J. Math. Anal. Appl. 350 (2009) 340–347

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Iterative methods for nonlinear complementarity problems on isotone projection cones ☆

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ARTICLE INFO

Article history:
Received 22 May 2008
Available online 2 October 2008
Submitted by J.D.M. Wright

Keywords:
Nonlinear complementarity problems
Projection onto cones
Isotone projection cones
Iterative methods
Generalized order monotone mappings
Generalized Lipschitz mappings

ABSTRACT

In this paper we present a recursion related to a nonlinear complementarity problem defined by a closed convex cone in a Hilbert space and a continuous mapping defined on the cone. If the recursion is convergent, then its limit is a solution of the nonlinear complementarity problem. In the case of isotone projection cones sufficient conditions are given for the mapping so that the recursion to be convergent.

1. Introduction

If $(H, \langle\cdot,\cdot\rangle)$ is a Hilbert space $K \subseteq H$ a closed convex cone, K^* the dual cone of K, and $\mathbf{f}: K \to H$ a mapping, then the nonlinear complementarity problem defined by K and $\mathbf{f}$ is the problem of finding an $\mathbf{x}^* \in K$ such that $\mathbf{f}(\mathbf{x}^*) \in K^*$ and $\langle \mathbf{x}^*, \mathbf{f}(\mathbf{x}^*)\rangle = 0$. Complementarity problems are used to model several problems of economics, physics and engineering and they occur in constraint qualifications for mathematical programming too. It is known that $\mathbf{x}^*$ is a solution of the nonlinear complementarity problem defined by K and $\mathbf{f}$ if and only if $\mathbf{x}^*$ is a fixed point of the mapping $K \ni \mathbf{x} \mapsto \mathbf{P}_K(\mathbf{x} - \mathbf{f}(\mathbf{x}))$, where $\mathbf{P}_K$ is the projection mapping onto K. Therefore, it is natural to consider the recursion

$$\mathbf{x}^{n+1} = \mathbf{P}_K\left(\mathbf{x}^n - \mathbf{f}(\mathbf{x}^n)\right), \tag{1}$$

where n is a nonnegative integer and $\mathbf{x}^0 \in K$. If the sequence $\{\mathbf{x}^n\}_{n\in\mathbb{N}}$ is convergent to $\mathbf{x}^* \in K$ and the mapping $\mathbf{f}$ is continuous, then taking the limit in the recursion (1) as n approaches infinity, we obtain that $\mathbf{x}^*$ is a fixed point of the mapping $K \ni \mathbf{x} \mapsto \mathbf{P}_K(\mathbf{x} - \mathbf{f}(\mathbf{x}))$ and therefore a solution of the nonlinear complementarity problem defined by K and $\mathbf{f}$. In this paper we will always suppose that $\mathbf{f}$ is continuous. The central issue of the paper will be to find conditions under which the sequence $\{\mathbf{x}^n\}_{n\in\mathbb{N}}$ is convergent. In the section "Preliminaries" we will recall several definitions and fix the notations. In particular, we will define classes of cones and mappings which will be used in the section "Main results" to generate solutions of nonlinear complementarity problems. The main result of this paper complements the results of Section 6 of [1] and is related to the results of [2] too. We will also consider recursion (1) with $\mathbf{f}$ replaced by a positive scaling of $\mathbf{f}$. As a particular case, we will consider the problem of finding the zeros of $\mathbf{f}$. Recursions for complementarity problems, variational inequalities and optimization problems, similar to (1), were considered in several other works, for example [3–13]. However, neither of these works used the order induced by the cone for analyzing the convergence. Instead, they used the

☆ The author was supported by the Hungarian Research Grant OTKA 60480.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.09.066

Banach fixed point theorem based approach, assuming Kachurovskii–Minty–Browder type monotonicity (see [14–17]) and global Lipschitz properties for $\mathbf{f}$.

2. Preliminaries

Let H be a Hilbert space and $K \subseteq H$. K is called a *closed convex cone*, if it is a closed convex set and for any $\lambda > 0$ and $\mathbf{x} \in K$, $\lambda\mathbf{x} \in K$. A closed convex cone K is called *pointed* if $K \cap (-K) = \{0\}$.

If $K \subseteq H$ is a closed convex cone, then

$$K^* = \{\mathbf{y} \in H\colon\ \langle \mathbf{x}, \mathbf{y}\rangle \geqslant 0 \text{ for all } \mathbf{x} \in K\}$$

is called the *dual cone* of K.

A relation ρ on H is called *reflexive* if $\mathbf{x}\rho\mathbf{x}$ for all $\mathbf{x} \in H$. A relation ρ on H is called *transitive* if $\mathbf{x}\rho\mathbf{y}$ and $\mathbf{y}\rho\mathbf{z}$ imply $\mathbf{x}\rho\mathbf{z}$. A relation ρ is called *antisymmetric* if $\mathbf{x}\rho y$ and $\mathbf{y}\rho\mathbf{x}$ imply $\mathbf{x} = \mathbf{y}$. A relation ρ on H is called a *preorder* if it is reflexive and transitive. A preorder is called *order* if it is antisymmetric. A relation ρ on H is called *translation invariant* if $\mathbf{x}\rho\mathbf{y}$ implies $(\mathbf{x}+\mathbf{z})\rho(\mathbf{y}+\mathbf{z})$ for any $\mathbf{z} \in H$. A relation ρ on H is called *scale invariant* if $\mathbf{x}\rho\mathbf{y}$ implies $(\lambda\mathbf{x})\rho(\lambda\mathbf{y})$ for any $\lambda > 0$. A relation ρ on H is called *continuous* if for any two convergent sequences $\{\mathbf{x}^n\}_{n\in\mathbb{N}}$ and $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ with $\mathbf{x}^n\rho\mathbf{y}^n$ for all $n \in \mathbb{N}$ we have $\mathbf{x}^*\rho\mathbf{y}^*$, where $\mathbf{x}^*$ and $\mathbf{y}^*$ are the limits of $\{\mathbf{x}^n\}_{n\in\mathbb{N}}$ and $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$, respectively.

