Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 344 (2008) 417–428

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Sign-changing solutions on a kind of fourth-order Neumann boundary value problem ☆

Fuyi Li *, Yanbiao Zhang, Yuhua Li

School of Mathematical Sciences, Shanxi University, Taiyuan 030006, People's Republic of China

Received 28 June 2007

Available online 7 March 2008

Submitted by D. O'Regan

Abstract

In this paper, we consider the fourth-order Neumann boundary value problem $u^{(4)}(t) - 2u''(t) + u(t) = f(t, u(t))$ for all $t \in [0, 1]$ and subject to $u'(0) = u'(1) = u'''(0) = u'''(1) = 0$. Using the fixed point index and the critical group, we establish the existence theorem of solutions that guarantees the problem has at least one positive solution and two sign-changing solutions under certain conditions.

Keywords: Sign-changing solution; Neumann boundary value problem; Fixed point index; Critical group

1. Introduction

It is well known that Neumann boundary value problem for the ordinary differential equations and elliptic equations is an important kind of boundary value problems. By using fixed point theorems in cone, in [1,10,11], the authors discussed the existence of positive solutions to ordinary differential equation Neumann boundary value problems. In [6], the authors discussed $2m$th order ordinary differential equation Neumann boundary value problems by using the critical point theory and the monotone operator principle, and obtained the existence of one nontrivial solution, infinitely many solutions and a unique solution under certain conditions, respectively. There are also papers which study nonlinear elliptic equation Neumann boundary value problems, see [5,12]. However, there are few papers which study fourth-order Neumann boundary value problems by Morse theory. In this paper, motivated by Liu and Sun [8,9], using the fixed point index and the critical group, we discuss the existence of sign-changing solutions and positive solutions to the following nonlinear fourth-order Neumann boundary value problem (BVP):

$$\begin{cases} u^{(4)}(t) - 2u''(t) + u(t) = f\big(t, u(t)\big), & t \in [0, 1], \\ u'(0) = u'(1) = u'''(0) = u'''(1) = 0. \end{cases} \tag{1.1}$$

☆ Project supported by the National Science Foundation of China (Grant No. 10771128) and the NSF of Shanxi Province (2006011002).

* Corresponding author.
E-mail address: fyli@sxu.edu.cn (F. Li).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.02.050

Assumptions on f are listed below.

(f_1) $f \in C^1([0,1]\times\mathbb{R},\mathbb{R})$, and $f(t,\cdot)$ is increasing on $\mathbb{R}$ for each $t\in[0,1]$;
(f_2) $f(t,0)=0$ and $\lambda_{k_0-1} < f'_x(t,0) < \lambda_{k_0}$ for all $t\in[0,1]$ and some $k_0 \geqslant 2$, where $\lambda_k=(k^2\pi^2+1)^2$ for all $k=0,1,2,\ldots;$
(f_3) $\limsup_{x\to+\infty} f(t,x)/x < \lambda_0$ uniformly in $t\in[0,1]$;
(f_4) $\lambda_0 < \liminf_{x\to-\infty} f(t,x)/x \leqslant \limsup_{x\to-\infty} f(t,x)/x < +\infty$ uniformly in $t\in[0,1]$.

The following theorem is the main result of this paper.

Theorem 1.1. *Suppose that the conditions* (f_1)–(f_4) *hold. Then the BVP* (1.1) *has at least one positive solution and two sign-changing solutions.*

2. Proof of main result

It is well known that the solution of BVP (1.1) in $C^4[0,1]$ is equivalent to the solution of integral equation

$$u(t)=\int_0^1 G(t,\tau)\int_0^1 G(\tau,s) f\big(s,u(s)\big)\,ds\,d\tau,\quad t\in[0,1], \tag{2.1}$$

in $C[0,1]$, where

$$G(t,s)=\frac{1}{\sinh 1}\begin{cases}\cosh t\cdot\cosh(1-s), & 0\leqslant t\leqslant s\leqslant 1,\\ \cosh s\cdot\cosh(1-t), & 0\leqslant s\leqslant t\leqslant 1.\end{cases}$$

Let $C[0,1]$ be the usual real Banach space with norm $\|u\|_0=\max_{t\in[0,1]}|u(t)|$ for all $u\in C[0,1]$ and $P=\{u\in C[0,1]\colon u(t)\geqslant 0,\ t\in[0,1]\}$. Then P is a solid cone in $C[0,1]$. Let $L^2[0,1]$ be the usual real Hilbert space with inner product $(u,v)=\int_0^1 u(t)v(t)\,dt$ for all $u,v\in L^2[0,1]$, the corresponding norm denoted by $\|\cdot\|$, and $Q=\{u\in L^2[0,1]\colon u(t)\geqslant 0,\ \text{a.e. } t\in[0,1]\}$. Then Q is a cone in $L^2[0,1]$.

We now define operators K on $L^2[0,1]$ and $\mathbf{f}$ on $C[0,1]$ as follows:

$$Ku(t)=\int_0^1 G(t,s)u(s)\,ds,\quad t\in[0,1],\ \forall u\in L^2[0,1],$$

$$\mathbf{f}u(t)=f\big(t,u(t)\big),\quad t\in[0,1],\ \forall u\in C[0,1].$$

Then the solution of integral equation (2.1) in $C[0,1]$ is equivalent to the solution of operator equation

$$K^2\mathbf{f}u=u$$

in $C[0,1]$. For the linear operator K, it is easy to see that

Lemma 2.1.

(i) $K: L^2[0,1]\to C[0,1]\hookrightarrow L^2[0,1]$ *is compact symmetric and strongly increasing*;
(ii) $Ku\geqslant\sigma\|Ku\|_0 e_0$ *for all* $u\in Q$, *where* $\sigma=m_0/M_0$, $m_0=\min_{t,s\in[0,1]}G(t,s)$, $M_0=\max_{t,s\in[0,1]}G(t,s)$, $e_0=1$;
(iii) *all the eigenvalues of* K *are* $\{1/\mu_k\}$, *where* $\mu_k=k^2\pi^2+1$ *for all* $k=0,1,2,\ldots,$ *and corresponding orthonormal eigenvectors are* $\{e_k\}$, *where* $e_0=1$, $e_k=\sqrt{2}\cos k\pi t$ *for all* $k=1,2,\ldots.$

Lemma 2.2.

(i) *Let* $f\in C([0,1]\times\mathbb{R},\mathbb{R})$. *Then the existence, uniqueness and multiplicity of solution on both equations* $u=K^2\mathbf{f}u$ *in* $C[0,1]$ *and* $v=K\mathbf{f}Kv$ *in* $L^2[0,1]$ *are equivalent, respectively.*
(ii) *Assume that* $f\in C^1([0,1]\times\mathbb{R},\mathbb{R})$, $f(t,\cdot)$ *is increasing on* $\mathbb{R}$ *for each* $t\in[0,1]$ *and* $f(t,0)=0$ *for all* $t\in[0,1]$. *Then the existence of positive solution, negative solution and sign-changing solution is also equivalent for these two equations, respectively.*

Proof. The proof of (i) refers to [7, Lemma 3.1]. Now we only prove the equivalence for the sign-changing solution. The proofs for the positive and negative solution are similar. Let u be a sign-changing solution of the equation $u = K^2\mathbf{f}u$ in $C[0,1]$. Then $K\mathbf{f}u = K\mathbf{f}K(K\mathbf{f}u)$, so $v = K\mathbf{f}u$ is a solution of $v = K\mathbf{f}Kv$. If $v \in Q \cup (-Q)$, then it follows from Lemma 2.1 that $u = K^2\mathbf{f}u = Kv$ lies in $P \cup (-P)$. This is a contradiction. Therefore, $v = K\mathbf{f}u$ is also a sign-changing solution of the equation $v = K\mathbf{f}Kv$.

