



On meromorphic extendibility

Josip Globevnik

Institute of Mathematics, Physics and Mechanics, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia

ARTICLE INFO

Article history:

Received 8 July 2008

Available online 8 October 2008

Submitted by D. Khavinson

Keywords:

Meromorphic extensions

Argument principle

ABSTRACT

Let D be a bounded domain in the complex plane whose boundary consists of finitely many pairwise disjoint real-analytic simple closed curves. Let f be an integrable function on bD . In the paper we show how to compute the candidates for poles of a meromorphic extension of f through D and thus reduce the question of meromorphic extendibility to the question of holomorphic extendibility. Let $A(D)$ be the algebra of all continuous functions on \bar{D} which are holomorphic on D . We prove that a continuous function f on bD extends meromorphically through D if and only if there is an $N \in \mathbb{N} \cup \{0\}$ such that the change of argument of $Pf + Q$ along bD is bounded below by $-2\pi N$ for all $P, Q \in A(D)$ such that $Pf + Q \neq 0$ on bD . If this is the case then the meromorphic extension of f has at most N poles in D , counting multiplicity.

© 2008 Elsevier Inc. All rights reserved.

1. Introduction

Let $D \subset \mathbb{C}$ be a bounded domain whose boundary consists of a finite number of pairwise disjoint, real-analytic simple closed curves. Let $A(D)$ be the algebra of all continuous functions on \bar{D} which are holomorphic on D . Denote by $H^1(D)$ the space of all holomorphic functions on D such that $z \mapsto |h(z)|$ ($z \in D$) has a harmonic majorant [6]. Every $h \in H^1(D)$ has nontangential boundary values h^* almost everywhere on bD , $h^* \in L^1(bD)$ and

$$h(z) = \frac{1}{2\pi i} \int_{bD} \frac{h^*(\zeta) d\zeta}{\zeta - z} \quad (z \in bD).$$

We say that $f \in L^1(bD)$ extends holomorphically through D if there is $h \in H^1(D)$ such that $h^* = f$ almost everywhere on bD . We say that $f \in L^1(bD)$ extends meromorphically through D if there are a function $h \in H^1(D)$ and a nonzero polynomial Q with all zeros contained in D such that $f = h^*/Q$ almost everywhere on bD , or, equivalently, if Qf extends holomorphically through D .

A function $f \in L^1(bD)$ extends holomorphically through D if and only if

$$\int_{bD} f(\zeta) \omega(\zeta) d\zeta = 0$$

for each $\omega \in A(D)$ [6], which, since rational functions with poles outside \bar{D} are dense in $A(D)$, is equivalent to

$$\frac{1}{2\pi i} \int_{bD} \frac{f(\zeta) d\zeta}{\zeta - z} \equiv 0 \quad (z \in \mathbb{C} \setminus \bar{D}).$$

E-mail address: josip.globevnik@fmf.uni-lj.si.

There is no such simple test for meromorphic extendibility. If we happen to know the potential poles and their multiplicities, that is, if we know Q then to check the meromorphic extendibility is easy as we simply check whether Qf extends holomorphically through D . The problem becomes more difficult if we do not know in advance where the poles are. In this paper we show that if we know an upper bound for the number of poles then all possible candidates for Q can be easily determined in advance so the question about meromorphic extendibility can be easily reduced to the question about holomorphic extendibility.

2. Poles of meromorphic extensions

Let $f \in L^1(bD)$. For large z we have

$$\frac{1}{2\pi i} \int_{bD} \frac{f(\zeta) d\zeta}{\zeta - z} = \frac{c_1(f)}{z} + \frac{c_2(f)}{z^2} + \dots$$

where

$$c_j(f) = -\frac{1}{2\pi i} \int_{bD} \zeta^{j-1} f(\zeta) d\zeta \quad (j \in \mathbb{N}). \quad (2.1)$$