The relation ρ on H is a continuous, translation and scale invariant preorder if and only if it is induced by a closed convex cone $K \subseteq H$; that is, $\rho = \leqslant_K$, where $\mathbf{x} \leqslant_K \mathbf{y}$ if and only if $\mathbf{y} - \mathbf{x} \in K$. For simplicity we will denote "$\leqslant_K$" by "$\leqslant$". The closed convex cone K can be written as $K = \{\mathbf{x} \in H\colon\ 0 \leqslant \mathbf{x}\}$ and it is called the positive cone of the preorder "$\leqslant$". The triplet $(H, \langle\cdot,\cdot\rangle, K)$ is called an *ordered vector space*. If the closed convex cone K is pointed, then the preorder "$\leqslant$" becomes an order. A closed convex cone K is called *regular* if every decreasing sequence of elements in K is convergent.

The ordered vector space $(H, \langle\cdot,\cdot\rangle, K)$ is called a *vector lattice* if for every $\mathbf{x}, \mathbf{y} \in H$ there exist $\mathbf{x} \wedge \mathbf{y} := \inf\{\mathbf{x}, \mathbf{y}\}$ and $\mathbf{x} \vee \mathbf{y} := \sup\{\mathbf{x}, \mathbf{y}\}$. In this case we say that the cone K is *latticial* and for each $\mathbf{x} \in H$ we denote $\mathbf{x}^+ = \mathbf{0} \vee \mathbf{x}$, $\mathbf{x}^- = \mathbf{0} \vee (-\mathbf{x})$ and $|\mathbf{x}| = \mathbf{x} \vee (-\mathbf{x})$. Then, $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$ and $|\mathbf{x}| = \mathbf{x}^+ + \mathbf{x}^-$.

Recall that the pointed closed convex cone $K \subset H$ is called an *isotone projection cone* (see [1,2,18–20]) if from $\mathbf{y} - \mathbf{x} \in K$ it follows that $\mathbf{P}_K(\mathbf{y}) - \mathbf{P}_K(\mathbf{x}) \in K$, where $\mathbf{P}_K$ is the projection onto K. By using the order relation defined by K, this condition can be written as $\mathbf{x} \leqslant \mathbf{y} \Rightarrow \mathbf{P}_K(\mathbf{x}) \leqslant \mathbf{P}_K(\mathbf{y})$ [20]. Every isotone projection cone is latticial and regular [2]. A closed generating cone in $\mathbb{R}^n$ is an isotone projection cone if and only if it is polyhedral and correct (this result and the corresponding notions can be found in [20]). Such cones are used for abstract convex programming problems in [21] (without making the connection with the ordering induced by the cone). Let K be a closed convex cone and $\mathbf{f}\colon K \to H$. The nonlinear complementarity problem defined by K and $\mathbf{f}$ will be denoted $NCP(\mathbf{f}, K)$.

3. Main results

Let H be a Hilbert space, $K \subset H$ a closed convex cone and $\mathbf{f}\colon K \to H$ a continuous mapping. In this section we will consider the recursion (1). Our main goal is to find conditions for K and $\mathbf{f}$ such that the sequence $\{\mathbf{x}^n\}_{n\in\mathbb{N}}$ to be convergent. If $\{\mathbf{x}^n\}_{n\in\mathbb{N}}$ is convergent, then its limit $\mathbf{x}^*$ is a solution of the nonlinear complementarity problem $NCP(\mathbf{f}, K)$. First we state two lemmas on which our main results are based.

Lemma 1. *Let H be a Hilbert space, $K \subset H$ a closed convex cone and $\mathbf{f}\colon K \to H$ a continuous mapping. Consider the recursion* (1)*. If the sequence $\{\mathbf{x}^n\}_{n\in\mathbb{N}}$ is convergent and $\mathbf{x}^*$ is its limit, then $\mathbf{x}^*$ is a solution of the nonlinear complementarity problem $NCP(\mathbf{f}, K)$.*

Proof. Indeed, taking the limit in (1), it follows that $\mathbf{x}^*$ is a fixed point of the operator $K \ni \mathbf{x} \mapsto \mathbf{P}_K(\mathbf{x} - \mathbf{f}(\mathbf{x}))$. It is known that in this case $\mathbf{x}^*$ is a solution of the nonlinear complementarity problem $NCP(\mathbf{f}, K)$. $\square$

Lemma 2. *Let H be a Hilbert space, $K \subset H$ a closed convex regular cone and $\mathbf{f}\colon K \to H$ a continuous mapping. Consider the recursion* (1)*. If the sequence $\{\mathbf{x}^n\}_{n\in\mathbb{N}}$ is monotone decreasing, then it is convergent and its limit $\mathbf{x}^*$ is a solution of the nonlinear complementarity problem $NCP(\mathbf{f}, K)$.*

Proof. Since K is regular, the sequence $\{\mathbf{x}^n\}_{n\in\mathbb{N}}$ is convergent. Hence, the remaining assertion follows from Lemma 1. $\square$

The following notion is inspired by the notion of pseudomonotonicity defined by Karamardian and Schaible in [22]. If $H = \mathbb{R}$, then the notion of pseudomonotonicity defined for f coincides with the notion of pseudomonotone decreasing for $-f$ defined here.

Definition 1. Let H be a Hilbert space, $K \subseteq H$ a closed convex cone and $\leqslant$ the preorder generated by K. The mapping $\mathbf{f}\colon K \to H$ is called *pseudomonotone decreasing* if for every $\mathbf{x}, \mathbf{y} \in K$

$$\mathbf{x} \leqslant \mathbf{y} \quad \text{and} \quad 0 \leqslant \mathbf{f}(\mathbf{y}) \quad \text{implies} \quad 0 \leqslant \mathbf{f}(\mathbf{x}).$$

Remark 1.

(1) If $\mathbf{f}$ is monotone decreasing, then it is pseudomonotone decreasing.
(2) If $\mathbf{f}(K) \subseteq K$, then $\mathbf{f}$ is pseudomonotone decreasing.
(3) If $\mathbf{f}(K) \subseteq H \setminus K$, then $\mathbf{f}$ is pseudomonotone decreasing.

