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Non-spectral self-affine measure problem on the plane domain

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ABSTRACT

The self-affine measure $\mu_{M,D}$ corresponding to an expanding integer matrix

$$M=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad D=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

is supported on the attractor (or invariant set) of the iterated function system $\{\phi_d(x)=M^{-1}(x+d)\}_{d\in D}$. In the present paper we show that if $(a+d)^2=4(ad-bc)$ and $ad-bc$ is not a multiple of 3, then there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 3 is the best. This extends several known results on the non-spectral self-affine measure problem. The proof of such result depends on the characterization of the zero set of the Fourier transform $\hat{\mu}_{M,D}$, and provides a way of dealing with the non-spectral problem.

1. Introduction

Invariant measures, such as self-similar measures, have recently found wide use in the theory of fractals, in dynamics, in harmonic analysis and in quasi-crystals (see [12,24]). A measure μ is self-similar if it is a convex combination of a given set S of transformations applied to the measure itself. In the literature, one usually restricts attention to the case where the set S is finite. Then, an iterated function system (IFS) results, and varying S yields a rich family of measures μ. To get a manageable problem, further restrictions are placed on the transformation from S. For example, that they are contractive, and that they fall in a definite class, such as conformal maps (giving equilibrium measures on Julia sets), or affine mappings. Here the affine case is considered.

Let $M\in M_n(\mathbb{Z})$ be an expanding integer matrix, that is, one with all eigenvalues $|\lambda_i(M)|>1$ and let $D\subseteq\mathbb{Z}^n$ be a finite subset of cardinality $|D|$. Associated with iterated function system (IFS) $\{\phi_d(x)=M^{-1}(x+d)\}_{d\in D}$, there exists a unique probability measure $\mu:=\mu_{M,D}$ satisfying the self-affine identity (see [16])

$$\mu=\frac{1}{|D|}\sum_{d\in D}\mu\circ\phi_d^{-1}. \tag{1.1}$$

Such μ is supported on $T(M,D)$ and is called self-affine measure.

The invariant set $T(M,D)$ includes complicated geometries, and the invariant measure $\mu_{M,D}$ which is also called self-affine measure includes restrictions of n-dimensional Lebesgue measure. So for $n=1$, in the way of examples, there are Cantor set and Cantor measure on the line; and for $n=2$ there is a rich variety of geometries, of which the best known example is the Sierpinski gasket. The problem considered below started with a discovery in an earlier paper of Jorgensen and Pedersen [26] where it was proved that certain IFS fractals have Fourier bases. And furthermore that the question of

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counting orthogonal Fourier frequencies (or orthogonal exponentials in $L^2(\mu_{M,D})$) for a fixed fractal involves an intrinsic arithmetic of the finite set of functions making up the IFS $\{\phi_d(x)\}_{d\in D}$ under consideration. For example if $M=3$ and $D=\{0,2\}$ is the middle-third Cantor example on the line, there cannot be more than two orthogonal Fourier frequencies [26, Theorem 6.1], while a similar Cantor example using instead a subdivision scale $M=4$, turns out to have an ONB in $L^2(\mu_{M,D})$ consisting of Fourier frequencies [26, Theorem 3.4].

Since this affine case includes restrictions of n-dimensional Lebesgue measure, Cantor measures, and IFS fractal measures, say on Sierpinski gaskets, it is natural to ask for Fourier duality. Can one get some kind of Fourier representation for $\mu_{M,D}$? We know from prior research on $L^2(\mu_{M,D})$ that a naive notion of orthogonal Fourier series is not feasible in general for affine IFSs. For example, the familiar middle-third Cantor set $T(M,D)$ corresponding to $M=3$ and $D=\{0,2\}$. In the case when $M=p$, $p>1$, is odd and $D=\{0,1\}$, Dutkay and Jorgensen [11, Theorem 5.1(i)] proved that there are no 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. In this paper we will explore plane affine IFS-examples when the obstruction to getting a Fourier basis is extreme.

Recall that for a probability measure μ of compact support on $\mathbb{R}^n$, we call μ a spectral measure if there exists a discrete set $\Lambda\subseteq\mathbb{R}^n$ such that $E_\Lambda:=\{e^{2\pi i\langle\lambda,x\rangle}\colon \lambda\in\Lambda\}$ forms an orthogonal basis for $L^2(\mu)$. The set Λ is then called a spectrum for μ. Spectral measure is a natural generalization of spectral set introduced by Fuglede [7] whose famous conjecture and its related problems have received much attention in the recent years (see [12,15]). The spectral self-affine measure problem at the present day consists in determining conditions under which $\mu_{M,D}$ is a spectral measure, and has been studied in the papers [9,10,14,17,19,20,26,28] (see also [29,30] for the main goal). The non-spectral self-affine measure problem originated from the Lebesgue measure case (see [3–5,7,8,18,23] and [1,2] where the conjecture that the disk has no more than three orthogonal exponentials is still unsolved) usually consists of the following two classes:

(I) There is at most a finite number of orthogonal exponentials in $L^2(\mu_{M,D})$, that is, $\mu_{M,D}$-orthogonal exponentials contains at most finite elements. The main questions here are to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them (see [10]).

(II) There are natural infinite families of orthogonal exponentials, but none of them forms an orthogonal basis in $L^2(\mu_{M,D})$. The main question is whether some of these families can be combined to form larger collections of orthogonal exponentials. The other questions concerning this class can be found in [25].

A fractal F is a set which admits a system of scale transformations; intuitively they have the property that F looks the same as the scaling is varied. Typically a fractal comes equipped with an invariant measure. However as is illustrated by such familiar cases as the Cantor set and its invariant measure, or one of the Sierpinski examples, one must pass to a limit, and the limit typically allows intricate non-linearities. A popular representation of a class of fractals is realized with a finite set of affine transformations in Euclidean space, and this is the setting for the present paper. Now classical Fourier series relies on linearity, and so asking for Fourier series in the context of fractals is a new framework. The result below indicate the limits one encounters in such an endeavor.

Except the case that there might be no more than two orthogonal exponentials, the problem on non-spectral measure $\mu_{M,D}$ in fact falls into one of the above two classes. Nevertheless, the first problem we meet is how to determine a measure $\mu_{M,D}$ being non-spectral. There are some results in this direction, such as [11, Theorem 3.1], but we are still far from settling this problem. Relating to the questions of the class (I), we first recall the following related conclusions.

(i) The plane Sierpinski gasket $T(M,D)$ corresponds to

$$M=\begin{bmatrix}2&0\\0&2\end{bmatrix}\quad\text{and}\quad D=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\},\tag{1.2}$$

Dutkay and Jorgensen [11, Theorem 5.1(ii)] proved that $\mu_{M,D}$-orthogonal exponentials contain at most 3 elements and found such 3-elements orthogonal exponentials.

(ii) The generalized plane Sierpinski gasket $T(M,D)$ corresponds to

$$M=\begin{bmatrix}2&1\\0&2\end{bmatrix}\quad\text{and}\quad D=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\},\tag{1.3}$$

see Fig. 3 and Example 3.1 in [11], by applying [11, Theorem 3.1], Dutkay and Jorgensen proved that any set of $\mu_{M,D}$-orthogonal exponentials contains at most 7 elements. In [33], Yuan obtained that any set of $\mu_{M,D}$-orthogonal exponentials contains at most 3 elements and find it.

(iii) The generalized plane Sierpinski gasket $T(M,D)$ corresponds to

$$M=\begin{bmatrix}2&b\\0&2\end{bmatrix}\quad\text{and}\quad D=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\},\tag{1.4}$$

J.-L. Li [21] proved that any set of $\mu_{M,D}$-orthogonal exponentials contains at most 3 elements, and the number 3 is the best. More recently, J.-L. Li [22] proved that for the self-affine measure $\mu_{M,D}$ corresponding to

$$M=\begin{bmatrix}a&b\\d&c\end{bmatrix}\quad\text{and}\quad D=\left\{\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\},\tag{1.5}$$

where $a, b, c \in \mathbb{Z}$, $|a| > 1$, $|c| > 1$ and $ac \in \mathbb{Z} \setminus 3\mathbb{Z}$, there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 3 is the best.

Conjecture. *(See [22].) For an expanding integer matrix $M \in M_n(\mathbb{Z})$ and a finite digit set $D \subseteq \mathbb{Z}^n$, if $|D| \notin W(m)$, then $\mu_{M,D}$ is a non-spectral measure and the non-spectral problem on this $\mu_{M,D}$ falls in the class* (I).

In the plane, the above set D (usually called the *digit* set) which consists of the canonical vectors in $\mathbb{R}^n$ is fundamental, many digit sets can be obtained from this set. From (1.2), (1.3), (1.4) and (1.5), we see that the condition $|D| \notin W(m)$ is always satisfied or assumed, where $|\det(M)| = m = p_1^{b_1} p_2^{b_2} \cdots p_r^{b_r}$ ($p_1 < p_2 < \cdots < p_r$ are prime numbers, $b_j > 0$) is the standard prime factorization and $W(m)$ denotes the non-negative integer combination of $p_1, p_2, \ldots, p_r$ (see [17, Section 4.2], [20, Section 3]).

Motivated by the previous research, especially the above conjecture, when $|D| \in W(m)$, we study non-spectral self-affine measure problem on the plane domain. Our main results are the following three theorems.