Given $N \in \mathbb{N}$ there is a nonzero polynomial $P(z) = D_0 + D_1(z) + \dots + D_N z^N$ such that for large z we have

$$P(z) \frac{1}{2\pi i} \int_{bD} \frac{f(\zeta) d\zeta}{\zeta - z} = \left[\frac{d_{N+1}}{z^{N+1}} + \frac{d_{N+2}}{z^{N+2}} + \dots \right] + R(z)$$

where R is a polynomial and d_j , $j \geq N+1$, are complex numbers. To get such P we have to solve the system

$$\left. \begin{aligned} c_1(f)D_0 + c_2(f)D_1 + \dots + c_{N+1}(f)D_N &= 0, \\ \vdots \\ c_N(f)D_0 + c_{N+1}(f)D_1 + \dots + c_{2N}(f)D_N &= 0. \end{aligned} \right\} \quad (2.2)$$

This is a homogeneous system of N linear equations with $N+1$ unknowns which always has a nontrivial solution.

Theorem 2.1. Let $f \in L^1(bD)$ and let $c_j(f)$, $j \in \mathbb{N}$, be as in (2.1). Let $N \in \mathbb{N}$ and let $P(z) = D_0 + D_1 z + \dots + D_N z^N$ where D_0, D_1, \dots, D_N is a nontrivial solution of the system (2.2). The function f extends meromorphically through D with the extension having at most N poles in D , counting multiplicity, if and only if the function $z \mapsto P(z)f(z)$ extends holomorphically through D .

Thus, if we are asking whether f has a meromorphic extension through D with at most N poles in D , counting multiplicity, then the only candidates for the poles are the zeros of P . For an analogous result for continuous functions on the unit circle see [1].

Suppose that $z \mapsto P(z)f(z)$ ($z \in bD$) extends holomorphically through D . If P has no zero on bD then f extends meromorphically through D for in this case we can write $P = QS$ where the polynomial Q has all its zeros on D and the polynomial S has all its zeros in $\mathbb{C} \setminus \bar{D}$. So if $Pf = h^*$ on bD where $h \in H^1(D)$ then

$$f = \frac{h^*/S}{Q} \quad \text{almost everywhere on } bD$$

where $h/S \in H^1(D)$. Even in the case when P has zeros on bD the function f extends meromorphically through D . This follows from the following

Lemma 2.2. Let $f \in L^1(bD)$, let $a \in bD$ and assume that $z \mapsto (z-a)f(z)$ ($z \in bD$) extends holomorphically through D . Then f extends holomorphically through D .

In other words, if $g \in H^1(D)$ and if the function $z \mapsto g^*(z)/(z-a)$ ($z \in bD$) belongs to $L^1(bD)$ then there is a function $h \in H^1(bD)$ such that $h^*(z) = g^*(z)/(z-a)$ a.e. on bD .

3. Some facts about the Cauchy integrals

Our bD consists of m pairwise disjoint curves $\Gamma_1, \dots, \Gamma_m$ where Γ_m is the boundary of the unbounded component D_m of $\mathbb{C} \setminus \bar{D}$. For each j , $1 \leq j \leq m-1$, let D_j be the domain bounded by Γ_j .

For each j , $1 \leq j \leq m-1$, let $G_j \in A(D_j)$, let $G \in A(D)$ and let G_m be a continuous function on \bar{D}_m , holomorphic on D_m and vanishing at infinity. Assume that G and G_j , $1 \leq j \leq m$, all have smooth boundary values so that the function g , defined on bD by

$$g(z) = G(z) - G_j(z) \quad (1 \leq j \leq m) \quad (3.1)$$

is smooth. The function g determines the functions G and G_j , $1 \leq j \leq m$, uniquely. For, if $H_j \in A(D_j)$, $1 \leq j \leq m-1$, and $H \in A(D)$ are functions with smooth boundary values and H_m is smooth on $\overline{D_m}$, holomorphic on D_m and vanishing at infinity such that

$$g(z) = H(z) - H_j(z) \quad (1 \leq j \leq m) \quad (3.2)$$

then the function

$$\Phi(z) = \begin{cases} G(z) - H(z) & (z \in \overline{D}), \\ G_j(z) - H_j(z) & (z \in \overline{D_j}, 1 \leq j \leq m) \end{cases}$$

is, by (3.1) and (3.2), well defined, continuous on \mathbb{C} and holomorphic on $\mathbb{C} \setminus bD$ and vanishing at infinity. So Φ is an entire function vanishing at infinity hence $\Phi \equiv 0$ so $G \equiv H$ and $G_j \equiv H_j$ ($1 \leq j \leq m$). In fact, by the Plemelj jump formulas for the Cauchy integrals we have

$$\frac{1}{2\pi i} \int_{bD} \frac{g(\zeta) d\zeta}{\zeta - z} = \begin{cases} G(z) & (z \in D), \\ G_j(z) & (z \in D_j, 1 \leq j \leq m). \end{cases}$$