Definition 2. Let H be a Hilbert space, $K \subseteq H$ a closed convex cone and $\leqslant$ the preorder generated by K. The function $f: K \to \mathbb{R}$ is called *pseudomonotone decreasing* if for every $\mathbf{x}, \mathbf{y} \in K$

$$\mathbf{x} \leqslant \mathbf{y} \quad \text{and} \quad 0 \leqslant f(\mathbf{y}) \quad \text{implies} \quad 0 \leqslant f(\mathbf{x}).$$

Remark 2. Let $H = \mathbb{R}^m$, $K = \mathbb{R}^m_+$ and $\mathbf{f} = (f_1, \dots, f_n): \mathbb{R}^m_+ \to \mathbb{R}^m$. Then, $\mathbf{f}$ is a pseudomonotone decreasing mapping if and only if f_i are pseudomonotone decreasing functions for all $i \in \{1, \dots, m\}$.

Theorem 1. *Let H be a Hilbert space, $K \subset H$ an isotone projection cone and $\mathbf{f}: K \to H$ a continuous mapping such that $K \cap \mathbf{f}^{-1}(K) \neq \emptyset$. Consider the recursion* (1) *starting from an $\mathbf{x}^0 \in K \cap \mathbf{f}^{-1}(K)$. If $\mathbf{f}$ is pseudomonotone decreasing, then the sequence $\{\mathbf{x}_n\}_{n \in \mathbb{N}}$ is convergent and its limit $\mathbf{x}^*$ is a solution of the nonlinear complementarity problem NCP($\mathbf{f}$, K).*

Proof. Since every isotone projection cone is regular, by Lemma 2, it is enough to prove that the sequence $\{\mathbf{x}^n\}_{n \in \mathbb{N}}$ is monotone decreasing. Moreover, it is enough to prove that $\mathbf{f}(\mathbf{x}^n) \in K$ for all $n \in \mathbb{N}$. Indeed, since $\mathbf{x}^n - \mathbf{f}(\mathbf{x}^n) \leqslant \mathbf{x}^n$, we have

$$\mathbf{x}^{n+1} = \mathbf{P}_K\big(\mathbf{x}^n - \mathbf{f}\big(\mathbf{x}^n\big)\big) \leqslant \mathbf{P}_K\big(\mathbf{x}^n\big) = \mathbf{x}^n. \tag{2}$$

Hence, the sequence $\{\mathbf{x}^n\}_{n \in \mathbb{N}}$ is monotone decreasing. We will prove the proposition

$$(\Pi_n) \quad \mathbf{f}\big(\mathbf{x}^n\big) \in K \quad \forall n \in \mathbb{N}$$

by induction. (Π_0) is obviously true. We suppose that (Π_n) is true and prove that (Π_{n+1}) is also true. Since $\mathbf{f}(\mathbf{x}^n) \in K$, by relation (2) we have that $\mathbf{x}^{n+1} \leqslant \mathbf{x}^n$. Since $\mathbf{f}$ is pseudomonotone decreasing we have $\mathbf{f}(\mathbf{x}^{n+1}) \in K$; that is, (Π_{n+1}) is true. $\square$

Example 1. Let $H = \mathbb{R}^m$ and $K = \mathbb{R}^m_+$ where m is a positive integer. Then, K is an isotone projection cone [18]. Let $P_1, \dots, P_m$ be polynomial functions of $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m_+$ with positive coefficients such that $P_1(\mathbf{0}) = \dots = P_m(\mathbf{0}) = 0$ and $a_1, \dots, a_m \in \mathbb{R}_+$ nonnegative constants. Then, the mapping $\mathbf{f}: \mathbb{R}^m_+ \to \mathbb{R}^m$ defined by

$$\mathbf{f}(\mathbf{x}) = \big(a_1 - P_1(\mathbf{x}), \dots, a_m - P_m(\mathbf{x})\big)$$

is monotone decreasing. Moreover, if $\mathbf{x} \in \mathbb{R}^m_+$ is sufficiently close to the origin, then $\mathbf{0} \leqslant \mathbf{f}(\mathbf{x})$. Hence, $K \cap \mathbf{f}^{-1}(K) \neq \emptyset$ and we can define the recursion given in Theorem 1 which is convergent to a solution of the nonlinear complementarity problem *NCP*($\mathbf{f}$, $\mathbb{R}^m_+$).

We remark that the projection onto $K = \mathbb{R}^m_+$ can be easily obtained: $\mathbf{P}_K(\mathbf{x}) = \mathbf{y}$, where $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{y} = (y_1, \dots, y_m)$ and

$$y_i = \begin{cases} x_i & \text{if } x_i \geqslant 0, \\ 0 & \text{if } x_i < 0 \end{cases} \tag{3}$$

for all $i \in \{1, \dots, m\}$. Relations (3) can simply written as

$$y_i = \max\{x_i, 0\} \tag{4}$$

for all $i \in \{1, \dots, m\}$.

Proposition 1. *Let $H = \mathbb{R}^m$ and $K = \mathbb{R}^m_+$ where m is a positive integer. Let $\mathbf{f} = (f_1, \dots, f_m): \mathbb{R}^m_+ \to \mathbb{R}^m$ be a pseudomonotone decreasing mapping and $g_i: \mathbb{R}^m_+ \to \,]0, +\infty[$ arbitrary continuous functions. Then, the mapping $\mathbf{h} = (f_1 g_1, \dots, f_m g_m): \mathbb{R}^m_+ \to \mathbb{R}^m$ defined by*

$$\mathbf{h}(\mathbf{x}) = \big(f_1(\mathbf{x}) g_1(\mathbf{x}), \dots, f_m(\mathbf{x}) g_m(\mathbf{x})\big)$$

is pseudomonotone decreasing.

