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Nonlinear problems arising in electrostatic actuation in MEMS

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ABSTRACT

In this paper we study the nonlinear problem arising in electrostatic actuation of MEMS. We show that the existence and non-existence of the solution of this problem depend on the value of the physical parameters of the equation. In addition we consider the corresponding initial value problem and we derived the existence of periodic solution, stability of steady states and the ω-limit set.

1. Introduction

In 1959, Feynman [8,9] proposed a new field of science, the manufacture and control of micro-devices. Recently, this new field of science micro-electro-mechanical systems (MEMS) or nano-electro-mechanical systems (NEMS), has been developing rapidly. Applications of this emerging technology can be found in many industries, including airbag systems in automobiles, protection systems of computer hard drives, and motion control devices in video games. Therefore, it is important to understand the behavior of such devices through mathematical modeling. Numerous previous studies have examined this topic, including C. Cercignani [3], Chatterjee and Aluru [4], Chen et al. [5], Elata et al. [6], Guo et al. [7,10,11], Hung and Senturia [12], Lin and Yang [14], Pelesko and Bernstein [15], see also Legtenberg et al. [13], Wang and Hadaegh [16], Ye et al. [17].

In this research we investigate the behavior of the solution of the mathematical model of electrostatic actuation applied in variety of MEMS devices. The structure of electrostatic actuation consists of an elastic plate suspended above a ground plate. Both of the plates are made of conductive materials, and a dielectric medium fills the gap between them. The applied voltage causes the plate to deform. The range of the input voltage must thus be limited, or the elastic plate will be electronically damaged. The physicals laws that describe the behavior of such devices are combination of elastic theory and electrostatic Coulomb's law. In general, the Coulomb force follows inverse square law. Therefore, the deformation of the elastic plate obeys the following singular nonlinear equation

$$T\Delta u - D\Delta^2 u = \frac{\lambda}{(L+u)^2} \tag{1}$$

where the parameter λ affected by a number of factors, such as the input voltage V_i, capacitance C, the gap between the elastic and ground plates and the dielectric medium constant between the plates. Lin and Yang [14] show how to derive the above equation. Moreover, they demonstrate the behavior of the solutions with respect to parameters T, D, and λ.

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In this note we scale the spatial variable by $x' = \alpha x$ where $\alpha = \sqrt{\frac{T}{D}}$ then Eq. (1) can be simplified to an equation with parameters μ and L as follows

$$\Delta u - \Delta^2 u = \frac{\mu}{(L+u)^2}. \tag{2}$$

We shall mention that when the size of MEMS devices is relatively small then the ratio between the surface area and volume becomes large. Therefore, the induced physics phenomenon become much more complicated. For this reason, we generalize the nonlinear term to $\frac{\mu}{f(u)}$ where $f(u)$ includes all crucial properties of an inverse-square functions. In this note, we also extend problem (2) to time dependent problems [10]. In [10], Guo et al. demonstrate a thorough study of the semi-linear parabolic equation of problem (2) with varying dielectric properties, but they neglect the case of the existence of periodic solutions of the problem. Our results contain the condition of the non-existence of generalized problem (2). The results show that the parameter μ has a minimum μ_* such that Eq. (9) has no solution for $\mu > \mu_*$. This μ_* corresponds to the threshold of "pull-in" voltage which causes the devices break down. Note that; the parameter μ affected by a number of factors, including the input voltage, which can be a time dependent periodic function. Therefore, it is reasonable to consider the existence of time periodic solutions. Further, the existence of a periodic solution for a time dependent system is itself an interesting mathematical topics [1,2].

In this note we will apply the method of monotone iteration scheme [18] to study Eq. (2). The iteration scheme is based on the existence of upper and lower solution together with the following maximum principle.

Lemma 1. *If u is a classical solution satisfying equation*

$$-\Delta^2 u \geqslant 0,$$

or

$$-u_t - \Delta^2 u \geqslant 0, \quad u(0,x) = u_0 \leqslant 0$$

and boundary condition

$$u|_{\partial\Omega} \leqslant 0, \quad \text{and} \quad \Delta u|_{\partial\Omega} \geqslant 0, \tag{3}$$

then $u \leqslant 0$ and $\Delta u \geqslant 0$ in Ω.

We call a function ϕ an *upper solution* of problem (4)

$$\begin{cases} -\Delta^2 u = f(x,u,\Delta u), \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \end{cases} \tag{4}$$

provided that ϕ is smooth satisfying

$$\begin{cases} -\Delta^2 \phi \leqslant f(x,\phi,\Delta\phi), \\ \phi|_{\partial\Omega} \geqslant 0, \quad \Delta\phi|_{\partial\Omega} \leqslant 0, \end{cases} \tag{5}$$

and ψ a *lower solution* of problem (4) provided that ψ reverse the inequality sign of Eq. (5).