Suppose now that $g(z) = R(z)/S(z)$ ($z \in bD$) where S is a polynomial with all zeros contained in D and R is a polynomial, $\deg R < \deg S$. Then the functions

$$\begin{aligned} G(z) &= 0 \quad (z \in \overline{D}), \\ G_j(z) &= -R(z)/S(z) \quad (z \in \overline{D_j}, 1 \leq j \leq m) \end{aligned}$$

have the properties above so

$$\frac{1}{2\pi i} \int_{bD} \frac{g(\zeta) d\zeta}{\zeta - z} = -R(z)/S(z) \quad (z \in D_m).$$

4. Proof of Theorem 2.1

We first prove Lemma 2.2.

The assumption implies that for each $z \in \mathbb{C} \setminus \overline{D}$ we have

$$0 = \frac{1}{2\pi i} \int_{bD} \frac{(\zeta - a)f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{bD} f(\zeta) d\zeta + (z - a) \frac{1}{2\pi i} \int_{bD} \frac{f(\zeta) d\zeta}{\zeta - z}$$

hence

$$\frac{1}{2\pi i} \int_{bD} \frac{f(\zeta) d\zeta}{\zeta - z} = -\frac{1}{z - a} \left[\frac{1}{2\pi i} \int_{bD} f(\zeta) d\zeta \right] \quad (z \in \mathbb{C} \setminus \overline{D}).$$

We shall show that this implies that $\int_{bD} f(\zeta) d\zeta = 0$ so

$$\frac{1}{2\pi i} \int_{bD} \frac{f(\zeta) d\zeta}{\zeta - z} = 0 \quad (z \in \mathbb{C} \setminus \overline{D}) \quad (4.1)$$

which we want to prove. Suppose, contrarily to what we want to prove, that there is an $L \neq 0$ such that

$$\frac{1}{2\pi i} \int_{bD} \frac{f(\zeta) d\zeta}{\zeta - z} = \frac{L}{z - a} \quad (z \in \mathbb{C} \setminus \overline{D}).$$

It follows that

$$\lim_{z \rightarrow a, z \in \mathbb{C} \setminus \overline{D}} (z - a) \int_{\Delta} \frac{f(\zeta) d\zeta}{\zeta - z} = L \quad (4.2)$$

for each arc $\Delta \subset bD$ containing a in its interior. Denote by Δ the open unit disc. Since bD is real-analytic there are a map Φ mapping a disc Ω centered at the origin biholomorphically onto a neighbourhood of a , and C , $0 < C < \infty$, such that $\Phi(0) = a$, such that Φ maps $[-T, T] \subset \Omega$ onto an arc $\Delta \subset bD$, the upper half of $T\Delta$ to $\mathbb{C} \setminus \overline{D}$, the lower half of $T\Delta$ to D , and such that

$$\frac{1}{C} |z - w| \leq |\Phi(z) - \Phi(w)| \leq C |z - w| \quad (z, w \in \Omega).$$

Now, (4.2) implies that

$$\lim_{t \searrow 0} [\Phi(it) - \Phi(0)] \int_{-T}^T \frac{f(\Phi(x))\Phi'(x) dx}{\Phi(x) - \Phi(it)} \neq 0. \quad (4.3)$$

Write $g(x) = f(\Phi(x))\Phi'(x)$. Let $\varepsilon > 0$. Since f is integrable it follows that g is integrable so there is an $M < \infty$ such that

$$\int_{\{x: |g(x)| \geq M\}} |g(x)| dx < \varepsilon.$$

Let

$$A_M = \{x \in [-T, T]: |g(x)| < M\} \quad B_M = \{x \in [-T, T]: |g(x)| \geq M\}.$$