Proof. Suppose that $0 \leqslant \mathbf{h}(\mathbf{x})$ and $\mathbf{x} \leqslant \mathbf{y}$. Then, $0 \leqslant f_i(\mathbf{x}) g_i(\mathbf{x})$ for all $i \in \{1, \dots, m\}$. Hence, $0 \leqslant f_i(\mathbf{x})$ for all $i \in \{1, \dots, m\}$, or equivalently $0 \leqslant \mathbf{f}(\mathbf{x})$. Since f is pseudomonotone decreasing, $0 \leqslant \mathbf{f}(\mathbf{y})$, or equivalently

$$0 \leqslant f_i(\mathbf{y}) \tag{5}$$

for all $i \in \{1, \dots, m\}$. Multiplying relations (5) by $g_i(\mathbf{y}) > 0$, we get $0 \leqslant f_i(\mathbf{y}) g_i(\mathbf{y})$ for all $i \in \{1, \dots, m\}$. Hence, $0 \leqslant \mathbf{h}(\mathbf{y})$. Therefore, $\mathbf{h}$ is pseudomonotone decreasing. $\square$

By slightly modifying Example 1, we get

Example 2. Let $H=\mathbb{R}^m$ and $K=\mathbb{R}^m_+$ where m is a positive integer. Let $P_1,\dots,P_m$ be polynomial functions of $\mathbf{x}=(x_1,\dots,x_m)\in\mathbb{R}^m_+$ with positive coefficients such that $P_1(\mathbf{0})=\cdots=P_m(\mathbf{0})=0$, $a_1,\dots,a_m\in\mathbb{R}_+$ nonnegative constants; and $g_1,\dots,g_m:\mathbb{R}^m_+\to\,]0,+\infty[$ arbitrary continuous functions. Then, by Proposition 1, the mapping $\mathbf{f}:\mathbb{R}^m_+\to\mathbb{R}^m$ defined by

$$\mathbf{f}(\mathbf{x})=\big(\big(a_1-P_1(\mathbf{x})\big)g_1(\mathbf{x}),\dots,\big(a_m-P_m(\mathbf{x})\big)g_m(\mathbf{x})\big) \tag{6}$$

is pseudomonotone decreasing. Moreover, if $\mathbf{x}\in\mathbb{R}^m_+$ is sufficiently close to the origin, then $\mathbf{0}\leqslant\mathbf{f}(\mathbf{x})$. Hence, $K\cap\mathbf{f}^{-1}(K)\neq\emptyset$ and we can define the recursion given in Theorem 1 which is convergent to a solution of the nonlinear complementarity problem $NCP(\mathbf{f},\mathbb{R}^m_+)$.

We remark that it is easy to construct continuous functions

$$g:\mathbb{R}^m_+\to\,]0,+\infty[.$$

For example, let $P:\mathbb{R}^m_+\to\mathbb{R}$ be a polynomial function of nonnegative coefficients and $\phi,\psi,\chi:\mathbb{R}^m_+\to\mathbb{R}$ arbitrary continuous functions. Then, the function $P|\phi|e^\psi+e^\chi:\mathbb{R}^m_+\to\mathbb{R}$, defined by

$$\big(P|\phi|e^\psi+e^\chi\big)(\mathbf{x})=P(\mathbf{x})\big|\phi(\mathbf{x})\big|e^{\psi(\mathbf{x})}+e^{\chi(\mathbf{x})}$$

takes only positive values. Moreover, any linear combination with positive coefficients of functions of the above type has the same property. Based on these remarks we can construct the following numerical example.

Example 3. Let $\mathbf{f}=(f_1,f_2,f_3):\mathbb{R}^3_+\to\mathbb{R}$ defined by

$$f_1(\mathbf{x})=(9-x_1x_2x_3)e^{x_1+x_2+x_3},$$
$$f_2(\mathbf{x})=(6-x_2x_3)\big(1+e^{x_1}(x_2+x_3)\big),$$
$$f_3(\mathbf{x})=(5-x_2-x_3)\big(1+x_1^2+x_2^2+x_3^2-3x_1x_2x_3\big).$$

Since

$$x_1^3+x_2^3+x_3^3-3x_1x_2x_3=\frac{1}{2}(x_1+x_2+x_3)\big|(x_1-x_2)^2+(x_1-x_3)^2+(x_2-x_3)^2\big|$$

it can be seen that $\mathbf{f}$ is of the type (6). If $\mathbf{0}\leqslant\mathbf{f}(\mathbf{x}^0)$, then the recursion given in Theorem 1 is convergent to a solution of the nonlinear complementarity problem $NCP(\mathbf{f},\mathbb{R}^3_+)$. By using relation (4), we can write the recursion explicitly as

$$\begin{cases} x_1^{n+1}=\max\big\{x_1^n-\big(9-x_1^nx_2^nx_3^n\big)e^{x_1^n+x_2^n+x_3^n},0\big\},\\ x_2^{n+1}=\max\big\{x_2^n-\big(6-x_2^nx_3^n\big)\big(1+e^{x_1^n}\big(x_2^n+x_3^n\big)\big),0\big\},\\ x_3^{n+1}=\max\big\{x_3^n-\big(5-x_2^n-x_3^n\big)\big(1+\big(x_1^n\big)^2+\big(x_2^n\big)^2+\big(x_3^n\big)^2-3x_1^nx_2^nx_3^n\big),0\big\}, \end{cases}$$

where f_1, f_2 and f_3 are defined above. It can be easily checked that any

$$\mathbf{x}^*\in\big\{(0,0,0),(0,2,3),(0,3,2),(1.5,2,3),(1.5,3,2)\big\}$$

is a solution of the complementarity $NCP(\mathbf{f},\mathbb{R}^3_+)$. We have written a Scilab script for this recursion. The stopping criterion we used is $|x_i^{n+1}-x_i^n|\leqslant 10^{-5}$ for all $i\in\{1,2,3\}$.

- If we start the algorithm from $\mathbf{x}^0=(1.4998747291,2,3)$, then it stops at the third step with the solution $\mathbf{x}^*=(0,2,3)$.
- If we start the algorithm from $\mathbf{x}^0=(1.4998747291,3,2)$, then it stops at the third step with the solution $\mathbf{x}^*=(0,3,2)$.
- If we start the algorithm from $\mathbf{x}^0=(1,2,3)$, then it stops at the second step with the solution $\mathbf{x}^*=(0,2,3)$.
- If we start the algorithm from $\mathbf{x}^0=(1,2,2.9999999)$, then it stops at the fifth step with the solution $\mathbf{x}^*=(0,0,0)$.

From the above starting points it can be seen that very slight alterations of the starting points of the algorithm can lead to very different solutions.

