



# Compressible Euler equations with second sound: Asymptotics of discontinuous solutions

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## ABSTRACT

We consider the compressible Euler equations in three space dimensions where heat conduction is modeled by Cattaneo's law instead of Fourier's law. For the arising purely hyperbolic system, the asymptotic behavior of discontinuous solutions to the linearized Cauchy problem is investigated. We give a description of the behavior as time tends to infinity and, in particular, as the relaxation parameter tends to zero. The latter corresponds to the singular limit and a formal convergence to the classical (i.e. Fourier law for the heat flux–temperature relation) Euler system. We recover a phenomenon observed for hyperbolic thermoelasticity, namely the dependence of the asymptotic behavior on the mean curvature of the initial surface of discontinuity; in addition, we observe a more complex behavior in general.

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## 1. Introduction

The compressible Euler system with Cattaneo's law for heat conduction consists of the equations

$$\rho_t + \operatorname{div}(\rho u) = 0, \quad (1.1)$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u + \rho I) = 0, \quad (1.2)$$

$$(\rho E)_t + \operatorname{div}((\rho E + p)u + q) = 0, \quad (1.3)$$

$$\tau q_t + K \nabla e + q = 0, \quad (1.4)$$

where  $\rho, u = (u_1, u_2, u_3)^T, E, p, q = (q_1, q_2, q_3)^T, e$  represent density, velocity, total energy, pressure, heat flux, and internal energy, respectively; cf. [1]. Functions depend on the time variable  $t \in \mathbb{R}_+$  and on the space variable  $x \in \mathbb{R}^3$  (Cauchy problem).  $I$  is the identity matrix,  $K$  is a given positive definite matrix which we assume to equal  $\kappa I$  with  $\kappa > 0$ , and  $\tau > 0$  is the relaxation parameter. The relations

$$p = (\gamma - 1)\rho e, \quad E = \frac{1}{2}|u|^2 + e \quad (1.5)$$

are assumed to hold with a constant  $\gamma > 1$  (polytropic gases). The case  $\tau = 0$  corresponds to Fourier's law of heat conduction, while  $\tau > 0$  represents Cattaneo's law.

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We have initial conditions

$$\rho(t=0) = \rho^0, \quad u(t=0) = u^0, \quad e(t=0) = e^0, \quad q(t=0) = q^0, \quad (1.6)$$

and we shall be interested in the asymptotic behavior of solutions to the corresponding *linearized* system (cf. (2.1)–(2.8) below) with initial data that may have jumps on an initial surface  $\sigma$  given by

$$\sigma = \{x \in \Omega_0 \subset \mathbb{R}^3 \mid \Phi^0(x) = 0\} \quad (1.7)$$

as the level set of a  $C^2$ -function

$$\Phi^0 : \Omega_0 \subset \mathbb{R}^3 \longrightarrow \mathbb{R}, \quad (1.8)$$

with

$$|\nabla \Phi^0| \equiv 1 \quad \text{on } \Omega_0, \quad (1.9)$$

defined on an open set  $\Omega_0 \subset \mathbb{R}^3$ , which is a neighborhood of the surface  $\sigma$ . Assumption (1.9) is made without loss of generality,  $\Phi^0$  may be taken as the distance function  $\Phi^0(x) = \text{dist}(x, \sigma)$ , cp. [2].

The classical Euler system corresponding to  $\tau = 0$  has been widely investigated (see for example [3,4] and the references therein). We are interested in describing the propagation of initial jumps in the data as time  $t$  tends to infinity, and, in particular, in giving information on the behavior of the propagating jumps as  $\tau \rightarrow 0$ . That is, we shall describe the effect of the singular limit of the hyperbolic system to the usual hyperbolic–parabolic one.

This kind of analysis has been carried out for the system of hyperbolic resp. hyperbolic–parabolic thermoelasticity in [5] in one space dimension, and in [6] in three (or two) space dimensions (for an extension to compressible Navier–Stokes equations, cf. [7]). In the latter case, an interesting relation between the geometry of the initial surface of discontinuity  $\sigma$  and the asymptotic behavior was found saying that the behavior strongly depends on the mean curvature of  $\sigma$ .

We shall demonstrate that, again, the mean curvature may determine the specific asymptotic behavior; but, in addition, we shall investigate the, in general, more complex behavior. The linearized system (2.1)–(2.8) is obviously more complicated. Indeed, the linearized equations of thermoelasticity in [6] can be considered as a special case of (2.1)–(2.8) with  $u \equiv 0$ , under the notation correspondence indicated in Section 3.2. However, it turns out that the dominating terms describing the asymptotic behavior of the propagation of the initial jumps are basically the same as the ones for thermoelasticity in [6], and the mean curvature of the surface of initial discontinuities plays an important role.