Since $\Phi'(0) \neq 0$, (4.3) implies that

$$\lim_{t \searrow 0} t \int_{-T}^T \frac{g(x)}{x - it} \frac{x - it}{\Phi(x) - \Phi(it)} dx \neq 0. \quad (4.4)$$

We have

$$\left| t \int_{B_M} \frac{g(x)}{x - it} \frac{x - it}{\Phi(x) - \Phi(it)} dx \right| \leq |t| \frac{1}{|t|} C \int_{B_M} |g(x)| dx \leq C\varepsilon.$$

Further,

$$\left| t \int_{A_M} \frac{g(x)}{x - it} \frac{x - it}{\Phi(x) - \Phi(it)} dx \right| = \left| t \int_{A_M} g(x) \frac{x + it}{x^2 + t^2} \cdot \frac{x - it}{\Phi(x) - \Phi(it)} dx \right| \leq |t| \int_{-T}^T MC \frac{|x|}{x^2 + t^2} dx + |t|^2 \int_{-T}^T \frac{MC dx}{x^2 + t^2}.$$

It is easy to see that both terms in the last expression tend to zero as $t \searrow 0$ so

$$\left| \int_{-T}^T \frac{g(x)}{\Phi(x) - \Phi(it)} dx \right| \leq 2C\varepsilon$$

provided that $t > 0$ is small enough which contradicts (4.4) since ε can be chosen arbitrarily small. This completes the proof of Lemma 2.2.

We now turn to the proof of the theorem. Let P be as in Theorem 2.1 and assume that Pf extends holomorphically through D . If $a \in bD$ is a zero of P then by Lemma 2.2 the function $z \mapsto [P(z)/(z - a)]f(z)$ extends holomorphically through D . Thus, factoring out zeros of P contained in bD we conclude that there is a polynomial Q having no zero on bD and having at most N zeros in D such that $z \mapsto Q(z)f(z)$ extends holomorphically through D so the function f extends meromorphically through D and the extension has at most N poles in D , counting multiplicity. This completes the proof of the if part.

To prove the only if part of Theorem 2.1 observe first that if $h \in H^1(D)$ and $a \in D$ then

$$\frac{h(z)}{(z - a)^m} = \frac{h(a)}{(z - a)^{m-1}} + \cdots + \frac{h^{(m-1)}(a)}{(m-1)!(z - a)} + w(z)$$

where $w \in H^1(D)$. Assume now that f extends meromorphically through D with the extension having at most N poles in D , counting multiplicity. This means that either f extends holomorphically through D – there is nothing to prove in this case – or there are $a_1, \dots, a_J \in D$, and $k_1, \dots, k_J \in \mathbb{N}$, such that $k_1 + \cdots + k_J \leq N$ and that

$$f(z) = \frac{G(z)}{(z - a_1)^{k_1} \cdots (z - a_J)^{k_J}} \quad (z \in bD)$$

where $G \in H^1(D)$. Decomposing

$$\frac{1}{(z - a_1)^{k_1} \cdots (z - a_J)^{k_J}}$$

into partial fractions the preceding discussion implies that

$$f(z) = H^*(z) + T(z) \quad (z \in bD)$$

where $H \in H^1(D)$ and

$$T(z) = \frac{R(z)}{S(z)} \quad (z \in bD)$$

where R, S are polynomials with no common zeros, $\deg R < \deg S \leq N$ and all zeros of S are contained in D . Let P be as in Theorem 1.1. We want to prove that $Pf = PH^* + PT$ extends holomorphically through D . Since $PH \in H^1(D)$ it is enough to prove that PT extends holomorphically through D .

