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Wong's comparison theorem for second order linear dynamic equations on time scales

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ABSTRACT

We obtain Wong-type comparison theorems for second order linear dynamic equations on a time scale. The results obtained extend and are motivated by Wong's comparison theorems. As a particular application of our results, we show that the difference equation

$$\Delta^2 x(n) + b\frac{(-1)^n}{n^c}x(n+1) = 0$$

where $c < 1$, $b \neq 0$, is oscillatory.

1. Introduction

In a fundamental paper [15], Wong extended and improved oscillation criteria and comparison theorem due to many earlier authors for the differential equation

$$x'' + p(t)x = 0$$

in the cases when $p(t)$ is not eventually of one sign. His work also surveyed earlier results of Wintner [16], Fite [11], Hille [13], and Hartman [12] for the cases when $\int^\infty p(s)\,ds$ exists. In this paper we obtain a 'Wong-type' comparison theorem for dynamic equations on time scales by means of a second-level Riccati integral equation on time scales (see [1,2]) which Wong [15] refers to as a new Riccati integral equation in the continuous case (see [3] for the discrete case). Using this approach, one is able to handle various critical cases. These ideas are of particular importance in treating the case when $P(t) := \int_t^\infty p(s)\,ds$ is not of one sign for large t.

Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup\mathbb{T} = \infty$. Consider the second order dynamic equation on time scale

$$\left[x^\Delta(t)\right]^\Delta + p(t)x^\sigma(t) = 0, \tag{1.1}$$

$$\left[x^\Delta(t)\right]^\Delta + q(t)x^\sigma(t) = 0, \tag{1.2}$$

where p, q are right-dense continuous functions on $\mathbb{T}$ and $\int_{t_0}^\infty p(s)\,\Delta s$ and $\int_{t_0}^\infty q(s)\,\Delta s$ are convergent.

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For $\mathbb{T}=\mathbb{R}$, suppose that $\lim_{t\to\infty}\int_{t_0}^t p(s)\,ds$ and $\lim_{t\to\infty}\int_{t_0}^t q(s)\,ds$ exist. Define $P(t)=\int_t^\infty p(s)\,ds$, $Q(t)=\int_t^\infty q(s)\,ds$. Wong [15] proved the following

Wong's Comparison Theorem. *Suppose that the improper integral $\int_{t_0}^\infty P(t)\,dt$ converges, and in addition we have*

$$\bar{P}(t)\geqslant\bar{Q}(t)\quad\text{and}\quad P(t)+\bar{P}(t)\geqslant Q(t)+\bar{Q}(t),$$

for large t, where

$$\bar{P}(t)=\int_t^\infty P^2(s)E_P(s,t)\,ds,\qquad E_P(s,t)=\exp\left(2\int_t^s P(\tau)\,d\tau\right).$$

$$\bar{Q}(t)=\int_t^\infty Q^2(s)F_Q(s,t)\,ds,\qquad F_Q(s,t)=\exp\left(2\int_t^s Q(\tau)\,d\tau\right).$$

Then if $x''+p(t)x=0$ is nonoscillatory, $x''+q(t)x=0$ is also nonoscillatory.

In [14], Willett considered the equation

$$x''+\frac{b\sin\lambda t}{t^c}x=0 \tag{1.3}$$

and proved that (1.3) is oscillatory when $c<1$, $\lambda b\neq 0$ and showed that 1 is a critical value, i.e., (1.3) is nonoscillatory when $c>1$.

One can show that (1.3) is oscillatory when $c<1$, $\lambda b\neq 0$, by using Wong's comparison theorem and the following Erbe's comparison theorem [6]. Willett [14] used the Riccati integral equation and weighted averaging technique to establish oscillation in this case.

Erbe's Comparison Theorem. *Assume that $a(t)\in C^1[t_0,\infty)$ satisfies*

$$a(t)\geqslant 1\quad\text{and}\quad a''(t)\leqslant 0.$$

Then $x''+p(t)x=0$ is oscillatory implies $x''+a(t)p(t)x=0$ is oscillatory.

To see how Wong's comparison theorem and Erbe's comparison theorem can be used directly to obtain the oscillation result in the case when $c<1$, $\lambda b\neq 0$, let

$$x''+p(t)x=0 \tag{1.4}$$

where $p(t)=\frac{b\sin\lambda t}{t^c}$, $c<1$, $\lambda b\neq 0$,

$$x''+q(t)x=0 \tag{1.5}$$

where $q(t)=\frac{a}{t^2}$, $\frac{1}{4}<a<\frac{1}{2}$.

Assume first that $\frac{1}{2}<c<1$. It is easy to see that

$$P(t)=\int_t^\infty p(s)\,ds=\frac{b\cos\lambda t}{\lambda t^c}+O\left(\frac{1}{t^{c+1}}\right),\qquad Q(t)=\frac{a}{t}. \tag{1.6}$$

So $\int_t^\infty P(t)\,dt$ converges. Therefore, given $0<\epsilon<1$, there exists $T>0$ such that $\exp(2\int_t^s P(\tau)\,d\tau)\geqslant 1-\epsilon$ if $T\leqslant t<s$. Hence,

$$\bar{P}(t)=\int_t^\infty P^2(s)\exp\left(2\int_t^s P(\tau)\,d\tau\right)ds\geqslant(1-\epsilon)\int_t^\infty\left[\frac{b\cos\lambda s}{\lambda s^c}+O\left(\frac{1}{s^{c+1}}\right)\right]^2ds.$$

Since

$$\int_t^\infty\frac{\cos 2\lambda s}{s^{2c}}\,ds=-\frac{\sin 2\lambda t}{2\lambda t^{2c}}+O\left(\frac{1}{t^{2c+1}}\right),$$

we have

$$\bar{P}(t) \geqslant (1-\epsilon)\left[\frac{b^2}{2\lambda^2(2c-1)t^{2c-1}} + O\left(\frac{1}{t^{2c}}\right)\right], \quad \text{for large } t \geqslant T > 0, \tag{1.7}$$

and

$$\bar{Q}(t) = \int_t^{\infty} Q^2(s)\exp\left(2\int_t^s Q(\tau)\,d\tau\right)ds = \frac{a^2}{(1-2a)t}. \tag{1.8}$$

Note that $0 < 2c-1 < c < 1$, so we have, for large t

$$\frac{1}{t^{2c-1}} > \frac{1}{t^c} > \frac{1}{t}.$$

Therefore from (1.6)–(1.8), we get, for large t

$$\bar{P}(t) \geqslant \bar{Q}(t) \quad \text{and} \quad P(t) + \bar{P}(t) \geqslant Q(t) + \bar{Q}(t).$$

By Hille's theorem [13], the Euler equation (1.5) is oscillatory, for $\frac{1}{4} < a < \frac{1}{2}$. So by Wong's comparison theorem, (1.4) is oscillatory, for $\frac{1}{2} < c < 1$, $\lambda b \neq 0$.