For the initial value problem (6)

$$\begin{cases} -u_t - \Delta^2 u = f(x,u,\Delta u), \\ u|_{\partial\Omega} = 0, \quad \Delta u|_{\partial\Omega} = 0, \quad u(0,x) = u_0 \end{cases} \tag{6}$$

we denote $\Gamma_\tau = \Omega \times (0,\tau)$. We call ϕ an *upper solution* on Γ_τ of problem (6) provide

$$\begin{cases} -\phi_t - \Delta^2 \phi \leqslant f(x,\phi,\Delta\phi), \\ \phi|_{\partial\Gamma_\tau} \geqslant 0, \quad \Delta\phi|_{\partial\Gamma_\tau} \leqslant 0, \quad \phi(0,x) \geqslant u_0 \end{cases} \tag{7}$$

and ψ a *lower solution* on Γ_τ provided ψ reverses the inequality sign of Eq. (7).

Notice that the solutions $\hat{u}$, $\bar{u}$ that we obtained by iteration method starting from upper solution ϕ and lower solution ψ are called the maximal and minimal solution respectively.

For an ω-periodic solution of problem (6), we call ϕ an *ω-upper solution* on Γ_τ of problem (6) provided $\tau > \omega$, and

$$\begin{cases} -\phi_t - \Delta^2 \phi \leqslant f(x,\phi,\Delta\phi), \\ \phi|_{\partial\Gamma_\tau} \geqslant 0, \quad \Delta\phi|_{\partial\Gamma_\tau} \leqslant 0, \quad \phi(0,x) \geqslant \phi(\omega,x) \end{cases} \tag{8}$$

and an *ω-lower solution* ψ on Γ_τ of problem (6) provided ψ reverses the inequality sign of Eq. (8).

2. Spatial problems under pinned boundary condition

We start our discussion with the following equation

$$\Delta u - \Delta^2 u = \frac{\mu}{f(u)}, \tag{9}$$

satisfying the pinned boundary condition given below

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0. \tag{10}$$

This boundary condition means the device is hinged along its edges such that it can be rotated freely and is subject neither to torque nor to bending moment on its edges.

As mentioned in the previous section that the parameter μ of the nonlinear term $\frac{\mu}{f(u)}$ of Eq. (9) relates to the controlling effects of this device such as the input voltage, capacity, the gap between membrane and the parameters of deflection.

If we consider further by adding the electrostatic force contribute by the changing of capacitance after deformation then the nonlinear term in the above equation needs to be adjusted. Suppose the input voltage V_i changes to V, the actuator deforms and induces the changing of capacitance which follows the relation below

$$V = \frac{V_i}{1 + \frac{C}{C_d}}. \tag{11}$$

In Eq. (11), C_d and V are the capacitance and the voltage of the actuator after deforming. C is the normal capacitance. Let $g(u)$ be the relation between deformation and the electrostatic force then we have

$$V = \frac{V_i}{1 + \chi \int_\Omega g(u)\,dx}.$$

Here χ is a constant relates to the circuit series capacitance (cf. Lin and Yang [14], Pelesko and Bernstein [15]) and hence we have the non-local equation of MEMS

$$\Delta u - \Delta^2 u = \frac{\mu}{f(u)\cdot(1 + \chi \int_\Omega g(u)\,dx)}. \tag{12}$$

Before we start to prove our results, we introduce the notation that we need in this article. We use λ_i, ξ_i to denote the ith eigen-value and corresponding eigen-function of Laplacian subject to boundary condition (10) on domain Ω.

We shall mention that ξ_1 never achieves 0 except at the boundary of Ω. In the sequel we denote $\Lambda = \lambda_1 + \lambda_1^2$ since it will appear frequently.

Throughout this article we assume f satisfies the following conditions:

(C-1) $f(\cdot)$, $f'(\cdot)$ are all non-negative, non-decreasing, differentiable functions.
(C-2) $f(u) \to 0$, as $u \to -1^+$.
(C-3) $(\frac{1}{f(u)})'' > 0$.

We first show that there exists a constant $\mu_* = \Lambda f(0)$ such that Eq. (9) has no solution for $\mu > \mu_*$.

