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Permutable entire functions satisfying algebraic differential equations

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ABSTRACT

In this paper we characterize the relation between two entire functions which are permutable and satisfy certain algebraic differential equations.

1. Introduction and Main Results

Let $f(z)$ be a transcendental entire function on $\mathbb{C}$. The order and lower order of $f(z)$ are defined, respectively, by

$$\lambda(f)=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r}\quad\text{and}\quad \mu(f)=\liminf_{r\to\infty}\frac{\log\log M(r,f)}{\log r},$$

where $M(r,f)=\max\{|f(z)|:\ |z|=r\}$. Two entire functions f and g are said to be permutable if they satisfy

$$f\big(g(z)\big)=g\big(f(z)\big)$$

for all $z\in\mathbb{C}$. The study of permutability of two entire functions has attracted many researchers, see [1,4,5,7–10]. In this paper, we consider permutability of two entire functions one of which is a solution of an algebraic differential equation.

For each multi-index $\lambda=(i_0,i_1,\dots,i_n)$ with $i_j\in\mathbb{N}\cup\{0\}$, set

$$M_\lambda[w](z)=w^{i_0}(w')^{i_1}\cdots\big(w^{(n)}\big)^{i_n},$$

and $D(\lambda)=i_0+i_1+\cdots+i_n$. Then a differential polynomial $P[w]$ is an expression of the following form

$$P[w](z)=\sum_{\lambda\in J}a_\lambda(z)M_\lambda[w](z)$$

where J is a finite set of multi-indices, $a_\lambda(z)$ a polynomial and put $D[P]=\max\{D(\lambda):\ \lambda\in J\}$. An equation of the form $P[w](z)=0$ is called an algebraic differential equation.

In [9], Zheng and Yang established the following, which extends the results in [4] and [10].

Theorem A. *Let $f(z)$ and $g(z)$ be permutable entire functions of finite order and $f(z)$ of positive lower order. If $f(z)$ satisfies an algebraic differential equation $P[w](z)=0$, then $g(z)$ also satisfies an algebraic differential equation $Q[w](z)=0$ with $D[Q]\leqslant D[P]$.*

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Recently, Bergweiler [1] proved Theorem A excluding $D[Q] \leqslant D[P]$ without the hypothesis about the growth of $f(z)$ and $g(z)$ in terms of a result of Ostrowski about a composition of two analytic functions satisfying an algebraic differential equation. It is obvious that in his result the inequality $D[Q] \leqslant D[P]$ may not hold, for instance, if $f(z)$ satisfies a linear differential equation with polynomial coefficients, then $f(f(z))$ can in no way satisfy such a linear differential equation, for $f(f(z))$ is of infinite growth order.

In light of Theorem A, we determine the relationship between two permutable entire functions satisfying certain differential equations and establish the following.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be permutable transcendental entire functions of finite order and of positive lower order. Assume that $f(z)$ satisfies the following algebraic differential equation*

$$a_0(z)w^{(n)} + \sum_{\lambda\in J} a_\lambda(z)M_\lambda[w](z) = 0 \tag{1}$$

where J is a finite set consisting of multi-indices with the form: for $n \geqslant 2$, $\lambda = (i_0, i_1, \ldots, i_{n-2})$; for $n = 1$, $\lambda = (i_0)$ with $i_0 \neq 0$ and $a_0(z)(\not\equiv 0)$ and $a_\lambda(z)$ are polynomials. If $f(z)$ cannot satisfy any algebraic differential equation of order less than n, then

$$g(z) = af(z) + b$$

for two complex numbers $a(\neq 0)$ and b.

There exists a special case, that is, $w' = \alpha(z)w + \beta(z)$ for a pair of rational functions $\alpha(z)$ and $\beta(z)(\not\equiv 0)$, not contained in Theorem 1.1. However for this case we can get a pair of rational functions $A(z)$ and $B(z)$ such that $A(f(z)) = B(g(z))$. Actually, when one of $\alpha(z)$ and $\beta(z)$ is a constant, we can still deduce $g(z) = af(z) + b$.

We have a consequence of Theorem 1.1.