Theorem 1.1. *Let $a, b, d \in \mathbb{Z}$, $|a| > 1$, $|d| > 1$ and $a \in \mathbb{Z} \setminus 3\mathbb{Z}$. For the self-affine measure $\mu_{M,D}$ corresponding to*

$$M = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \quad \textit{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \tag{1.6}$$

there exist at most 3 *mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number* 3 *is the best.*

Theorem 1.2. *Let $a, c, d \in \mathbb{Z}$, $|a| > 1$, $|d| > 1$ and $d \in \mathbb{Z} \setminus 3\mathbb{Z}$. For the self-affine measure $\mu_{M,D}$ corresponding to*

$$M = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \quad \textit{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \tag{1.7}$$

there exist at most 3 *mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number* 3 *is the best.*

Theorem 1.3. *For self-affine measure $\mu_{M,D}$ corresponding to the expanding integer matrix*

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \textit{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \tag{1.8}$$

if $(a + d)^2 = 4(ad - bc)$ and $a + d$ is not a multiple of 3, *then there exist at most* 3 *mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number* 3 *is the best.*

This extends the above mentioned some results on the non-spectral self-affine measure problem. We first prove Theorem 1.1 and Theorem 1.2 in Section 2. The proof of Theorem 1.3 depends mainly on the characterization of the zero set $Z(\hat{\mu}_{M,D})$ of the Fourier transform $\hat{\mu}_{M,D}$. We find more inclusion relations inside the zero set $Z(\hat{\mu}_{M,D})$. Some facts concerning this zero set are given in Section 3. Based on these established facts, we prove Theorem 1.3 in Section 4. It is worth noting this is different from the method of [22,34]. But it is difficult to find any general principles for dealing with similar non-spectral questions. Finally we give some examples and remarks on a related question.

2. Proofs of Theorem 1.1 and Theorem 1.2

We divided the proof of Theorem 1.1 into two parts:

(1) There exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$.
(2) The number 3 is the best.

Proof of Theorem 1.1. (1) For the general expanding matrix $M \in M_n(\mathbb{Z})$ and finite subset $D \subset \mathbb{Z}^n$, the Fourier transform of the self-affine measure $\mu_{M,D}$ is

$$\hat{\mu}_{M,D}(\xi) = \int e^{2\pi i \langle \xi, t \rangle} \, d\mu_{M,D}(t) \quad \left(\xi \in \mathbb{R}^n\right). \tag{2.1}$$

From (1.1), we have

$$\hat{\mu}_{M,D}(\xi) = m_D\left(M^{*-1}\xi\right)\hat{\mu}_{M,D}\left(M^{*-1}\xi\right) \quad \left(\xi \in \mathbb{R}^n\right), \tag{2.2}$$

which yields

$$\hat{\mu}_{M,D}(\xi) = \prod_{j=1}^{\infty} m_D\left(M^{*-j}\xi\right), \tag{2.3}$$

by iteration, where

$$m_D(t) := \frac{1}{|D|}\sum_{d\in D} e^{2\pi i\langle d,t\rangle}, \tag{2.4}$$

and M^* denotes the conjugated transpose of M, in fact $M^* = M^T$.

For any $\lambda_1, \lambda_2 \in \mathbb{R}^n$, $\lambda_1 \neq \lambda_2$, the orthogonality condition

$$\begin{aligned}\langle e^{2\pi i\langle\lambda_1,x\rangle}, e^{2\pi i\langle\lambda_2,x\rangle}\rangle_{L^2(\mu_{M,D})} &= \int e^{2\pi i\langle\lambda_1-\lambda_2,x\rangle}\,d\mu_{M,D}\\ &= \hat{\mu}_{M,D}(\lambda_1-\lambda_2) = 0\end{aligned} \tag{2.5}$$

directly relates to the zero set $Z(\hat{\mu}_{M,D})$ of $\hat{\mu}_{M,D}$. From (2.3), we have

$$Z(\hat{\mu}_{M,D}) = \{\xi \in \mathbb{R}^n\colon\ \exists j\in\mathbb{N} \text{ such that } m_D(M^{*-j}\xi) = 0\}. \tag{2.6}$$

For the given M and D in (1.6), we first have

$$m_D(M^{*-j}t) = \frac{1}{3}\{1 + e^{2\pi i a^{-j}t_1} + e^{2\pi i(d^{-j}t_2 - \frac{bt_1(a^{j-1}+d^{j-1}+a^{j-2}d+d^{j-2}a+\cdots)}{2d^ja^j})}\}, \tag{2.7}$$

where $t = (t_1,t_2)^T \in \mathbb{R}^2$. Relating to the zero set of the function m_D, it is known that if $1+w_1+w_2 = 0$ and $|w_1| = |w_2| = 1$, then

$$\{w_1, w_2\} = \{e^{2\pi i\cdot\frac{1}{3}}, e^{2\pi i\cdot\frac{2}{3}}\}. \tag{2.8}$$

If $t = (t_1,t_2)^T \in \mathbb{R}^2$ is the zero point of (2.3), then there exists some $j\in\mathbb{N}$ such that (2.7) is equal to 0. It follows from (2.7) and (2.8) that

$$\begin{cases} a^{-j}t_1 = \dfrac{1}{3} + k_1,\\ d^{-j}t_2 - \dfrac{bt_1(a^{j-1}+d^{j-1}+a^{j-2}d+d^{j-2}a+\cdots)}{2d^ja^j} = \dfrac{2}{3}+k_2\end{cases}$$

or

$$\begin{cases} a^{-j}t_1 = \dfrac{2}{3} + \tilde{k}_1,\\ d^{-j}t_2 - \dfrac{bt_1(a^{j-1}+d^{j-1}+a^{j-2}d+d^{j-2}a+\cdots)}{2d^ja^j} = \dfrac{1}{3}+\tilde{k}_2,\end{cases} \tag{2.9}$$

hence we always have

$$t_1 = \frac{a^jk}{3} \notin \mathbb{Z} \quad (k\in\mathbb{Z}). \tag{2.10}$$

If there is a set of the self-affine measure $\mu_{M,D}$ that contains four elements, denoted by Λ, we always may assume that $(0,0)^T \in \Lambda$ by taking some $\lambda_0 \in \Lambda$ and replacing Λ by $\Lambda - \lambda_0$. Λ may be denoted as follows:

$$\Lambda = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\lambda_1\\ \xi_1\end{pmatrix}, \begin{pmatrix}\lambda_2\\ \xi_2\end{pmatrix}, \begin{pmatrix}\lambda_3\\ \xi_3\end{pmatrix}\right\}. \tag{2.11}$$

Then for any $\lambda, \beta \in \Lambda$, $\lambda \neq \beta$, $\lambda - \beta$ is the zero point of $\hat{\mu}_{M,D}$. Further, from (2.10), we have

$$3\lambda_1 = a^{j_1}k_1, \qquad 3\lambda_2 = a^{j_2}k_2, \qquad 3\lambda_3 = a^{j_3}k_3, \tag{2.12}$$

since a is not a multiple of 3, thus

$$a^{j_1}k_1 \neq 3l_1, \qquad a^{j_2}k_2 \neq 3l_2, \qquad a^{j_3}k_3 \neq 3l_3 \quad (l_1, l_2, l_3 \in \mathbb{Z}). \tag{2.13}$$

Therefore, there are two of $a^{j_1}k_1, a^{j_2}k_2$ and $a^{j_3}k_3$ in the same mod(3). Without loss of generality, we assume $a^{j_1}k_1 \equiv a^{j_2}k_2$ mod(3), then $3/(a^{j_1}k_1 - a^{j_2}k_2)$, namely

$$\lambda_1 - \lambda_2 = \frac{a^{j_1}k_1 - a^{j_2}k_2}{3} \in \mathbb{Z}, \tag{2.14}$$

which contradicts (2.10). Hence there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$.

In order to complete our proof, we only need taking $a = b = 2$, $m = 1$, finding out all the orthogonal exponential functions. It follows from (2.6) and (2.7) that

$$Z(\hat{\mu}_{M,D}) = \{Z_j \text{ or } \tilde{Z}_j\colon\ j \in \mathbb{N}\}, \tag{2.15}$$

where

$$Z_j = \left\{\left(\frac{2^{j+1}}{3}, \frac{2^j(1+j)}{3}\right)^T + \left(2^j k_2, 2^j k_1 + jk_2 2^{j-1}\right)^T\colon\ k_1, k_2 \in \mathbb{Z}\right\} \subseteq \mathbb{R}^2 \tag{2.16}$$

and

$$\tilde{Z}_j = \left\{\left(\frac{2^j}{3}, \frac{2^{j-1}(4+j)}{3}\right)^T + \left(2^j \tilde{k}_2, 2^j \tilde{k}_1 + j\tilde{k}_2 2^{j-1}\right)^T\colon\ \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z}\right\} \subseteq \mathbb{R}^2. \tag{2.17}$$

Then, one can verify that

$$Z_j \subset \tilde{Z}_{j-3} \quad \text{and} \quad \tilde{Z}_j \subset Z_{j-3} \quad (j \in \mathbb{N} \text{ and } j \geqslant 4)$$

hold. Hence, we have the following.

Proposition 2.1. *Let $a = d = 2$, $b = 1$. For the self-affine measure $\mu_{M,D}$ corresponding to* (1.6)*, the zero set $Z(\hat{\mu}_{M,D})$ is given by*

$$Z(\hat{\mu}_{M,D}) = Z_1 \cup Z_2 \cup Z_3 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3$$

where

$$Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3 \text{ are mutually disjoint} \quad \text{and} \quad \bigcup_{j=1}^{3}(Z_j \cup \tilde{Z}_j) \cap \mathbb{Z}^2 = \emptyset,$$

where Z_j and $\tilde{Z}_j$ $(j = 1, 2, 3)$ are given by (2.16) *and* (2.17) *respectively.*

(2) By the above Proposition 2.1, one can obtain many such orthogonal systems which contain three elements, for example, Λ given by

$$\Lambda = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{8}{3} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{4}{3} \\ -4 \end{pmatrix}\right\}, \tag{2.18}$$

is a three-elements orthogonal system in $L^2(\mu_{M,D})$. This shows that the number 3 is the best. The proof of Theorem 1.1 is complete. Proof of Theorem 1.2 is similar to Theorem 1.1. $\square$

Remark 2.1. Note that b may be a multiple of 3 in Theorem 1.1. If $|D| \in W(m)$, then we still get that $\mu_{M,D}$-orthogonal exponentials contain at most 3 elements, where M and D are given by (1.6).

3. Characterization of the zero set $Z(\hat{\mu}_{M,D})$

The self-affine measure $\mu_{M,D}$ and its Fourier transform $\hat{\mu}_{M,D}$ given by (2.3) play an important role in analysis and geometry. Previous research on such measure and its Fourier transform revealed some surprising connections with a number of areas in mathematics, such as harmonic analysis, dynamical systems, number theory, and others, see [6,13,27,31,32] and references cited there in. Here we are interested in the zero set $Z(\hat{\mu}_{M,D})$ of $\hat{\mu}_{M,D}$ which is highly important to the spectral and non-spectral problems on the self-affine measures.