On the other hand, let v be a sign-changing solution of the equation $v = K\mathbf{f}Kv$ in $L^2[0,1]$. Then $Kv = K^2\mathbf{f}Kv$, so $u = Kv$ is a solution of $u = K^2\mathbf{f}u$ in $C[0,1]$. If $u \in P \cup (-P)$, then it follows from Lemma 2.1 that $K\mathbf{f}u = K\mathbf{f}Kv = v$ lies in $P \cup (-P) \subset Q \cup (-Q)$. This is a contradiction. Therefore, $u = Kv$ is also a sign-changing solution of the equation $u = K^2\mathbf{f}u$. The proof is completed. $\square$

Let $F(t,x) = \int_0^x f(t,y)\,dy$ for all $t \in [0,1]$ and $x \in \mathbb{R}$. Define the functional on $L^2[0,1]$:

$$J(u) = \frac{1}{2}\|u\|^2 - \int_0^1 F\big(t, Ku(t)\big)\,dt, \quad u \in L^2[0,1].$$

Since $f \in C^1([0,1]\times\mathbb{R},\mathbb{R})$, J is Fréchet differentiable on $L^2[0,1]$, and $J' = I - K\mathbf{f}K := I - T$ is continuous on $L^2[0,1]$. Consequently, the solution of equation $u = K\mathbf{f}Ku$ is equivalent to the critical point of J in $L^2[0,1]$.

In what follows, we always assume that conditions (f_1)–(f_4) hold.

Lemma 2.3. *The equation $u = K\mathbf{f}Ku$ has a positive solution $u^* \in C[0,1]$ with $u^* \in P^\circ$.*

Proof. By the condition (f_3), there exist $\delta \in (0,1)$ and $C > 0$ such that

$$f(t,x) \leqslant \lambda_0(1-\delta)x + C, \quad t \in [0,1],\ x \geqslant 0. \tag{2.2}$$

We now prove that $u \neq tK\mathbf{f}Ku$ for all $t \in [0,1]$ and $u \in P$ with $\|u\|_0 = R > R_0 := C/(\sigma\delta)$. In fact, if there exist $t_0 \in [0,1]$ and $u_0 \in P$ with $\|u_0\|_0 = R$ such that $u_0 = t_0K\mathbf{f}Ku_0$, then it follows from (2.2) that

$$\begin{aligned}(u_0,e_0) &= t_0(K\mathbf{f}Ku_0,e_0) \leqslant (\mathbf{f}Ku_0,Ke_0) \leqslant \lambda_0(1-\delta)(Ku_0,Ke_0) + C\\ &= \lambda_0(1-\delta)\big(u_0,K^2e_0\big) + C = (1-\delta)(u_0,e_0) + C.\end{aligned}$$

This implies that $(u_0,e_0) \leqslant C/\delta$.

On the other hand, $u_0 = t_0K\mathbf{f}Ku_0 \in K(Q)$, so $u_0 \geqslant \sigma\|u_0\|_0e_0$. This implies that $(u_0,e_0) \geqslant \sigma\|u_0\|_0 = \sigma R$. Therefore, $R \leqslant C/(\sigma\delta)$. This is a contradiction. Thus, the fixed point index $i(T,P_R,P) = 1$, where $P_R = \{u \in P\colon \|u\|_0 < R\}$.

By the condition (f_2), there exist $\delta_1 \in (0,1)$ and $r_0 \in (0,R_0)$ such that

$$f(t,x) \geqslant \lambda_0(1+\delta_1)x, \quad t \in [0,1],\ x \in [0,r_0]. \tag{2.3}$$

We now prove that $u \neq K\mathbf{f}Ku + te_0$ for all $t \geqslant 0$ and $u \in P$ with $\|u\|_0 = r < r_0$. In fact, if there exist $t_1 \geqslant 0$ and $u_1 \in P$ with $\|u_1\|_0 = r$ such that $u_1 = K\mathbf{f}Ku_1 + t_1e_0$, then we have from (2.3) that

$$(u_1,e_0) = (K\mathbf{f}Ku_1,e_0) + t_1(e_0,e_0) \geqslant (\mathbf{f}Ku_1,Ke_0) \geqslant \lambda_0(1+\delta_1)(Ku_1,Ke_0) = (1+\delta_1)(u_1,e_0).$$

This implies that $(u_1,e_0) \leqslant 0$. This is a contradiction. Thus, the fixed point index $i(T,P_r,P) = 0$. Therefore, $i(T,P_R\setminus\bar{P}_r,P) = 1$, and then T has at least a fixed point $u^* \in P_R\setminus\bar{P}_r$, where $\bar{P}_r = \{u \in P\colon \|u\| \leqslant r\}$. Obviously, Lemma 2.1 implies that $u^* = Tu^* \in P^\circ$. The proof is completed. $\square$

We can see from the proof of Lemma 2.3 that if $u \in P$ is a fixed point of T, then $\|u\|_0 \leqslant C/(\sigma\delta) := M$. Now let $w = Me_0$. Then $u \leqslant w$. It follows from (2.2) that

$$Tw = K\mathbf{f}Kw \leqslant \lambda_0(1-\delta)Me_0 + Ce_0 \leqslant Me_0 = w.$$

Therefore, w is a super solution of the equation $u = Tu$.

Let $X = \{u \in C[0,1]\colon u \leqslant w\}$. Then X is a closed convex subset of $C[0,1]$. Now we consider the auxiliary equation

$$u = K\mathbf{f}Ku - te_0, \tag{2.4}$$

where $t \geqslant 0$ is a parameter.

Lemma 2.4. *There exists $R > 0$ such that if $u \in X$ is a solution of Eq.* (2.4) *with some $t \geqslant 0$ then $\|u\|_0 < R$.*

Proof. Notice that $M = \|w\|_0 > 0$. By the condition (f_4), there exist $\varepsilon > 0$ and $C > 0$ such that

$$f(t,x) \leqslant (\lambda_0 + \varepsilon)x + C, \quad t \in [0,1],\ x \leqslant M. \tag{2.5}$$

Suppose that $u \in X$ is a solution of Eq. (2.4) with some $t \geqslant 0$, then $u \leqslant w$, so Lemma 2.1 and the increasing property of $\mathbf{f}$ imply that

$$v := Tw - u = K[\mathbf{f}Kw - \mathbf{f}Ku + te_0] \geqslant \sigma\|v\|_0 e_0.$$

Taking the inner product to Eq. (2.4) with e_0, we have from (2.5) that

$$\begin{aligned}(u, e_0) &= (K\mathbf{f}Ku, e_0) - t(e_0, e_0) \leqslant (\lambda_0 + \varepsilon)(Ku, Ke_0) + C(1, Ke_0) - t \\ &= (\lambda_0 + \varepsilon)\big(u, K^2 e_0\big) + C - t \leqslant (\lambda_0 + \varepsilon)(u, e_0) + C.\end{aligned}$$

Consequently, $\varepsilon(u, e_0) + C = \varepsilon(Tw - v, e_0) + C \geqslant 0$, so that $\varepsilon(Tw, e_0) + C \geqslant \varepsilon(v, e_0) \geqslant \varepsilon\sigma\|v\|_0$. Therefore, $\|v\|_0 \leqslant (\varepsilon(Tw, e_0) + C)/(\varepsilon\sigma)$. Thus, $\|u\|_0 \leqslant M + (\varepsilon(Tw, e_0) + C)/(\varepsilon\sigma) := R$. The proof is completed. □

By the condition (f_2), there exists $\delta_0 > 0$ such that $f(t,x)/x > \lambda_0$ for all $t \in [0,1]$ and $|x| \in (0, \delta_0]$. Then for any $\delta \in (0, \delta_0]$, we have from Lemma 2.1 that

$$\begin{aligned}&T(\delta e_0) = K\mathbf{f}K(\delta e_0) \gg \lambda_0 K(\delta e_0) = \delta e_0, \\ &T(-\delta e_0) = K\mathbf{f}K(-\delta e_0) \ll \lambda_0 K(-\delta e_0) = -\delta e_0.\end{aligned} \tag{2.6}$$

By again the condition (f_2), 0 is an isolated fixed point of T, i.e., there exists $r \in (0, \delta_0/\sigma)$ such that 0 is the unique fixed point of T in $B(0,r) := \{u \in C[0,1]\colon\ \|u\|_0 < r\}$. Therefore, if $u \in P \setminus \{0\}$ is a fixed point of T, then $u \geqslant \sigma\|u\|_0 e_0 \geqslant \sigma r e_0 = \delta e_0$, where $\delta = \sigma r \in (0, \delta_0]$, so it follows from (2.6) that $u = Tu \geqslant T(\delta e_0) \gg \delta e_0$. Similarly, if $u \in -P \setminus \{0\}$ is a fixed point of T, then $u \ll -\delta e_0$.