Corollary 1.1. *Let $f(z)$ and $g(z)$ be permutable transcendental entire functions and $g(z)$ is of finite order. Assume that $f(z)$ satisfies the following linear differential equation with $n > 1$*

$$a_n(z)w^{(n)} + a_{n-2}(z)w^{(n-2)} + \cdots + a_0(z)w = P(z) \tag{2}$$

where each $a_j(z)$ is a polynomial and $P(z)$ a polynomial but does not satisfy any linear differential equation of order less than n. Then

$$g(z) = af(z) + b$$

for two complex numbers $a(\neq 0)$ and b.

We make remarks on Corollary 1.1. In the corollary, actually $f(z)$ is of finite order and of positive lower order, for $f(z)$ is assumed to satisfy Eq. (2) (see Laine [3]). In view of Theorem A, $g(z)$ also satisfies such an equation and then $g(z)$ is of positive lower order. Thus, Corollary 1.1 follows from the Theorem 1.1.

As a special case, we consider the second order linear differential equation

$$w'' + P(z)w = 0 \tag{3}$$

with polynomial $P(z)$. If $f(z)$ satisfies (3) but not the equation $w' + R(z)w = 0$ with $R(z)$ being a rational function, then in view of Corollary 1.1 we have $g(z) = af(z) + b$. Assume now that $f(z)$ satisfies such an equation $w' + R(z)w = 0$ and therefore $\frac{f'(z)}{f(z)}$ is a rational function. This immediately implies that $f(z)$ has only finitely many zeros and so we can write

$$f(z) = A(z)e^{B(z)}$$

for a pair of polynomials $A(z)$ and $B(z)$ where $B(z)$ is not a constant. From Theorem 3 of [10] (see Lemma 2.3 in the sequel), we have $g(z) = af(z) + b$ for two complex numbers $a(\neq 0)$ and b. Thus, we have proved the following result.

Corollary 1.2. *Let $f(z)$ and $g(z)$ be as in Corollary 1.1. Assume that $f(z)$ satisfies (3). Then $g(z) = af(z) + b$ for two complex numbers $a(\neq 0)$ and b.*

What we should mention is that there are several classes of important functions satisfying Eq. (3), for examples, the Airy function satisfying $w'' - zw = 0$ and $s^{1/s}\sqrt{z}J_{1/s}(\frac{2z^{s/2}}{s})$ satisfying $w'' + z^{s-2}w = 0$ with $s \geqslant 2$ where J stands for the Bessel function.

2. Proofs of our results

In order to prove Theorem 1.1, we first of all establish a result which is also of independent significance. To this end, we need Theorem 1 of Yanagihara [6], which is stated as a lemma as follows.

Lemma 2.1. *Let $P(z)$ and $Q(z)$ be polynomials of degree greater than one. If the equation*

$$P\big(f(z)\big) = f\big(Q(z)\big)$$

admits a meromorphic solution $f(z)$, then $\deg P = \deg Q$ and the solution $f(z)$ is not transcendental.

Now we establish the desired result.

Lemma 2.2. *Let $f(z)$ and $g(z)$ be permutable transcendental entire functions. If $f(z) = R(g(z))$ for a non-constant rational function $R(z)$, then $g(z) = af(z) + b$.*

Proof. From the equality $f(g(z)) = g(f(z))$, we have

$$R(g) \circ g(z) = f(g) = g(f) = g \circ R\big(g(z)\big)$$

so that

$$R\big(g(z)\big) = g\big(R(z)\big).$$

Since $R(g(z))$, and hence $g(R(z))$ from the above equality, is meromorphic, $R(z)$ must be a polynomial. Thus, by applying Lemma 2.1 yields that $R(z)$ is linear, that is, $R(z) = az + b$. Lemma 2.2 follows.

Lemma 2.2 leads us to consider a general case. Let $P(u, v)$ be an irreducible non-constant complex polynomial in u and v.

Question. *Is $g(z)$ a linear expression of $f(z)$ or do there exist a pair of rational functions $A(z)$ and $B(z)$ with degree two such that $A(f(z)) = B(g(z))$, if $f(z)$ and $g(z)$ are permutable and $P(f(z), g(z)) \equiv 0$?*

Possibility of the second case in the Question can be confirmed by Example 1 of Ng [5]. Actually, for $a, c \in \mathbb{C}$ with $e^{4a} = -1$ and $c \neq 0$,

$$f(z) = 2ci \cos\left(\frac{az^2}{2c^2}\right) \quad \text{and} \quad g(z) = 2ci \sin\left(\frac{az^2}{2c^2}\right)$$

are permutable and $f^2 = -g^2 - 2c^2$.