From Section 3 we see that

$$\frac{1}{2\pi i} \int_{bD} \frac{f(\zeta) d\zeta}{\zeta - z} = \frac{1}{2\pi i} \int_{bD} \frac{H^*(\zeta) d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{bD} \frac{T(\zeta) d\zeta}{\zeta - z} = 0 - R(z)/S(z) \quad (z \in D_m)$$

which implies that for large z the polynomial P satisfies

$$P(z) \left(-\frac{R(z)}{S(z)} \right) = Q(z) + \frac{c_{N+1}}{z^{N+1}} + \frac{c_{N+2}}{z^{N+2}} + \dots$$

where Q is a polynomial. There are arbitrarily small $\alpha \in \mathbb{C}$ such that

$$P(z) \left(\frac{R(z)}{S(z)} \right) + Q(z) + \alpha \neq 0 \quad (z \in bD). \quad (4.5)$$

Assume for a moment that not all c_j , $j \geq N+1$, vanish. Then the left side of (4.1) equals

$$\alpha - \frac{c_{N+1}}{z^{N+1}} - \dots$$

and by the argument principle the change of argument along bD_m oriented as part of bD is less than or equal to $-2\pi(N+1)$ provided that α is small enough. Thus, the change of argument of the left side of (4.5) along the part of bD that coincides with bD_m is less than or equal to $-(N+1)2\pi$. Further, the left side of (4.5) is holomorphic on each D_j , $1 \leq j \leq m-1$, so by the argument principle its change of argument along the part of bD that coincides with bD_j is nonpositive, $1 \leq j \leq m-1$. Thus, the change of argument of the left side of (4.5) along bD is less than or equal to $-(N+1)2\pi$ which contradicts the fact that S has at most N zeros. This proves that $c_j = 0$ ($j \geq N+1$) and so

$$P(z)T(z) = -Q(z) \quad (z \in bD)$$

so PT extends holomorphically through D . The proof is complete.

5. Meromorphic extendibility of continuous functions

We show that for continuous functions on bD the meromorphic extendibility can be expressed in terms of the argument principle.

Given a continuous function $\varphi : bD \rightarrow \mathbb{C} \setminus \{0\}$ we denote by $\mathcal{W}(\varphi)$ the winding number of φ (around the origin). So $2\pi\mathcal{W}(\varphi)$ equals the change of argument of $\varphi(z)$ as z runs along bD following the standard orientation.

If a continuous function $f : bD \rightarrow \mathbb{C} \setminus \{0\}$ extends meromorphically through D then $\mathcal{W}(f) \geq -N$ where N is the number of poles of the meromorphic extension \tilde{f} (counted with multiplicity). Indeed, by the argument principle,

$$\mathcal{W}(f) = v_0(\tilde{f}) - v_p(\tilde{f}) = v_0(\tilde{f}) - N \geq -N$$

where $v_0(\tilde{f})$ is the number of zeros of \tilde{f} on D and $v_p(\tilde{f})$ is the number of poles of \tilde{f} on D .

Let $f : bD \rightarrow \mathbb{C}$ be a continuous function which extends meromorphically through D and whose meromorphic extension \tilde{f} has N poles on D . Then $\mathcal{W}(Pf + Q) \geq -N$ for all functions P, Q in $A(D)$ such that $Pf + Q \neq 0$ on bD . Indeed, $P\tilde{f} + Q$, the meromorphic extension of $Pf + Q$, has no other poles than \tilde{f} and therefore, by the argument principle, $\mathcal{W}(Pf + Q) \geq -N$. This property characterizes meromorphic extendibility:

Theorem 5.1. Let D be a bounded domain in \mathbb{C} whose boundary consists of a finite number of pairwise disjoint simple closed curves. A continuous function $f : bD \rightarrow \mathbb{C}$ extends meromorphically through D if and only if there is an $N \in \mathbb{N} \cup \{0\}$ such that

$$\mathcal{W}(Pf + Q) \geq -N \quad (5.1)$$

for all $P, Q \in A(D)$ such that $Pf + Q \neq 0$ on bD . If this is the case then the meromorphic extension of f has at most N poles in D , counting multiplicity.

If $N = 0$ there are no poles so we have holomorphic extendibility – when D is the unit disc a better theorem (with $P \equiv 1$) was proved by the author in [4] and a simpler proof was obtained by D. Khavinson in [5], and if D is a multiply connected domain, such a theorem was proved by the author in [2]. If D is the open unit disc the theorem was proved for general N by the author in [1] using harmonic analysis.

