Div}(n \times \phi) \, dS \\ &= 0. \qquad \square\end{aligned}$$

3. Helmholtz equation

In this section, we apply the domain dependent objects to derive the representation formulae for the solutions of the Helmholtz equation

$$\Delta u + k^2 u = 0. \tag{18}$$

We show that function Φ_x (1) is the domain dependent fundamental solution for the domain dependent Helmholtz operator, by which we mean that

$$-(\Delta^D + k^2 \chi_D)\Phi_x = \delta_x^D, \quad x \in \mathbb{R}^3. \tag{19}$$

After we have (19), the global representation formula follows from the definitions of Δ^D and δ_x^D.

Theorem 3.1. *Function Φ_x (1) is the domain dependent fundamental solution of the domain dependent Helmholtz operator*

$$-(\Delta^D + k^2 \chi_D), \quad k \geqslant 0.$$

Proof. Let $x \in \mathbb{R}^3$. Now each component of $\nabla\Phi_x$ is in $L^1_\partial(D)$. Denote

$$D_r = D_r(x),$$

see (6). For a test function ϕ,

$$\langle -(\Delta^D + k^2\chi_D)\Phi_x, \phi\rangle = -\int_{\partial D} n\cdot\nabla\Phi_x\phi\, dS + \int_D \nabla\Phi_x\cdot\nabla\phi\, dy - k^2\int_D \Phi_x\phi\, dy \tag{20}$$

$$= I_{D\setminus D_r} + I_{D_r} + J_{D_r}, \tag{21}$$

where

$$I_{D\setminus D_r} = -\int_{\partial(D\setminus D_r)} n\cdot\nabla\Phi_x\phi\, dS + \int_{D\setminus D_r} \left(\nabla\Phi_x\cdot\nabla\phi - k^2\Phi_x\phi\right) dy,$$

$$I_{D_r} = -\int_{\partial D_r} n\cdot\nabla\Phi_x\phi\, dS, \qquad J_{D_r} = \int_{D_r}\left(\nabla\Phi_x\cdot\nabla\phi - k^2\Phi_x\phi\right) dy.$$

We take the limit $r \to 0$. In $D\setminus D_r$, Φ_x is a smooth function with

$$\left(\Delta + k^2\right)\Phi_x = 0$$

and so

$$I_{D\setminus D_r} = -\int_{D\setminus D_r} \left(\Delta + k^2\right)\Phi_x\phi\, dy = 0.$$

Also,

$$J_{D_r} \to 0 \quad \text{as } r\to 0,$$

because the integrand is an integrable function in D. The integral kernel of I_{D_r} is

$$-n\cdot\nabla\Phi_x = \frac{1}{4\pi}\frac{n(y)\cdot(y-x)}{|y-x|^3} + \mathcal{O}\left(|x-y|^{-1}\right)$$

and so

$$I_{D_r} = \frac{\phi(x)}{4\pi}\int_{\partial D_r} \frac{n(y)\cdot(y-x)}{|y-x|^3}\, dS(y) + \mathcal{O}(r) \to \Omega_D(x)\phi(x) \quad \text{as } r\to 0.$$

Hence,

$$\langle -(\Delta^D + k^2\chi_D)\Phi_x, \phi\rangle = \Omega_D(x)\phi(x) = \langle \delta_x^D, \phi\rangle. \qquad \square$$

Corollary 3.2. *Suppose D is bounded. If $u \in C^2(D)\cap C^1(\overline{D})$ solves the Helmholtz equation* (18) *in D, then*

$$\Omega_D(x)u(x) = \int_{\partial D} \Phi_x\frac{\partial u}{\partial n}\, dS - \int_{\partial D}\frac{\partial\Phi_x}{\partial n}u\, dS, \quad x\in\mathbb{R}^3. \tag{22}$$