For any given $\delta \in (0, \delta_0]$, we now define two subsets of X:

$$\begin{aligned}&U_1^* = \{u \in X\colon\ \delta e_0 \ll u \leqslant w\}, \\ &U_2^* = \{u \in X\colon\ u \ll -\delta e_0\},\end{aligned}$$

then U_1^* and U_2^* are both open convex subsets of X, and U_1^* contains all the positive solutions of $u = Tu$ and U_2^* contains all the possible negative solutions $u = Tu$. Thus, nontrivial solutions found in $L^2[0,1] \setminus (U_1^* \cup U_2^*)$ must be sign-changing solutions. We may assume that there is only a finite number of solutions in $L^2[0,1] \setminus (U_1^* \cup U_2^*)$.

In addition, it is obvious that $T(X) \subset X, T(\operatorname{clos}_X U_i^*) \subset U_i^*, i = 1, 2,$ and U_1^* is bounded. Therefore, $i(T, U_1^*, X) = 1$.

In what follows, C_i will be used to represent positive constants. For any $u \in L^2[0,1]$, we denote $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$.

Lemma 2.5. *The functional J satisfies P.S. condition on $L^2[0,1]$.*

Proof. Suppose that $\{u_n\} \subset L^2[0,1]$ satisfies $J'(u_n) \to 0$ as $n \to \infty$ and $|J(u_n)| \leqslant C_1$ for all $n \in \mathbb{N} := \{1, 2, \ldots\}$ and some $C_1 > 0$. Obviously, $u = -(Ku)'' + Ku$ for all $u \in L^2[0,1]$. Now let $v^+ = -[(Ku)^+]'' + (Ku)^+$, $v^- = -[(Ku)^-]'' + (Ku)^-$ for all $u \in L^2[0,1]$. Then $u = v^+ + v^-$, $(v^+, v^-) = 0$, $Kv^+ = (Ku)^+$, and $Kv^- = (Ku)^-$ for all $u \in L^2[0,1]$. Taking the inner product of $J'(u_n)$ and v_n^+, we have from (2.2) that

$$\begin{aligned}\big(J'(u_n), v_n^+\big) &= \|v_n^+\|^2 - \int_0^1 f\big(t, Ku_n(t)\big)Kv_n^+(t)\,dt \\ &= \|v_n^+\|^2 - \int_0^1 f\big(t, (Ku_n)^+(t) + (Ku_n)^-(t)\big)(Ku_n)^+(t)\,dt\end{aligned}$$

$$= \|v_n^+\|^2 - \int_0^1 f\big(t, (Ku_n)^+(t)\big)(Ku_n)^+(t)\,dt$$
$$= \|v_n^+\|^2 - \int_0^1 f\big(t, Kv_n^+(t)\big)Kv_n^+(t)\,dt$$
$$\geqslant \|v_n^+\|^2 - \lambda_0(1-\delta)\|Kv_n^+\|^2 - C\int_0^1 Kv_n^+(t)\,dt$$
$$\geqslant \|v_n^+\|^2 - \frac{\lambda_0(1-\delta)}{\lambda_0}\|v_n^+\|^2 - \frac{C}{\mu_0}\|v_n^+\|$$
$$= \delta\|v_n^+\|^2 - C\|v_n^+\|, \quad n \in \mathbb{N}.$$

It follows that

$$o(1)\|v_n^+\| \geqslant \delta\|v_n^+\|^2 - C\|v_n^+\|, \quad n \in \mathbb{N}.$$

Therefore, $\{v_n^+\}$ is bounded. Then $\{J(v_n^+)\}$ is bounded. Since

$$J(u_n) = \frac{1}{2}\|u_n\|^2 - \int_0^1 F\big(t, Ku_n(t)\big)\,dt$$
$$= \frac{1}{2}\|v_n^+\|^2 + \frac{1}{2}\|v_n^-\|^2 - \int_0^1 F\big(t, (Ku_n)^+(t) + (Ku_n)^-(t)\big)\,dt$$
$$= \frac{1}{2}\|v_n^+\|^2 + \frac{1}{2}\|v_n^-\|^2 - \int_0^1 F\big(t, (Ku_n)^+(t)\big)\,dt - \int_0^1 F\big(t, (Ku_n)^-(t)\big)\,dt$$
$$= \frac{1}{2}\|v_n^+\|^2 + \frac{1}{2}\|v_n^-\|^2 - \int_0^1 F\big(t, Kv_n^+(t)\big)\,dt - \int_0^1 F\big(t, Kv_n^-(t)\big)\,dt$$
$$= J\big(v_n^+\big) + J(v_n^-),$$

it follows that $\{J(v_n^-)\}$ is bounded. In order to find a bound for $\{v_n^-\}$, we use a contradiction and assume that $\|v_n^-\| \to \infty$ as $n \to \infty$. Defining $w_n = v_n^-/\|v_n^-\|$, $n \in \mathbb{N}$, and selecting a subsequence if necessary, we have $w_n \to w_0$ weakly in $L^2[0,1]$, and so $Kw_n \to Kw_0$ strongly in $C[0,1] \hookrightarrow L^2[0,1]$. By conditions ($f_2$) and ($f_4$), there exists $C_2 > 0$ such that $F(t,x) \leqslant C_2|x|^2$ for all $t \in [0,1]$ and $x \leqslant 0$. Since $\{J(v_n^-)\}$ is bounded, we have

$$\frac{1}{2} = \frac{J(v_n^-)}{\|v_n^-\|^2} + \int_0^1 \frac{F(t, Kv_n^-(t))}{\|v_n^-\|^2}\,dt \leqslant o(1) + C_2\int_0^1 \frac{|Kv_n^-(t)|^2}{\|v_n^-\|^2}\,dt = o(1) + C_2\|Kw_n\|^2. \tag{2.7}$$

Let $n \to \infty$, by (2.7), we have that $1/2 \leqslant C_2\|Kw_0\|^2$, and then $Kw_0 \neq 0$. Since $Kw_n = (Ku_n)^-/\|v_n^-\| \leqslant 0$ for all $n \in \mathbb{N}$, $Kw_0 \leqslant 0$.