In the following, we will restrict our discussion on the special M and D given by (1.8), and find out some characteristic properties on the set $Z(\hat{\mu}_{M,D})$, where $bc \neq 0$.

Lemma 3.1. *For the given M in* (1.8)*, then there exists a non-singular matrix P such that*

$$M = P\begin{pmatrix} \Delta_1 & m \\ 0 & \Delta_2 \end{pmatrix} P^{-1}$$

where

$$\Delta_1 = \frac{a + d + \sqrt{(a-d)^2 + 4bc}}{2},$$
$$\Delta_2 = \frac{a + d - \sqrt{(a-d)^2 + 4bc}}{2}$$

and $m \in \{0, 1\}$.

From the condition $(a+d)^2 = 4(ad-bc)$ and Lemma 3.1, we first have

$$M^{-j} = \begin{pmatrix} (1+mj)\Delta^{-j} - amj\Delta^{-j-1} & -bmj\Delta^{-j-1} \\ -cmj\Delta^{-j-1} & (1-mj)\Delta^{-j} + amj\Delta^{-j-1} \end{pmatrix} \quad (j = 1, 2, \ldots)$$

and

$$m_D\bigl(M^{*-j}\xi\bigr) = \frac{1}{3}\bigl\{1 + e^{2\pi i\cdot p_j} + e^{2\pi i\cdot q_j}\bigr\}, \tag{3.1}$$

where

$$\xi = (\xi_1, \xi_2)^T \in \mathbb{R}^2, \qquad p_j = (1+mj)\Delta^{-j}\xi_1 - amj\Delta^{-j-1}\xi_1 - cmj\Delta^{-j-1}\xi_2$$

and

$$q_j = -bmj\Delta^{-j-1}\xi_1 + (1-mj)\Delta^{-j}\xi_2 + amj\Delta^{-j-1}\xi_2 \quad \left(bc \in \mathbb{Z}\setminus\{0\},\ \Delta = \frac{a+d}{2}\right).$$

Then, we get from (2.6) and (3.1) that

$$Z(\hat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty}(Z_j \cup \tilde{Z}_j), \tag{3.2}$$

where

$$Z_j = \left\{\begin{pmatrix} \frac{\Delta^j + (-\Delta+a+2c)mj\Delta^{j-1}}{3} \\ \frac{2\Delta^j + (2\Delta+b-2a)mj\Delta^{j-1}}{3} \end{pmatrix} + \begin{pmatrix} (1-mj)\Delta k_1 + amjk_1 + cmjk_2 \\ (1+mj)\Delta k_2 - amjk_2 + bmjk_1 \end{pmatrix} \colon k_1, k_2 \in \mathbb{Z}\right\} \tag{3.3}$$

and

$$\tilde{Z}_j = \left\{\begin{pmatrix} \frac{2\Delta^j + (-2\Delta+2a+c)mj\Delta^{j-1}}{3} \\ \frac{\Delta^j + (\Delta+2b-a)mj\Delta^{j-1}}{3} \end{pmatrix} + \begin{pmatrix} (1-mj)\Delta\tilde{k}_2 + amj\tilde{k}_2 + cmj\tilde{k}_1 \\ (1+mj)\Delta\tilde{k}_1 - amj\tilde{k}_1 + bmj\tilde{k}_2 \end{pmatrix} \colon \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z}\right\}. \tag{3.4}$$

From (3.3) and (3.4), we first have the following facts.

Proposition 3.2. *The sets Z_j and $\tilde{Z}_j$ given by* (3.3) *and* (3.4) *satisfy the following properties*:

(1) $(x, y)^T \in Z_j \Leftrightarrow (-x, -y)^T \in \tilde{Z}_j$, *that is*, $Z_j = \tilde{Z}_j$ *or* $\tilde{Z}_j = -Z_j$ $(j = 1, 2, \ldots)$;
(2) $Z_j - Z_j \subseteq \mathbb{Z}^2$ *and* $\tilde{Z}_j - \tilde{Z}_j \subseteq \mathbb{Z}^2$ $(j = 1, 2, \ldots)$;
(3) $Z_j + Z_j \subseteq \tilde{Z}_j$ *and* $\tilde{Z}_j + \tilde{Z}_j \subseteq Z_j$ $(j = 1, 2, \ldots)$.

In order to find more relations inside the zero set $Z(\hat{\mu}_{M,D})$, we will reduce the fractional expressions in (3.3) and (3.4) to their lowest terms. The denominator of all such fractional expressions is the number 3. So we consider the integers a, b, c and Δ according to the residue class modulo 3 where these integers belong.

Firstly, we discuss the case $m = 1$. From (3.3) and (3.4), we have

$$Z_j = \left\{\begin{pmatrix} \frac{\Delta^j + (-\Delta+a+2c)j\Delta^{j-1}}{3} \\ \frac{2\Delta^j + (2\Delta+b-2a)j\Delta^{j-1}}{3} \end{pmatrix} + \begin{pmatrix} (1-j)\Delta k_1 + ajk_1 + cjk_2 \\ (1+j)\Delta k_2 - ajk_2 + bjk_1 \end{pmatrix} \colon k_1, k_2 \in \mathbb{Z}\right\} \tag{3.5}$$

and

$$\tilde{Z}_j = \left\{\begin{pmatrix} \frac{2\Delta^j + (-2\Delta+2a+c)j\Delta^{j-1}}{3} \\ \frac{\Delta^j + (\Delta+2b-a)j\Delta^{j-1}}{3} \end{pmatrix} + \begin{pmatrix} (1-j)\Delta\tilde{k}_2 + aj\tilde{k}_2 + cj\tilde{k}_1 \\ (1+j)\Delta\tilde{k}_1 - aj\tilde{k}_1 + bj\tilde{k}_2 \end{pmatrix} \colon \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z}\right\}. \tag{3.6}$$

The condition $\Delta = \frac{a+d}{2} \in \mathbb{Z}\setminus 3\mathbb{Z}$ can be divided into the following two cases:

$$\Delta = 3g+1 \quad \bigl(g \in \mathbb{Z}\setminus\{0\}\bigr); \qquad \Delta = 3g+2 \quad (g \in \mathbb{Z}). \tag{3.7}$$

The assumption that $a, b, c \in \mathbb{Z}$ and $bc \in \mathbb{Z}\setminus\{0\}$ implies that a, b and c satisfy one of the following twelve cases:

(A) $a = 3l$ $(l \in \mathbb{Z})$, $b = 3l_1 + 1$ $(l_1 \in \mathbb{Z})$ and $c = 3l_2 + 2$ $(l_2 \in \mathbb{Z})$;
(B) $a = 3l$ $(l \in \mathbb{Z})$, $b = 3l_1 + 2$ $(l_1 \in \mathbb{Z})$ and $c = 3l_2 + 1$ $(l_2 \in \mathbb{Z})$;
(C) $a = 3l + 1$ $(l \in \mathbb{Z})$, $b = 3l_1$ $(l_1 \in \mathbb{Z}\setminus\{0\})$ and $c = 3l_2 + 1$ or $c = 3l_2 + 2$ $(l_2 \in \mathbb{Z})$;
(D) $a = 3l + 1$ $(l \in \mathbb{Z})$, $b = 3l_1 + 1$ $(l_1 \in \mathbb{Z})$ and $c = 3l_2$ $(l_2 \in \mathbb{Z}\setminus\{0\})$;
(E) $a = 3l + 1$ $(l \in \mathbb{Z})$, $b = 3l_1 + 2$ $(l_1 \in \mathbb{Z})$ and $c = 3l_2$ $(l_2 \in \mathbb{Z}\setminus\{0\})$;

(F) $a=3l+2$ $(l\in\mathbb{Z})$, $b=3l_1+1$ $(l_1\in\mathbb{Z})$ and $c=3l_2+2$ $(l_2\in\mathbb{Z})$;
(G) $a=3l+2$ $(l\in\mathbb{Z})$, $b=3l_1+2$ $(l_1\in\mathbb{Z})$ and $c=3l_2+1$ $(l_2\in\mathbb{Z})$;
(H) $a=3l+1$ $(l\in\mathbb{Z})$, $b=3l_1+1$ $(l_1\in\mathbb{Z})$ and $c=3l_2+2$ $(l_2\in\mathbb{Z})$;
(I) $a=3l+1$ $(l\in\mathbb{Z})$, $b=3l_1+2$ $(l_1\in\mathbb{Z})$ and $c=3l_2+1$ $(l_2\in\mathbb{Z})$;
(J) $a=3l+2$ $(l\in\mathbb{Z})$, $b=3l_1$ $(l_1\in\mathbb{Z}\setminus\{0\})$ and $c=3l_2+1$ or $c=3l_2+2$ $(l_2\in\mathbb{Z})$;
(K) $a=3l+2$ $(l\in\mathbb{Z})$, $b=3l_1+1$ $(l_1\in\mathbb{Z})$ and $c=3l_2$ $(l_2\in\mathbb{Z}\setminus\{0\})$;
(L) $a=3l+2$ $(l\in\mathbb{Z})$, $b=3l_1+2$ $(l_1\in\mathbb{Z})$ and $c=3l_2$ $(l_2\in\mathbb{Z}\setminus\{0\})$.

We therefore divide our discussion into the following two sections according to (3.7). Section 3.1 is the case $\Delta=3g+1$ $(g\in\mathbb{Z}\setminus\{0\})$ and Section 3.2 is the case $\Delta=3g+2$ $(g\in\mathbb{Z})$. In each section, we will discuss Z_j and $\tilde{Z}_j$ according to the above twelve cases. The main goal of each section is to simplify the expression of the zero set $Z(\hat{\mu}_{M,D})$ in (2.6). The detailed process is given in Sections 3.1.1, 3.1.2, 3.2.1 and 3.2.2, the other subsections are presented briefly.