Lemma 2. *If $\mu \geqslant \mu_* = \Lambda f(0)$ and f satisfies assumption* (C-1) *then problem* (9) *has no solution.*

Proof. Notice that

$$\Delta\xi_1 - \Delta^2\xi_1 = -\Lambda\xi_1.$$

Multiply ξ_1 to Eq. (9) and integrate over Ω, it yields

$$0 = \int_\Omega \left[\Lambda u + \frac{\mu}{f(u)}\right]\xi_1. \tag{13}$$

By our assumption of $f \geqslant 0$ and Lemma 1, we see that if Eq. (9) has a solution then it satisfies $-1 < u \leqslant 0$.
Let

$$F(\mu, u) = \Lambda u + \frac{\mu}{f(u)}. \tag{14}$$

If $\mu > \mu_*$ then by assumption (C-1) we have

$$F(\mu, u) \geqslant F(\mu_0, u) = \Lambda u + \frac{\Lambda f(0)}{f(u)} \geqslant \Lambda(u+1) \geqslant 0$$

contradict to Eq. (13). $\square$

We demonstrate the existence of solution of Eq. (9) by constructing its upper and lower solutions.

Theorem 3. *If $\mu \in (0, \mu_*)$ and f satisfies assumptions* (C-1), (C-2), (C-3) *then problem* (9) *has at least two solutions.*

Proof. First we will show that $\frac{1}{f(u)}$ has a minimum in $[-1, 0]$. Let us consider $F(\mu, u)$ of (14). The critical point of F occurs either $\Lambda = \frac{\mu f'(u_0)}{f^2(u_0)}$ or equivalently $\mu = \frac{\Lambda f^2(u_0)}{f'(u_0)}$. By assumption (C-3) u_0 is a minimal. To solve u_0, we substitute μ back to F. The assumption $(\frac{1}{f(u)})'' > 0$ of (C-3) then implies that u_0 is the minimum of F.

To prove Eq. (9) has a solution we need to construct an upper and a lower solution.

Exam Eq. (9) we see that 0 is an upper solution of Eq. (9), since $\mu = \frac{\mu}{f(0)} > \Delta 0 - \Delta^2 0$.

Let $v_1 = 0$ and $v_i = T(v_{i-1})$ where $T(v_{i-1})$ satisfies

$$\Delta v_i - \Delta^2 v_i = \frac{\mu}{f(v_{i-1})}, \tag{15}$$

then by Lemma 1, $v_2 \leqslant 0$ and hence $v_2 \leqslant v_1$ we obtained a monotone decreasing sequences

$$0 = v_0 \geqslant v_1 \geqslant v_2 \geqslant \cdots \geqslant \psi.$$

Let $v(x) = \lim_{i\to\infty} v_i(x)$ then $v(x)$ is a solution of Eq. (9), moreover, $v(x)$ is a maximal solution.

To construct lower solution of Eq. (9) we let $x_0 = \min_{x\in[-1,0]} u$ and Ω_1 be a domain larger then Ω. Let ξ_1' be the first eigen-function of Δ on Ω_1 and we choose an ϵ small enough so that $\epsilon\xi_1'|_{\partial\Omega} \geqslant |x_0|$ then

$$\begin{aligned}(\Delta - \Delta^2)(-\epsilon\xi_1') &= \Lambda\epsilon\xi_1' \\ &\geqslant \Lambda|x_0| \\ &\geqslant \frac{\mu}{f(x_0)}.\end{aligned}$$

Since $\frac{\mu}{f(x_0)} \geqslant \frac{\mu}{f(-\epsilon\xi_1')}$, we have $(\Delta - \Delta^2)(-\epsilon\xi_1) \geqslant \frac{1}{f(-\epsilon\xi_1)}$. Hence $-\epsilon\xi_1' = \psi$ is a lower solution of Eq. (9). Similarly let $u_0 = -\epsilon\xi_1'$ then by definition (15) we get a monotone sequences $\{u_i\}$. Let $u(x) = \lim_{i\to\infty} u_i(x)$, we get minimal solution of Eq. (9).

To demonstrate the regularity of Eq. (9) we let

$$I_\mu = \{\mu \mid \text{Eq. (9) has classical solution}\},$$

then the previous proof show that $I_\mu \neq \emptyset$. Let $(0, \mu'] \subset I_\mu$ then $\forall \mu \in (0, \mu']$ we have $u_{\mu'}$ a lower solution to u_μ, where u_μ and $u_{\mu'}$ are the solution to Eq. (9) corresponding to the parameter μ and μ'. Therefore, $\mu \in I_\mu$. In fact, if $\mu' \leqslant \mu''$ be elements in I_μ and let u' and u'' be the corresponding maximal solution of Eq. (9) corresponding to μ' and μ'' then

$$u' > u'' \quad \text{in } \Omega. \tag{16}$$

We complete the proof. $\square$

From the monotone results (16) and the uniform boundness of the solutions we conclude that $U = \lim_{\mu\to\mu_*} u_\mu$ exists. However, the nonlinear term is singular at -1 therefore, we discuss further the regularity of the limiting case of solution to Eq. (9).