On the other hand, since $\{v_n^+\}$ is bounded, it follows from $(Ku_n)^+ = Kv_n^+$ that $\{\mathbf{f}(Ku_n)^+\}$ is also bounded. Taking the inner product of $J'(u_n)/\|v_n^-\|$ with e_0, we see from (2.5) that

$$\left(\frac{v_n^-}{\|v_n^-\|}, e_0\right) = \left(\frac{J'(u_n)}{\|v_n^-\|}, e_0\right) - \left(\frac{v_n^+}{\|v_n^-\|}, e_0\right) + \left(\frac{\mathbf{f}(Ku_n)^+}{\|v_n^-\|}, Ke_0\right) + \left(\frac{\mathbf{f}(Ku_n)^-}{\|v_n^-\|}, Ke_0\right)$$
$$\leqslant o(1) + \frac{\lambda_0 + \varepsilon}{\lambda_0}\left(\frac{v_n^-}{\|v_n^-\|}, e_0\right).$$

This implies that $(w_0, e_0) \geqslant 0$, i.e., $(Kw_0, e_0) \geqslant 0$, which is a contradiction with $Kw_0 \leqslant 0$ and $Kw_0 \neq 0$. Consequently, $\{v_n^-\}$ is bounded in $L^2[0,1]$. Thus $\{u_n\}$ is bounded in $L^2[0,1]$. It follows from the fact $u_n - Tu_n \to 0$ and T is completely continuous that $\{u_n\}$ has a convergent subsequence. Therefore, J satisfies P.S. condition. The proof is completed. □

For definition and properties of critical groups of a functional at an isolated point, we refer to [3]. Similar to [3, Corollary 1.2, p. 144], using the Palais theorem (see [3, Theorem 1.3, p. 14]), we can prove that critical groups of J defined on $L^2[0,1]$ are the same as critical groups of J constrained on $C[0,1]$. Therefore, by the condition (f_2), according to [3, Theorem 4.1, p. 34], we easily obtain the following theorem.

Theorem 2.6. 0 *is a nondegenerate critical point of* J *with Morse index* k_0, *so the critical groups of* J *at* 0 *with coefficient* $\mathbb{R}$ *are given by* $C_q(J,0) \cong \delta_{qk_0}\mathbb{R}$ *for all* $q \in \mathbb{N}$, *and there is* $r_0 > 0$ *such that* $B(0,r_0) \cap \operatorname{clos}_X(U_1^* \cup U_2^*) = \emptyset$, $B(0,r_0) \subset X$, *and* $i(T, B(0,r_0), X) = (-1)^{k_0}$, *where* $B(0,r_0) = \{u \in C[0,1]\colon \|u\|_0 < r_0\}$.

Now we define the negative gradient flow of J. By the condition (f_1), T is Fréchet differentiable in $L^2[0,1]$ and locally Lipschitz continuous on $L^2[0,1]$, and then it is also locally Lipschitz continuous on $C[0,1]$. For each $u \in L^2[0,1]$, we can define $\phi(t,u)$, $t \in [0,\tau(u))$, is the unique solution of the initial value problem

$$\begin{cases} \dfrac{d}{dt}\phi(t,u) = -\phi(t,u) + T\phi(t,u), \\ \phi(0,u) = u, \end{cases} \tag{2.8}$$

where $[0,\tau(u))$ is the right maximal interval of existence of $\phi(\cdot,u)$ in $L^2[0,1]$.

Lemma 2.7. *If* $u \in C[0,1]$, *then* $\phi(\cdot,u)$ *is continuous from* $[0,\tau(u))$ *into* $C[0,1]$ *and* $\phi(t,u)$, $t \in [0,\tau(u))$ *is also the solution of* (2.8) *considered in* $C[0,1]$.

Proof. We consider the initial value problem (2.8). Multiplying e^s and integrating in $L^2[0,1]$, we get

$$\int_0^t e^s \frac{d\phi(s,u)}{ds}\,ds = -\int_0^t e^s\phi(s,u)\,ds + \int_0^t e^s T\phi(s,u)\,ds,$$

so

$$\phi(t,u) = e^{-t}u + \int_0^t e^{-t+s}T\phi(s,u)\,ds. \tag{2.9}$$

Since $\phi(s,u) \in L^2[0,1]$ yields $T\phi(s,u) \in C[0,1]$ for all $s \in [0,\tau(u))$, and $u \in C[0,1]$, it follows from (2.9) that $\phi(t,u) \in C[0,1]$ for all $t \in [0,\tau(u))$. Furthermore, since $\phi(s,u)$ is continuous on s in $L^2[0,1]$, $T\phi(s,u)$ is continuous on s in $C[0,1]$. Therefore, $\int_0^t e^{-t+s}T\phi(s,u)\,ds$ is continuous on t in $C[0,1]$. Consequently, it follows from (2.9) that $\phi(t,u)$ is continuous on t in $C[0,1]$. Thus, $\phi(t,u)$, $t \in [0,\tau(u))$ is also the solution of the initial value problem (2.8) in $C[0,1]$. The proof is completed. □

We need the following lemma which can be found in [4, Theorem 4.1].

Lemma 2.8. *Suppose that* E *is a real Banach space,* M *is a closed convex subset of* E, $H : M \to E$ *is locally Lipschitz continuous, and for* $u \in M$,

$$\lim_{\beta \to 0^+} \frac{\operatorname{dist}(u + \beta Hu, M)}{\beta} = 0,$$

then there exists $\delta > 0$ *such that*

$$\begin{cases} \dfrac{d\phi(t,u)}{dt} = H\phi(t,u), \\ \phi(0,u) = u \end{cases}$$

has a unique solution $\phi(t,u)$, $t \in [0,\delta)$, *and it satisfies* $\phi(t,u) \in M$ *for all* $t \in [0,\delta)$.

Lemma 2.9.

(i) *For each $u \in X$, the solution of* (2.8) $\phi(t,u) \in X$ *for all* $t \in [0,\tau(u))$.
(ii) *For each $u \in \operatorname{clos}_X U_1^*$, $\phi(t,u) \in U_1^*$ for all $t \in (0,\tau(u))$.*
(iii) *For each $u \in \operatorname{clos}_X U_2^*$, $\phi(t,u) \in U_2^*$ for all $t \in (0,\tau(u))$.*

Proof. We only prove (i), proofs of (ii) and (iii) are similar. Let $u \in X$. Since X is a closed convex subset of $C[0,1]$ and $T(X) \subset X$, it follows that

$$\lim_{\beta\to 0^+} \frac{\operatorname{dist}(v-\beta v+\beta Tv, X)}{\beta} = \lim_{\beta\to 0^+} \frac{\operatorname{dist}((1-\beta)v+\beta Tv, X)}{\beta} = 0, \quad v \in X. \tag{2.10}$$

We now consider (i). If there exists $t_2 \in (0,\tau(u))$ such that $\phi(t_2,u) \in X^c$, then there exists $t_1 \in (0,t_2)$ such that $\phi(t_1,u) \in \partial X$, and $\phi(t,u) \in X^c$ for all $t \in (t_1,t_2]$. Consider the following initial value problem:

$$\begin{cases} \dfrac{d\phi(t,\phi(t_1,u))}{dt} = -\phi\big(t,\phi(t_1,u)\big) + T\phi\big(t,\phi(t_1,u)\big), \\ \phi\big(0,\phi(t_1,u)\big) = \phi(t_1,u). \end{cases}$$

It follows from (2.10) and Lemma 2.8 that there exists $\delta > 0$ such that $\phi(t,u) \in X$ for all $t \in [t_1,t_1+\delta)$, which is a contradiction. Therefore, for each $u \in X$, we have $\phi(t,u) \in X$ for all $t \in [0,\tau(u))$. The proof is completed. □