3.1. The case $\Delta=3g+1$ $(g\in\mathbb{Z}\setminus\{0\})$

In the case when $\Delta=3g+1$ $(g\in\mathbb{Z}\setminus\{0\})$, a,b and c satisfy one of the seven conditions (A), (B), (C), (D), (E), (F) and (G), we will find some interesting inclusion relations between Z_j and $\tilde{Z}_j$. Therefore, we further divide our discussion into the following seven subsections according to (A), (B), (C), (D), (E), (F) and (G).

3.1.1. The case $\Delta=3g+1$ $(g\in\mathbb{Z}\setminus\{0\})$ and (A)

Under the conditions $\Delta=3g+1$ $(g\in\mathbb{Z}\setminus\{0\})$ and (A), that is

$$\Delta=3g+1\quad\bigl(g\in\mathbb{Z}\setminus\{0\}\bigr),\qquad a=3l\quad(l\in\mathbb{Z}),\qquad b=3l_1+1\quad(l_1\in\mathbb{Z})\quad\text{and}\quad c=3l_2+2\quad(l_2\in\mathbb{Z}),$$

we can rewrite Z_j in (3.5) as

$$Z_j=\left\{\begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\end{pmatrix}+\begin{pmatrix}x(l,l_2,g,j;k_1,k_2)\\ y(l,l_1,g,j;k_1,k_2)\end{pmatrix}:\ k_1,k_2\in\mathbb{Z}\right\}\subseteq\mathbb{R}^2 \tag{3.8}$$

where

$$\begin{aligned}x(l,l_2,g,j;k_1,k_2)={}&\frac{(3g+1)^j-1}{3}+(l+2l_2-g+1)j(3g+1)^{j-1}\\ &+(1-j)k_1\Delta^j+ajk_1\Delta^{j-1}+cjk_2\Delta^{j-1}\in\mathbb{Z}\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}y(l,l_1,g,j;k_1,k_2)={}&\frac{2(3g+1)^j-2}{3}+(2g-2l+l_1+1)j(3g+1)^{j-1}\\ &+(1+j)k_2\Delta^j-ajk_2\Delta^{j-1}+bjk_1\Delta^{j-1}\in\mathbb{Z}.\end{aligned} \tag{3.10}$$

The case $j=1$ plays an important role in (3.9) and (3.10). In fact, we find, from (3.9) and (3.10), that there exist $k_1'\in\mathbb{Z}$, $k_2'\in\mathbb{Z}$ such that

$$\begin{aligned}&x(l,l_2,g,j;k_1,k_2)=x\bigl(l,l_2,g,1;k_1',k_2'\bigr),\\ &y(l,l_1,g,j;k_1,k_2)=y\bigl(l,l_1,g,1;k_1',k_2'\bigr).\end{aligned} \tag{3.11}$$

This shows that

$$Z_j\subseteq Z_1\quad\text{for } j\geqslant 1. \tag{3.12}$$

In the same way, we can rewrite $\tilde{Z}_j$ in (3.6) as

$$\tilde{Z}_j=\left\{\begin{pmatrix}\frac{2}{3}\\ \frac{1}{3}\end{pmatrix}+\begin{pmatrix}\tilde{x}(l,l_2,g,j;\tilde{k}_1,\tilde{k}_2)\\ \tilde{y}(l,l_1,g,j;\tilde{k}_1,\tilde{k}_2)\end{pmatrix}:\ \tilde{k}_1,\tilde{k}_2\in\mathbb{Z}\right\}\subseteq\mathbb{R}^2 \tag{3.13}$$

where

$$\begin{aligned}\tilde{x}(l,l_2,g,j;\tilde{k}_1,\tilde{k}_2)={}&\frac{2(3g+1)^j-2}{3}+(2l+l_2-2g)j(3g+1)^{j-1}\\ &+(1-j)\tilde{k}_2\Delta^j+aj\tilde{k}_2\Delta^{j-1}+cj\tilde{k}_1\Delta^{j-1}\in\mathbb{Z}\end{aligned} \tag{3.14}$$

and

$$\tilde{y}(l,l_1,g,j;k_1,k_2) = \frac{(3g+1)^j-1}{3} + (g+l+2l_1+1)j(3g+1)^{j-1} + (1+j)\tilde{k}_1\Delta^j - aj\tilde{k}_1\Delta^{j-1} + bj\tilde{k}_2\Delta^{j-1} \in \mathbb{Z}. \tag{3.15}$$

Then, one can verify that there exist $\tilde{k}_1' \in \mathbb{Z}$, $\tilde{k}_2' \in \mathbb{Z}$ such that

$$\tilde{x}(l,l_2,g,j;\tilde{k}_1,\tilde{k}_2) = \tilde{x}\big(l,l_2,g,1;\tilde{k}_1',\tilde{k}_2'\big),$$
$$\tilde{y}(l,l_1,g,j;\tilde{k}_1,\tilde{k}_2) = \tilde{y}\big(l,l_1,g,1;\tilde{k}_1',\tilde{k}_2'\big). \tag{3.16}$$

This also shows that

$$\tilde{Z}_j \subseteq \tilde{Z}_1 \quad \text{for } j \geqslant 1. \tag{3.17}$$

Hence, from (3.2), (3.12) and (3.17), we have the following.

Proposition 3.3. *Let $\Delta = 3g+1$ ($g \in \mathbb{Z}\setminus\{0\}$), $a = 3l$ ($l \in \mathbb{Z}$), $b = 3l_1+1$ ($l_1 \in \mathbb{Z}$) and $c = 3l_2+2$ ($l_2 \in \mathbb{Z}$). For the self-affine measure $\mu_{M,D}$ corresponding to* (1.8)*, the zero set $Z(\hat{\mu}_{M,D})$ is given by*

$$Z(\hat{\mu}_{M,D}) = Z_1 \cup \tilde{Z}_1 \tag{3.18}$$

with

$$Z_1 \cap \tilde{Z}_1 = (Z_1 \cup \tilde{Z}_1) \cap \mathbb{Z}^2 = \emptyset, \tag{3.19}$$

where Z_1 and $\tilde{Z}_1$ are given by (3.8) *and* (3.13) *respectively.*

3.1.2. The case $\Delta = 3g+1$ ($g \in \mathbb{Z}\setminus\{0\}$) and (B)

Under the conditions $\Delta = 3g+1$ ($g \in \mathbb{Z}\setminus\{0\}$) and (B), that is

$$\Delta = 3g+1 \quad \big(g \in \mathbb{Z}\setminus\{0\}\big), \qquad a = 3l \quad (l \in \mathbb{Z}), \qquad b = 3l_1+2 \quad (l_1 \in \mathbb{Z}) \quad \text{and} \quad c = 3l_2+1 \quad (l_2 \in \mathbb{Z}),$$

we can rewrite Z_j in (3.5) as

$$Z_j = \left\{ \begin{pmatrix} \frac{1+j}{3} \\ \frac{2+4j}{3} \end{pmatrix} + \begin{pmatrix} x(l,l_2,g,j;k_1,k_2) \\ y(l,l_1,g,j;k_1,k_2) \end{pmatrix} :\ k_1,k_2 \in \mathbb{Z} \right\} \subseteq \mathbb{R}^2 \tag{3.20}$$

where

$$x(l,l_2,g,j;k_1,k_2) = \frac{(3g+1)^j-1}{3} + \frac{j(3g+1)^{j-1}-j}{3} + (l+2l_2-g)j(3g+1)^{j-1} + (1-j)k_1\Delta^j + ajk_1\Delta^{j-1} + cjk_2\Delta^{j-1} \in \mathbb{Z} \tag{3.21}$$

and

$$y(l,l_1,g,j;k_1,k_2) = \frac{2(3g+1)^j-2}{3} + \frac{4j(3g+1)^{j-1}-4j}{3} + (2g-2l+l_1)j(3g+1)^{j-1} + (1+j)k_2\Delta^j - ajk_2\Delta^{j-1} + bjk_1\Delta^{j-1} \in \mathbb{Z}. \tag{3.22}$$

A little difference from the above case, we find, from (3.21) and (3.22), that there exist $k_1' \in \mathbb{Z}$, $k_2' \in \mathbb{Z}$ such that

$$1 + x(l,l_2,g,j+3;k_1,k_2) = x\big(l,l_2,g,j;k_1',k_2'\big),$$
$$4 + y(l,l_1,g,j+3;k_1,k_2) = y\big(l,l_1,g,j;k_1',k_2'\big). \tag{3.23}$$

This shows that

$$Z_{j+3} \subseteq Z_j \quad \text{for } j \geqslant 1. \tag{3.24}$$

In the same way, we can rewrite $\tilde{Z}_j$ in (3.6) as

$$\tilde{Z}_j = \left\{ \begin{pmatrix} \frac{2+j}{3} \\ \frac{1+5j}{3} \end{pmatrix} + \begin{pmatrix} \tilde{x}(l,l_2,g,j;\tilde{k}_1,\tilde{k}_2) \\ \tilde{y}(l,l_1,g,j;\tilde{k}_1,\tilde{k}_2) \end{pmatrix} :\ \tilde{k}_1,\tilde{k}_2 \in \mathbb{Z} \right\} \subseteq \mathbb{R}^2 \tag{3.25}$$

where

$$\tilde{x}(l,l_2,g,j;\tilde{k}_1,\tilde{k}_2) = \frac{2(3g+1)^j-2}{3} - \frac{j(3g+1)^j+j}{3} + (l+l_2-2g)j(3g+1)^{j-1} + (1-j)\tilde{k}_2\Delta^j + aj\tilde{k}_2\Delta^{j-1} + cj\tilde{k}_1\Delta^{j-1} \in \mathbb{Z} \tag{3.26}$$

and

$$\tilde{y}(l, l_1, g, j; k_1, k_2) = \frac{(3g+1)^j - 1}{3} + \frac{5j(3g+1)^j - 5j}{3} + (g - l + 2l_1)j(3g+1)^{j-1} + (1+j)\tilde{k}_1\Delta^j - aj\tilde{k}_1\Delta^{j-1} + bj\tilde{k}_2\Delta^{j-1} \in \mathbb{Z}. \tag{3.27}$$

Then, one can verify that there exist $\tilde{k}_1' \in \mathbb{Z}$, $\tilde{k}_2' \in \mathbb{Z}$ such that

$$\begin{aligned} &1 + \tilde{x}(l, l_2, g, j; \tilde{k}_1, \tilde{k}_2) = \tilde{x}\big(l, l_2, g, 1; \tilde{k}_1', \tilde{k}_2'\big), \\ &5 + \tilde{y}(l, l_1, g, j; \tilde{k}_1, \tilde{k}_2) = \tilde{y}\big(l, l_1, g, 1; \tilde{k}_1', \tilde{k}_2'\big). \end{aligned} \tag{3.28}$$

This also shows that

$$\tilde{Z}_{j+3} \subseteq \tilde{Z}_j \quad \text{for } j \geqslant 1. \tag{3.29}$$

Hence, from (3.2), (3.24) and (3.29), we have the following.