Theorem 4. $U = \lim_{\mu\to\mu_*} u_\mu$ *is a weak solution.*

Proof. Equality (13) is essential of understanding the behavior of solution of Eq. (9). In fact, from (13) we have

$$\int_\Omega \frac{\xi_1}{f(u)} \leqslant C, \tag{17}$$

and hence for any compact subset $D \subset \Omega$ we have

$$\int_D \frac{\mu}{f(u)} \leqslant \frac{\Lambda|\Omega|}{\inf_{x\in D} \xi_1(x)} = C(D). \tag{18}$$

Let $v = 1 + u$ then $0 \leqslant v < 1$ and by maximum principle we have $v - \Delta v < 1$. Let

$$w = v + \frac{x^2}{2N}, \tag{19}$$

then

$$\Delta w > v > 0. \tag{20}$$

Thus w is a positive sub-harmonic function in Ω and therefore $\int_D |\nabla w|^2\,dx$ is bounded locally. Let $\Omega_0 \Subset \Omega$ and we insert Ω_1 such that $\Omega_0 \Subset \Omega_1 \Subset \Omega$ and we let η be a smooth cut off function satisfying

$$\eta = \begin{cases} 1, & x \in \Omega_0, \\ 0 < \eta < 1, & x \in \Omega_1 \setminus \Omega_0, \\ 0, & \text{otherwise.} \end{cases}$$

Multiplying $\eta^2 w$ to (20) and integrating over Ω we get

$$\int_\Omega \eta^2 |\nabla w|^2\,dx < -2 \int_\Omega \eta w \nabla\eta \cdot \nabla w\,dx. \tag{21}$$

By Schwarz's inequality and by the uniform boundness of w we get

$$\int_\Omega \eta^2 |\nabla u|^2\,dx \leqslant C(\eta). \tag{22}$$

Multiplying $\eta^4 u$ to Eq. (9) and applying integration by part twice we get

$$\begin{aligned} &-\int_\Omega \left(\eta^4 |\nabla u|^2 + 4\eta^3 u \nabla\eta \cdot \nabla u\right) dx - \int_\Omega \frac{\mu \eta^4 u}{f(u)}\,dx \\ &\quad = \int_\Omega \left(\eta^4 |\Delta u|^2 + 8\eta^3 \nabla\eta \cdot \nabla u \Delta u + 12\eta^2 u |\nabla\eta|^2 \Delta u + 4\eta^3 u \Delta\eta \Delta u\right) dx. \end{aligned} \tag{23}$$

By Young's inequality, Eq. (13) and the bound of $\eta^2 \nabla u$ of inequality (22) we get

$$\int_\Omega \eta^4 |\Delta u|^2 \leqslant C_1(\eta). \tag{24}$$

From (24) we see that for any $\Omega_0 \Subset \Omega$ we have

$$\|u_\mu\|_{W^{2,2}(\Omega_0)} \leqslant C(\Omega_0). \tag{25}$$

Thus there exists a $U \in W^{2,2}_{\mathrm{loc}}(\Omega)$ such that

$$\lim_{\mu \to \mu_*} u_\mu = U, \quad \text{weakly in } W^{2,2}_{\mathrm{loc}}(\Omega). \tag{26}$$

Hence U is a weak solution to Eq. (9) when $\mu = \mu_*$. $\square$

Next, we consider non-local equation

$$\Delta u - \Delta^2 u = \frac{\mu}{f(u) \cdot (1 + \chi \int_\Omega g(u)\,dx)}. \tag{27}$$

According to the Coulomb law and (11) $g(u) \approx \frac{1}{u}$ thus we assume the following

(N-1) $g(\cdot)$ is a positive decreasing function and without lost of generality we assume $g(0) = 1$.