We now denote $K = \{u \in L^2[0,1]\colon J'(u)=0\}$, $K_\delta = \{u \in L^2[0,1]\colon \operatorname{dist}(u,K) < \delta\}$ for all $\delta > 0$. It follows from Lemma 2.1 that $K \subset C[0,1]$. We denote also for any number c,

$$J_X^c := \big\{u \in X\colon J(u) \leqslant c\big\}$$

and

$$K_X^c := \big\{u \in X\colon J'(u)=0,\ J(u)=c\big\}.$$

Lemma 2.10. *Let A be a bounded closed subset of $L^2[0,1]$. If $K \cap A = \emptyset$, then there exists $\varepsilon > 0$ such that*

$$\big\|J'(u)\big\| \geqslant \varepsilon, \quad u \in A.$$

Proof. Suppose that there exists a sequence $\{u_n\} \subset A$ such that $\|J'(u_n)\| < 1/n$ for all $n \in \mathbb{N}$, then $\{u_n\}$ is bounded, so $\{J(u_n)\}$ is bounded. It follows from P.S. condition that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to u \in L^2[0,1]$ as $k \to \infty$. Therefore, $u \in A$ and $J'(u)=0$. This is a contradiction. The proof is completed. □

Lemma 2.11. *Let $u \notin K$ and $\phi(t,u)$, $t \in [0,+\infty)$, be the unique solution of* (2.8). *Suppose that $\{J(\phi(t,u))\colon t \in [0,+\infty)\}$ is bounded below. If there exists a sequence of positive numbers $\{t_n\}$ with $t_n \to +\infty$ such that $\phi(t_n,u) \to \tilde u$ in $L^2[0,1]$ and $\tilde u \in K$, then $\{\phi(t,u)\colon t \in [0,\tau(u))\}$ is bounded in $L^2[0,1]$. Furthermore, if $\tilde u$ is an isolated critical point, then* $\lim_{t\to+\infty}\phi(t,u) = \tilde u$ *in* $L^2[0,1]$.

Proof. (i) According to Lemma 2.4, $K \cap \operatorname{clos}_X(U_1^* \cup U_2^*)$ is bounded in $C[0,1] \hookrightarrow L^2[0,1]$. By the assumption, $K \setminus \operatorname{clos}_X(U_1^* \cup U_2^*)$ is a finite set. Then K is a bounded set in $L^2[0,1]$. Thus, for any $\delta > 0$, $\overline{K_{2\delta}} \setminus K_\delta$ is bounded closed set and $K \cap (\overline{K_{2\delta}} \setminus K_\delta) = \emptyset$. It follows from Lemma 2.10 that there exists $\varepsilon > 0$ such that

$$\big\|J'(u)\big\| \geqslant \varepsilon, \quad u \in \overline{K_{2\delta}} \setminus K_\delta.$$

Suppose that the set $\{\phi(t,u)\colon t \in [0,\tau(u))\}$ is not bounded, then $\limsup_{t\to+\infty} \operatorname{dist}(\phi(t,u),K) = +\infty$. Since $\lim_{n\to\infty}\|\phi(t_n,u)-\tilde u\| = 0$, we can choose two sequences $\{t_n'\}$, $\{t_n''\}$, and a subsequence of $\{t_n\}$, without loss of generality, we can assume that this subsequence is $\{t_n\}$ itself, such that $t_n < t_n' < t_n''$, and

$$\operatorname{dist}\big(\phi\big(t_n',u\big),K\big) = \delta, \qquad \operatorname{dist}\big(\phi\big(t_n'',u\big),K\big) = 2\delta, \qquad \delta \leqslant \operatorname{dist}\big(\phi\big(t_n',u\big),K\big) \leqslant 2\delta, \quad t \in \big[t_n',t_n''\big].$$

This implies that

$$\big\|J'\big(\phi(t,u)\big)\big\| \geqslant \varepsilon, \quad t \in \big[t_n',t_n''\big].$$

Therefore,

$$\delta \leqslant \|\phi(t_n'', u) - \phi(t_n', u)\| = \left\| \int_{t_n'}^{t_n''} \frac{d}{dt}\phi(t,u)\,dt \right\| = \left\| \int_{t_n'}^{t_n''} J'\big(\phi(t,u)\big)\,dt \right\| \leqslant \frac{1}{\varepsilon} \int_{t_n'}^{t_n''} \big\|J'\big(\phi(t,u)\big)\big\|^2\,dt$$

$$= \frac{1}{\varepsilon} \int_{t_n'}^{t_n''} -\frac{d}{dt} J\big(\phi(t,u)\big)\,dt = \frac{1}{\varepsilon}\big[J\big(\phi(t_n',u)\big) - J\big(\phi(t_n'',u)\big)\big].$$

Since the limit $\lim_{t\to+\infty} J(\phi(t,u))$ exists, and $t_n' \to +\infty$, $t_n'' \to +\infty$, then $\delta \leqslant 0$, which is a contradiction. Therefore, $\{\phi(t,u)\colon t \in [0, \tau(u))\}$ is bounded.

(ii) Let $\tilde{u}$ be an isolated critical point of J. If the limit $\lim_{t\to+\infty}\phi(t,u) \neq \tilde{u}$ in $L^2[0,1]$, then $\limsup_{t\to+\infty} \|\phi(t,u) - \tilde{u}\| \geqslant 2\delta_0 > 0$ for some $\delta_0 > 0$. Choose $\delta_1 \in (0, \delta_0)$ such that $K \cap \{u \in L^2[0,1]\colon \delta_1 \leqslant \|u - \tilde{u}\| \leqslant 2\delta_1\} = \emptyset$. It follows from Lemma 2.10 that there exists $\varepsilon_1 > 0$ such that

$$\big\|J'(u)\big\| \geqslant \varepsilon_1, \quad u \in \big\{u \in L^2[0,1]\colon \delta_1 \leqslant \|u - \tilde{u}\| \leqslant 2\delta_1\big\}.$$

Since $\lim_{n\to\infty} \|\phi(t_n,u) - \tilde{u}\| = 0$, we can choose two sequences $\{t_n'\}$, $\{t_n''\}$, and a subsequence of $\{t_n\}$, without loss of generality, we can assume that this subsequence is $\{t_n\}$ itself, such that $t_n < t_n' < t_n''$, and

$$\big\|\phi(t_n',u) - \tilde{u}\big\| = \delta_1, \qquad \big\|\phi(t_n'',u) - \tilde{u}\big\| = 2\delta_1, \qquad \big\|J'\big(\phi(t,u)\big)\big\| \geqslant \varepsilon_1, \quad t \in [t_n', t_n''].$$

Thus,

$$\delta_1 \leqslant \|\phi(t_n'', u) - \phi(t_n', u)\| = \left\| \int_{t_n'}^{t_n''} J'\big(\phi(t,u)\big)\,dt \right\| \leqslant \frac{1}{\varepsilon_1} \int_{t_n'}^{t_n''} \big\|J'\big(\phi(t,u)\big)\big\|^2\,dt = \frac{1}{\varepsilon_1} \int_{t_n'}^{t_n''} -\frac{d}{dt} J\big(\phi(t,u)\big)\,dt$$

$$= \frac{1}{\varepsilon_1}\big[J\big(\phi(t_n',u)\big) - J\big(\phi(t_n'',u)\big)\big].$$

Since the limit $\lim_{t\to+\infty} J(\phi(t,u))$ exists, and $t_n' \to +\infty$, $t_n'' \to +\infty$, then $\delta_1 \leqslant 0$, which is a contradiction. Therefore, $\phi(t,u) \to \tilde{u}$ in $L^2[0,1]$. The proof is completed. □

Let a and b with $a < b$ be two numbers. For each $u \in J_X^b \setminus (K_X^b \cup J_X^a \cup \operatorname{clos}_X(U_1^* \cup U_2^*))$, we define a number $\eta(u) > 0$ to be the supremum of all $\tau \in (0, \tau(u))$ such that $\phi(t,u) \in J_X^b \setminus (K_X^b \cup J_X^a \cup \operatorname{clos}_X(U_1^* \cup U_2^*))$ for all $t \in [0, \tau]$. Obviously, we have $0 < \eta(u) \leqslant \tau(u)$ for all $u \in J_X^b \setminus (K_X^b \cup J_X^a \cup \operatorname{clos}_X(U_1^* \cup U_2^*))$. In order to construct a deformation between two different levels of J in X and outside of $U_1^* \cup U_2^*$ in the space $C[0,1]$, we need the following assumption:

$$(J^*) \qquad \bigcup_{c\in(a,b)} \big(K_X^c \setminus \operatorname{clos}_X\big(U_1^* \cup U_2^*\big)\big) = \emptyset.$$

We shall follow a similar argument developed in [8] to construct a deformation from $(J_X^b \cup \operatorname{clos}_X(U_1^* \cup U_2^*)) \setminus K_X^b$ to $J_X^a \cup \operatorname{clos}_X(U_1^* \cup U_2^*)$. First we prove the existence of the limit $\lim_{t\to\eta(u)^-} \phi(t,u)$.