Assume, in addition, that $f(z)$ is left-side prime, that is, if $f(z)$ can be factorized into the form $S(h(z))$ for a meromorphic function $S(z)$ and a transcendental entire function $h(z)$, then $S(z)$ is a fractional linear transformation. We can confirm the above question.

Actually, by employing the argument in the proof of Lemma 1 of Fuchs and Song [2] to $P(f(z), g(z)) \equiv 0$, we have an entire function $h(z)$ and a pair of rational functions $U(z)$ and $V(z)$ with at most one (possibly multiple) pole such that

$$f(z) = U\big(h(z)\big) \quad \text{and} \quad g(z) = V\big(h(z)\big).$$

Then $U(z)$ is a fractional linear transformation and so

$$g(z) = V\big(h(z)\big) = V\big(U^{-1}\big) \circ f(z)$$

where U^{-1} is the inverse of $U(z)$ and it is also a fractional linear transformation, and $V(U^{-1})(z)$ is a rational function. The result that $g = af + b$ follows from Lemma 2.2.

The following is Theorem 3 of [10]. □

Lemma 2.3. *Let $f(z) = Q + He^P$, where Q and $H(\not\equiv 0)$ are polynomials and P is a non-constant polynomial and let $g(z)$ be a non-linear entire function of finite order, permutable with f. Then $g(z) = af(z) + b$.*

In the proof of Theorem 1.1, we need the Nevanlinna theory, which the reader is assumed to be familiar with. By $T(r, f)$ and $m(r, f)$ we mean in the standard way the characteristic and proximity functions of a meromorphic function f and by $N(r, f = a)$ and $\overline{N}(r, f = a)$ for $a \in \mathbb{C} \cup \{\infty\}$ the integrated counting functions for the roots counting multiplicities and distinct roots of $f(z) = a$ in $\{|z| < r\}$, respectively.

When $f(z)$ is entire, then $T(r, f) = m(r, f)$. Furthermore, $m(r, f'/f) = O(\log r)$, provided that $f(z)$ is of finite order; and $T(r, f) = O(\log r)$ if and only if f is rational. The reader is referred to [3] for more details.

Now we are in position to prove Theorem 1.1.

Proof of Theorem 1.1. Let us begin our proof with the case $n > 1$. Differentiating both sides of $f(g(z)) = g(f(z))$ step by step, we obtain

$$f'(g) = \frac{f'}{g'} g'(f)$$

and

$$f''(g)g' = \left(\frac{f'}{g'}\right) f' g''(f) + \left(\frac{f'}{g'}\right)' g'(f)$$

or equivalently

$$f''(g) = \left(\frac{f'}{g'}\right)^2 g''(f) + \frac{1}{g'}\left(\frac{f'}{g'}\right)' g'(f).$$

Assume by induction that for $k \geqslant 1$ we have

$$f^{(k)}(g) = \left(\frac{f'}{g'}\right)^k g^{(k)}(f) + \frac{k(k-1)}{2}\frac{1}{g'}\left(\frac{f'}{g'}\right)^{k-2}\left(\frac{f'}{g'}\right)' g^{(k-1)}(f) + R_k, \tag{4}$$

where

$$R_k = \sum_{j=0}^{k-2} \phi_{k,j} g^{(j)}(f)$$

with coefficients $\phi_{k,j}$ being rational functions in $f', \dots, f^{(k)}$ and $g', \dots, g^{(k)}$. We differentiate both sides of (4) to obtain

$$\begin{aligned} f^{(k+1)}(g)g' &= \left(\frac{f'}{g'}\right)^k f' g^{(k+1)}(f) + k\left(\frac{f'}{g'}\right)^{k-1}\left(\frac{f'}{g'}\right)' g^{(k)}(f) + \frac{k(k-1)}{2}\left(\frac{f'}{g'}\right)^{k-1}\left(\frac{f'}{g'}\right)' g^{(k)}(f) \\ &\quad + \frac{k(k-1)}{2}\left[\frac{1}{g'}\left(\frac{f'}{g'}\right)^{k-2}\left(\frac{f'}{g'}\right)'\right]' g^{(k-1)}(f) + R_k' \\ &= \left(\frac{f'}{g'}\right)^k f' g^{(k+1)}(f) + \frac{k(k+1)}{2}\left(\frac{f'}{g'}\right)^{k-1}\left(\frac{f'}{g'}\right)' g^{(k)}(f) \\ &\quad + \frac{k(k-1)}{2}\left[\frac{1}{g'}\left(\frac{f'}{g'}\right)^{k-2}\left(\frac{f'}{g'}\right)'\right]' g^{(k-1)}(f) + R_k'. \end{aligned}$$