Analyzing Theorem 1, it can be seen that for any f satisfying the conditions of this theorem the nonlinear complementarity problem $NCP(\mathbf{f},K)$ has the trivial solution $\mathbf{x}^*=\mathbf{0}$. Indeed, since $\mathbf{0}\leqslant\mathbf{x}^0$, $\mathbf{0}\leqslant\mathbf{f}(\mathbf{x}_0)$ and $\mathbf{f}$ is pseudomonotone decreasing, it follows that $0\leqslant\mathbf{f}(0)$, or equivalently $\mathbf{f}(0)\in K$. But every isotone projection cone is *subdual*; that is, $K\subseteq K^*$ [1]. Therefore, $\mathbf{f}(0)\in K^*$; that is, $\mathbf{x}^*=\mathbf{0}$ is a solution of $NCP(\mathbf{f},K)$.

Hence, depending on the starting point, the recursion may be convergent to the trivial solution $\mathbf{x}^*=\mathbf{0}$. In practical problems it is important to find nonzero solutions for a complementarity problem. For the mappings similar to those of the type given in Example 2 we have the following result:

Proposition 2. *Let $H = \mathbb{R}^m$ and $K = \mathbb{R}^m_+$ where m is a positive integer. Let $P_1, \dots, P_m$ be polynomial functions of $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{R}^m_+$ with positive coefficients such that $P_1(\mathbf{0}) = \cdots = P_m(\mathbf{0}) = 0$ and for all $i \in \{1, \dots, m\}$. Let $a_1, \dots, a_m \in \mathbb{R}_+$ be nonnegative constants and $g_1, \dots, g_m : \mathbb{R}^m_+ \to]0, +\infty[$ arbitrary continuous functions. Then, the mapping $\mathbf{f} : \mathbb{R}^m_+ \to \mathbb{R}^m$ defined by*

$$\mathbf{f}(\mathbf{x}) = \big(\big(a_1 - P_1(\mathbf{x})\big)g_1(\mathbf{x}), \dots, \big(a_m - P_m(\mathbf{x})\big)g_m(\mathbf{x})\big) \tag{7}$$

is pseudomonotone decreasing. Moreover, if $\mathbf{x} \in \mathbb{R}^m_+$ is sufficiently close to the origin, then $\mathbf{0} \leqslant \mathbf{f}(\mathbf{x})$. Hence, $K \cap \mathbf{f}^{-1}(K) \neq \emptyset$ and we can define the recursion given in Theorem 1 which is convergent to a solution of $\mathbf{x}^*$ of the nonlinear complementarity problem NCP($\mathbf{f}, \mathbb{R}^m_+$). Let

$$I = \big\{i \in \{1, \dots, m\}:\ 0 < x_i^*\big\},$$

and $\mathbf{x}^0$ be the starting point of the recursion. Then, we have $x_i^0 = x_i^*$ for all $i \in I$ with $\partial P_i / \partial x_i$ a nonzero polynomial.

Proof. Without loss of generality we can assume that $I \neq \emptyset$ and there is an $i \in I$ such that $\partial P_i/\partial x_i$ is a nonzero polynomial. Fix such an i. Suppose that $x_i^0 < x_i^*$. Then, from the proof of Theorem 1 we get $x_i^n \leqslant x_i^0 < x_i^*$ for all $n \in \mathbb{N}$. Tending with n to infinity it follows that $x_i^* \leqslant x_i^0 < x_i^*$ which is absurd. Hence, $x_i^* \leqslant x_i^0$. Suppose that $x_i^* < x_i^0$. Since x^* is a solution of the nonlinear complementarity problem *NCP*($\mathbf{f}, \mathbb{R}^m_+$) and $0 < x_i^*$ we have $f_i(\mathbf{x}^*) = 0$. By using again the proof of Theorem 1, we obtain $\mathbf{x}^n \leqslant \mathbf{x}^0$ for all $n \in \mathbb{N}$, and taking the limit as n tends to infinity we obtain $\mathbf{x}^* \leqslant \mathbf{x}^0$. Since $x_i^* < x_i^0$ and $\partial P_i/\partial x_i$ is a nonzero polynomial,

$$a_i - P_i\big(\mathbf{x}^0\big) < a_i - P_i(\mathbf{x}^*) = \frac{f_i(\mathbf{x}^*)}{g_i(\mathbf{x}^*)} = 0.$$

It follows that $f_i(\mathbf{x}^0) = (a_i - P_i(\mathbf{x}^0))g^i(\mathbf{x}^0) < 0$, which is a contradiction. Thus, $x_i^0 = x_i^*$. □

The next proposition gives another class of pseudomonotone decreasing mappings.

Proposition 3. *Let $P_1, \dots, P_m : \mathbb{R}_+ \to \mathbb{R}$ be polynomial functions of one variable and $\mathbf{f} : \mathbb{R}^m_+ \to \mathbb{R}^m$ be a mapping defined by*

$$\mathbf{f}(\mathbf{x}) = \big(P_1(x_1), \dots, P_m(x_m)\big).$$

Then, there are constants $\varepsilon_i \in \{-1, 1\}$ and $a_i \in \mathbb{R}_+$ such that the mapping $\mathbf{g} : \mathbb{R}^m_+ \to \mathbb{R}^m$ defined by

$$\mathbf{g}(\mathbf{x}) = \big(\varepsilon_1 P_1(x_1) + a_1, \dots, \varepsilon_m P_m(x_m) + a_m\big)$$

is pseudomonotone decreasing.

Proof. If $\lim_{t \to -\infty} P_i(t) < 0$, then put $\varepsilon_i = -1$, otherwise put $\varepsilon_i = 1$ for all $i \in \{1, \dots, m\}$. For all $i \in \{1, \dots, m\}$ we can choose $a_i \in \mathbb{R}_+$ sufficiently large such that the equation $\varepsilon_i P_i(t) + a_i = 0$ has at most one solution. Hence, it follows that if $0 \leqslant \varepsilon_i P_i(x_i) + a_i$ and $x_i \leqslant y_i$, then $0 \leqslant \varepsilon_i P_i(y_i) + a_i$ for all $i \in \{1, \dots, m\}$. It follows that $\mathbf{g}$ is pseudomonotone decreasing. □

By using Proposition 1, it follows that if we multiply each component of the mapping $\mathbf{g}$ from Proposition 3 by a positive function, then we get another pseudomonotone decreasing mapping. In this way we can generate several other pseudomonotone mappings. This is similar to how Example 2 was obtained from Example 1.

The next theorem gives a sufficient condition for the recursion (1) to be convergent to a nonzero solution.