Proof. Because D is bounded, the solution u can be used as a test function. Now, by (8) and (19),

$$\begin{aligned}\Omega_D(x)u(x) &= -\langle \nabla^D\cdot\nabla\Phi_x + k^2\Phi_x\chi_D, u\rangle \\ &= -\int_{\partial D}\frac{\partial\Phi_x}{\partial n}u\, dS + \int_D \nabla\Phi_x\cdot\nabla u\, dy - k^2\int_D \Phi_x u\, dy \\ &= -\int_{\partial D}\frac{\partial\Phi_x}{\partial n}u\, dS + \int_{\partial D}\Phi_x\frac{\partial u}{\partial n}\, dS - \int_D \Phi_x\left(\Delta u + k^2 u\right) dy,\end{aligned}$$

which proves the claim. $\square$

Corollary 3.3. *Suppose D is bounded and let $D^e = \mathbb{R}^3\setminus\overline{D}$ be the unbounded exterior domain. If $u\in C^2(D^e)\cap C^1(\overline{D^e})$ solves the Helmholtz equation* (18) *in D^e, and u satisfies the Sommerfeld radiation condition* [2]

$$\hat{x}\cdot\nabla u(x) - iku(x) = o\left(\frac{1}{|x|}\right), \quad \hat{x} = \frac{x}{|x|}, \tag{23}$$

as $|x| \to \infty$, *then*

$$\Omega_{D^e}(x)u(x) = -\int_{\partial D} \Phi_x \frac{\partial u}{\partial n}\,dS + \int_{\partial D} \frac{\partial \Phi_x}{\partial n} u\,dS, \quad x \in \mathbb{R}^3. \tag{24}$$

Proof. Fix $x \in \mathbb{R}^3$. Let $R > |x|$ be large. We apply (22) in domain

$$D_R^e = D^e \cap B(0, R)$$

to get

$$\Omega_{D_R^e}(x)u(x) = -\int_{\partial D} \left(\Phi_x \frac{\partial u}{\partial n} - \frac{\partial \Phi_x}{\partial n} u\right) dS + \int_{\partial B(0,R)} \left(\Phi_x \frac{\partial u}{\partial n} - \frac{\partial \Phi_x}{\partial n} u\right) dS.$$

The integral on $\partial B(0, R)$ vanishes by the Sommerfeld radiation condition (23), see [2]. By definition,

$$\Omega_{D_R^e}(x) = \Omega_{D^e}(x)$$

for a fixed x. This proves the claim. □

4. The time-harmonic Maxwell system

The normalized homogeneous time-harmonic Maxwell system with the wave number $k > 0$ is

$$\begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix}\begin{pmatrix} E \\ H \end{pmatrix} + ik\begin{pmatrix} E \\ H \end{pmatrix} = 0. \tag{25}$$

From (25), it follows

$$\nabla \cdot E = 0 = \nabla \cdot H. \tag{26}$$

The divergence equations (26) are, however, lost when $k = 0$, and this causes the low frequency problems to the boundary integral equations derived from (25), see [7].

Let

$$A(\nabla) = \begin{pmatrix} 0 & \nabla\cdot & 0 & 0 \\ \nabla & 0 & \nabla\times & 0 \\ 0 & -\nabla\times & 0 & \nabla \\ 0 & 0 & \nabla\cdot & 0 \end{pmatrix}.$$

Note that A is symmetric as a matrix, because

$$\xi\times = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix} = -(\xi\times)^T, \quad \xi\cdot = \xi^T.$$

The extended Maxwell system is [4,5]

$$\big(A(\nabla) + ikI\big)U = 0, \quad U = (\phi, E, H, \psi)^T,\ k \geqslant 0, \tag{27}$$

where I is an identity matrix, E, H are vector fields and ϕ, ψ are scalar fields. The Maxwell system (25) and the extended Maxwell system (27) have the following connection.

Lemma 4.1. *Suppose* $U = (\phi, E, H, \psi)^T$ *solves the extended Maxwell system* (27)*. Then* (E, H) *solves the Maxwell system* (25) *if and only if*

$$\phi \equiv 0 \equiv \psi.$$

Note that the electromagnetic fields E and H solving the extended Maxwell system with $\phi \equiv 0 \equiv \psi$ still have the divergence condition (26) valid also at $k = 0$. As a consequence, the boundary integral equations that arise from the extended Maxwell system survive well with the low frequencies [7].