Theorem 5. *If $\mu \in (0, \mu_*)$, f satisfies assumptions* (C-1), (C-2), (C-3) *and if g satisfies* (N-1) *then non-local problem* (27) *has a solution.*

Proof. To show non-local equation (27) has a solution, we consider solutions from two auxiliary problems, namely

$$\Delta\psi - \Delta^2\psi = \frac{\mu}{f(\psi) \cdot (1 + \chi |\Omega|)}, \tag{28}$$

and

$$\Delta\psi - \Delta^2\psi = \frac{\mu}{f(\psi) \cdot (1 + \chi \int_\Omega g(w)\,dx)}. \tag{29}$$

Since $\frac{\mu}{(1+\chi|\Omega|)} \leqslant \mu < \mu_*$, the solution $u(x)$ of Eq. (9) exists and is a lower solution to the first auxiliary problem (28). Thus there exists a classical maximal solution ψ of Eq. (28). Using ψ, we define the following closed convex set

$$\mathcal{S} = \left\{ w \in L^2(\Omega) \;\middle|\; \psi \leqslant w \leqslant 0, \text{ a.e. on } \Omega \right\}. \tag{30}$$

If $w \in \mathcal{S}$ then by condition (N-1) we have $\frac{\mu}{1+\chi|\Omega|} \geqslant \frac{\mu}{1+\chi \int_\Omega g(w)\,dx} \geqslant 0$, and hence

$$\Delta\psi - \Delta^2\psi = \frac{\mu}{f(\psi)\cdot(1+\chi|\Omega|)} \geqslant \frac{\mu}{f(\psi)\cdot(1+\chi\int_\Omega g(w)\,dx)}.$$

Thus the solution to the first auxiliary problem ψ is a lower solution to our second auxiliary problem (29) provide that $w \in \mathcal{S}$. We may then apply monotone iteration scheme again to the second auxiliary problem and obtain a maximal solution ψ_w satisfying $\psi \leqslant \psi_w \leqslant 0$ in Ω. Therefore, we may define a map $T:\mathcal{S} \to \mathcal{S}$ by $T(w) = \psi_w$. Notice that the function in $T(\mathcal{S})$ is uniformly bounded away from -1, we can show by L^2 estimating that $T(\mathcal{S})$ is bounded subset of $W^{4,2}(\Omega)$. Consequently, T is a completely continuous map and therefore, in $\mathcal{S}$, T has a fixed point which by standard theory of elliptic equation is a classical solution of (27).

If we consider χ as a parameter then the family of solutions $\{u_\chi\}$ of Eq. (27) is bounded uniformly in $W^{4,2}(\Omega)$ and hence u_χ converges weakly to $\hat{u}$ as $\chi \to 0$, where $\hat{u}$ satisfies Eq. (9). Notice that Eq. (27) is reduced to (9) as $\chi \to 0$. Therefore, it implies that $\hat{u}$ recovers the solution of (27). □

3. Stability and existence of periodic solutions

In this section we begin with the discussion of the existence of initial value problem

$$\begin{cases} -u_t + \Delta u - \Delta^2 u = \dfrac{\mu}{f(u)}, & x \in \Gamma_\tau, \\ u|_{\partial\Gamma_\tau} = \Delta u|_{\partial\Gamma_\tau} = 0, \end{cases} \tag{31}$$

and the stability of the steady states. At the end of this section we will discuss the existence of ω-periodic solution

$$\begin{cases} -u_t + \Delta u - \Delta^2 u = \dfrac{\mu}{f(u)}, & x \in \Gamma_\tau, \\ u|_{\partial\Gamma_\tau} = \Delta u|_{\partial\Gamma_\tau} = 0, \end{cases} \tag{32}$$

and their stability as well.

Since the upper and lower solution of steady states (9) subject to boundary condition (10) is also the upper and lower solution to the corresponding initial value problem, we may derive the existence of global time solution to initial value problem (31) provided that f satisfies conditions (C-1), (C-2) and (C-3).

Theorem 6. *If $\mu \in (0, \mu_*)$ and f satisfies assumptions* (C-1), (C-2), (C-3) *let ψ_μ be the lower solution of steady state* (9) *then problem* (31) *has at least two global time solutions if the initial data u_0 satisfies $\psi_\mu \leqslant u_0 \leqslant 0$.*

Proof. From the previous proof 0 and ψ_μ are upper and lower solutions of (9). We let $u_i = T(u_{i-1})$ and start iteration from upper solution 0 then u_i satisfies

$$-u_{it} + \Delta u_i - \Delta^2 u_i = \frac{\mu}{f(u_{i-1})},$$

and boundary condition

$$u_i|_{\partial\Gamma_\tau} = \Delta u_i|_{\partial\Gamma_\tau} = 0.$$

We get a monotone decreasing sequence

$$\psi_\mu \leqslant \cdots \leqslant u_2 \leqslant u_1 \leqslant 0,$$

hence u_i are uniformly bounded. Let $u(t,x) = \lim_{n\to\infty} u_i(t,x)$ then $u(t,x)$ is a solution of (31). In fact, $u(t,x)$ is a maximal solution. Similarly, if we start iteration from ψ_μ, we will get a minimal solution. To see that solution $u(t,x)$ is a classical solution we consider Lyapunov functions