Lemma 2.12. *Assume the condition* (J^*) *holds. Then for each* $u \in J_X^b \setminus (K_X^b \cup J_X^a \cup \operatorname{clos}_X(U_1^* \cup U_2^*))$*, the limit* $\sigma(u) = \lim_{t\to\eta(u)^-} \phi(t,u)$ *exists in* $C[0,1]$ *and at least one of the three holds*:

(i) $\sigma(u) \in \partial_X U_1^* \cup \partial_X U_2^*$;
(ii) $J(\sigma(u)) = a$ *and* $J'(\sigma(u)) \neq 0$;
(iii) $J(\sigma(u)) = a$ *and* $J'(\sigma(u)) = 0$.

Moreover, either (i) *or* (ii) *holds if* $\eta(u) < \tau(u)$*, and* (iii) *holds if* $\eta(u) = \tau(u)$.

Proof. If $\eta(u) < \tau(u)$, then as a direct consequence of Lemma 2.7 the limit $\sigma(u) = \lim_{t\to\eta(u)^-} \phi(t,u)$ exists in $C[0,1]$ and either (i) or (ii) holds.

Now we assume $\eta(u) = \tau(u)$. Then the definition of $\eta(u)$ implies that $\phi(t,u) \in J_X^b \setminus (K_X^b \cup J_X^a \cup \mathrm{clos}_X(U_1^* \cup U_2^*))$ for all $t \in [0,\tau(u))$. First we claim that $\tau(u) = +\infty$. Indeed, for any $t_1, t_2 \in [0,\tau(u))$ with $t_1 < t_2$, it follows that

$$\begin{aligned}\|\phi(t_2,u) - \phi(t_1,u)\| &= \left\| \int_{t_1}^{t_2} \frac{d}{dt}\phi(t,u)\,dt \right\| = \left\| \int_{t_1}^{t_2} J'\big(\phi(t,u)\big)\,dt \right\| \leqslant \left(\int_{t_1}^{t_2} \big\|J'\big(\phi(t,u)\big)\big\|^2\,dt \right)^{1/2} (t_2-t_1)^{1/2} \\ &= \left(-\int_{t_1}^{t_2} \frac{d}{dt} J\big(\phi(t,u)\big)\,dt \right)^{1/2} (t_2-t_1)^{1/2} \leqslant (b-a)^{1/2}(t_2-t_1)^{1/2}.\end{aligned}$$

If $\tau(u) < +\infty$, then the limit $u^* = \lim_{t\to\tau(u)^-} \phi(t,u)$ exists and then $\phi(t,u)$ can be extended to a larger interval $[0,\tau(u)+\tau(u^*))$, which contradicts the maximality of the interval $[0,\tau(u))$. Thus, $\eta(u) = \tau(u) = +\infty$. Since

$$\int_0^{+\infty} \big\|J'\big(\phi(t,u)\big)\big\|^2\,dt \leqslant b-a,$$

there exists a sequence of positive numbers $\{t_n\}$ with $t_n \to +\infty$ such that $J'(\phi(t_n,u)) \to 0$ as $n\to\infty$. By Lemma 2.5, J satisfies P.S. condition on $L^2[0,1]$, so we may assume that $\phi(t_n,u) \to \tilde u$ and $J'(\tilde u) = 0$. If $\tilde u \in \mathrm{clos}_X(U_1^* \cup U_2^*)$, then the fact that $T(\mathrm{clos}_X U_1^*) \subset U_1^*$ and $T(\mathrm{clos}_X U_2^*) \subset U_2^*$ would imply $\tilde u \in U_1^* \cup U_2^*$. According to Lemma 2.11, $\{\phi(t,u)\colon t\in[0,+\infty)\}$ is bounded in $L^2[0,1]$, so $\{T\phi(t,u)\colon t\in[0,+\infty)\}$ and $\{\int_0^t e^{-t+s}T\phi(s,u)\,ds\colon t\in[0,+\infty)\}$ are both relatively compact in $C[0,1]$. It follows from (2.9) that $\{\phi(t,u)\colon t\in[0,+\infty)\}$ is relatively compact in $C[0,1]$. Thus there exists a subsequence $\{\phi(t_{n_k},u)\}$ such that $\phi(t_{n_k},u)\to\tilde u$ in $C[0,1]$. This implies for k large enough, $\phi(t_{n_k},u) \in U_1^*\cup U_2^*$, which is impossible and thus $\tilde u \notin \mathrm{clos}_X(U_1^*\cup U_2^*)$. Then it follows from the assumption (J^*) that $J(\tilde u) = a$. Using the assumption that $K_X^a \setminus (\mathrm{clos}_X(U_1^*\cup U_2^*))$ is a finite set, $\tilde u$ is an isolated critical point. Then it follows from Lemma 2.11 that $\lim_{t\to+\infty}\phi(t,u) = \tilde u$ in $L^2[0,1]$. Therefore, it follows from (2.9) that $\lim_{t\to+\infty}\phi(t,u) = \tilde u$ in $C[0,1]$. The proof is completed. □

By Lemma 2.12, the limit $\lim_{t\to\eta(u)^-}\phi(t,u)$ exists in $C[0,1]$, and then defines a map $\sigma : J_X^b \setminus (K_X^b \cup J_X^a \cup \mathrm{clos}_X(U_1^*\cup U_2^*)) \to C[0,1]$. We now prove the continuity of σ.

Lemma 2.13. *Assume the condition (J^*) holds. Then $\sigma : J_X^b \setminus (K_X^b \cup J_X^a \cup \mathrm{clos}_X(U_1^*\cup U_2^*)) \to C[0,1]$ is continuous.*

Proof. Let $u \in J_X^b \setminus (K_X^b \cup J_X^a \cup \mathrm{clos}_X(U_1^*\cup U_2^*))$. If $\eta(u) < \tau(u)$ then either (i) or (ii) in Lemma 2.12 occurs. By the definition of $\eta(u)$, $\phi(t,u) \notin J_X^a \cup \mathrm{clos}(U_1^*\cup U_2^*)$ for all $t\in[0,\eta(u))$. In case (i) it follows from Lemma 2.9 that $\phi(t,u)\in U_1^*\cup U_2^*$ for all $t\in(\eta(u),\tau(u))$, while in case (ii) $J(\phi(t,u)) < a$ for all $t\in(\eta(u),\tau(u))$. In either case the continuous dependence of $\phi(t,u)$ on initial data implies that σ is continuous at u.