It follows that

$$f^{(k+1)}(g) = \left(\frac{f'}{g'}\right)^{k+1} g^{(k+1)}(f) + \frac{k(k+1)}{2}\frac{1}{g'}\left(\frac{f'}{g'}\right)^{k-1}\left(\frac{f'}{g'}\right)' g^{(k)}(f) + R_{k+1}$$

where

$$R_{k+1} = \sum_{j=0}^{k-1} \phi_{k+1,j} g^{(j)}(f)$$

with

$$\phi_{k+1,k-1} = \frac{k(k-1)}{2}\frac{1}{g'}\left[\frac{1}{g'}\left(\frac{f'}{g'}\right)^{k-2}\left(\frac{f'}{g'}\right)'\right]' + \phi_{k,k-2} f'/g'$$

and $\phi_{k+1,j} = (\phi_{k,j}' + \phi_{k,j-1} f')/g'$ $(0 \leqslant j \leqslant k-2)$, $\phi_{k,-1} \equiv 0$, $\phi_{k,k-1}' \equiv 0$. Then the coefficients $\phi_{k+1,j}$ $(0 \leqslant j \leqslant k-1)$ in the expression of R_{k+1} are rational functions in $f', \dots, f^{(k+1)}$ and $g', \dots, g^{(k+1)}$. Thus, in view of the induction principle, the equality (4) holds for any natural number k.

From the assumptions of Theorem 1.1, we have

$$a_0(z) f^{(n)}(z) + \sum_{\lambda \in J} a_\lambda(z) M_\lambda[f](z) = 0. \tag{5}$$

From (5), with $g(z)$ in place of z, we get

$$a_0\big(g(z)\big) f^{(n)}\big(g(z)\big) + \sum_{\lambda \in J} a_\lambda\big(g(z)\big) M_\lambda[f]\big(g(z)\big) = 0. \tag{6}$$

Substituting (4) into (6) yields

$$a_0(g)\left[\left(\frac{f'}{g'}\right)^n g^{(n)}(f) + \frac{n(n-1)}{2}\frac{1}{g'}\left(\frac{f'}{g'}\right)^{n-2}\left(\frac{f'}{g'}\right)' g^{(n-1)}(f)\right] + \sum_{\lambda \in I} \psi_\lambda M_\lambda[g](f) = 0 \tag{7}$$

where I is a finite set of multi-indices with the form $\lambda = (i_0, i_1, \ldots, i_{n-2})$ and ψ_λ is a rational function in $f', \ldots, f^{(n)}$ and $g', \ldots, g^{(n)}$. In view of the argument in the proof of Theorem A, there exist polynomials $b_0(z)$, $b_1(z)$ and $b_\lambda(z)$ for each $\lambda \in I$, not all zero, such that

$$b_0(z)g^{(n)} + b_1(z)g^{(n-1)} + \sum_{\lambda\in I} b_\lambda(z)M_\lambda[g](z) = 0. \tag{8}$$

It is easy to see that $b_0(z) \not\equiv 0$, otherwise $g(z)$ shall satisfy an algebraic differential equation with order less than n and then in view of Theorem A, $f(z)$ also has to satisfy such an equation, which contradicts our hypothesis.

With $f(z)$ in place of z in (8), we have

$$b_0(f)g^{(n)}(f) + b_1(f)g^{(n-1)}(f) + \sum_{\lambda\in I} b_\lambda(f)M_\lambda[g](f) = 0. \tag{9}$$

Comparing (7) with (9) yields

$$\frac{n(n-1)}{2}\frac{1}{g'}\Big(\frac{f'}{g'}\Big)^{n-2}\left(\frac{f'}{g'}\right)' b_0(f) = \left(\frac{f'}{g'}\right)^n b_1(f)$$

so that

$$\frac{n(n-1)}{2}\left(\frac{f'}{g'}\right)^{-1}\left(\frac{f'}{g'}\right)' = \frac{b_1(f)}{b_0(f)} f'. \tag{10}$$

Otherwise, we have

$$g^{(n-1)}(f) + \sum_{\lambda\in I} \Gamma_\lambda M_\lambda[g](f) = 0,$$

where every Γ_λ is a rational function in $f', \ldots, f^{(n)}$ and $g', \ldots, g^{(n)}$, for $a_0(z), b_0(z), b_1(z)$ and $b_\lambda(z)$ are polynomials. Thus $g(z)$, and hence $f(z)$ from Theorem A, shall satisfy an algebraic differential equation with order less than n and then this contradicts our hypothesis.