The equality

$$\big(A(\nabla) - ikI\big)\big(A(\nabla) + ikI\big) = \big(\Delta + k^2\big)I$$

is the basis in obtaining the fundamental solution for the extended Maxwell system. Define

$$G_x = -\big(A(\nabla) - ikI\big)(\Phi_x I). \tag{28}$$

Because A is symmetric as a matrix,

$$G_x^T = G_x.$$

Interchanging x and y leads to the reciprocity relation

$$G_y(x) = -\big(A\big(\nabla_x \Phi_y(x)\big) - ik\Phi_y(x)I\big) = \big(A(\nabla_y) + ikI\big)\big(\Phi_x(y)I\big). \tag{29}$$

Theorem 4.2. *Matrix G_x is the domain dependent fundamental solution of the domain dependent extended Maxwell system*

$$\big(A\big(\nabla^D\big) + ik\chi_D I\big)G_x = \delta_x^D, \quad x \in \mathbb{R}^3, \tag{30}$$

with the reciprocity relation

$$-\big(A\big(\nabla_y^D\big) - ik\chi_D I\big)G_y(x) = \delta_x^D(y). \tag{31}$$

Proof. Now,

$$\begin{aligned}\big(A\big(\nabla^D\big) + ik\chi_D I\big)G_x &= -\big(A\big(\nabla^D\big) + ik\chi_D I\big)\big(A(\nabla) - ikI\big)(\Phi_x I)\\ &= -\big(A\big(\nabla^D\big)A(\nabla) - ik\chi_D A(\nabla) + ikA\big(\nabla^D\big) + k^2\chi_D I\big)(\Phi_x I).\end{aligned}$$

Because $\nabla\Phi_x$ is a function

$$A\big(\nabla^D\big)(\Phi_x I) = \chi_D A(\nabla)(\Phi_x I).$$

By Lemma 2.3,

$$A\big(\nabla^D\big)A(\nabla) = \Delta^D I, \tag{32}$$

and hence,

$$\big(A\big(\nabla^D\big) + ik\chi_D I\big)G_x = -\big(\Delta^D + k^2\chi_D I\big)\Phi_x.$$

This proves (30) by (19). The proof for Eq. (31) is similar. □

Similarly to the Helmholtz equation, the global representation formulae for the extended Maxwell system follow by applying the domain dependent fundamental solution G_x.

Let

$$S\phi(x) = \int_{\partial D} \Phi_y(x)\phi(y)\,dS(y), \quad x \in \mathbb{R}^3, \tag{33}$$

be the single layer operator of ∂D.

Corollary 4.3. *Suppose D is bounded. If $U \in [C(\overline{D}) \cap C^1(D)]^8$ solves the extended Maxwell system* (27) *in D, then*

$$\Omega_D(x)U(x) = \big(A(\nabla) - ikI\big)S\big[A(n)U\big](x), \quad x \in \mathbb{R}^3. \tag{34}$$

Proof. By (8), (14), and (31),

$$\begin{aligned}\Omega_D(x)U(x) &= \langle -(A(\nabla_y^D) - ik\chi_D I)G_y(x), U\rangle \\ &= -\int_{\partial D}\big[A(n)G_y(x)\big]^T U\,dS(y) + \int_D G_y(x)^T A(\nabla)^T U\,dy + ik\int_D G_y(x)^T U\,dy \\ &= -\int_{\partial D} G_y(x)A(n)U\,dS(y) + \int_D G_y(x)\big[A(\nabla)U + ikU\big]\,dy,\end{aligned}$$

which proves the claim by (28) since G_x and $A(\nabla)$ are symmetric matrices. □

Corollary 4.4. *Suppose D is bounded and let $D^e = \mathbb{R}^3 \setminus \overline{D}$ be the unbounded exterior domain. If $U \in [C(\overline{D^e}) \cap C^1(D^e)]^8$ solves the extended Maxwell system* (27) *in D^e with the radiation condition*

$$\big(A(\hat{x}) + I\big)U(x) = o\left(\frac{1}{|x|}\right), \quad \hat{x} = \frac{x}{|x|}, \tag{35}$$

as $|x| \to \infty$, then

$$\Omega_{D^e}(x)U(x) = -\big(A(\nabla) - ikI\big)S\big[A(n)U\big](x), \quad x \in \mathbb{R}^3. \tag{36}$$

Proof. As in the proof of Corollary 3.3, we need to show that the integral arising from the representation formula on the boundary $\partial B(0,R)$ vanishes as $R \to \infty$.