To show further that (1.4) is oscillatory for $c < 1$, we take $a(t) = t^{\alpha}$, $0 < \alpha < 1$, then we have $a(t) \geqslant 1$, $a''(t) \leqslant 0$, for large t. Using Erbe's comparison theorem repeatedly and the fact that $x'' + b\frac{\sin\lambda t}{t^{\frac{3}{4}}}x = 0$ is oscillatory, we get that

$$x'' + t^{\beta} b \frac{\sin\lambda t}{t^{\frac{3}{4}}} x = 0$$

is oscillatory, for large t and all $\beta > 0$, $\lambda b \neq 0$. So the equation

$$x'' + b\frac{\sin\lambda t}{t^{\frac{3}{4}-\beta}}x = 0$$

is oscillatory, for large t and all $\beta > 0$, $\lambda b \neq 0$. This means that the equation

$$x'' + b\frac{\sin\lambda t}{t^c}x = 0$$

is oscillatory, for large t and all $c < \frac{3}{4}$, $\lambda b \neq 0$.

In addition to the above proof that (1.4) is oscillatory for $\frac{1}{2} < c < 1$, $\lambda b \neq 0$, we get that (1.4) is oscillatory for $c < 1$, $\lambda b \neq 0$.

From the above example, we have the following

Remark 1. The importance of Wong's comparison theorem is that by comparing Eq. (1.4) where $p(t)$ is not nonnegative to the oscillatory Euler equation (1.5) where $q(t)$ is positive, we get that (1.4) is oscillatory, for $\frac{1}{2} < c < 1$, $\lambda b \neq 0$.

Remark 2. When we use Erbe's comparison theorem, we need to look for an appropriate oscillatory equation as a good criterion. Here Wong's comparison theorem supplies such a criterion. In place of Willett's weighted averaging technique, here we use Erbe's comparison theorem. In [1], we give some interesting applications of the time scale version [9] of Erbe's comparison theorem.

Willett [14] proved that $x'' + b\frac{\sin\lambda t}{t}x = 0$, $|\frac{b}{\lambda}| > \frac{1}{\sqrt{2}}$, is oscillatory. In [2], by using Wong's oscillation theorem [15, Theorem 2], we can also get Willett's result. Here if we choose the oscillatory equation $x'' \pm \lambda\frac{\sin\lambda t}{t}x = 0$, as a criterion of Erbe's theorem and take $a(t) = At^{\alpha}$, $A > 0$, $0 < \alpha < 1$, by repeatedly using Erbe's theorem, it is easy to obtain that (1.4) is oscillatory, for $c < 1$, $\lambda b \neq 0$.

Remark 3. Kwong [8] showed that Erbe's comparison theorem is still true for a wider class of function $a(t)$. See also [9] for the time scales extension of these results.

In this paper, we obtain a 'Wong-type' comparison theorem for dynamic equations on time scales by means of a second-level Riccati integral equation on time scales (see [1,2]). As a special application, we get that the difference equation

$$\Delta^2 x(n) + b\frac{(-1)^n}{n^c}x(n+1) = 0 \tag{1.9}$$

where $c < 1$, $b \neq 0$, is oscillatory. From [1], it follows that 1 is the critical value, i.e. (1.9) is nonoscillatory when $c > 1$.

For completeness (see [4] and [5] for elementary results for the time scale calculus), we recall some basic results for dynamic equations and the calculus on time scales. The forward jump operator is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T}\colon\ s > t\},$$

and the backward jump operator is defined by

$$\rho(t) = \sup\{s \in \mathbb{T}\colon\ s < t\},$$

where $\inf\emptyset = \sup\mathbb{T}$, where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ we say t is right-dense, while if $\rho(t) = t$ and $t \neq \inf\mathbb{T}$ we say t is left-dense. Given an interval $[c,d] := \{t \in \mathbb{T}\colon\ c \leqslant t \leqslant d\}$ in $\mathbb{T}$ the notation $[c,d]^\kappa$ denotes the interval $[c,d]$ in case $\rho(d) = d$ and denotes the interval $[c,d)$ in case $\rho(d) < d$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by C_{rd}. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are delta differentiable on $[c,d]^\kappa$ and whose delta derivative is rd-continuous on $[c,d]^\kappa$ is denoted by C^1_{rd}.

We recall that a solution of Eq. (1.1) is said to be oscillatory on $[a,\infty)$ in case it is neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory. Eq. (1.1) is said to be oscillatory in case all of its solutions are oscillatory.

We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided that

$$1 + \mu(t)p(t) \neq 0, \quad t \in \mathbb{T}.$$

We denote the set of all $f : \mathbb{T} \to \mathbb{R}$ which are right-dense continuous and regressive by $\Re$. If $p \in \Re$, then we can define the exponential function by

$$e_p(ts) = \exp\left(\int_s^t \xi_{\mu(\tau)}\big(p(\tau)\big)\,\Delta\tau\right)$$

for $t \in \mathbb{T}$, $s \in \mathbb{T}^k$, where $\xi_h(z)$ is the cylinder transformation, which is given by

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

2. Notations and lemmas

Lemmas 2.1–2.4 and the definitions of Condition C and Condition D were introduced in [1] and [2].