Proposition 3.4. *Let $\Delta = 3g+1$ ($g \in \mathbb{Z} \setminus \{0\}$), $a = 3l$ ($l \in \mathbb{Z}$), $b = 3l_1 + 2$ ($l_1 \in \mathbb{Z}$) and $c = 3l_2 + 1$ ($l_2 \in \mathbb{Z}$). For the self-affine measure $\mu_{M,D}$ corresponding to* (1.8), *the zero set $Z(\hat{\mu}_{M,D})$ is given by*

$$Z(\hat{\mu}_{M,D}) = Z_1 \cup Z_2 \cup Z_3 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3 \tag{3.30}$$

where

$$Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3 \text{ are mutually disjoint} \quad \text{and} \quad \bigcup_{j=1}^{3} Z_j \cap \tilde{Z}_j \cap \mathbb{Z}^2 = \emptyset, \tag{3.31}$$

where Z_1, Z_2 and Z_3 are given by (3.20), *$\tilde{Z}_1$, $\tilde{Z}_2$ and $\tilde{Z}_3$ are given by* (3.25).

3.1.3. The case $\Delta = 3g + 1$ ($g \in \mathbb{Z} \setminus \{0\}$) and (C)

Under the conditions $\Delta = 3g + 1$ ($g \in \mathbb{Z} \setminus \{0\}$) and (C), that is

$$\begin{aligned} &\Delta = 3g+1 \quad \big(g \in \mathbb{Z} \setminus \{0\}\big), \qquad a = 3l+1 \quad (l \in \mathbb{Z}), \qquad b = 3l_1 \quad \big(l_1 \in \mathbb{Z} \setminus \{0\}\big) \quad \text{and} \\ &c = 3l_2 + 1 \quad \text{or} \quad c = 3l_2 + 2 \quad (l_2 \in \mathbb{Z}), \end{aligned} \tag{3.32}$$

we find that the following inclusion relations

$$Z_{j+3} \subseteq Z_j \quad \text{and} \quad \tilde{Z}_{j+3} \subseteq \tilde{Z}_j \quad (j = 1, 2, \ldots) \tag{3.33}$$

hold. Hence, from (3.2) and (3.33), we have the following.

Proposition 3.5. *Let $\Delta = 3g + 1$ ($g \in \mathbb{Z} \setminus \{0\}$), $a = 3l + 1$ ($l \in \mathbb{Z}$), $b = 3l_1$ ($l_1 \in \mathbb{Z} \setminus \{0\}$) and $c = 3l_2 + 1$ or $c = 3l_2 + 2$ ($l_2 \in \mathbb{Z}$). For the self-affine measure $\mu_{M,D}$ corresponding to* (1.8), *the zero set $Z(\hat{\mu}_{M,D})$ satisfies* (3.30) *and* (3.31), *where Z_j and $\tilde{Z}_j$ ($j = 1, 2, 3$) are given by* (3.5) *and* (3.6) *respectively with Δ, a, b and c given by* (3.32).

3.1.4. The case $\Delta = 3g + 1$ ($g \in \mathbb{Z} \setminus \{0\}$) and (D)

Under the conditions $\Delta = 3g + 1$ ($g \in \mathbb{Z} \setminus \{0\}$) and (D), that is

$$\begin{aligned} &\Delta = 3g+1 \quad \big(g \in \mathbb{Z} \setminus \{0\}\big), \qquad a = 3l+1 \quad (l \in \mathbb{Z}), \qquad b = 3l_1 + 1 \quad (l_1 \in \mathbb{Z}) \quad \text{and} \\ &c = 3l_2 \quad \big(l_2 \in \mathbb{Z} \setminus \{0\}\big), \end{aligned} \tag{3.34}$$

we find that the following inclusion relations

$$Z_{j+3} \subseteq Z_j \quad \text{and} \quad \tilde{Z}_{j+3} \subseteq \tilde{Z}_j \quad \text{for } j \geqslant 1 \tag{3.35}$$

hold. Hence, from (3.2) and (3.35), we have the following.

Proposition 3.6. *Let $\Delta = 3g + 1$ ($g \in \mathbb{Z} \setminus \{0\}$), $a = 3l + 1$ ($l \in \mathbb{Z}$), $b = 3l_1 + 1$ ($l_1 \in \mathbb{Z}$) and $c = 3l_2$ ($l_2 \in \mathbb{Z} \setminus \{0\}$). For the self-affine measure $\mu_{M,D}$ corresponding to* (1.8), *the zero set $Z(\hat{\mu}_{M,D})$ satisfies* (3.30) *and* (3.31), *where Z_j and $\tilde{Z}_j$ ($j = 1, 2, 3$) are given by* (3.5) *and* (3.6) *respectively with Δ, a, b and c given by* (3.34).

3.1.5. The case $\Delta = 3g+1$ ($g \in \mathbb{Z} \setminus \{0\}$) and (E)

Under the conditions $\Delta = 3g+1$ ($g \in \mathbb{Z} \setminus \{0\}$) and (E), that is

$$\Delta = 3g+1 \quad \big(g \in \mathbb{Z} \setminus \{0\}\big), \qquad a = 3l+1 \quad (l \in \mathbb{Z}), \qquad b = 3l_1+2 \quad (l_1 \in \mathbb{Z}) \quad \text{and}$$
$$c = 3l_2 \quad \big(l_2 \in \mathbb{Z} \setminus \{0\}\big), \tag{3.36}$$

we find that the following inclusion relations

$$Z_{j+3} \subseteq Z_j \quad \text{and} \quad \tilde{Z}_{j+3} \subseteq \tilde{Z}_j \quad (j = 1, 2, \ldots) \tag{3.37}$$

hold. Hence, from (3.2) and (3.37), we have the following facts.

Proposition 3.7. *Let $\Delta = 3g+1$ ($g \in \mathbb{Z} \setminus \{0\}$), $a = 3l+1$ ($l \in \mathbb{Z}$), $b = 3l_1+2$ ($l_1 \in \mathbb{Z}$) and $c = 3l_2$ ($l_2 \in \mathbb{Z} \setminus \{0\}$). For the self-affine measure $\mu_{M,D}$ corresponding to* (1.8)*, the zero set $Z(\hat{\mu}_{M,D})$ satisfies* (3.30) *and* (3.31)*, where Z_j and $\tilde{Z}_j$ ($j = 1, 2, 3$) are given by* (3.5) *and* (3.6) *respectively with Δ, a, b and c given by* (3.36)*.*

3.1.6. The case $\Delta = 3g+1$ ($g \in \mathbb{Z} \setminus \{0\}$) and (F)

Under the conditions $\Delta = 3g+1$ ($g \in \mathbb{Z} \setminus \{0\}$) and (F), that is

$$\Delta = 3g+1 \quad \big(g \in \mathbb{Z} \setminus \{0\}\big), \qquad a = 3l+2 \quad (l \in \mathbb{Z}), \qquad b = 3l_1+1 \quad (l_1 \in \mathbb{Z}) \quad \text{and}$$
$$c = 3l_2+2 \quad (l_2 \in \mathbb{Z}), \tag{3.38}$$

we still have that the following inclusion relations

$$Z_{j+3} \subseteq Z_j \quad \text{and} \quad \tilde{Z}_{j+3} \subseteq \tilde{Z}_j \quad (j = 1, 2, \ldots) \tag{3.39}$$

hold. Hence, from (3.2) and (3.39), we have the following facts.

Proposition 3.8. *Let $\Delta = 3g+1$ ($g \in \mathbb{Z} \setminus \{0\}$), $a = 3l+2$ ($l \in \mathbb{Z}$), $b = 3l_1+1$ ($l_1 \in \mathbb{Z}$) and $c = 3l_2+2$ ($l_2 \in \mathbb{Z}$). For the self-affine measure $\mu_{M,D}$ corresponding to* (1.8)*, the zero set $Z(\hat{\mu}_{M,D})$ satisfies* (3.30) *and* (3.31)*, where Z_j and $\tilde{Z}_j$ ($j = 1, 2, 3$) are given by* (3.5) *and* (3.6) *respectively with Δ, a, b and c given by* (3.38)*.*

3.1.7. The case $\Delta = 3g+1$ ($g \in \mathbb{Z} \setminus \{0\}$) and (G)

Under the conditions $\Delta = 3g+1$ ($g \in \mathbb{Z} \setminus \{0\}$) and (G), that is

$$\Delta = 3g+1 \quad \big(g \in \mathbb{Z} \setminus \{0\}\big), \qquad a = 3l+2 \quad (l \in \mathbb{Z}), \qquad b = 3l_1+2 \quad (l_1 \in \mathbb{Z}) \quad \text{and}$$
$$c = 3l_2+1 \quad (l_2 \in \mathbb{Z}), \tag{3.40}$$

we find that the following inclusion relations

$$Z_j \subseteq Z_1 \quad \text{and} \quad \tilde{Z}_j \subseteq \tilde{Z}_1 \quad \text{for } j \geqslant 1 \tag{3.41}$$

hold. Hence, from (3.2) and (3.41), we have the following facts.