If $b_1(z) \equiv 0$, then from (10), f'/g' is a constant and so $f(z) = ag(z) + b$ for two complex numbers a and b. Now assume that $b_1(z) \not\equiv 0$ so that f'/g' is not a constant and we shall complete our proof by deriving a contradiction. Set $R(z) = \frac{2}{n(n-1)}\frac{b_1(z)}{b_0(z)}$. It is well known that we can write

$$R(z) = P(z) + \sum_{j=1}^{p} \frac{A_j}{(z-c_j)^{n_j}} + \sum_{i=1}^{q} \frac{B_i}{z-d_i}$$

where $P(z)$ is a polynomial and $n_j > 1$ and A_j, B_i, c_j, d_i are complex numbers (in fact this is a basic property of a rational function). From (10) we easily know that $R(f)f'$ has only simple poles at which the residues are integers. In view of Lemma 2.3, we can assume without any loss of generality that $f(z)$ has no Picard exceptional values at all and from the following equation

$$R(f)f' = P(f)f' + \sum_{j=1}^{p} A_j \frac{f'}{(f-c_j)^{n_j}} + \sum_{i=1}^{q} B_i \frac{f'}{f-d_i},$$

we have that all A_j vanish and all B_i are rational numbers, for the residues of $f'/(f-d_i)$ at roots of $f = d_i$ are also integers. In view of (10), in any simply connected domain which does not contain the zeros of $f - d_i$ $(i = 1, 2, \ldots, q)$, we have

$$\frac{f'}{g'} = \prod_{i=1}^{q} (f-d_i)^{B_i} e^{Q(f)}$$

where $Q(z) = \int P(z)dz$ is a polynomial. Noting that $(\frac{f'}{g'})\exp(-Q(f))$ is a meromorphic function, we have therefore that $\prod_{i=1}^{q}(f-d_i)^{B_i}$ is meromorphic on $\mathbb{C}$. Since $(\frac{f'}{g'})\prod_{i=1}^{q}(f-d_i)^{-B_i}$ is of finite order, this implies $Q(z)$ must be a constant. We can write

$$g' = H(f)f', \qquad H(z) = c\prod_{i=1}^{q}(z-d_i)^{-B_i}. \tag{11}$$

Assume that all $B_i \neq 0$. Now we want to prove that each B_i is a negative integer. First of all we claim that if B_i is an integer, then it is negative. Suppose the claim fails, that is, there exists a positive integer B_{i_0}. It is easy to see that the zeros of $f(z) - d_{i_0}$ will be the poles of $f'(f-d_{i_0})^{-B_{i_0}}$ and so of $H(f)f' = g'$. It follows from this and (11) that $f(z) - d_{i_0}$ has no zeros at all and so has the form $f(z) = d_{i_0} + e^{A(z)}$ for a non-constant polynomial $A(z)$. In view of Lemma 2.3,

$g(z) = af(z) + b$ and so f'/g' is a constant, a contradiction is derived. Thus we have shown our claim. Below we need to treat two cases.

(I) $B_1 = \frac{s}{t}$ is not an integer with $t > 1$ and B_i $(2 \leqslant i \leqslant q)$ are negative integers (the following argument is also valid for $q = 1$). From (11), $(f - d_1)^{B_1}$ must be meromorphic on $\mathbb{C}$ and thus we can write

$$f(z) = d_1 + h^t(z)$$

for an entire function $h(z)$ and we can assume that $h(z)$ has at least one zero from Lemma 2.3. Substituting the expression of $f(z)$ into (11) yields

$$g' = cth^{t-s-1} \prod_{i=2}^{q} \left(d_1 - d_i + h^t\right)^{-B_i} h'.$$

It is easily seen that $t > s$, otherwise the zeros of $h(z)$ must be the poles of $g'(z)$, and further we have $g = B(h)$, where $B(z) = \int [ctz^{t-s-1} \prod_{i=2}^{q} (d_1 - d_i + z^t)^{-B_i}]\,dz$ is a polynomial with the degree $\deg(B) = t - s + \sum_{i=2}^{q}(-tB_i)$. It follows from $f(g(z)) = g(f(z))$ that