Now we consider case (iii) in Lemma 2.12. If σ is not continuous at some u, then there exists a sequence $\{u_n\}\subset J_X^b \setminus (K_X^b \cup J_X^a \cup \mathrm{clos}_X(U_1^*\cup U_2^*))$ satisfying $\|u_n - u\|_0 \to 0$ as $n\to\infty$ but $\inf_{n\in\mathbb{N}}\|\sigma(u_n)-\sigma(u)\|_0 > 0$. Since $\eta(u) = +\infty$, Lemma 2.12 implies that there is a sequence of positive numbers $\{t_n\}$ with $t_n\to+\infty$ such that

$$\inf_{n\in\mathbb{N}}\big\|\phi(t_n,u_n)-\sigma(u)\big\|_0 > 0. \tag{2.11}$$

Since $\lim_{k\to\infty}\|\phi(t_k,u)-\sigma(u)\|_0 = 0$, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\lim_{k\to\infty}\|\phi(t_k,u_{n_k}) - \sigma(u)\|_0 = 0$. We denote $\{u_{n_k}\}$ by $\{v_k\}$, then $\lim_{n\to\infty}\|\phi(t_n,v_n)-\sigma(u)\|_0 = 0$. Consider the set $L = \{\phi(t,v_n)\colon t\in[0,\eta(v_n)),\ n\in\mathbb{N}\}$. Since $\eta(u)=\tau(u)=+\infty$, Lemma 2.12 and the continuous dependence of $\phi(t,u)$ on initial data imply that for any $\alpha>0$, there exists $N_\alpha\in\mathbb{N}$ such that $\{\phi(t,v_n)\colon t\in[0,\alpha],\ n\geqslant N_\alpha\}$ is bounded. If L is not bounded, then there is a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ and a sequence of positive numbers $\{t_{n_k}\}$ with $t_{n_k}\to+\infty$ as $k\to\infty$ such that $\|\phi(t_{n_k},v_{n_k})-\sigma(u)\| > k$ for all $k\in\mathbb{N}$. Since $\sigma(u)$ is an isolated critical point, there exist $\delta>0$, $\varepsilon>0$, a subsequence of $\{t_{n_k}\}$, without loss of generality, we may assume that this subsequence is $\{t_{n_k}\}$ itself, and two sequences $\{t'_{n_k}\}$, $\{t''_{n_k}\}$ such that $t_{n_k} < t'_{n_k} < t''_{n_k} < \eta(v_{n_k})$ and

$$\big\|\phi\big(t'_{n_k},v_{n_k}\big)-\sigma(u)\big\| = \delta, \qquad \big\|\phi\big(t''_{n_k},v_{n_k}\big)-\sigma(u)\big\| = 2\delta, \qquad \big\|J'\big(\phi(t,v_{n_k})\big)\big\| \geqslant \varepsilon, \quad t\in\big[t'_{n_k},t''_{n_k}\big].$$

Using the way of proof of Lemma 2.11, we have

$$\delta \leqslant \left\|\phi\left(t''_{n_k}, v_{n_k}\right) - \phi\left(t'_{n_k}, v_{n_k}\right)\right\| \leqslant 1/\varepsilon \cdot \left[J\left(\phi(t_{n_k}, v_{n_k})\right) - a\right].$$

Letting $k \to \infty$, we have that $\delta \leqslant 0$, which is a contradiction. Therefore, the set $L = \{\phi(t, v_n):\ t \in [0, \eta(v_n)),\ n \in \mathbb{N}\}$ is bounded in $L^2[0,1]$. Notice that

$$\phi(t, v_n) = e^{-t} v_n + \int_0^t e^{-t+s} T\phi(s, v_n)\, ds$$

implies that L is relatively compact in $C[0,1]$. Then there exists a subsequence $\{(t_{n_k}, v_{n_k})\}$ of $\{(t_n, v_n)\}$ such that $\lim_{k\to\infty} \|\phi(t_{n_k}, v_{n_k}) - \sigma(u)\|_0 = 0$, which is a contradiction with (2.11). Then σ is continuous. The proof is completed. □

Using η and σ we are ready to construct a deformation from $(J_X^b \setminus K_X^b) \cup \mathrm{clos}_X(U_1^* \cup U_2^*)$ to $J_X^a \cup \mathrm{clos}_X(U_1^* \cup U_2^*)$.

Lemma 2.14. *Assume the condition (J^*) holds. Then $J_X^a \cup \mathrm{clos}_X(U_1^* \cup U_2^*)$ is a strong deformation retract of $(J_X^b \setminus K_X^b) \cup \mathrm{clos}_X(U_1^* \cup U_2^*)$.*

Proof. Let $u \in J_X^b \setminus (K_X^b \cup J_X^a \cup \mathrm{clos}_X(U_1^* \cup U_2^*))$ and define

$$\eta_1(u) := J(u) - \lim_{t\to\eta(u)^-} J\left(\phi(t,u)\right) = J(u) - J\left(\sigma(u)\right).$$

We will need the set O defined by

$$O := \left\{(s,u):\ s \in \left[0, \eta_1(u)\right),\ u \in J_X^b \Big\backslash \left(K_X^b \cup J_X^a \cup \mathrm{clos}_X\left(U_1^* \cup U_2^*\right)\right)\right\}.$$

The function $s = T_u(t) := J(u) - J(\phi(t,u))$ being increasing in $t \in [0, \eta(u))$ admits an inverse function $t = T_u^{-1}(s)$. Define $\psi : O \to (J_X^b \setminus K_X^b) \cup \mathrm{clos}_X(U_1^* \cup U_2^*)$ as

$$\psi(s,u) = \phi\left(T_u^{-1}(s), u\right), \quad (s,u) \in O.$$

Define $H : [0,1] \times ((J_X^b \setminus K_X^b) \cup \mathrm{clos}_X(U_1^* \cup U_2^*)) \to (J_X^b \setminus K_X^b) \cup \mathrm{clos}_X(U_1^* \cup U_2^*)$ as

$$H(s,u) = \begin{cases} u, & (s,u) \in [0,1] \times (J_X^a \cup \mathrm{clos}_X(U_1^* \cup U_2^*)), \\ \psi(s\eta_1(u), u), & (s,u) \in [0,1) \times (J_X^b \setminus (K_X^b \cup J_X^a \cup \mathrm{clos}_X(U_1^* \cup U_2^*))), \\ \sigma(u), & (s,u) \in \{1\} \times (J_X^b \setminus (K_X^b \cup J_X^a \cup \mathrm{clos}_X(U_1^* \cup U_2^*))). \end{cases}$$

Lemma 2.13 together with an argument similar to its proof shows that H is continuous. Here we omit the details. Thus $J_X^a \cup \mathrm{clos}_X(U_1^* \cup U_2^*)$ is a strong deformation retract of $(J_X^b \setminus K_X^b) \cup \mathrm{clos}_X(U_1^* \cup U_2^*)$. The proof is completed. □

To prove the existence of the first sign-changing solution of the mountain pass type, we define

$$\Gamma := \left\{h \in C\left([0,1], X\right):\ h(0) \in \mathrm{clos}_X U_1^*,\ h(1) \in \mathrm{clos}_X U_2^*\right\}$$

and

$$c := \inf_{h\in\Gamma} \sup_{h(t)\in X\setminus \mathrm{clos}_X(U_1^*\cup U_2^*)} J\left(h(t)\right).$$

Since J is bounded from below on $\mathrm{clos}_X U_1^*$, c is a finite number.