Let $\hat{\mathbb{T}} := \{t \in \mathbb{T}\colon\ \mu(t) > 0\}$ and let χ denote the characteristic function of $\hat{\mathbb{T}}$. The following condition, which will be needed later in Section 3, imposes a lower bound on the graininess function $\mu(t)$, for $t \in \hat{\mathbb{T}}$. More precisely, we introduce the following (see [7]):

Condition C. We say that $\mathbb{T}$ satisfies Condition C if there exists an $M > 0$ such that

$$\chi(t) \leqslant M\mu(t), \quad t \in \mathbb{T}.$$

Lemma 2.1. *Assume that $\mathbb{T}$ satisfies Condition* C *and suppose that Eq.* (1.1) *is nonoscillatory. Let $x(t)$ is a solution of* (1.1) *with $x(t) > 0$ on $[t_0,\infty)$. Then $z(t) = \frac{x^\Delta(t)}{x(t)}$ is a solution of the Riccati equation*

$$z^\Delta + p(t) + \frac{z^2}{1 + \mu(t)z} = 0$$

on $[t_0,\infty)$. Moreover, if $\int_{t_0}^\infty p(t)\,\Delta t$ is convergent, then $\int_{t_0}^\infty \frac{z^2(s)}{1+\mu(s)z(s)}\,\Delta s$ is also convergent and $\lim_{t\to\infty} z(t) = 0$.

We will also need below conditions which guarantee that $\int_1^t \frac{1}{s}\,\Delta s$ does not grow faster than $M\ln t$, for some $M > 0$. For a time scale $\mathbb{T}$, the following example shows that the inequality $\int_1^t \frac{1}{s}\,\Delta s \leqslant M\ln t$, for $M > 1$, does not hold in general without some additional restrictions.

Example. Consider the time scale

$$\mathbb{T} = \big\{2^{2^k}\colon\ k \in \mathbb{N}_0\big\}.$$

It is easy to see from the definition of the integral that for $t_k = 2^{2^k}$

$$\lim_{k\to\infty}\frac{\int_{t_0}^{t_k}\frac{1}{s}\,\Delta s}{\ln t_k}=\frac{1}{\ln 2}\lim_{k\to\infty}\frac{1}{2^k}\sum_{j=0}^{k-1}\left(2^{2^j}-1\right)=\infty.$$

So we shall impose an additional assumption on the time scale $\mathbb{T}$ to obtain Lemma 2.2. We note first that if $\mathbb{T}$ satisfies Condition C, then the set

$$\check{\mathbb{T}}=\{t\in\mathbb{T}:\ t>0 \text{ is isolated or right scattered or left scattered}\}$$

is necessarily countable since a bounded real interval can contain only finitely many elements of $\check{\mathbb{T}}$.

We introduce the following

Condition D. Suppose that $\mathbb{T}$ satisfies Condition C and let

$$\check{\mathbb{T}}=\{t_0,t_1,t_2,\dots,t_k,\dots\},$$

where

$$0<t_0<t_1<t_2<\cdots<t_k<\cdots.$$

Then we say $\mathbb{T}$ satisfies *Condition* D. If there is a constant $K>1$ such that

$$\max_{k\in\mathbb{N}}\left\{\frac{t_{k+1}-t_k}{t_k-t_{k-1}}\right\}\leqslant K,\quad \text{for all } k\geqslant 1. \tag{2.1}$$

Lemma 2.2. *Assume that* (1.1) *is nonoscillatory, $x(t)>0$ is a solution of* (1.1). *$\mathbb{T}$ satisfies Condition* D. *Then we have, for $t\in\mathbb{T}$, $t>t_1$,*

$$\ln\frac{x(t)}{x(t_1)}\leqslant\int_{t_1}^{t}\frac{x^{\Delta}(t)}{x(t)}\,\Delta t\quad \textit{and}\quad \int_{t_1}^{t}\frac{1}{s}\,\Delta s\leqslant K\ln\frac{t}{t_0}.$$

Lemma 2.3. *Assume that $\int_{t_0}^{\infty}p(t)\,\Delta t$ is convergent, $P(t)=\int_t^{\infty}p(s)\,\Delta s$, $\mu(t)$ is bounded and satisfies Condition* D. *If* (1.1) *is nonoscillatory, then there is a $T\in[t_0,\infty)$ such that*

$$\int_T^{\infty}P^2(t)\times\frac{e_P(t,T)}{e_{-P}(t,T)}\,\Delta t<\infty.$$

Also, if $x(t)>0$ is a positive solution of (1.1) *on $[T,\infty)$ and $z(t):=\frac{x^{\Delta}(t)}{x(t)}$, then $z(t)$ is a solution of the Riccati equation*

$$z^{\Delta}+p(t)+\frac{z^2}{1+\mu(t)z}=0$$

on $[T,\infty)$, with $1+\mu(t)z(t)>0$, on $[T,\infty)$. Furthermore,

$$v(t)=\int_t^{\infty}\frac{z^2(s)}{1+\mu(t)z(s)}\,\Delta s>0$$

satisfies the integral equation

$$v(t)=\frac{e_{-P}(t,T)}{e_P(t,T)}\int_t^{\infty}\frac{e_P(s,T)}{e_{-P}(\sigma(s),T)}\left[P^2(s)+v(s)v\big(\sigma(s)\big)\right]\Delta s \tag{2.2}$$

for large $t\in[T,\infty)$, where $e_P(t,T)$ and $e_{-P}(t,T)$ are the exponential functions.

Lemma 2.4. *Assume that $\int_{t_0}^{\infty}p(t)\,\Delta t$ is convergent, $P(t)=\int_t^{\infty}p(s)\,\Delta s$, $1\pm\mu(t)P(t)>0$, for large t. If $\int_T^{\infty}P^2(t)\times\frac{e_P(t,T)}{e_{-P}(t,T)}\,\Delta t$ converges and there exists a function $v(t)>0$, for large t, satisfying*

$$v(t)\geqslant\frac{e_{-P}(t,T)}{e_P(t,T)}\int_t^{\infty}\frac{e_P(s,T)}{e_{-P}(\sigma(s),T)}\left[P^2(s)+v(s)v\big(\sigma(s)\big)\right]\Delta s \tag{2.3}$$

for large t, then (1.1) *is nonoscillatory.*

The following lemma appears in [9].