Proposition 3.9. *Let $\Delta = 3g+1$ ($g \in \mathbb{Z} \setminus \{0\}$), $a = 3l+2$ ($l \in \mathbb{Z}$), $b = 3l_1+2$ ($l_1 \in \mathbb{Z}$) and $c = 3l_2+1$ ($l_2 \in \mathbb{Z}$). For the self-affine measure $\mu_{M,D}$ corresponding to* (1.8)*, the zero set $Z(\hat{\mu}_{M,D})$ satisfies* (3.18) *and* (3.19)*, where Z_1 and $\tilde{Z}_1$ are given by* (3.5) *and* (3.6) *respectively with Δ, a, b and c given by* (3.40)*.*

3.2. The case $\Delta = 3g+2$ ($g \in \mathbb{Z}$)

In the case when $\Delta = 3g+2$ ($g \in \mathbb{Z}$), a, b and c satisfy one of the seven conditions (A), (B), (H), (I), (J), (K) and (L), we will find certain inclusion relations between Z_j and $\tilde{Z}_j$ (a little difference from Section 3.1) by applying the same technique as Section 3.1.

3.2.1. The case $\Delta = 3g+2$ ($g \in \mathbb{Z}$) and (A)

Under the conditions $\Delta = 3g+2$ ($g \in \mathbb{Z}$) and (A), that is

$$\Delta = 3g+2 \quad (g \in \mathbb{Z}), \qquad a = 3l \quad (l \in \mathbb{Z}), \qquad b = 3l_1+1 \quad (l_1 \in \mathbb{Z}) \quad \text{and} \quad c = 3l_2+2 \quad (l_2 \in \mathbb{Z}), \tag{3.42}$$

as in the above Section 3.1.1, we first rewrite Z_j in (3.5) and $\tilde{Z}_j$ in (3.6) as

$$Z_j = \left\{ \begin{pmatrix} \frac{(1+j)2^j}{3} \\ \frac{(4+5j)2^{j-1}}{3} \end{pmatrix} + \begin{pmatrix} x(l,l_2,g,j;k_1,k_2) \\ y(l,l_1,g,j;k_1,k_2) \end{pmatrix} : k_1,k_2 \in \mathbb{Z} \right\} \subseteq \mathbb{R}^2, \tag{3.43}$$

$$\tilde{Z}_j = \left\{ \begin{pmatrix} \frac{(2+j)2^j}{3} \\ \frac{(1+2j)2^j}{3} \end{pmatrix} + \begin{pmatrix} \tilde{x}(l,l_2,g,j;\tilde{k}_1,\tilde{k}_2) \\ \tilde{y}(l,l_1,g,j;\tilde{k}_1,\tilde{k}_2) \end{pmatrix} : \tilde{k}_1,\tilde{k}_2 \in \mathbb{Z} \right\} \subseteq \mathbb{R}^2, \tag{3.44}$$

where

$$\begin{aligned} x(l,l_2,g,j;k_1,k_2) = {} & \frac{(3g+2)^j - 2^j}{3} + \frac{2j(3g+2)^{j-1} - 2j2^{j-1}}{3} + (l+2l_2-g)j(3g+2)^{j-1} \\ & + (1-j)k_1\Delta^j + ajk_1\Delta^{j-1} + cjk_2\Delta^{j-1} \in \mathbb{Z}, \end{aligned} \tag{3.45}$$

$$\begin{aligned} y(l,l_1,g,j;k_1,k_2) = {} & \frac{2(3g+2)^j - 2^{j+1}}{3} + \frac{5j(3g+2)^{j-1} - 5j2^{j-1}}{3} + (2g-2l+l_1)j(3g+2)^{j-1} \\ & + (1+j)k_2\Delta^j - ajk_2\Delta^{j-1} + bjk_1\Delta^{j-1} \in \mathbb{Z}, \end{aligned} \tag{3.46}$$

$$\begin{aligned} \tilde{x}(l,l_2,g,j;\tilde{k}_1,\tilde{k}_2) = {} & \frac{2(3g+2)^j - 2^{j+1}}{3} + \frac{2j(3g+2)^{j-1} - 2j2^{j-1}}{3} + (2l+l_2-2g)j(3g+2)^{j-1} \\ & + (1-j)\tilde{k}_2\Delta^j + aj\tilde{k}_2\Delta^{j-1} + cj\tilde{k}_1\Delta^{j-1} \in \mathbb{Z}, \end{aligned} \tag{3.47}$$

and

$$\begin{aligned} \tilde{y}(l,l_1,g,j;k_1,k_2) = {} & \frac{(3g+2)^j - 2^j}{3} + \frac{4j(3g+2)^{j-1} - j2^{j+1}}{3} + (g-l+2l_1)j(3g+2)^{j-1} \\ & + (1+j)\tilde{k}_1\Delta^j - aj\tilde{k}_1\Delta^{j-1} + bj\tilde{k}_2\Delta^{j-1} \in \mathbb{Z}. \end{aligned} \tag{3.48}$$

Then, one can verify that

$$Z_{j+3} \subseteq \tilde{Z}_j \quad \text{and} \quad \tilde{Z}_{j+3} \subseteq Z_j \quad (j = 1,2,\ldots). \tag{3.49}$$

Hence, from (3.2) and (3.49), we have the following.

Proposition 3.10. *Let $\Delta = 3g+2$ ($g \in \mathbb{Z}$), $a = 3l$ ($l \in \mathbb{Z}$), $b = 3l_1 + 1$ ($l_1 \in \mathbb{Z}$) and $c = 3l_2 + 2$ ($l_2 \in \mathbb{Z}$). For the self-affine measure $\mu_{M,D}$ corresponding to* (1.8)*, the zero set $Z(\hat{\mu}_{M,D})$ satisfies* (3.30) *and* (3.31)*, where Z_j and $\tilde{Z}_j$ ($j = 1,2,3$) are given by* (3.43) *and* (3.44) *respectively with Δ, a, b and c given by* (3.42)*.*

3.2.2. The case $\Delta = 3g+2$ ($g \in \mathbb{Z}$) and (B)

Under the conditions $\Delta = 3g+2$ ($g \in \mathbb{Z}$) and (B), that is

$$\Delta = 3g+2 \quad (g \in \mathbb{Z}), \qquad a = 3l \quad (l \in \mathbb{Z}), \qquad b = 3l_1 + 2 \quad (l_1 \in \mathbb{Z}) \quad \text{and} \quad c = 3l_2 + 1 \quad (l_2 \in \mathbb{Z}), \tag{3.50}$$

as in the above Section 3.2.1, we first rewrite Z_j in (3.5) and $\tilde{Z}_j$ in (3.6) as

$$Z_j = \left\{ \begin{pmatrix} \frac{2^j}{3} \\ \frac{2^{j+1}}{3} \end{pmatrix} + \begin{pmatrix} x(l,l_2,g,j;k_1,k_2) \\ y(l,l_1,g,j;k_1,k_2) \end{pmatrix} : k_1,k_2 \in \mathbb{Z} \right\} \subseteq \mathbb{R}^2, \tag{3.51}$$

$$\tilde{Z}_j = \left\{ \begin{pmatrix} \frac{2^{j+1}}{3} \\ \frac{2^j}{3} \end{pmatrix} + \begin{pmatrix} \tilde{x}(l,l_2,g,j;\tilde{k}_1,\tilde{k}_2) \\ \tilde{y}(l,l_1,g,j;\tilde{k}_1,\tilde{k}_2) \end{pmatrix} : \tilde{k}_1,\tilde{k}_2 \in \mathbb{Z} \right\} \subseteq \mathbb{R}^2, \tag{3.52}$$

where

$$\begin{aligned} x(l,l_2,g,j;k_1,k_2) = {} & \frac{(3g+2)^j - 2^j}{3} + (l+2l_2-g)j(3g+2)^{j-1} \\ & + (1-j)k_1\Delta^j + ajk_1\Delta^{j-1} + cjk_2\Delta^{j-1} \in \mathbb{Z}, \end{aligned} \tag{3.53}$$

$$\begin{aligned} y(l,l_1,g,j;k_1,k_2) = {} & \frac{2(3g+2)^j - 2^{j+1}}{3} + (2g-2l+l_1+2)j(3g+2)^{j-1} \\ & + (1+j)k_2\Delta^j - ajk_2\Delta^{j-1} + bjk_1\Delta^{j-1} \in \mathbb{Z}, \end{aligned} \tag{3.54}$$

$$\begin{aligned} \tilde{x}(l,l_2,g,j;\tilde{k}_1,\tilde{k}_2) = {} & \frac{2(3g+2)^j - 2^{j+1}}{3} + (2l+l_2-2g-1)j(3g+2)^{j-1} \\ & + (1-j)\tilde{k}_2\Delta^j + aj\tilde{k}_2\Delta^{j-1} + cj\tilde{k}_1\Delta^{j-1} \in \mathbb{Z} \end{aligned} \tag{3.55}$$

and

$$\tilde{y}(l, l_1, g, j; k_1, k_2) = \frac{(3g+2)^j - 2^j}{3} + (g - l + 2l_1 + 2)j(3g+2)^{j-1} + (1+j)\tilde{k}_1\Delta^j - aj\tilde{k}_1\Delta^{j-1} + bj\tilde{k}_2\Delta^{j-1} \in \mathbb{Z}. \tag{3.56}$$

Then, we find, with a little difference from the above cases, that the following inclusion relations

$$Z_{j+1} \subseteq \tilde{Z}_j \quad \text{and} \quad \tilde{Z}_{j+1} \subseteq Z_j \quad (j = 1, 2, \ldots) \tag{3.57}$$

hold. Hence, from (3.2) and (3.57), we have the following.

Proposition 3.11. *Let $\Delta = 3g + 2$ ($g \in \mathbb{Z}$), $a = 3l$ ($l \in \mathbb{Z}$), $b = 3l_1 + 2$ ($l_1 \in \mathbb{Z}$) and $c = 3l_2 + 1$ ($l_2 \in \mathbb{Z}$). For the self-affine measure $\mu_{M,D}$ corresponding to* (1.8)*, the zero set $Z(\hat{\mu}_{M,D})$ satisfies* (3.18) *and* (3.19)*, where Z_1 and $\tilde{Z}_1$ are given by* (3.51) *and* (3.52) *respectively with Δ, a, b and c given by* (3.50)*.*

3.2.3. The case $\Delta = 3g + 2$ ($g \in \mathbb{Z}$) and (H)

Under the conditions $\Delta = 3g + 2$ ($g \in \mathbb{Z}$) and (H), that is

$$\Delta = 3g + 2 \quad (g \in \mathbb{Z}), \qquad a = 3l + 1 \quad (l \in \mathbb{Z}), \qquad b = 3l_1 + 1 \quad (l_1 \in \mathbb{Z}) \quad \text{and} \quad c = 3l_2 + 2 \quad (l_2 \in \mathbb{Z}), \tag{3.58}$$

we find that the following inclusion relations

$$\tilde{Z}_{j+1} \subseteq Z_j \quad \text{and} \quad Z_{j+1} \subseteq \tilde{Z}_j \quad (j = 1, 2, \ldots) \tag{3.59}$$

hold. Hence, from (3.2) and (3.59), we have the following.