$$I = \frac{1}{2}\int_\Omega u^2\,dx, \tag{33}$$

$$J = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx, \tag{34}$$

and

$$K = \frac{1}{2}\int_\Omega |\Delta u|^2\,dx, \tag{35}$$

then by Poincaré's inequality and Young's inequality it yields

$$\frac{dI}{dt} = \int_\Omega -|\nabla u|^2 - |\Delta u|^2 - \frac{\mu u}{f(u)} \leqslant -a_1 I + c_1\big(f(\psi)\big), \tag{36}$$

$$\frac{dJ}{dt} = \int_\Omega -|\nabla u|^2 - |\Delta u|^2 - \frac{\mu u}{f(u)} \leqslant -a_2 J + c_2\big(f(\psi)\big), \tag{37}$$

$$\frac{dK}{dt} = \int_\Omega -|\nabla u|^2 - |\Delta u|^2 - \frac{\mu u}{f(u)} \leqslant -a_3 K + c_3\big(f(\psi)\big). \tag{38}$$

Thus by the standard embedding theory of elliptic equations, the solution $u(t,x)$ is classical and $u \in C([0,\infty], C^2(\bar{\Omega})) \cap C^1((0,\infty], C^4(\bar{\Omega}))$. $\square$

Next, we consider initial value problem of non-local equation

$$\begin{cases} -u_t + \Delta u - \Delta^2 u = \dfrac{\mu}{f(u)\cdot(1+\chi\int_\Omega g(u)\,dx)}, & x \in \Gamma_\tau, \\ u|_{\partial\Gamma_\tau} = \Delta u|_{\partial\Gamma_\tau} = 0, \quad u(0,x) = u_0. \end{cases} \tag{39}$$

Theorem 7. *If $\mu \in (0,\mu_*)$ and f satisfies assumptions* (C-1), (C-2), (C-3), *g satisfies* (N-1), *ψ is the lower solution of steady state* (27) *and if the initial data satisfies $\psi \leqslant u_0 \leqslant 0$ then problem* (39) *has a solution.*

Proof. Similar to Theorem 5, we consider solutions from two auxiliary problems, namely

$$-u_t + \Delta u - \Delta^2 u = \frac{\mu}{f(u)\cdot(1+\chi|\Omega|)}, \tag{40}$$

where $|\Omega|$ is the measure of Ω and

$$-u_t + \Delta u - \Delta^2 u = \frac{\mu}{f(u)\cdot(1+\chi\int_\Omega g(w)\,dx)}. \tag{41}$$

Since $\frac{\mu}{1+\chi|\Omega|} < \mu < \mu_*$, Theorem 6 shows that Eq. (40) has a maximal solution ψ_τ. Using ψ_τ, we define convex set

$$\mathcal{S}_\tau = \big\{ w \in C\big([0,\tau), L^2(\Omega)\big) \cap C^1\big((0,\tau), L^2(\Omega)\big) \bigm| \psi_\tau \leqslant w \leqslant 0,\ \text{a.e. on } \Omega \big\} \tag{42}$$

then for all $w \in \mathcal{S}_\tau$, we have $-1 < \psi_\tau \leqslant w \leqslant 0$ and

$$-\psi_{\tau t} + \Delta\psi_\tau - \Delta^2\psi_\tau = \frac{\mu}{f(\psi_\tau)\cdot(1+\chi|\Omega|)} \geqslant \frac{\mu}{f(\psi_\tau)\cdot(1+\chi\int_\Omega g(w)\,dx)}, \tag{43}$$

thus ψ_τ is a lower solution to non-local equation (39) subject to boundary condition (10). Thus for each $w \in \mathcal{S}_\tau$ there is a maximal solution u_w satisfying $\psi_\tau \leqslant u_w \leqslant 0$ on Γ_τ. Therefore, we may define map $T: \mathcal{S}_\tau \to \mathcal{S}_\tau$ by $T(w) = u_w$. Since the functions in $T(\mathcal{S}_\tau)$ are uniformly bounded away from -1, $T(\mathcal{S}_\tau)$ is a bounded subset in $C([0,\tau), W^{4,2}(\Omega)) \cap C^1((0,\tau), W^{4,2}(\Omega))$. Consider Lyapunov functions (33)–(35), by Poincaré's inequality and Young's inequality it yields

$$\frac{dI}{dt} = \int_\Omega -|\nabla u|^2 - |\Delta u|^2 - \frac{\mu u}{f(u)(1+\chi\int_\Omega g(u)\,dx)} \leqslant -b_0 I + k\big(f(\psi_\tau), g(\psi_\tau)\big), \tag{44}$$

$$\frac{dJ}{dt} \leqslant -b_1 I + k\big(f(\psi_\tau), g(\psi_\tau)\big), \tag{45}$$

$$\frac{dK}{dt} \leqslant -b_2 I + k\big(f(\psi_\tau), g(\psi_\tau)\big). \tag{46}$$