In using expansions with respect to the relaxation parameter  $\tau$ , we shall obtain that the jumps evolving on the characteristic surfaces go to zero for the internal energy  $e$  (or, equivalently, the temperature for polytropic gases) as  $\tau \rightarrow 0$ , a mirror of the singular limit and the approach of the hyperbolic–parabolic system.

As in thermoelasticity [6], we notice that the decay of the jumps is faster for smaller heat conductive coefficient which is similar to a phenomenon observed for discontinuous solutions to the compressible Navier–Stokes equations by Hoff [8].

We remark that our three-dimensional discussion immediately carries over to the one- and two-dimensional cases. Of course, for the one-dimensional case, the initial singularity only lies in the origin, so that there will be no mean curvature terms in the dominating terms describing the asymptotic behavior, which is similar as in thermoelasticity [5].

The paper is organized as follows. In Section 2 we shall give the setting in a normalized form of a first-order system, and Section 3 presents the discussion of the asymptotics, the main result being summarized in Theorem 3.7.

## 2. Linearization and decomposition

We rewrite the equation for the conservation of energy (1.3), using (1.5), as

$$\rho u \partial_t u + \frac{1}{2} |u|^2 \partial_t \rho + \partial_t (\rho e) + \frac{1}{2} |u|^2 \text{div}(\rho u) + \nabla \left( \frac{1}{2} |u|^2 \right) \rho u + \text{div}(\rho e u) + p \text{div} u + u \nabla p + \text{div} q = 0.$$

Using (1.1) and (1.2), we arrive at

$$\partial_t (\rho e) + \text{div}(\rho e u) + p \text{div} u + \text{div} q = 0,$$

or

$$\partial_t e + u \nabla e + (\gamma - 1) e \text{div} u + \frac{1}{\rho} \text{div} q = 0.$$

Now we linearize Eqs. (1.1)–(1.3) to the following system for the unknowns  $\tilde{\rho}, \tilde{u}, \tilde{e}, \tilde{q}$ , with (now) constant  $\rho, u, e, q$ :

$$\partial_t \tilde{\rho} + u_1 \partial_{x_1} \tilde{\rho} + \rho \partial_{x_1} \tilde{u}_1 + u_2 \partial_{x_2} \tilde{\rho} + \rho \partial_{x_2} \tilde{u}_2 + u_3 \partial_{x_3} \tilde{\rho} + \rho \partial_{x_3} \tilde{u}_3 = 0, \quad (2.1)$$

$$\partial_t \tilde{u}_1 + u_1 \partial_{x_1} \tilde{u}_1 + u_2 \partial_{x_2} \tilde{u}_1 + u_3 \partial_{x_3} \tilde{u}_1 + (\gamma - 1) \frac{e}{\rho} \partial_{x_1} \tilde{\rho} + (\gamma - 1) \partial_{x_1} \tilde{e} = 0, \quad (2.2)$$

$$\partial_t \tilde{u}_2 + u_1 \partial_{x_1} \tilde{u}_2 + u_2 \partial_{x_2} \tilde{u}_2 + u_3 \partial_{x_3} \tilde{u}_2 + (\gamma - 1) \frac{e}{\rho} \partial_{x_2} \tilde{\rho} + (\gamma - 1) \partial_{x_2} \tilde{e} = 0, \quad (2.3)$$

$$\partial_t \tilde{u}_3 + u_1 \partial_{x_1} \tilde{u}_3 + u_2 \partial_{x_2} \tilde{u}_3 + u_3 \partial_{x_3} \tilde{u}_3 + (\gamma - 1) \frac{e}{\rho} \partial_{x_3} \tilde{\rho} + (\gamma - 1) \partial_{x_3} \tilde{e} = 0, \quad (2.4)$$

$$\partial_t \tilde{e} + u_1 \partial_{x_1} \tilde{e} + u_2 \partial_{x_2} \tilde{e} + u_3 \partial_{x_3} \tilde{e} + (\gamma - 1) e (\partial_{x_1} \tilde{u}_1 + \partial_{x_2} \tilde{u}_2 + \partial_{x_3} \tilde{u}_3) + \frac{1}{\rho} (\partial_{x_1} \tilde{q}_1 + \partial_{x_2} \tilde{q}_2 + \partial_{x_3} \tilde{q}_3) = 0, \quad (2.5)$$

$$\partial_t \tilde{q}_1 + \frac{\kappa}{\tau} \partial_{x_1} \tilde{e} + \frac{1}{\tau} \tilde{q}_1 = 0, \quad (2.6)$$

$$\partial_t \tilde{q}_2 + \frac{\kappa}{\tau} \partial_{x_2} \tilde{e} + \frac{1}{\tau} \tilde{q}_2 = 0, \quad (2.7)$$

$$\partial_t \tilde{q}_3 + \frac{\kappa}{\tau} \partial_{x_3} \tilde{e} + \frac{1}{\tau} \tilde{q}_3 = 0. \quad (2.8)$$