Proposition 3.12. *Let $\Delta = 3g + 2$ ($g \in \mathbb{Z}$), $a = 3l + 1$ ($l \in \mathbb{Z}$), $b = 3l_1 + 1$ ($l_1 \in \mathbb{Z}$) and $c = 3l_2 + 2$ ($l_2 \in \mathbb{Z}$). For the self-affine measure $\mu_{M,D}$ corresponding to* (1.8)*, the zero set $Z(\hat{\mu}_{M,D})$ satisfies* (3.18) *and* (3.19)*, where Z_1 and $\tilde{Z}_1$ are given by* (3.5) *and* (3.6) *respectively with Δ, a, b and c given by* (3.58)*.*

3.2.4. The case $\Delta = 3g + 2$ ($g \in \mathbb{Z}$) and (I)

Under the conditions $\Delta = 3g + 2$ ($g \in \mathbb{Z}$) and (I), that is

$$\Delta = 3g + 2 \quad (g \in \mathbb{Z}), \qquad a = 3l + 1 \quad (l \in \mathbb{Z}), \qquad b = 3l_1 + 2 \quad (l_1 \in \mathbb{Z}) \quad \text{and} \quad c = 3l_2 + 1 \quad (l_2 \in \mathbb{Z}), \tag{3.60}$$

we find that the following inclusion relations

$$\tilde{Z}_{j+3} \subseteq Z_j \quad \text{and} \quad Z_{j+3} \subseteq \tilde{Z}_j \quad (j = 1, 2, \ldots) \tag{3.61}$$

hold. Hence, from (3.2) and (3.61), we have the following.

Proposition 3.13. *Let $\Delta = 3g + 2$ ($g \in \mathbb{Z}$), $a = 3l + 1$ ($l \in \mathbb{Z}$), $b = 3l_1 + 2$ ($l_1 \in \mathbb{Z}$) and $c = 3l_2 + 1$ ($l_2 \in \mathbb{Z}$). For the self-affine measure $\mu_{M,D}$ corresponding to* (1.8)*, the zero set $Z(\hat{\mu}_{M,D})$ satisfies* (3.30) *and* (3.31)*, where Z_1, Z_2 and Z_3 are given by* (3.5)*, $\tilde{Z}_1$, $\tilde{Z}_2$ and $\tilde{Z}_3$ are given by* (3.6)*.*

3.2.5. The other three cases

Under the other three cases, that is

Case 1. $\Delta = 3g + 2$ ($g \in \mathbb{Z}$) and (J),
Case 2. $\Delta = 3g + 2$ ($g \in \mathbb{Z}$) and (K),
Case 3. $\Delta = 3g + 2$ ($g \in \mathbb{Z}$) and (L),

we find that the following inclusion relations

$$\tilde{Z}_{j+3} \subseteq Z_j \quad \text{and} \quad Z_{j+3} \subseteq \tilde{Z}_j \quad (j = 1, 2, \ldots) \tag{3.62}$$

hold. Hence we have the following.

Proposition 3.14. *If Δ, a, b and c hold one of the above three cases, then for the self-affine measure $\mu_{M,D}$ corresponding to* (1.8)*, the zero set $Z(\hat{\mu}_{M,D})$ satisfies* (3.30) *and* (3.31)*, where Z_1, Z_2 and Z_3 are given by* (3.5)*, $\tilde{Z}_1$, $\tilde{Z}_2$ and $\tilde{Z}_3$ are given by* (3.6)*.*

Lastly, when $m = 0$, we can rewrite the plane sets Z_j and $\tilde{Z}_j$ in (3.3) and (3.4) as

$$Z_j = \left\{ \begin{pmatrix} \frac{\Delta^j}{3} \\ \frac{2\Delta^j}{3} \end{pmatrix} + \begin{pmatrix} k_1\Delta^j \\ k_2\Delta^j \end{pmatrix} : k_1, k_2 \in \mathbb{Z} \right\} \subseteq \mathbb{R}^2 \tag{3.63}$$

and

$$\tilde{Z}_j = \left\{ \begin{pmatrix} \frac{2\Delta^j}{3} \\ \frac{\Delta^j}{3} \end{pmatrix} + \begin{pmatrix} \tilde{k}_2 \Delta^j \\ \tilde{k}_1 \Delta^j \end{pmatrix} : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\} \subseteq \mathbb{R}^2. \tag{3.64}$$

The condition $\Delta \in \mathbb{Z}$ can be divide into the following two cases:

$$\Delta = 3g+1 \quad \big(g \in \mathbb{Z} \setminus \{0\}\big); \qquad \Delta = 3g+2 \quad (g \in \mathbb{Z}). \tag{3.65}$$

3.3. The case $\Delta = 3g+1$ ($g \in \mathbb{Z} \setminus \{0\}$) or $\Delta = 3g+2$ ($g \in \mathbb{Z}$)

Under the conditions $\Delta = 3g+1$ ($g \in \mathbb{Z} \setminus \{0\}$) or $\Delta = 3g+2$ ($g \in \mathbb{Z}$), one can verify that the following inclusion relations

$$\tilde{Z}_{j+1} \subseteq Z_j \quad \text{and} \quad Z_{j+1} \subseteq \tilde{Z}_j \quad \text{for } j \geqslant 1 \tag{3.66}$$

hold. Hence, from (3.2) and (3.66), we have the following facts.

Proposition 3.15. *If $\Delta = 3g+1$ ($g \in \mathbb{Z} \setminus \{0\}$) or $\Delta = 3g+2$ ($g \in \mathbb{Z}$), then for the self-affine measure $\mu_{M,D}$ corresponding to* (1.8)*, the zero set $Z(\hat{\mu}_{M,D})$ satisfies* (3.18) *and* (3.19)*, where Z_1 and $\tilde{Z}_1$ are given by* (3.63) *and* (3.64) *respectively.*

3.4. Summary of the above cases (thirteen subcases)

The above discussion involves the four cases:

$$\Delta = 3g+1 \quad \big(g \in \mathbb{Z} \setminus \{0\},\ m=1\big) \quad (\text{Section 3.1}), \qquad \Delta = 3g+2 \quad (g \in \mathbb{Z},\ m=1) \quad (\text{Section 3.2}),$$
$$\Delta = 3g+1 \quad \big(g \in \mathbb{Z} \setminus \{0\},\ m=0\big) \quad (\text{Section 3.3}) \quad \text{and} \quad \Delta = 3g+2 \quad (g \in \mathbb{Z},\ m=0) \quad (\text{Section 3.3}).$$

Propositions 3.3–3.15 correspond to the thirteen subsections. These established propositions characterize the zero set $Z(\hat{\mu}_{M,D})$. They can be divided into two typical cases:

Typical case 1. Propositions 3.3, 3.9, 3.11, 3.12 and 3.15 illustrate that $Z(\hat{\mu}_{M,D})$ satisfies (3.18) and (3.19).

Typical case 2. Propositions 3.4, 3.5, 3.6, 3.7, 3.8, 3.10, 3.13 and 3.14 illustrate that $Z(\hat{\mu}_{M,D})$ satisfies (3.30) and (3.31).

The above two typical cases correspond to two kinds of representations for $Z(\hat{\mu}_{M,D})$ which will help us to prove Theorem 1.3 in the next section.

4. Proof of Theorem 1.3

If $c=0$, then Theorem 1.3 fall into Theorem 1.1. If $b=0$, then Theorem 1.3 fall into Theorem 1.2. In the following we will discuss the case $bc \neq 0$.

If λ_j $(j=1,2,3,4) \in \mathbb{R}^2$ are such that the exponential functions

$$e^{2\pi i\langle \lambda_1, x\rangle},\ e^{2\pi i\langle \lambda_2, x\rangle},\ e^{2\pi i\langle \lambda_3, x\rangle},\ e^{2\pi i\langle \lambda_4, x\rangle}$$

are mutually orthogonal in $L^2(\mu_{M,D})$, then the differences $\lambda_j - \lambda_k$ $(1 \leqslant j \neq k \leqslant 4)$ are in the zero set $Z(\hat{\mu}_{M,D})$. That is, we have

$$\lambda_j - \lambda_k \in Z(\hat{\mu}_{M,D}) \quad (1 \leqslant j \neq k \leqslant 4). \tag{4.1}$$

We will use the above established facts on the zero set $Z(\hat{\mu}_{M,D})$ to deduce a contradiction. The proof will divide into two sections according to Typical cases 1–2.

Typical case 1. From (3.18) and (4.1) we have

$$\lambda_j - \lambda_k \in Z_1 \cup \tilde{Z}_1 \quad (1 \leqslant j \neq k \leqslant 4) \tag{4.2}$$

and (3.19) hold. Especially the following three differences

$$\lambda_1 - \lambda_2,\ \lambda_1 - \lambda_3,\ \lambda_1 - \lambda_4 \tag{4.3}$$

are in $Z_1 \cup \tilde{Z}_1$. Combined with (3.19), (4.2) and Proposition 3.2(2), we immediately deduce a contradiction, since any two of three differences in (4.3) cannot belong to the same set Z_1 or $\tilde{Z}_1$. For example, if $\lambda_1 - \lambda_2 \in \tilde{Z}_1$ and $\lambda_1 - \lambda_4 \in \tilde{Z}_1$, by Proposition 3.2(2), then

$$\lambda_4 - \lambda_2 = (\lambda_1 - \lambda_2) - (\lambda_1 - \lambda_4) \in \tilde{Z}_1 - \tilde{Z}_1 \subseteq \mathbb{Z}^2$$

which contradicts (3.19) and (4.2). Hence any set of $\mu_{M,D}$-orthogonal exponentials contains at most 3 elements. We can find many such orthogonal systems which contain three elements. For instance, the exponential function system E_Λ with Λ given by

$$\Lambda = \{0, s_1, s_2\} \subseteq \mathbb{R}^2 \tag{4.4}$$

is a three-elements orthogonal system in $L^2(\mu_{M,D})$, where $s_1 \in Z_1$ and $s_2 \in \tilde{Z}_1$. This shows that the number 3 is the best.