Proof of Theorem 5.1. The only if part follows from the argument principle as we have shown above. To prove the if part, assume that (5.1) holds for all $P, Q \in A(D)$ such that $Pf + Q \neq 0$ on bD . Passing to a smaller N if necessary we may assume that there are $P_0, Q_0 \in A(D)$ such that $P_0f + Q_0 \neq 0$ on bD and such that $\mathcal{W}(P_0f + Q_0) = -N$. Write $P_0f + Q_0 = F$. Obviously $F \neq 0$ on bD . Given $P, Q \in A(D)$ we have $PF + Q = P(P_0f + Q_0) + Q = PP_0f + (PQ_0 + Q)$ where $PP_0 \in A(D)$ and $PQ_0 + Q \in A(D)$ so (5.1) implies that $\mathcal{W}(PF + Q) \geq -N$ whenever $P, Q \in A(D)$ are such that $PF + Q \neq 0$ on bD . It follows that

$$\mathcal{W}\left(P + Q \frac{1}{F}\right) = \mathcal{W}\left(\frac{PF + Q}{F}\right) = \mathcal{W}(PF + Q) - \mathcal{W}(F) \geq -N - (-N) = 0$$

whenever $P, Q \in A(D)$ are such that $P + Q \frac{1}{F} \neq 0$ on bD . The main result of [2] implies that $1/F$ extends holomorphically through D , that is, $1/F = G$ where $G \in A(D)$. Obviously G has no zero on bD and since $\mathcal{W}(F) = -N$ it follows that G has precisely N zeros on D counting multiplicity. Thus $1/G$ is a meromorphic extension of F through D which has no zero on D and has exactly N poles on D , counting multiplicity. If α is suitably chosen small number then by the same process we see that $F_1 = (P_0 + \alpha)f + Q_0$ has also a meromorphic extension through D with exactly N poles in D . Thus, $f = (F_1 - F)/\alpha$ extends meromorphically through D . We can choose α so that $P + \alpha \neq 0$ at the poles of the meromorphic extension of f so this meromorphic extension has the same number of poles as the meromorphic extension of F_1 has the same number of poles as F_1 which is N . \square

For holomorphic extendibility ($N = 0$) this theorem holds also for open Riemann surfaces [3] and therefore, repeating the proof, we see that Theorem 5.1 holds for open Riemann surfaces as well.

We have already mentioned that there is a better version of Theorem 5.1 in the case when $N = 0$: *A continuous function f on bD extends holomorphically through D if and only if $\mathcal{W}(f + Q) \geq 0$ for all $Q \in A(D)$ such that $f + Q \neq 0$ on bD* [2]. That is, in the case when $N = 0$ one can take $P \equiv 1$. Whether this holds for $N \geq 1$ remains to be seen:

Question. Let $N \in \mathbb{N}$ and suppose that f is a continuous function on bD such that $\mathcal{W}(f + Q) \geq -N$ whenever $Q \in A(D)$ is such that $f + Q \neq 0$ on bD . Does it follow that f extends meromorphically through D ?

The answer is not known even in the case when D is a disc.

Acknowledgment

This work was supported in part by the Ministry of Higher Education, Science and Technology of Slovenia through the research program Analysis and Geometry, Contract No. P1-0291.

References

- [1] J. Globevnik, Meromorphic extendibility and the argument principle, Publ. Mat. 52 (2008) 171–188.
- [2] J. Globevnik, The argument principle and holomorphic extendibility, J. Anal. Math. 94 (2004) 385–395.
- [3] J. Globevnik, The argument principle and holomorphic extendibility to finite Riemann surfaces, Math. Z. 253 (2006) 219–225.
- [4] J. Globevnik, Holomorphic extendibility and the argument principle, in: Contemp. Math., vol. 382, 2005, pp. 171–175.
- [5] D. Khavinson, A note on a theorem of J. Globevnik, in: Contemp. Math., vol. 382, 2005, pp. 227–228.
- [6] W. Rudin, Analytic functions of class H_p , Trans. Amer. Math. Soc. 78 (1955) 46–66.