Lemma 2.5 *(Riccati technique). Eq.* (1) *is nonoscillatory if and only if there exist* $T \in [\tau, \infty)$ *and a function* u *satisfying the Riccati dynamic inequality*

$$u^{\Delta}(t) + p(t) + \frac{u^2(t)}{1+\mu(t)u(t)} \leqslant 0$$

with $1 + \mu(t)u(t) > 0$ *for* $t \in [T, \infty)$.

3. Main theorem

Theorem 3.1. *Assume that* $\int^{\infty} p(t)\,\Delta t$ *and* $\int^{\infty} q(t)\,\Delta t$ *are convergent. Let*

$$P(t) = \int_t^{\infty} p(s)\,\Delta s, \qquad Q(t) = \int_t^{\infty} q(s)\,\Delta s.$$

Assume that

$$\int^{\infty} P(t)\,\Delta t \quad \text{and} \quad \int^{\infty} P^2(t)\,\Delta t \tag{3.1}$$

are convergent, $\mu(t)$ *is bounded and satisfies Condition* D. *Let*

$$\bar{P}(t) := \frac{e_{-P}(t,T)}{e_P(t,T)} \int_t^{\infty} \frac{e_P(s,T)}{e_{-P}(\sigma(s),T)} P^2(s)\,\Delta s, \tag{3.2}$$

$$\bar{Q}(t) := \frac{e_{-Q}(t,T)}{e_Q(t,T)} \int_t^{\infty} \frac{e_Q(s,T)}{e_{-Q}(\sigma(s),T)} Q^2(s)\,\Delta s. \tag{3.3}$$

If

$$\frac{2P(t) + \bar{P}(t) + \bar{P}(\sigma(t))}{1 - \mu(t)P(t) - \mu(t)\bar{P}(\sigma(t))} \geqslant \frac{2Q(t) + \bar{Q}(t) + \bar{Q}(\sigma(t))}{1 - \mu(t)Q(t) - \mu(t)\bar{Q}(\sigma(t))}, \tag{3.4}$$

$$\frac{\bar{P}(t)\bar{P}(\sigma(t))}{1 - \mu(t)P(t) - \mu(t)\bar{P}(\sigma(t))} \geqslant \frac{\bar{Q}(t)\bar{Q}(\sigma(t))}{1 - \mu(t)Q(t) - \mu(t)\bar{Q}(\sigma(t))}, \tag{3.5}$$

$$P(t) + \bar{P}\big(\sigma(t)\big) \geqslant Q(t) + \bar{Q}\big(\sigma(t)\big), \tag{3.6}$$

then if (1.1) *is nonoscillatory,* (1.2) *is also nonoscillatory.*

Remark. For $\mathbb{T} = \mathbb{R}$, the assumptions of Wong's comparison theorem stated earlier in Section 1 imply that $\int^{\infty} P^2(t)\,dt$ is convergent. Therefore, Theorem 3.1 may be considered as an extension of Wong's comparison theorem.

Proof. In the first place, we will prove that

$$\lim_{t\to\infty} \bar{P}(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \bar{Q}\big(\sigma(t)\big) = 0.$$

By the definition of $e_P(t,T)$ [5, p. 57], we have

$$e_P(t,T) = \exp\left(\int_T^t \xi_{\mu(\tau)}\big(P(\tau)\big)\,\Delta\tau\right),$$

where

$$\xi_h\big(P(\tau)\big) = \begin{cases} \frac{1}{h}\operatorname{Log}(1 + hP(\tau)), & h > 0, \\ 0, & h = 0 \end{cases}$$

and where Log is the principal logarithm function.

Note that $1 \pm \mu(t)P(t) > 0$. So for $\mu(\tau) > 0$, by Taylor's formula, we get that

$$\xi_{\mu(\tau)}\big(P(\tau)\big) = \frac{1}{\mu(\tau)} \ln\big(1 + \mu(\tau)P(\tau)\big) = P(\tau) - \frac{\mu(\tau)P^2(\tau)}{2} + \mu(\tau)o\big(P^2(\tau)\big).$$

This same formula holds when $\mu(\tau)=0$, since we have in this case that $\xi_{\mu(\tau)}(P(\tau))=P(\tau)$.

So in any case, we have for all $\mu(\tau)$ the formula

$$\xi_{\mu(\tau)}\big(P(\tau)\big)=P(\tau)-\frac{\mu(\tau)P^2(\tau)}{2}+\mu(\tau)o\big(P^2(\tau)\big).$$

Since $\int^\infty P(t)\,\Delta t$ and $\int^\infty P^2(t)\,\Delta t$ are both convergent, we get that

$$\int_T^\infty \xi_{\mu(\tau)}\big(P(\tau)\big)\,\Delta\tau$$

is convergent. Similarly we have $\int_T^\infty \xi_{\mu(\tau)}(-P(\tau))\,\Delta\tau$ is also convergent. Therefore by the definitions of $e_{-P}(t,T)$ and $e_P(t,T)$, we get that there exist constants $c_1>0$, $c_2>0$ such that

$$c_1\leqslant\frac{e_{-P}(t,T)}{e_P(t,T)}\leqslant c_2. \tag{3.7}$$

Note that $e_{-P}(\sigma(s),T)=[1-\mu(s)P(s)]e_{-P}(s,T)$ and using (3.1), (3.7) and the definition (3.2) of $\bar P(t)$, we get $\lim_{t\to\infty}\bar P(t)=0$.

From (3.6) and $P(t)\to 0$, $Q(t)\to 0$ as $t\to\infty$, we obtain $\lim_{t\to\infty}\bar Q(\sigma(t))=0$. So we have

$$1-\mu(t)P(t)-\mu(t)\bar P\big(\sigma(t)\big)>0,\qquad 1-\mu(t)Q(t)-\mu(t)\bar Q\big(\sigma(t)\big)>0 \tag{3.8}$$

for large t.