We have the initial values

$$\tilde{\rho}(t=0) = \tilde{\rho}^0, \quad \tilde{u}(t=0) = \tilde{u}^0, \quad \tilde{e}(t=0) = \tilde{e}^0, \quad \tilde{q}(t=0) = \tilde{q}^0, \quad (2.9)$$

which may have jumps on the initial surface  $\sigma$  given by (1.8).

Introducing the vector

$$V := (\tilde{\rho}, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{e}, \tilde{q}_1, \tilde{q}_2, \tilde{q}_3)^T$$

we may rewrite Eqs. (2.1)–(2.8) as the following first-order system

$$\partial_t V + A_1 \partial_{x_1} V + A_2 \partial_{x_2} V + A_3 \partial_{x_3} V + A_0 V = 0, \quad (2.10)$$

where

$$A_1 := \begin{bmatrix} u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ (\gamma-1)\frac{e}{\rho} & u_1 & 0 & 0 & (\gamma-1) & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 & 0 & 0 \\ 0 & (\gamma-1)e & 0 & 0 & u_1 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\kappa}{\tau} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 := \begin{bmatrix} u_2 & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ (\gamma-1)\frac{e}{\rho} & 0 & u_2 & 0 & (\gamma-1) & 0 & 0 & 0 \\ 0 & 0 & 0 & u_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\gamma-1)e & 0 & u_2 & 0 & \frac{1}{\rho} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\kappa}{\tau} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_3 := \begin{bmatrix} u_3 & 0 & 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & u_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_3 & 0 & 0 & 0 & 0 & 0 \\ (\gamma-1)\frac{e}{\rho} & 0 & 0 & u_3 & (\gamma-1) & 0 & 0 & 0 \\ 0 & 0 & 0 & (\gamma-1)e & u_3 & 0 & 0 & \frac{1}{\rho} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\kappa}{\tau} & 0 & 0 & 0 \end{bmatrix},$$

$$A_0 := \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\tau} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\tau} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\tau} \end{bmatrix}.$$

Finally, we decompose the vectors  $\tilde{u} := (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)^T$  and  $\tilde{q} := (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)^T$  into their potential and solenoidal parts,

$$\tilde{u} = \tilde{u}^p + \tilde{u}^s, \quad \tilde{q} = \tilde{q}^p + \tilde{q}^s,$$

with

$$\nabla \times \tilde{u}^p = \nabla \times \tilde{q}^p = 0, \quad \operatorname{div} \tilde{u}^s = \operatorname{div} \tilde{q}^s = 0.$$

Correspondingly, the initial data  $\tilde{u}^0$  and  $\tilde{q}^0$  are also decomposed,

$$\tilde{u}^0 = \tilde{u}^{0,p} + \tilde{u}^{0,s}, \quad \tilde{q}^0 = \tilde{q}^{0,p} + \tilde{q}^{0,s}.$$

Since  $u$  is a constant vector, we have the identities

$$\nabla \times ((u \cdot \nabla) \tilde{u}^p) = (u \cdot \nabla)(\nabla \times \tilde{u}^p) = 0, \quad \operatorname{div} ((u \cdot \nabla) \tilde{u}^s) = 0,$$

which imply a decomposition of Eqs. (2.1)–(2.8) into a system for  $(\tilde{\rho}, \tilde{u}^p, \tilde{e}, \tilde{q}^p)$  that has exactly the same structure as (2.1)–(2.8), and the following system for  $(\tilde{u}^s, \tilde{q}^s)$ ,

$$\partial_t \tilde{u}^s + (u \cdot \nabla) \tilde{u}^s = 0, \tag{2.11}$$

$$\partial_t \tilde{q}^s + \frac{1}{\tau} \tilde{q}^s = 0. \tag{2.12}$$

As initial conditions we have

$$\tilde{\rho}(t=0) = \tilde{\rho}^0, \quad \tilde{u}^p(t=0) = \tilde{u}^{0,p}, \quad \tilde{e}(t=0) = \tilde{e}^0, \quad \tilde{q}^p(t=0) = \tilde{q}^{0,p}, \tag{2.13}$$

and

$$\tilde{u}^s(t=0) = \tilde{u}^{0,s}, \quad \tilde{q}^s(t=0) = \tilde{q}^{0,s}. \tag{2.14}$$