Typical case 2. We obtain from (3.30) and (4.1) that

$$\lambda_j - \lambda_k \in Z_1 \cup Z_2 \cup Z_3 \cup \tilde{Z}_1 \cup \tilde{Z}_2 \cup \tilde{Z}_3 \quad (1 \leqslant j \neq k \leqslant 4) \tag{4.5}$$

and (3.31) hold. We will use Proposition 3.2, (3.31) and (4.5) to deduce a contradiction.

Observe that the following six differences

$$\begin{array}{lll} \lambda_1 - \lambda_2, & \lambda_1 - \lambda_3, & \lambda_1 - \lambda_4, \\ & \lambda_2 - \lambda_3, & \lambda_2 - \lambda_4, \\ & & \lambda_3 - \lambda_4, \end{array} \tag{4.6}$$

belong to the six sets $Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$. By Proposition 3.2 and (3.31), the differences in each row of (4.6) (except the final row) and the differences in each column of (4.6) (except the first column) cannot belong to the same set. Especially, the following three differences in the first row of (4.6)

$$\lambda_1 - \lambda_2,\ \lambda_1 - \lambda_3,\ \lambda_1 - \lambda_4$$

will be in the three different sets of the six sets $Z_1, Z_2, Z_3, \tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3$. There are 120 distribution methods. One can use the method presented in [21] to deal with each case. For completeness, we use this method to deal with the following one of three typical cases:

Case 1. $\lambda_1 - \lambda_2 \in \tilde{Z}_1, \lambda_1 - \lambda_3 \in \tilde{Z}_2, \lambda_1 - \lambda_4 \in \tilde{Z}_3$.
Case 2. $\lambda_1 - \lambda_2 \in Z_1, \lambda_1 - \lambda_3 \in Z_2, \lambda_1 - \lambda_4 \in \tilde{Z}_3$.
Case 3. $\lambda_1 - \lambda_2 \in Z_1, \lambda_1 - \lambda_3 \in Z_2, \lambda_1 - \lambda_4 \in \tilde{Z}_1$.

In the sequel we only discuss Case 1, the other cases may be proved in the same manner.

Case 1. By Proposition 3.2(1), we first have the following fact holds:

$$\lambda_2 - \lambda_3 \text{ cannot belong to the sets (or small boxes) } Z_1, Z_2, \tilde{Z}_1, \tilde{Z}_2. \tag{4.7}$$

The reason is following.

(1) If $\lambda_2 - \lambda_3 \in Z_1$, by Proposition 3.2(2), then

$$\lambda_3 - \lambda_1 = (\lambda_2 - \lambda_1) - (\lambda_2 - \lambda_3) \in Z_1 - Z_1 \subseteq \mathbb{Z}^2, \tag{4.8}$$

which contradicts (3.31) and $\lambda_3 - \lambda_1 \in Z_2$. Similarly, we show that $\lambda_2 - \lambda_3 \notin \tilde{Z}_2$.

(2) If $\lambda_2 - \lambda_3 \in Z_2$, then from Proposition 3.2(3), we get that

$$\lambda_2 - \lambda_1 = (\lambda_2 - \lambda_3) + (\lambda_3 - \lambda_1) \in Z_2 + Z_2 \subseteq \tilde{Z}_2, \tag{4.9}$$

which contradicts (3.31) and $\lambda_2 - \lambda_1 \in Z_1$. The same reason shows that $\lambda_2 - \lambda_3 \notin \tilde{Z}_1$.

Similarly, the following facts hold:

$$\lambda_2 - \lambda_4 \text{ cannot belong to the sets (or small boxes) } Z_1, Z_3, \tilde{Z}_1, \tilde{Z}_3; \tag{4.10}$$

$$\lambda_3 - \lambda_4 \text{ cannot belong to the sets (or small boxes) } Z_2, Z_3, \tilde{Z}_2, \tilde{Z}_3. \tag{4.11}$$

Hence, from (4.7), (4.10) and (4.11), we have

$$\lambda_2 - \lambda_3 \in Z_3 \quad \text{or} \quad \tilde{Z}_3; \qquad \lambda_2 - \lambda_4 \in Z_2 \quad \text{or} \quad \tilde{Z}_2; \qquad \lambda_3 - \lambda_4 \in Z_1 \quad \text{or} \quad \tilde{Z}_1 \tag{4.12}$$

which is impossible. To see this, we only consider the following two typical cases:

(1′) If

$$\lambda_2 - \lambda_3 \in Z_3, \qquad \lambda_2 - \lambda_4 \in Z_2, \qquad \lambda_3 - \lambda_4 \in Z_1,$$

then by Proposition 3.2(1)–(3), since

$$(\lambda_2 - \lambda_1) - (\lambda_3 - \lambda_4) = (\lambda_4 - \lambda_1) + (\lambda_2 - \lambda_3),$$

the left-hand side is in $Z_1 - Z_1 \subseteq \mathbb{Z}^2$ and the right-hand side is in $Z_3 + Z_3 \subseteq \tilde{Z}_3$, which leads to a contradiction by (3.31). Also, by Proposition 3.2(3), the differences in Z_1 and Z_2 have the character that

$$(\lambda_2 - \lambda_1) + (\lambda_3 - \lambda_4) = (\lambda_3 - \lambda_1) + (\lambda_2 - \lambda_4) \in \tilde{Z}_1 \cap \tilde{Z}_2,$$

which contradicts (3.31).

(2′) If

$$\lambda_2 - \lambda_3 \in \tilde{Z}_3, \qquad \lambda_2 - \lambda_4 \in \tilde{Z}_2, \qquad \lambda_3 - \lambda_4 \in Z_1,$$

then, by Proposition 3.2(1) and Propositions 3.2(3), the elements in Z_2 and Z_3 (or $\tilde{Z}_2$ and $\tilde{Z}_3$) have the character that

$$(\lambda_3 - \lambda_1) + (\lambda_4 - \lambda_2) = (\lambda_4 - \lambda_1) + (\lambda_3 - \lambda_2) \in \tilde{Z}_2 \cap \tilde{Z}_3,$$

which contradicts (3.31). Another way to deduce a contradiction is to apply Propositions 3.2(2), 3.2(3) on the elements of sets Z_1 and Z_2 (or $\tilde{Z}_1$ and $\tilde{Z}_2$) respectively. Since

$$(\lambda_2 - \lambda_1) + (\lambda_3 - \lambda_4) = (\lambda_3 - \lambda_1) - (\lambda_4 - \lambda_2),$$

the left-hand side is in $Z_1 + Z_1 \subseteq \tilde{Z}_1$ and the right-hand side is in $Z_2 - Z_2 \subseteq \mathbb{Z}^2$, which also leads to a contradiction by (3.31). This completes the proof of Case 1.

Hence any set of $\mu_{M,D}$-orthogonal exponentials contains at most 3 elements. For instance, the exponential function system E_Λ with Λ given by (4.4) or Λ given by

$$\Lambda = \{0, s_1, s_2\} \subseteq \mathbb{R}^2 \quad \text{for each } s_1 \in Z_2 \text{ and } s_2 \in \tilde{Z}_2 \tag{4.13}$$

or with Λ given by

$$\Lambda = \{0, s_1, s_2\} \subseteq \mathbb{R}^2 \quad \text{for each } s_1 \in Z_3 \text{ and } s_2 \in \tilde{Z}_3 \tag{4.14}$$

is also the three-elements orthogonal system in $L^2(\mu_{M,D})$. This shows that the number 3 is the best. The proof of Theorem 1.3 is complete.

Remark 4.1. Note that in the above each type, Z_j and $\tilde{Z}_j$ have different representations according to the corresponding Propositions 3.3–3.15 or thirteen subcases.

5. Concluding remarks and examples

The non-spectral self-affine measure problem mentioned in Section 1 depends fundamentally on the characterization of the zero $Z(\hat{\mu}_{m,D})$. For any finite set $D \subseteq \mathbb{R}^n$ of the cardinality $|D| = 3$ or 4, one can obtain the certain expression for the set $Z(\hat{\mu}_{m,D})$ similar to (3.2). But it is more difficult to obtain some characteristic properties on this set.

Example 5.1. The self-affine measure $\mu_{M,D}$ corresponds to

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, \tag{5.1}$$

from Theorem 1.1, we get that there exist at most 3 mutually orthogonal exponential functions in $L^2(\tilde{\mu}_{M,D})$, but $|D| = 3 \in W(6)$.

Here the number 3 matches the cardinality of $|D|$, and we need not to divide $|\det(M)|$ or $|D|$ into the two cases: $|D| < |\det(M)|$ and $|D| > |\det(M)|$. The all known results on the non-spectral self-affine measure problem are in the case $|D| < |\det(M)|$. In the IFS $\{\phi\}_{d\in D}$, the condition $|D| \geqslant |\det(M)|$ is necessary for $T(M,D)$ to have positive Lebesgue measure. For the integral self-affine tile $T(M,D)$, there are infinite families of orthogonal exponentials in $L^2(\tilde{\mu}_{M,D})$ (see [20]). However this conclusion does not hold in the case when $|D| > |\det(M)|$, even if $T(M,D)$ has positive Lebesgue measure.

Example 5.2. (See [22].) The pair (M,D) is given by

$$M = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}. \tag{5.2}$$

We see that $T(M,D)$ has positive Lebesgue measure and $|D| > |\det(M)|$, but there are at most 3 mutually orthogonal exponentials in $L^2(\tilde{\mu}_{M,D})$, and the number 3 is the best.

Finally, it should be pointed out that we only consider the case $(a+d)^2 = 4(ad - bc)$ in Theorem 1.3. When $(a+d)^2 \neq 4(ad - bc)$, the method here may be provide a way to deal with such question.

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