Assume (1.1) is nonoscillatory, $x(t)>0$ is a solution of (1.1), and $z(t)=\frac{x^\Delta(t)}{x(t)}$ is a solution of the Riccati equation

$$z^\Delta+p(t)+\frac{z^2}{1+\mu(t)z}=0$$

on $[T,\infty)$. By Lemma 2.1, integrating the Riccati equation from t to ∞, we get that

$$z(t)=\int_t^\infty p(s)\,\Delta s+\int_t^\infty R(s)\,\Delta s,$$

where $R(s)=\frac{z^2}{1+\mu(t)z}$.

Define $v(t)=\int_t^\infty R(s)\,\Delta s>0$. We have $z(t)=P(t)+v(t)$. From Lemma 2.3, we have

$$v(t)=\frac{e_{-P}(t,T)}{e_P(t,T)}\int_t^\infty\frac{e_P(s,T)}{e_{-P}(\sigma(s),T)}\big[P^2(s)+v(s)v\big(\sigma(s)\big)\big]\,\Delta s \tag{3.9}$$

for large t, where $e_P(t,T)$ and $e_{-P}(t,T)$ are the exponential functions.

Let

$$\rho(t)=\frac{e_{-P}(t,T)}{e_P(t,T)}\int_t^\infty\frac{e_P(s,T)}{e_{-P}(\sigma(s),T)}v(s)v\big(\sigma(s)\big)\,\Delta s. \tag{3.10}$$

So

$$v(t)=\bar P(t)+\rho(t).$$

Using the product rule and $\rho(t)=\rho(\sigma(t))-\mu(t)\rho^\Delta(t)$, we get that

$$\begin{aligned}\rho^\Delta(t)&=\frac{-2P(t)\rho(\sigma(t))}{1-\mu(t)P(t)}-\frac{v(t)v(\sigma(t))}{1-\mu(t)P(t)}\\
&=\frac{-2P(t)\rho(\sigma(t))}{1-\mu(t)P(t)}-\frac{[\bar P(t)+\rho(t)][\bar P(\sigma(t))+\rho(\sigma(t))]}{1-\mu(t)P(t)} &(3.11)\\
&=\frac{-2P(t)\rho(\sigma(t))}{1-\mu(t)P(t)}-\frac{[\bar P(t)\bar P(\sigma(t))+\rho(t)\rho(\sigma(t))]}{1-\mu(t)P(t)} &(3.12)\\
&\quad-\frac{\bar P(t)\rho(\sigma(t))}{1-\mu(t)P(t)}-\frac{\bar P(\sigma(t))[\rho(\sigma(t))-\mu(t)\rho^\Delta(t)]}{1-\mu(t)P(t)}. &(3.13)\end{aligned}$$

By solving $\rho^\Delta(t)$ and noticing that (3.8) and $\bar P\geqslant 0$, $\rho\geqslant 0$, we get that

$$\rho^{\Delta}(t) = -\frac{[2P(t) + \bar{P}(t) + \bar{P}(\sigma(t))]\rho(\sigma(t))}{1 - \mu(t)P(t) - \mu(t)\bar{P}(\sigma(t))} \tag{3.14}$$

$$- \frac{\bar{P}(t)\bar{P}(\sigma(t)) + \rho(t)\rho(\sigma(t))}{1 - \mu(t)P(t) - \mu(t)\bar{P}(\sigma(t))}. \tag{3.15}$$

By (3.4)–(3.6), (3.14) and (3.15), we get

$$\rho^{\Delta}(t) \leqslant -\frac{[2Q(t) + \bar{Q}(t) + \bar{Q}(\sigma(t))]\rho(\sigma(t))}{1 - \mu(t)Q(t) - \mu(t)\bar{Q}(\sigma(t))} \tag{3.16}$$

$$- \frac{\bar{Q}(t)\bar{Q}(\sigma(t)) + \rho(t)\rho(\sigma(t))}{1 - \mu(t)Q(t) - \mu(t)\bar{Q}(\sigma(t))}. \tag{3.17}$$

So

$$\rho^{\Delta}(t)\big[1 - \mu(t)Q(t)\big] \leqslant -\big[2Q(t) + \bar{Q}(t) + \bar{Q}\big(\sigma(t)\big)\big]\rho\big(\sigma(t)\big) - \bar{Q}(t)\bar{Q}\big(\sigma(t)\big) - \rho(t)\rho\big(\sigma(t)\big) + \rho^{\Delta}(t)\mu(t)\bar{Q}\big(\sigma(t)\big).$$

Dividing by $1 - \mu(t)Q(t)$ and rearranging, we get that

$$\begin{aligned}
\rho^{\Delta}(t) &\leqslant \frac{-2Q(t)\rho(\sigma(t))}{1 - \mu(t)Q(t)} - \frac{\bar{Q}(t)\bar{Q}(\sigma(t)) + \rho(t)\rho(\sigma(t))}{1 - \mu(t)Q(t)} - \frac{\bar{Q}(t)\rho(\sigma(t))}{1 - \mu(t)Q(t)} - \frac{\bar{Q}(\sigma(t))[\rho(\sigma(t)) - \mu(t)\rho^{\Delta}(t)]}{1 - \mu(t)Q(t)} \\
&= \frac{-2Q(t)\rho(\sigma(t))}{1 - \mu(t)Q(t)} - \frac{\bar{Q}(t)\bar{Q}(\sigma(t)) + \rho(t)\rho(\sigma(t))}{1 - \mu(t)Q(t)} - \frac{\bar{Q}(t)\rho(\sigma(t))}{1 - \mu(t)Q(t)} - \frac{\bar{Q}(\sigma(t))\rho(t)}{1 - \mu(t)Q(t)} \\
&= \frac{-2Q(t)\rho(\sigma(t))}{1 - \mu(t)Q(t)} - \frac{[\bar{Q}(t) + \rho(t)][\bar{Q}(\sigma(t)) + \rho(\sigma(t))]}{1 - \mu(t)Q(t)}.
\end{aligned} \tag{3.18}$$