$$h^t \circ B = (B - d_1) \circ h \circ \left(d_1 + z^t\right).$$

Let the order of $h(z)$ be λ and clearly $\lambda = \lambda(f) > 0$. From the above equality we have

$$(\deg B)\lambda = \lambda\left(h^t \circ B\right) = \lambda\left((B - d_1) \circ h \circ \left(d_1 + z^t\right)\right) = t\lambda$$

or equivalently $t = \deg B = t - s + \sum_{i=2}^{q}(-tB_i)$, that is, $s = t \sum_{i=2}^{q}(-B_i) \geqslant t$, a contradiction is derived.

(II) $B_i = \frac{s_i}{t_i}$ $(i = 1, 2)$ with $t_i > 1$ are not integers and B_i $(3 \leqslant i \leqslant q)$ are negative integers (the following argument is also valid for $q = 2$). Then all zeros of $f - d_i$ $(i = 1, 2)$ have multiplicities at least t_i and in view of the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T(r, f) &\leqslant \overline{N}(r, f = d_1) + \overline{N}(r, f = d_2) + O(\log r) \\ &\leqslant \frac{1}{t_1} N(r, f = d_1) + \frac{1}{t_2} N(r, f = d_2) + O(\log r) \\ &\leqslant \left(\frac{1}{t_1} + \frac{1}{t_2}\right) T(r, f) + O(\log r). \end{aligned}$$

Since $f(z)$ is transcendental, the above inequality implies that $\frac{1}{t_1} + \frac{1}{t_2} \geqslant 1$ and so $t_i = 2$ $(i = 1, 2)$. We can write

$$f(z) = d_1 + h_1^2(z) = d_2 + h_2^2(z)$$

for two entire functions $h_i(z)$ $(i = 1, 2)$. To solve the equation, write $(h_2 - h_1)(h_2 + h_1) = \alpha^2$ for a complex number α with $\alpha^2 = d_1 - d_2 \neq 0$ and further, $h_2 + h_1$ has no zeros at all, so $h_2 + h_1 = \alpha e^{iW}$ for some polynomial W and $h_2 - h_1 = \alpha e^{-iW}$. Thus we get $h_1(z) = \alpha \sin W(z)$ for a non-zero complex number α and a polynomial $W(z)$, and a simple calculation implies that $f(z)$ satisfies

$$(f')^2 = \left(\alpha^4 - 4\left(f - d_1 - \frac{\alpha^2}{2}\right)^2\right)(W')^2.$$

This is a first order algebraic differential equation. A contradiction is derived, because $f(z)$ is assumed not to solve such an equation.

In view of the second fundamental theorem of Nevanlinna, there exist at most two B_i among $\{B_i\}_1^q$ which are not integers. Therefore, we have proved that all B_i are negative integers and $H(z)$ is a polynomial. In view of (11), we have $g(z) = E(f(z))$ for the polynomial $E(z) = \int H(z)\,dz$. It follows from Lemma 2.2 that $g(z) = af(z) + b$, a contradiction is derived. Thus we have proved Theorem 1.1 for $n > 1$.

Now we consider the case $n = 1$. In this case, we have

$$f'(z) = \alpha_p(z) f^p(z) + \cdots + \alpha_1(z) f(z)$$

where all $\alpha_j(z)$ are rational functions. We denote the function on the right side by $\Gamma(f)$. In view of a basic theorem in Nevanlinna theory (cf. Theorem 2.2.5 of [3]), we have

$$T\left(r, \Gamma(f)\right) = p\left(1 + o(1)\right) T(r, f).$$

On the other hand, we have

$$T(r, f') = m(r, f') \leqslant m(r, f) + m(r, f'/f) = \left(1 + o(1)\right) T(r, f).$$

Thus

$$p\left(1 + o(1)\right) T(r, f) \leqslant \left(1 + o(1)\right) T(r, f)$$

and this implies $p = 1$ or equivalently $f'(z) = \alpha_1(z) f(z)$. As in the implication before Corollary 1.2, we have $g(z) = af(z) + b$.

Thus we complete the proof of Theorem 1.1. □

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