For simplicity, suppose $x = 0$. From the definition (28) and the triangle inequality, we get an estimate

$$\left|\int_{\partial B_R} G_y(x)A(n)U(y)\,dS(y)\right| \leqslant k\int_{\partial B_R}\frac{1}{R}\big|\big(I + A(\hat{y})\big)U\big|\,dS + \int_{\partial B_R}\frac{1}{R^2}|U|\,dS. \tag{37}$$

The first term on the right-hand side vanishes by the radiation condition (35) as R grows. For the second term, we apply the Hölder inequality to get

$$\int_{\partial B_R}\frac{1}{R^2}|U|\,dS \leqslant \left(\int_{\partial B_R}\frac{1}{R^4}\,dS\right)^{1/2}\left(\int_{\partial B_R}|U|^2\,dS\right)^{1/2},$$

which proves the claim because

$$\lim_{R\to\infty}\int_{\partial B_R}|U|^2\,dS = \lim_{R\to\infty}\frac{1}{2}\int_{\partial B_R}\big|\big(A(\hat{y}) + I\big)U\big|^2\,dS - \int_{\partial\Omega} A(n)U\cdot\overline{U}\,dS = -\int_{\partial\Omega} A(n)U\cdot\overline{U}\,dS$$

is bounded. □

Corollary 4.5. *Suppose D is bounded. If $(E,H) \in [C(\overline{D}) \cap C^1(D)]^6$ solves the Maxwell system* (25) *in D with $k > 0$, then*

$$\begin{cases}\Omega_D(x)E(x) = \left(\frac{1}{ik}\nabla\nabla\cdot - ikI\right)S(n\times H)(x) - \nabla\times S(n\times E)(x), \\ \Omega_D(x)H(x) = -\nabla\times S(n\times H)(x) - \left(\frac{1}{ik}\nabla\nabla\cdot - ikI\right)S(n\times E)(x),\end{cases} \quad x \in \mathbb{R}^3. \tag{38}$$

Proof. Now,

$$U = (0, E, H, 0)^T$$

solves the extended Maxwell system in D. By substituting

$$A(n)U = \begin{pmatrix} n\cdot E \\ n\times H \\ -n\times E \\ n\cdot H\end{pmatrix}$$

in (34), the first and the last rows imply

$$0 = \begin{cases} \nabla \cdot S(n \times H) - ikS(n \cdot E), \\ -\nabla \cdot S(n \times E) - ikS(n \cdot H). \end{cases}$$

Hence,

$$\begin{aligned} \Omega_D(x) \begin{pmatrix} E \\ H \end{pmatrix} &= \begin{pmatrix} \nabla & 0 & \nabla\times & 0 \\ 0 & -\nabla\times & 0 & \nabla \end{pmatrix} S \begin{pmatrix} n \cdot E \\ n \times H \\ -n \times E \\ n \cdot H \end{pmatrix} - ikS \begin{pmatrix} n \times H \\ -n \times E \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{ik}\nabla\nabla\cdot & \nabla\times \\ -\nabla\times & +\frac{1}{ik}\nabla\nabla\cdot \end{pmatrix} S \begin{pmatrix} n \times H \\ -n \times E \end{pmatrix} - ikS \begin{pmatrix} n \times H \\ -n \times E \end{pmatrix}. \qquad \square \end{aligned}$$