Let $w(t) = \bar{Q}(t) + \rho(t)$. Note that

$$\bar{Q}^{\Delta}(t) = -\frac{2Q(t)\bar{Q}(\sigma(t))}{1 - \mu(t)Q(t)} - \frac{Q^2(t)}{1 - \mu(t)Q(t)}.$$

By (3.18), we get that

$$w^{\Delta}(t) \leqslant -\frac{2Q(t)w(\sigma(t))}{1 - \mu(t)Q(t)} - \frac{w(t)w(\sigma(t)) + Q^2(t)}{1 - \mu(t)Q(t)}. \tag{3.19}$$

Note that

$$\begin{aligned}
w^{\Delta}(t)\big[1 - \mu(t)Q(t)\big] + 2Q(t)w\big(\sigma(t)\big) &= w^{\Delta}(t) + \big[w\big(\sigma(t)\big) - w^{\Delta}(t)\mu(t)\big]Q(t) + Q(t)w\big(\sigma(t)\big) \\
&= w^{\Delta}(t) + Q(t)w(t) + Q(t)w\big(\sigma(t)\big),
\end{aligned}$$

and using (3.19), we get

$$w^{\Delta}(t) + Q(t)\big[w\big(\sigma(t)\big) + w(t)\big] + w(t)w\big(\sigma(t)\big) + Q^2(t) \leqslant 0. \tag{3.20}$$

Let

$$S(t) = \frac{e_Q(t, T)}{e_{-Q}(t, T)} w(t). \tag{3.21}$$

We have

$$S^{\Delta}(t) = \frac{2Q(t)e_Q(t, T)w(t)}{[1 - \mu(t)Q(t)]e_{-Q}(t, T)} + w^{\Delta}(t)\frac{e_Q(\sigma(t), T)}{e_{-Q}(\sigma(t), t)}.$$

Using (3.20), (3.21), we get that

$$S^{\Delta}(t) \leqslant Q(t)S(t) - Q(t)S\big(\sigma(t)\big) - \frac{e_{-Q}(t, T)}{e_Q(t, T)} S(t)S\big(\sigma(t)\big) - \frac{Q^2(t)e_Q(\sigma(t), T)}{e_{-Q}(\sigma(t), T)}. \tag{3.22}$$

Since $S(t) - S(\sigma(t)) = -\mu(t)S^{\Delta}(t)$, after rearranging, we get that

$$S^{\Delta}(t) \leqslant -\frac{e_{-Q}(t, T)S(t)S(\sigma(t))}{[1 + \mu(t)Q(t)]e_Q(t, T)} - \frac{Q^2(t)e_Q(\sigma(t), T)}{[1 + \mu(t)Q(t)]e_{-Q}(\sigma(t), T)}. \tag{3.23}$$

Note that $1 \pm \mu(t)P(t) > 0$, for large t. So $w(t) > 0$, $S(t) > 0$. From (3.23), we have $S^{\Delta}(t) \leqslant 0$. Suppose that $\lim_{t\to\infty} S(t) = A \geqslant 0$. Integrating from $\tau \geqslant T$ to t and rearranging, we get

$$S(t)+\int_{\tau}^{t}\frac{e_{-Q}(s,T)S(s)S(\sigma(s))}{[1+\mu(s)Q(s)]e_Q(s,T)}\,\Delta s+\int_{\tau}^{t}\frac{Q^2(s)e_Q(\sigma(s),T)}{[1+\mu(s)P(s)]e_{-Q}(\sigma(s),T)}\,\Delta s\leqslant S(\tau). \tag{3.24}$$

Let $t\to\infty$. Note that the integrands of the left integrals are positive. We get that

$$\int_{\tau}^{\infty}\frac{e_{-Q}(s,T)S(s)S(\sigma(s))}{[1+\mu(s)Q(s)]e_Q(s,T)}\,\Delta s+\int_{\tau}^{\infty}\frac{Q^2(s)e_Q(\sigma(s),T)}{[1+\mu(s)P(s)]e_{-Q}(\sigma(s),T)}\,\Delta s\leqslant S(\tau). \tag{3.25}$$

Noting $S(t)=\frac{e_Q(t,T)}{e_{-Q}(t,T)}w(t)$, we obtain that

$$\frac{e_{-Q}(\tau,T)}{e_Q(\tau,T)}\int_{\tau}^{\infty}\frac{e_Q(s,T)}{e_{-Q}(\sigma(s),T)}\big[Q^2(s)+w(s)w\big(\sigma(s)\big)\big]\,\Delta s\leqslant w(\tau). \tag{3.26}$$

Note that $e_{\pm Q}(\sigma(s),T)=[1\pm\mu(s)Q(s)]e_{\pm}Q(s,T)$ and $P(s)\to 0$, $Q(s)\to 0$ as $s\to\infty$. So by (3.25), we have

$$\int_{\tau}^{\infty}Q^2(s)\frac{e_Q(s,T)}{e_{-Q}(s,T)}\,\Delta s<\infty.$$

Note that (3.26) means that (2.3) holds. So from Lemma 2.4, we get that Eq. (1.2) is nonoscillatory. This completes the proof. □

For $\mathbb{T}=\mathbb{R}$, by Theorem 3.1, we get the following

Corollary 3.2. *Suppose that $\sum p_j$ and $\sum q_j$ are convergent. Let*

$$P_n=\sum_{j=n}^{\infty}p_j,\qquad Q_n=\sum_{j=n}^{\infty}q_j. \tag{3.27}$$

Assume that

$$\sum P_j\quad\text{and}\quad\sum P_j^2$$

are convergent.