Theorem 2. *Let H be a Hilbert space, $K \subset H$ an isotone projection cone and $\mathbf{f} : K \to H$ a pseudomonotone decreasing, continuous mapping such that $K \cap \mathbf{f}^{-1}(K) \neq \emptyset$. Let $\mathbf{J} : K \to H$ be the inclusion mapping defined by $\mathbf{J}(\mathbf{x}) = \mathbf{x}$ and $\mathbf{P}_K : H \to K$ the projection mapping onto K. If there are $\hat{\mathbf{x}} \in K \cap \mathbf{f}^{-1}(K)$ and $\mathbf{u} \in \hat{\mathbf{x}} + K$ such that*

$$\big(\mathbf{P}_K \circ (\mathbf{J} - \mathbf{f})\big)\big((\hat{\mathbf{x}} + K) \cap (\mathbf{u} - K) \cap \mathbf{f}^{-1}(K)\big) \subseteq \hat{\mathbf{x}} + K,$$

then $\hat{\mathbf{x}}$ is a solution of the nonlinear complementarity problem NCP($\mathbf{f}, K$) and for any $\mathbf{x}^0 \in (\hat{\mathbf{x}} + K) \cap (\mathbf{u} - K) \cap \mathbf{f}^{-1}(K)$ the recursion (1) starting from $\mathbf{x}^0$ is convergent and its limit $\mathbf{x}^*$ is a solution of the nonlinear complementarity problem NCP($\mathbf{f}, K$) such that $\hat{\mathbf{x}} \leqslant \mathbf{x}^* \leqslant \mathbf{u}$. In particular, if $\hat{\mathbf{x}} \neq \mathbf{0}$, then the recursion (1) is convergent to a nonzero solution.

Proof. Since $\hat{\mathbf{x}} - \mathbf{f}(\hat{\mathbf{x}}) \leqslant \hat{\mathbf{x}}$, $\hat{\mathbf{x}} \in (\hat{\mathbf{x}} + K) \cap (\mathbf{u} - K) \cap \mathbf{f}^{-1}(K)$ and

$$\big(\mathbf{P}_K \circ (\mathbf{J} - \mathbf{f})\big)\big((\hat{\mathbf{x}} + K) \cap (\mathbf{u} - K) \cap \mathbf{f}^{-1}(K)\big) \subseteq \hat{\mathbf{x}} + K,$$

we have $\hat{\mathbf{x}} \leqslant (\mathbf{P}_K \circ (\mathbf{J} - \mathbf{f}))(\hat{\mathbf{x}}) = \mathbf{P}_K(\hat{\mathbf{x}} - \mathbf{f}(\hat{\mathbf{x}})) \leqslant \mathbf{P}_K(\hat{\mathbf{x}}) = \hat{\mathbf{x}}$. Hence, $\hat{\mathbf{x}} = \mathbf{P}_K(\hat{\mathbf{x}} - \mathbf{f}(\hat{\mathbf{x}}))$; that is, $\hat{\mathbf{x}}$ is a solution of the nonlinear complementarity problem *NCP*($\mathbf{f}, K$). In the proof of Theorem 1 we have seen by induction that

$$\mathbf{x}^n \in K \cap \mathbf{f}^{-1}(K), \tag{8}$$

for all $n \in \mathbb{N}$. We prove by induction the proposition

$$(\Pi_n) \quad \hat{\mathbf{x}} \leqslant \mathbf{x}^n \leqslant \mathbf{u}, \tag{9}$$

for all $n \in \mathbb{N}$. Obviously, (Π_0) is true. Suppose that (Π_n) is true. Hence, by using relation (8), we have $\mathbf{x}^n \in (\hat{\mathbf{x}} + K) \cap (\mathbf{u} - K) \cap \mathbf{f}^{-1}(K)$. Thus,

$$\mathbf{x}^{n+1} = \big(\mathbf{P}_K \circ (\mathbf{J} - \mathbf{f})\big)\big(\mathbf{x}^n\big) \in \big(\mathbf{P}_K \circ (\mathbf{J} - \mathbf{f})\big)\big((\hat{\mathbf{x}} + K) \cap \mathbf{f}^{-1}(K)\big) \subseteq \hat{\mathbf{x}} + K. \tag{10}$$

On the other hand, by using relation (8), we have $\mathbf{x}^n - f(\mathbf{x}^n) \leqslant \mathbf{x}^n \leqslant \mathbf{u}$ which implies

$$\mathbf{x}^{n+1} = \mathbf{P}_K\big(\mathbf{x}^n - f\big(\mathbf{x}^n\big)\big) \leqslant \mathbf{P}_K(\mathbf{u}) = u. \tag{11}$$

Relations (10) and (11) imply that (Π_{n+1}) is also true. Taking the limit in relation (9), as n tends to infinity, we get $\hat{\mathbf{x}} \leqslant \mathbf{x}^* \leqslant \mathbf{u}$. $\square$

Definition 3. Let H be a Hilbert space, $K \subset H$ a closed convex cone, $\mathbf{f}: K \to H$ a mapping and $L > 0$. The mapping $\mathbf{f}$ is called *order weekly L-Lipschitz* if $\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y}) \leqslant \mathbf{L}(\mathbf{x} - \mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in K$ with $\mathbf{y} \leqslant \mathbf{x}$. If $L = 1$, then f is called *order weekly nonexpansive*.

The following proposition is an immediate consequence of Definition 3.

Proposition 4. *Let H be a Hilbert space, $K \subset H$ a closed convex cone, $\mathbf{f}: K \to H$ a mapping and $L > 0$. Then, the mapping $\mathbf{f}$ is order weekly L-Lipschitz if and only if the mapping $K \ni \mathbf{x} \mapsto L\mathbf{x} - \mathbf{f}(\mathbf{x})$ is monotone increasing.*

Definition 4. Let H be a Hilbert space, $K \subset H$ a closed convex cone, $\mathbf{f}: K \to H$ a mapping and $L > 0$. Then, the mapping $\mathbf{f}$ is called *projection order weekly L-Lipschitz* if the mapping $K \ni \mathbf{x} \mapsto \mathbf{P}_K(L\mathbf{x} - \mathbf{f}(\mathbf{x}))$ is monotone increasing where $\mathbf{P}_K$ is the projection mapping onto K. If $L = 1$ the mapping $\mathbf{f}$ is called *projection order weekly nonexpansive*.