Hence T maps $C([0,\tau), W^{4,2}(\Omega)) \cap C^1((0,\tau), W^{4,2}(\Omega))$ into itself. By standard theory of elliptic embedding T has a fixed point and again by standard elliptic theory the fixed point of T is a classical solution. $\square$

In Theorem 6 we use the Lyapunov functions (36)–(38) to derive the regularity of the solution of Eqs. (31) and (39). From inequalities (36) and (44) we see that the steady states of initial value problem (31) and (39) are stable if the initial data is proper. We define

$$I = \int_\Omega \frac{1}{2}\big(|\nabla u|^2 + |\Delta u|^2\big) + F(u)\,dx \tag{47}$$

where

$$F(u) = \int \frac{\mu}{f(w)}\,dw \tag{48}$$

for problem (31) and

$$F(u) = \int \frac{\mu}{f(w) \cdot (1 + \chi \int_\Omega g(w)\, dx)}\, dw \tag{49}$$

for problem (39). It is clear that

$$\frac{dI}{dt} = -\int_\Omega u_t^2 \leqslant 0. \tag{50}$$

Thus every steady state solution u of initial value problem (31) and (39) is stable equilibrium. Then we obtain the following theorem.

Theorem 8. *Every steady state of initial value problem* (31) *and* (39) *subject to boundary condition* (10) *is stable.*

By previous theorem we have at least 2 stable steady state solutions obtained by iteration starting from upper solution 0 and lower solution ψ of problem (31), that is, the minimal and maximal solution from iteration scheme. This fact indicates that there exists some unstable ω-limit set. Thus we study the existence of periodic solution of Eqs. (31) and (39) subject to boundary condition (10).

Theorem 9. *If $\mu \in (0, \mu_*)$ and f satisfies assumptions* (C-1), (C-2), (C-3) *and if $\frac{\mu f'(\xi)}{f^2(\xi)} - \lambda_1 - \lambda_1^2 \geqslant 0$ then there exists at least one ω-periodic solution $u(t,x)$ to problem* (31) *satisfying $\bar{u} \leqslant u(t,x) \leqslant \hat{u}$, moreover, the ω-limit set corresponding to the periodic solution is unstable.*

Proof. Let ψ be the lower solution of Eq. (9) and

$$\mathcal{F} = \left\{\xi \in C^2(\bar{\Omega}) \;\middle|\; \psi \leqslant \xi \leqslant 0,\ \text{and } \Delta\bar{u} \geqslant \Delta\xi \geqslant 0\right\}.$$

We define the Poincaré's map $Tu_0 = u(\omega, x)$ where $u_0 \in \mathcal{F}$ and $u(\cdot, x)$ is the solution to Eq. (31). By the previous proof, the solution $u(t,x)$ of Eq. (31) satisfies $u(t,x) \in C([0,\infty), C^2(\Omega)) \cap C^1((0,\infty), C^4(\Omega))$ thus $T : \mathcal{F} \to C^4(\Omega)$. Since $T(\mathcal{F})$ is uniformly bounded and $C^2(\Omega)$ is compactly embedded in $C^4(\Omega)$ then T is completely continuous. Therefore, it has a fixed point. The uniqueness of solution of Eq. (31) implies that $u(t,x)$ is periodic solution with period ω.

Let $u(t,x)$ be the ω-periodic solution to Eq. (31) and let

$$\mathcal{S} = \left\{u(t,x) \;\middle|\; 0 \leqslant t \leqslant \infty\right\}$$

be the invariant manifold of $u(t,x)$.

To show that ω-limit set $\mathcal{S}$ corresponding to periodic solution $u(t,x)$ is unstable, we use $d(x, A) = \min_{y\in A}\{\operatorname{dist}(x,y)\}$ to denote the distance between a point x and set A. Let ϕ be the solution to Eq. (31) satisfying $\phi_0 \in \mathcal{F}$ and $d(\phi_0, \mathcal{S}) \leqslant \epsilon$. Let $u(t_0, x) \in \mathcal{S}$ such that $d(\phi_0, \mathcal{S}) = d(\phi_0, u(t_0, x))$. Without loss of generality, we may assume $t_0 = 0$ otherwise by transformation $v(t,x) = u(t - t_0, x)$ and we still denote $v(t,x)$ by $u(t,x)$ which will then satisfy our assumption.