Let $N\geqslant 0$ be so large that

$$|P_n|<1,\qquad 1-P_n-\bar{P}_{n+1}>0,\qquad |Q_n|<1,\qquad 1-Q_n-\bar{Q}_{n+1}>0,$$

for $n\geqslant N$. Define

$$r_n=\prod_{j=N}^{n-1}\frac{1-P_j}{1+P_j},\qquad f(j;n)=r_nr_{j+1}^{-1}(1+P_j)^{-1}, \tag{3.28}$$

$$\bar{P}_n=\sum_{j=n}^{\infty}f(j;n)P_j^2,\quad\text{for } j\geqslant n\geqslant N. \tag{3.29}$$

Define

$$s_n=\prod_{j=N}^{n-1}\frac{1-Q_j}{1+Q_j},\qquad g(j;n)=s_ns_{j+1}^{-1}(1+Q_j)^{-1}, \tag{3.30}$$

$$\bar{Q}_n=\sum_{j=n}^{\infty}g(j;n)Q_j^2,\quad\text{for } j\geqslant n\geqslant N. \tag{3.31}$$

If for large n

$$\frac{2P_n+\bar{P}_n+\bar{P}_{n+1}}{1-P_n-\bar{P}_{n+1}}\geqslant\frac{2Q_n+\bar{Q}_n+\bar{Q}_{n+1}}{1-Q_n-\bar{Q}_{n+1}}, \tag{3.32}$$

$$\frac{\bar{P}_n\bar{P}_{n+1}}{1-P_n-\bar{P}_{n+1}}\geqslant\frac{\bar{Q}_n\bar{Q}_{n+1}}{1-Q_n-\bar{Q}_{n+1}}, \tag{3.33}$$

$$P_n+\bar{P}_{n+1}\geqslant Q_n+\bar{Q}_{n+1}, \tag{3.34}$$

then if $\Delta^2x(n)+p_nx(n+1)=0$ is nonoscillatory, $\Delta^2x(n)+q_nx(n+1)=0$ is also nonoscillatory.

4. Example

Consider the difference equation

$$\Delta^2 x(n) + p_n x(n+1) = 0, \tag{4.1}$$

where $p_n = b\frac{(-1)^n}{n^c}$, $c < 1$, $b \neq 0$ and

$$\Delta^2 x(n) + q_n x(n+1) = 0, \tag{4.2}$$

where $q_n = \frac{a}{n^2}$, $\frac{1}{4} < a < \frac{1}{2}$.

In the first place, we will prove that (4.1) is oscillatory, when $\frac{1}{2} < c < 1$.

Define that $P_j = P(j)$. Then

$$P(2k) = \frac{b}{(2k)^c} - \frac{b}{(2k+1)^c} + \frac{b}{(2k+2)^c} - \frac{b}{(2k+3)^c} + \cdots.$$

By an appropriate Taylor expansion, we get that

$$\frac{1}{(2k)^c} - \frac{1}{(2k+1)^c} = \frac{(1+\frac{1}{2k})^c - 1}{(2k+1)^c} = \frac{\frac{c}{2k}[1+o(1)]}{(2k+1)^c}.$$

So

$$P(2k) = \frac{\frac{bc}{2k}[1+o(1)]}{(2k+1)^c} + \frac{\frac{bc}{2k+2}[1+o(1)]}{(2k+3)^c} \sim \frac{bc}{(2k)(2k+1)^c} + \frac{bc}{(2k+2)(2k+3)^c} + \cdots \sim \frac{b}{2(2k)^c}.$$

Similarly, we have

$$P(2k+1) \sim -\frac{b}{2(2k+1)^c}.$$

So

$$P(n) \sim (-1)^n \frac{b}{2n^c}. \tag{4.3}$$

Therefore the series $\sum_{k=n}^{\infty} P_k$ converges. Since $\frac{1}{2} < c < 1$, we have $\sum_{k=n}^{\infty} P_k^2$ converges. Using $\ln(1+x) = x - \frac{1}{2}x^2 + o(x^2)$ as $x \to \infty$, we have for large j

$$\ln\left(1 - \frac{2P_j}{1+P_j}\right) = -\frac{2P_j}{1+P_j} - \frac{1}{2}\left(\frac{2P_j}{1+P_j}\right)^2 + o\left(\left(\frac{2P_j}{1+P_j}\right)^2\right) \tag{4.4}$$

as $\to \infty$. Also, we have

$$\frac{P_j}{1+P_j} = P_j\left[1 - P_j + O\left(P_j^2\right)\right]. \tag{4.5}$$

So from (4.3)–(4.5), we have

$$\sum_{j=N}^{\infty} \ln\left(\frac{1-P_j}{1+P_j}\right) = \sum_{j=N}^{\infty} \ln\left(1 - \frac{2P_j}{1+P_j}\right)$$

is convergent. So given $0 < \epsilon < 1$ (see (3.28))

$$r_n = \prod_{N}^{n-1} \frac{1-P_j}{1+P_j} = \exp\left(\sum_{j=N}^{n-1} \ln\left(1 - \frac{2P_j}{1+P_j}\right)\right) > 1 - \epsilon, \quad \text{for large } N.$$

We also have given $0 < \epsilon_1 < 1$

$$f(j,n) = \frac{r_n}{r_{j+1}(1+P_j)} \geqslant 1 - \epsilon_1, \quad j \geqslant n \geqslant N,$$

where we used $P_j \to 0$, $r_j \to 1$.

By (4.3), we get that given $0 < \epsilon_2, \epsilon_3 < 1$ (see (3.29))

$$\begin{aligned}\bar{P}_n &= \sum_{j=n}^{\infty} f(j,n)P_j^2 \geqslant (1-\epsilon_1)\sum_{j=n}^{\infty} P_j^2\\ &\geqslant (1-\epsilon_1)(1-\epsilon_2)\left(\frac{b^2}{4}\right)\left(\frac{1}{n^{2c}}+\frac{1}{(n+1)^{2c}}+\cdots\right)\\ &\geqslant (1-\epsilon_1)(1-\epsilon_2)(1-\epsilon_3)\frac{b^2}{4(2c-1)n^{2c-1}}. \end{aligned} \tag{4.6}$$

In the following, we will estimate $\bar{Q}_n$. Note that

$$\int_n^{\infty}\frac{1}{t^2}\,dt \leqslant \sum_{j=n}^{\infty}\frac{1}{j^2} \leqslant \int_{n-1}^{\infty}\frac{1}{t^2}\,dt.$$

We have

$$Q_n = \sum_{j=n}^{\infty}\frac{a}{j^2} = \frac{a}{n} + O\left(\frac{1}{n^2}\right). \tag{4.7}$$