It is easy to see that in case of isotone projection cones every order weekly L-Lipschitz mapping is projection order weekly L-Lipschitz and every order weekly nonexpansive mapping is projection order weekly nonexpansive.

Theorem 3. *Let H be a Hilbert space, $K \subset H$ an isotone projection cone, $L > 0$ and $\mathbf{f}: K \to H$ a pseudomonotone decreasing, projection order weekly L-Lipschitz, continuous mapping such that $K \cap \mathbf{f}^{-1}(K) \neq \emptyset$. Let $\hat{\mathbf{x}}$ be a solution of the nonlinear complementarity problem NCP($\mathbf{f}$, K). Then, for any $\mathbf{x}^0 \in (\hat{\mathbf{x}} + K) \cap \mathbf{f}^{-1}(K)$ the recursion*

$$\mathbf{x}^{n+1} = \mathbf{P}_K\left(\mathbf{x}^n - \frac{\mathbf{f}(\mathbf{x}^n)}{L}\right) \tag{12}$$

starting from $\mathbf{x}^0$ is convergent and its limit $\mathbf{x}^*$ is a solution of the nonlinear complementarity problem NCP($\mathbf{f}$, K) such that $\hat{\mathbf{x}} \leqslant \mathbf{x}^*$. In particular, if $\hat{\mathbf{x}} \neq \mathbf{0}$, then the recursion (12) *is convergent to a nonzero solution.*

Proof. We will use the following well-known property of the projection mapping $\mathbf{P}_\kappa$ onto a closed convex cone κ: $\mathbf{P}_\kappa(\lambda\mathbf{x}) = \lambda\mathbf{P}_\kappa(\mathbf{x})$ for all $\mathbf{x} \in H$ and $\lambda > 0$. We remark that the nonlinear complementarity problem *NCP*($\mathbf{f}$, K) is equivalent to the nonlinear complementarity problem *NCP*($\mathbf{f}/L$, K). Denote $\mathbf{g} = \mathbf{f}/L$. Then, the recursion (12) can be written in the form

$$\mathbf{x}^{n+1} = \mathbf{P}_K\big(\mathbf{x}^n - \mathbf{g}\big(\mathbf{x}^n\big)\big).$$

We will use Theorem 2 for the mapping $\mathbf{g}$. Let $\mathbf{J}: K \to H$ be the inclusion mapping defined by $\mathbf{J}(\mathbf{x}) = \mathbf{x}$ and $\mathbf{u} \in \hat{\mathbf{x}} + K$ arbitrary. Since any solution of the nonlinear complementarity problem *NCP*($\mathbf{g}$, K) is a solution of the nonlinear complementarity problem *NCP*($\mathbf{f}$, K) too, it is enough to check the relation

$$\big(\mathbf{P}_K \circ (\mathbf{J} - \mathbf{g})\big)\big((\hat{\mathbf{x}} + K) \cap (\mathbf{u} - K) \cap \mathbf{g}^{-1}(K)\big) \subseteq \hat{\mathbf{x}} + K. \tag{13}$$

We have

$$\mathbf{P}_K\big(\mathbf{x} - \mathbf{g}(\mathbf{x})\big) = \mathbf{P}_K\left(\frac{1}{L}\big(L\mathbf{x} - \mathbf{f}(\mathbf{x})\big)\right) = \frac{1}{L}\mathbf{P}_K\big(L\mathbf{x} - \mathbf{f}(\mathbf{x})\big), \tag{14}$$

for all $\mathbf{x} \in K$. Since the mapping f is projection order weekly L-Lipschitz, from relation (14) and the scale invariance of the ordering induced by K, it follows that the mapping $\mathbf{g}$ is projection order weekly nonexpansive. Hence, since $\hat{\mathbf{x}}$ is a solution of the nonlinear complementarity problem *NCP*($\mathbf{g}$, K), for each $\mathbf{x} \in (\hat{\mathbf{x}} + K) \cap (\mathbf{u} - K) \cap \mathbf{g}^{-1}(K)$ we have $\hat{\mathbf{x}} = \mathbf{P}_K(\hat{\mathbf{x}} - \mathbf{g}(\hat{\mathbf{x}})) \leqslant \mathbf{P}_K(\mathbf{x} - \mathbf{g}(\mathbf{x}))$. The previous relation can be rewritten as $(\mathbf{P}_K \circ (\mathbf{J} - \mathbf{g}))(\mathbf{x}) \in \hat{\mathbf{x}} + K$. Therefore, relation (13) holds. $\square$

The following result is a corollary of Theorem 2.

Corollary 1. *Let H be a Hilbert space, $K \subset H$ an isotone projection cone and $\mathbf{f}: K \to H$ a pseudomonotone decreasing, continuous mapping such that $K \cap \mathbf{f}^{-1}(K) \neq \emptyset$. Let $\mathbf{J}: K \to H$ be the inclusion mapping defined by $\mathbf{J}(\mathbf{x}) = \mathbf{x}$. If there are $\hat{\mathbf{x}} \in K \cap \mathbf{f}^{-1}(K)$ and $\mathbf{u} \in \hat{\mathbf{x}} + K$ such that*

$$(\mathbf{J} - \mathbf{f})\big((\hat{\mathbf{x}} + K) \cap (\mathbf{u} - K) \cap \mathbf{f}^{-1}(K)\big) \subseteq \hat{\mathbf{x}} + K,$$

then $\mathbf{f}(\hat{\mathbf{x}}) = 0$ and for any $\mathbf{x}^0 \in (\hat{\mathbf{x}} + K) \cap (\mathbf{u} - K) \cap \mathbf{f}^{-1}(K)$ the recursion $\mathbf{x}^{n+1} = \mathbf{x}^n - \mathbf{f}(\mathbf{x}^n)$ starting from $\mathbf{x}^0$ is convergent and its limit $\mathbf{x}^*$ satisfies the relations $\mathbf{f}(\mathbf{x}^*) = 0$ and $\hat{\mathbf{x}} \leqslant \mathbf{x}^* \leqslant \mathbf{u}$. In particular, if $\hat{\mathbf{x}} \neq \mathbf{0}$, then $\mathbf{x}^* \neq \mathbf{0}$.