We define Lyapunov function

$$I = \frac{1}{2}\int_\Omega (u-\phi)^2\, dx \tag{51}$$

then

$$\frac{dI}{dt} = \int_\Omega (u-\phi)\frac{d(u-\phi)}{dt}\, dx \tag{52}$$

$$= \int_\Omega -\big(\nabla(u-\phi)\big)^2 - \big(\Delta(u-\phi)\big)^2 + \mu(u-\phi)\left(\frac{1}{f(\phi)} - \frac{1}{f(u)}\right) dx \tag{53}$$

$$= \int_\Omega -\big(\nabla(u-\phi)\big)^2 - \big(\Delta(u-\phi)\big)^2 + \mu(u-\phi)^2\frac{f'(\xi)}{f^2(\xi)}\, dx \tag{54}$$

$$\geqslant \int_\Omega (u-\phi)^2\left[\frac{f'(\xi)}{f^2(\xi)} - \lambda_1 - \lambda_1^2\right] dx. \tag{55}$$

If f satisfies $\frac{\mu f'(\xi)}{f^2(\xi)} - \lambda_1 - \lambda_1^2 \geqslant 0$ then $\frac{dI}{dt} \geqslant 0$. Consequently, the ω-limit set corresponding to the periodic solution $u(t,x)$ is unstable. □

Theorem 10. *If $\mu \in (0, \mu_*)$ and f satisfies assumptions* (C-1), (C-2), (C-3) *and g satisfies* (N-1) *then there exists at least one ω-periodic solution $u(t,x)$ to problem* (31) *satisfying $\bar{u} \leqslant u(t,x) \leqslant \hat{u}$, moreover, the ω-limit set corresponding to the periodic solution is unstable provided that*

$$-\frac{d}{du}\left(\frac{\mu}{f(u)(1+\chi\int_\Omega g(u)\,dx)}\right) - \lambda_1 - \lambda_1^2 \geqslant 0.$$

Proof. By similar argument as Theorems 6 and 7 we can proof the existence of ω-periodic solution thus we only discuss the stability of ω-periodic solution.

Let $u(t,x)$ be the ω-periodic solution to Eq. (39) and $\mathcal{S}$ be the corresponding invariant manifold of $u(t,x)$. Suppose ϕ is a solution of Eq. (39) satisfying $\phi_0 \in \mathcal{F}$ and $d(\phi_0, \mathcal{S}) \leqslant \epsilon$. We define Lyapunov function

$$I = \frac{1}{2}\int_\Omega (u-\phi)^2\,dx \tag{56}$$

and to simplify the notation we denote $H(u) = \frac{\mu}{f(u)(1+\chi\int_\Omega g(u)\,dx)}$ then

$$\begin{aligned}\frac{dI}{dt} &= \int_\Omega (u-\phi)\frac{d(u-\phi)}{dt}\,dx \\ &= \int_\Omega -\big(\nabla(u-\phi)\big)^2 - \big(\Delta(u-\phi)^2\big) - (u-\phi)\big(H(u)-H(\phi)\big)\,dx.\end{aligned}$$

By mean value theorem there exist a $\xi = \theta u + (1-\theta)\phi$ and some $0 \leqslant \theta \leqslant 1$ such that $H(u) - H(\phi) = H'(\xi)(u-\phi)$. Thus

$$\begin{aligned}\frac{dI}{dt} &= \int_\Omega -\big(\nabla(u-\phi)\big)^2 - \big(\Delta(u-\phi)\big)^2 - (u-\phi)^2 H'(\xi) \\ &\geqslant \int_\Omega (u-\phi)^2\big[-H'(\xi) - \lambda_1 - \lambda_1^2\big]\,dx.\end{aligned}$$

By our assumption $-H'(\xi) - \lambda_1 - \lambda_1^2 \geqslant 0$, we complete the proof. □

We now give an example to end this paper.

Example 1. Consider $f(u) = (1+u)^2$ then Eq. (31) becomes

$$-u_t + \Delta u - \Delta^2 u = \frac{\mu}{(1+u)^2}. \tag{57}$$

Eq. (57) is a typical model. Many articles discuss the solution behavior of it. Since

$$\frac{\mu f'(\xi)}{f^2(\xi)} = \frac{2\mu}{(1+u)^3} \to \infty \quad \text{as } u \to -1,$$

Theorem 9 implies Eq. (57) has a periodic solution.

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