Corollary 4.6. *Suppose D is bounded and let $D^e = \mathbb{R}^3 \setminus \overline{D}$ be the unbounded exterior domain. If $(E, H) \in [C(\overline{D^e}) \cap C^1(D^e)]^6$ solves the Maxwell system* (25) *in D^e with $k > 0$ and with the radiation condition*

$$\left[\begin{pmatrix} 0 & \hat{x}\times \\ -\hat{x}\times & 0 \end{pmatrix} + I\right] \begin{pmatrix} E(x) \\ H(x) \end{pmatrix} = o\left(\frac{1}{|x|}\right), \quad \hat{x} = \frac{x}{|x|}, \tag{39}$$

as $|x| \to \infty$, then

$$\begin{cases} \Omega_{D^e}(x)E(x) = -\left(\frac{1}{ik}\nabla\nabla\cdot - ikI\right)S(n \times H)(x) + \nabla \times S(n \times E)(x), \\ \Omega_{D^e}(x)H(x) = \nabla \times S(n \times H)(x) + \left(\frac{1}{ik}\nabla\nabla\cdot - ikI\right)S(n \times E)(x), \end{cases} \quad x \in \mathbb{R}^3. \tag{40}$$

Proof. Now,

$$U = (0, E, H, 0)^T$$

is the solution for the extended Maxwell system (27) and U satisfies the radiation condition (35). Hence, the claim follows from (36) in the same way as in the interior case. $\square$

5. Conclusions

We defined the domain dependent Dirac's delta δ^D and derivatives ∂_j^D. For convenience, the definitions were given in $\mathbb{R}^3$. The space $L^1_\partial(D)$ in which the derivatives operate was chosen so that the definitions can be applied to the fundamental solution Φ_x.

We applied the domain dependent operators to get global representation formulae for the Helmholtz equation and for the Maxwell equations. The globality occurs so that the observation point x can be in whole $\mathbb{R}^3$ and not restricted away from the boundary. The key point was to show that the usual fundamental solution of the Helmholtz equation is also the domain dependent fundamental solution. From this the global representation formulae followed. The results of the Helmholtz equation case were applied to the extended Maxwell system to obtain their global representation formulas. The corresponding formulae for the Maxwell equations were then an immediate consequence. The representations for the extended Maxwell system were valid also for the zero frequency.

When obtaining the boundary integral equations by using the global representation formulae, one can think that the use of the domain dependent objects interchanges the order of the boundary limiting process and the integration by parts. Traditionally, one integrates by parts to obtain the representation formulae, and then takes the limits onto the boundary. When the domain dependent objects are used, the limiting process is already taken into count when defining these objects, and only the integration by parts is needed.

There are some directions to develop the theory of domain dependent objects. One is to weaken the assumptions for the domain and to define the domain dependent objects when the boundary is not smooth and the relative solid angle on the boundary is not necessarily $1/2$, see [8] for the definition of traces on such boundaries. Another direction is to define the domain dependent objects in more general spaces than L^1_∂. An obvious task is to define new domain dependent objects.

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References

[1] J.A. Stratton, L.J. Chu, Diffraction theory of electromagnetic waves, Phys. Rev. 56 (1) (1939) 99–107.
[2] D. Colton, R. Kress, Integral Equation Methods in Scattering Theory, John Wiley & Sons, New York, 1983.
[3] L. Hörmander, The Analysis of Linear Partial Differential Operators, I: Distribution Theory and Fourier Analysis, vol. 256, Springer-Verlag, Berlin, 1983.
[4] R. Picard, On the low frequency asymptotics in electromagnetic theory, J. Reine Angew. Math. 354 (1984) 50–73.
[5] R. Picard, On a structural observation in generalized electromagnetic theory, J. Math. Anal. Appl. 110 (1985) 247–264.
[6] E. Sarkola, A unified approach to direct and inverse scattering for acoustic and electromagnetic waves, Ann. Acad. Sci. Fenn. Math. Diss. 101 (1995).
[7] M. Taskinen, S. Vänskä, Current and charge integral equation formulations and Picard's extended Maxwell system, IEEE Trans. Antennas and Propagation 55 (12) (2007) 3495–3503.
[8] A. Buffa, M. Costabel, D. Sheen, On traces for $H(Curl, \Omega)$ in Lipschitz domains, J. Math. Anal. Appl. 276 (2002) 845–867.