Theorem 2.15. *The equation $u = Tu$ has a sign-changing solution u_2 such that the fixed point index of T at u_2 is -1. More precisely, there exists $r > 0$ such that $B(u_2, r) \cap \mathrm{clos}_X(U_1^* \cup U_2^*) = \emptyset$, $B(u_2, r) \subset X$, and $i(T, B(u_2, r), X) = -1$, where $B(u_2, r) = \{u \in C[0,1]:\ \|u - u_2\|_0 < r\}$.*

Proof. For any $\varepsilon > 0$, it follows from the definition of c that there is a path $h : [0,1] \to J_X^{c+\varepsilon} \cup \operatorname{clos}_X(U_1^* \cup U_2^*)$ joining $h(0) \in \operatorname{clos}_X U_1^*$ and $h(1) \in \operatorname{clos}_X U_2^*$. It follows also that $\operatorname{clos}_X U_1^*$ and $\operatorname{clos}_X U_2^*$ lie in different path components of $J_X^{c-\varepsilon} \cup \operatorname{clos}_X(U_1^* \cup U_2^*)$. Now we consider the long exact sequence of singular homology groups with coefficients in $\mathbb{R}$

$$\begin{aligned} H_1\big(J_X^{c+\varepsilon} \cup \operatorname{clos}_X\big(U_1^* \cup U_2^*\big), J_X^{c-\varepsilon} \cup \operatorname{clos}_X\big(U_1^* \cup U_2^*\big)\big) &\to^{\partial_*} H_0\big(J_X^{c-\varepsilon} \cup \operatorname{clos}_X\big(U_1^* \cup U_2^*\big)\big) \\ &\to^{i_*} H_0\big(J_X^{c+\varepsilon} \cup \operatorname{clos}_X\big(U_1^* \cup U_2^*\big)\big). \end{aligned}$$

Since

$$\operatorname{Im} \partial_* = \ker i_* \neq 0,$$

it implies that

$$H_1\big(J_X^{c+\varepsilon} \cup \operatorname{clos}_X\big(U_1^* \cup U_2^*\big), J_X^{c-\varepsilon} \cup \operatorname{clos}_X\big(U_1^* \cup U_2^*\big)\big) \neq 0.$$

Since we have assumed that the equation $u = Tu$ has only a finite number of solutions in $X \setminus (U_1^* \cup U_2^*)$, choosing an ε small enough, using Lemma 2.14 and the exactness, we arrive at

$$H_1\big(J_X^c \cup \operatorname{clos}_X\big(U_1^* \cup U_2^*\big), \big(J_X^c \setminus K_X^c\big) \cup \operatorname{clos}_X\big(U_1^* \cup U_2^*\big)\big) \neq 0.$$

The excision property of homology groups yields

$$H_1\big(J_X^c \setminus \operatorname{clos}_X\big(U_1^* \cup U_2^*\big), J_X^c \setminus \big(K_X^c \cup \operatorname{clos}_X\big(U_1^* \cup U_2^*\big)\big)\big) \neq 0,$$

which implies $K_X^c \setminus \operatorname{clos}_X(U_1^* \cup U_2^*) \neq \emptyset$. Assume $K_X^c \setminus \operatorname{clos}_X(U_1^* \cup U_2^*) = \{u_1^*, u_2^*, \ldots, u_m^*\}$. Then we have

$$H_1\big(J_X^c \setminus \operatorname{clos}_X\big(U_1^* \cup U_2^*\big), J_X^c \setminus \big(K_X^c \cup \operatorname{clos}_X\big(U_1^* \cup U_2^*\big)\big)\big) \cong \bigoplus_{i=1}^m C_1\big(J, u_i^*\big),$$

where $C_1(J, u_i^*)$ is the critical group of J at u_i^*. Thus there is $u_2 \in \{u_1^*, u_2^*, \ldots, u_m^*\}$ such that $C_1(J, u_2) \neq 0$. By [2, Proposition 3.3], we have $C_q(J, u_2) \cong \delta_{q1}\mathbb{R}$. Thus there exists $r > 0$ such that $B(u_2, r) \cap \operatorname{clos}_X(U_1^* \cup U_2^*) = \emptyset$, $B(u_2, r) \subset X$, and

$$i\big(T, B(u_2, r), X\big) = \sum_{q=0}^{\infty} (-1)^q \dim C_q(J, u_2) = -1.$$

Since $C_q(J, 0) \cong \delta_{qk_0}\mathbb{R}$ by Theorem 2.6 and $k_0 \geqslant 2$, $u_2 \neq 0$ is a sign-changing solution. The proof is completed. □

We are ready to prove the theorem.

Proof of Theorem 1.1. Let R be the number from Lemma 2.4. Set

$$\begin{aligned} U &:= \big\{u \in X: \|u\|_0 < R\big\}, \\ U_1 &:= \big\{u \in U_1^*: \|u\|_0 < R\big\}, \\ U_2 &:= \big\{u \in U_2^*: \|u\|_0 < R\big\}, \end{aligned}$$

and define $K_t : C[0,1] \to C[0,1]$ by $K_t u = Tu - te_0$, $u \in C[0,1]$. By Lemma 2.4, the fixed point index $i(K_t, U, X)$ is well defined for all $t \geqslant 0$. Choose $C > 0$ such that $\|Tu\|_0 < C$ for all $u \in C[0,1]$ with $\|u\|_0 \leqslant R$. If $u \in X$ and $t \geqslant 0$ satisfy $K_t u = u$, then Lemma 2.4 implies that $\|u\|_0 \leqslant R$, and then

$$t\|e_0\|_0 \leqslant R + C.$$

So, there is a t_0 large enough such that K_{t_0} has no fixed points in $\operatorname{clos}_X U$. The homotopy invariance of fixed point index implies

$$i(T, U, X) = i(K_0, U, X) = i(K_{t_0}, U, X) = 0.$$

Similarly,

$$i(T, U_2, X) = 0.$$

It follows from Lemma 2.4 that

$$i(T, U_1, X) = i\big(T, U_1^*, X\big) = 1.$$

If T has no fixed point in $U \setminus (U_1 \cup U_2)$ except 0 and u_2, then the additivity property of fixed point index implies

$$i(T, U, X) = i(T, U_1, X) + i(T, U_2, X) + i\big(T, B(0, r_0), X\big) + i\big(T, B(u_2, r), X\big),$$

that is, according to Theorems 2.6 and 2.15,

$$0 = 1 + 0 + (-1)^i - 1,$$

which is impossible. Therefore, T has a fixed point in $U \setminus (U_1 \cup U_2 \cup \{0, u_2\})$, which is a second sign-changing solution of $u = Tu$. □

References

[1] A. Bensedik, M. Bouchekif, Symmetry and uniqueness of positive solutions for a Neumann boundary value problem, Appl. Math. Lett. 20 (2007) 419–426.
[2] T. Bartsch, K.-C. Chang, Z.-Q. Wang, On the Morse indices of sign changing solutions of nonlinear elliptic problems, Math. Z. 233 (2000) 655–677.
[3] K.C. Chang, Infinite Dimensional Morse Theory and Multiple Solution Problems, Birkhäuser, Boston, 1993.
[4] K. Deimling, Ordinary Differential Equations in Banach Spaces, Lecture Notes in Math., vol. 596, Springer-Verlag, Berlin, 1977.
[5] C. Li, S. Li, Multiple solutions and sign-changing solutions of a class of nonlinear elliptic equations with Neumann boundary condition, J. Math. Anal. Appl. 298 (2004) 14–32.
[6] F. Li, Y. Li, Z. Liang, Existence and multiplicity of solutions to 2mth-order ordinary differential equations, J. Math. Anal. Appl. 331 (2007) 958–977.
[7] F. Li, Y. Li, Z. Liang, Existence of solutions to nonlinear Hammerstein integral equations and applications, J. Math. Anal. Appl. 323 (2006) 209–227.
[8] Z. Liu, J. Sun, An elliptic problem with jumping nonlinearities, Nonlinear Anal. 63 (2005) 1070–1082.
[9] Z. Liu, J. Sun, Invariant sets of descending flow in critical point theory with applications to nonlinear differential equations, J. Differential Equations 172 (2001) 257–299.
[10] J. Sun, W. Li, Multiple positive solutions to second-order Neumann boundary value problems, Appl. Math. Comput. 146 (2003) 187–194.
[11] J. Sun, W. Li, S. Cheng, Three positive solutions for second-order Neumann boundary value problems, Appl. Math. Lett. 17 (2004) 1079–1084.
[12] C. Tang, Some existence theorems for the sublinear Neumann boundary value problem, Nonlinear Anal. 48 (2002) 1003–1011.