It is easy to see that the result of Corollary 1 is true for an arbitrary closed convex cone K too.

Example 4. Let $H = \mathbb{R}^2$, $K = \mathbb{R}^2_+$ and $\mathbf{f}: \mathbb{R}^2_+ \to \mathbb{R}^2$ defined by

$$\mathbf{f}(x_1, x_2) = \big((1 - x_1)^2 e^{-x_1 - x_2}, x_2(1 - x_2)^2(2 - x_2)e^{-1-x_1}\big).$$

It can be seen that if $x_1 > 1$, then $f_1(x_1, x_2) \leqslant x_1 - 1$ and if $x_2 \in]1, 2[$, then $f_2(x_1, x_2) \leqslant x_2 - 1$.

(1) Let $\hat{\mathbf{x}} = (1, 1)$ and $\mathbf{u} \in]1, +\infty[\times \{2\}$ arbitrary. By using Corollary 1, for every starting point $\mathbf{x}^0 \in]1, +\infty[\times]1, 2[$ the recursion

$$\begin{cases} x_1^{n+1} = x_1^n - \big(1 - x_1^n\big)^2 e^{-x_1^n - x_2^n}, \\ x_2^{n+1} = x_2^n - x_2^n\big(1 - x_2^n\big)^2\big(2 - x_2^n\big)e^{-1-x_1^n} \end{cases}$$

is convergent to the solution $\mathbf{x}^* = \hat{\mathbf{x}} = (1, 1)$ of the system of equations

$$\begin{cases} (1 - x_1)^2 e^{-x_1 - x_2} = 0, \\ x_2(1 - x_2)^2(2 - x_2)e^{-1-x_1} = 0. \end{cases}$$

(2) Let $\hat{\mathbf{x}} = (0, 1)$ and $\mathbf{u} = (1, 2)$. By using Theorem 2, for every starting point $\mathbf{x}^0 \in]0, 1[\times]1, 2[$ the recursion

$$\begin{cases} x_1^{n+1} = \max\big\{x_1^n - \big(1 - x_1^n\big)^2 e^{-x_1^n - x_2^n}, 0\big\}, \\ x_2^{n+1} = \max\big\{x_2^n - x_2^n\big(1 - x_2^n\big)^2\big(2 - x_2^n\big)e^{-1-x_1^n}, 0\big\} \end{cases}$$

is convergent to the solution $\mathbf{x}^* = \hat{\mathbf{x}} = (0, 1)$ of the nonlinear complementarity problem $NCP(\mathbf{f}, \mathbb{R}^2_+)$.

The details are left to the reader.

We remark that similar results can be obtained if we replace the recursion (1) by the recursion $\mathbf{x}^{n+1} = \mathbf{P}_K(\mathbf{x}^n - \lambda\mathbf{f}(x^n))$, where $\lambda > 0$ is an arbitrary positive constant, because the nonlinear complementarity problems $NCP(\mathbf{f}, K)$ and $NCP(\lambda\mathbf{f}, K)$ are equivalent. A similar idea was exploited in Theorem 3.

4. Conclusions

In this paper we analyzed the convergence of the recursion

$$\mathbf{x}^{n+1} = P_K\big(\mathbf{x}^n - \mathbf{f}\big(\mathbf{x}^n\big)\big),$$

where K is a closed convex cone in a Hilbert space H and $\mathbf{f}: K \to H$ is a continuous mapping. If the sequence $\{\mathbf{x}^n\}_{n\in\mathbb{N}}$ is convergent to $\mathbf{x}^* \in K$, then $\mathbf{x}^*$ is a solution of the nonlinear complementarity problem defined by K and $\mathbf{f}$. In [1] a set of sufficient conditions is given for the convergence of $\{\mathbf{x}^n\}_{n\in\mathbb{N}}$. In general, it is not clear what are the necessary and/or sufficient conditions for $\{\mathbf{x}^n\}_{n\in\mathbb{N}}$ to be convergent. This paper presented another set of sufficient conditions for the convergence of $\{\mathbf{x}^n\}_{n\in\mathbb{N}}$. As a particular case, we considered the problem of finding the zeros of $\mathbf{f}$. We also considered the above recursion with $\mathbf{f}$ replaced by a positive scaling of $\mathbf{f}$.

The presented examples suggest that, for $H = \mathbb{R}^m$ and $K = \mathbb{R}^m_+$, there is a large number of mappings for which the recursion (1) is convergent to nonzero solutions of the corresponding nonlinear complementarity problem. Therefore, we can conclude that Theorem 1 and Theorem 2 are new and seemingly rather general results.

We remark that for $L = 1$ the recursion of Theorem 3 seems very similar to the one given in Theorem 6.3 of [1]. However, our extra condition of pseudomonotonicity implies that the recursion in Theorem 6.3 of [1] never leaves $\mathbf{0}$ if the mapping defining the complementarity problem is also pseudomonotone decreasing. Moreover, the recursion in Theorem 6.3 of [1] can find only one solution of the complementarity problem, because it always starts from $\mathbf{0}$, and if $\mathbf{0}$ is a solution, then the recursion in Theorem 6.3 of [1] never leaves the trivial solution $\mathbf{0}$.

We also remark that as far as we know the notions of pseudomonotone decreasing mappings, order weekly L-Lipschitz, projection order weekly L-Lipschitz, order weekly nonexpansive and projection order weekly nonexpansive were first considered in this paper. In the future we plan to find classes of projection order weekly L-Lipschitz mappings that are not order weekly L-Lipschitz, to extend the notion of pseudomonotone decreasing mappings to the notion of projection pseudomonotone decreasing mappings and to give classes of projection pseudomonotone decreasing mappings that are not pseudomonotone decreasing mappings. Accomplishing these goals would lead to more general results on more general cones. We also plan to analyze the extragradient iteration of the type $\mathbf{x}^{n+1} = \mathbf{P}_K(\mathbf{x}^n - \lambda\mathbf{f}(\mathbf{P}_K(\mathbf{x}^n - \lambda\mathbf{f}(\mathbf{x}^n))))$ considered in [6–9].

Acknowledgment

The author expresses his gratitude to Yunbin Zhao for many helpful conversations.

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