Using appropriate Taylor expansions, we have

$$\ln\left(1-\frac{2Q_j}{1+Q_j}\right) = -\frac{2Q_j}{1+Q_j} - \frac{1}{2}\left(\frac{2Q_j}{1+Q_j}\right)^2 + o\left(\left(\frac{2Q_j}{1+Q_j}\right)^2\right) \tag{4.8}$$

as $\to\infty$. Also, we have

$$\frac{Q_j}{1+Q_j} = Q_j\left[1-Q_j+O\left(Q_j^2\right)\right]. \tag{4.9}$$

From (4.7)–(4.9), we have

$$\sum_{j=n}^{\infty}\ln\left(\frac{1-Q_j}{1+Q_j}\right) = \sum_{j=n}^{\infty}\ln\left(1-\frac{2Q_j}{1+Q_j}\right) = -\sum_{j=n}^{\infty}\left(\frac{2a}{j}+O\left(\frac{1}{j^2}\right)\right). \tag{4.10}$$

Note that the series (4.10) is not convergent. From (4.10) and the inequality $\sum_{i=n}^{j}\frac{1}{i} \leqslant \int_{n-1}^{j}\frac{1}{t}\,dt = \ln\frac{j}{n-1}$, we get that given $0<\epsilon_4<1$ (see (3.30))

$$\frac{s_n}{s_{j+1}} = \exp\left(\sum_{i=n}^{j}(-1)\ln\left(1-\frac{2Q_i}{1+Q_i}\right)\right) \tag{4.11}$$

$$= \exp\sum_{i=n}^{j}\left(\frac{2a}{i}+O\left(\frac{1}{i^2}\right)\right) \leqslant (1+\epsilon_4)\left(\frac{j}{n-1}\right)^{2a}, \tag{4.12}$$

for large n. Using $(1+Q_j)^{-1} = 1+O(\frac{1}{n})$, we get that given $0<\epsilon_5<1$ (see (3.30))

$$g(j,n) = \frac{s_n}{s_{j+1}(1+Q_j)} \leqslant (1+\epsilon_4)(1+\epsilon_5)\left(\frac{j}{n-1}\right)^{2a},$$

for large n. Note that $Q_n \sim \frac{a}{n}$ and $\sum_{j=n}^{\infty}\frac{1}{j^{2-2a}} \sim \frac{1}{(1-2a)n^{1-2a}}$, we get that given $0<\epsilon_6,\epsilon_7,\epsilon_8<1$

$$\begin{aligned}\bar{Q}_n &= \sum_{j=n}^{\infty} g(j,n)Q_j^2\\ &\leqslant (1+\epsilon_4)(1+\epsilon_5)(1+\epsilon_6)\frac{a^2}{(n-1)^{2a}}\sum_{j=n}^{\infty}\frac{j^{2a}}{j^2}\\ &\leqslant (1+\epsilon_4)(1+\epsilon_5)(1+\epsilon_6)(1+\epsilon_7)\frac{a^2}{(1-2a)(n-1)^{2a}n^{1-2a}}\\ &\leqslant (1+\epsilon_4)(1+\epsilon_5)(1+\epsilon_6)(1+\epsilon_7)(1+\epsilon_8)\frac{a^2}{(1-2a)n}, \end{aligned} \tag{4.13}$$

where we use $(\frac{n}{n-1})^{2a} < 1+\epsilon_8$, for large n.

Since $0 < 2c - 1 < c < 1$, for $\frac{1}{2} < c < 1$, so we have for large n

$$\frac{1}{n^{2c-1}} > \frac{1}{n^c} > \frac{1}{n}.$$

Therefore from (4.3), (4.6), (4.7), (4.13), we obtain that (3.32)–(3.34) are satisfied, for large n. By Hille's theorem [10, p. 60], (4.2) is oscillatory for $\frac{1}{4} < a < \frac{1}{2}$. So by Corollary 3.2 (4.1) is oscillatory, for $\frac{1}{2} < c < 1$, $b \neq 0$.

To show that (4.1) is oscillatory, for all $c < 1$, we need the following useful comparison theorem [9] which is the time scales version of Erbe's comparison theorem stated earlier.

Theorem 4.1. *Assume $a(t) \in C^1_{\mathrm{rd}}$ and*

(i) $a(t) \geqslant 1$,
(ii) $\mu(t)a^{\Delta}(t) \geqslant 0$,
(iii) $a^{\Delta\Delta}(t) \leqslant 0$.

Then $x^{\Delta\Delta} + p(t)x^{\sigma} = 0$ is oscillatory on $[t_0, \infty)$ implies $x^{\Delta\Delta} + a(t)p(t)x^{\sigma} = 0$ is oscillatory on $[t_0, \infty)$.

By the above proof, the equation (note that $\frac{1}{2} < \frac{3}{4} < 1$)

$$\Delta^2 x(n) + b\frac{(-1)^n}{n^{\frac{3}{4}}}x(n+1) = 0$$

is oscillatory, for $b \neq 0$.

Let $a(n) = n^{\alpha}$, $0 < \alpha < 1$. We have $\Delta a(n) \geqslant 0$, $\Delta^2 a(n) \leqslant 0$ for large n. Using Theorem 4.1 repeatedly, we get that

$$\Delta^2 x(n) + bn^{\beta}\frac{(-1)^n}{n^{\frac{3}{4}}}x(n+1) = 0$$

is oscillatory, for $b \neq 0$, $\beta > 0$. So the equation

$$\Delta^2 x(n) + b\frac{(-1)^n}{n^{\frac{3}{4}-\beta}}x(n+1) = 0$$

is oscillatory, for $b \neq 0$, $\beta > 0$. This means that the equation

$$\Delta^2 x(n) + b\frac{(-1)^n}{n^c}x(n+1) = 0$$

is oscillatory, for $b \neq 0$, $c < \frac{3}{4}$. In addition to the above proof that (4.1) is oscillatory, for $\frac{1}{2} < c < 1$, $b \neq 0$, we obtain that (4.1) is oscillatory, for $c < 1$, $b \neq 0$.

The value 1 is a critical value, since in [1] we prove (4.1) is nonoscillatory for $c > 1$.

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