Since (2.12) is explicitly solvable, and since (2.11) is of a well-known type, cp. [9–11], the propagation of the jumps of the solenoidal parts  $\tilde{u}^s$  and  $\tilde{q}^s$  can be easily obtained. Thus we only look at the system for  $(\tilde{\rho}, \tilde{u}^p, \tilde{e}, \tilde{q}^p)$  given in (2.1)–(2.8) (with  $(\tilde{u}, \tilde{q})$  replaced by  $(\tilde{u}^p, \tilde{q}^p)$ ), with the initial data (2.13).

**Remark 2.1.** The decomposition will be used in the analysis of possible discontinuities for the eigenvalues  $\lambda = 0, \xi \cdot u$ .

### 3. Asymptotic behavior as $t \rightarrow \infty$ or $\tau \rightarrow 0$

In order to obtain an appropriate representation of solutions  $V = (\tilde{\rho}, \tilde{u}^p, \tilde{e}, \tilde{q}^p)^T$  to the first-order system (2.10) that allows a detailed description of the asymptotic behavior of the solutions as  $t \rightarrow \infty$  or  $\tau \rightarrow 0$ , we first determine, as in [6], the eigenvalues and eigenvectors of the associated matrix

$$\hat{A}(\xi) := \sum_{j=1}^3 \xi_j A_j,$$

where  $\xi = (\xi_1, \xi_2, \xi_3)^T \in \mathbb{R}^3$ . Then, expansions in the parameter  $\tau$  will be given, followed by the investigation of the evolution of jumps across the corresponding characteristic surfaces.

#### 3.1. The eigenvalues and their expansions in the parameter $\tau$

We compute the characteristic polynomial of  $\hat{A}(\xi)$ ,

$$P_3(\lambda, \xi; \tau) := \det(\lambda I - \hat{A}(\xi)) = \det \left( \lambda I - \sum_{j=1}^3 \xi_j A_j \right) \quad (\lambda \in \mathbb{C}). \tag{3.1}$$

With  $\xi u := \xi \cdot u$ , we directly have that

$$\lambda I - \sum_{j=1}^3 \xi_j A_j = \begin{bmatrix} \lambda - \xi u & -\rho \xi_1 & -\rho \xi_2 & -\rho \xi_3 & 0 & 0 & 0 & 0 \\ -(\gamma - 1) \frac{e}{\rho} \xi_1 & \lambda - \xi u & 0 & 0 & -(\gamma - 1) \xi_1 & 0 & 0 & 0 \\ -(\gamma - 1) \frac{e}{\rho} \xi_2 & 0 & \lambda - \xi u & 0 & -(\gamma - 1) \xi_2 & 0 & 0 & 0 \\ -(\gamma - 1) \frac{e}{\rho} \xi_3 & 0 & 0 & \lambda - \xi u & -(\gamma - 1) \xi_3 & 0 & 0 & 0 \\ 0 & -(\gamma - 1) e \xi_1 & -(\gamma - 1) e \xi_2 & -(\gamma - 1) e \xi_3 & \lambda - \xi u & -\frac{1}{\rho} \xi_1 & -\frac{1}{\rho} \xi_2 & -\frac{1}{\rho} \xi_3 \\ 0 & 0 & 0 & 0 & -\frac{\kappa}{\tau} \xi_1 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{\kappa}{\tau} \xi_2 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & -\frac{\kappa}{\tau} \xi_3 & 0 & 0 & \lambda \end{bmatrix}.$$

Let

$$B_1 := \begin{bmatrix} \lambda - \xi u & -\rho \xi_1 & -\rho \xi_2 & -\rho \xi_3 & 0 \\ -(\gamma - 1) \frac{e}{\rho} \xi_1 & \lambda - \xi u & 0 & 0 & -(\gamma - 1) \xi_1 \\ -(\gamma - 1) \frac{e}{\rho} \xi_2 & 0 & \lambda - \xi u & 0 & -(\gamma - 1) \xi_2 \\ -(\gamma - 1) \frac{e}{\rho} \xi_3 & 0 & 0 & \lambda - \xi u & -(\gamma - 1) \xi_3 \\ 0 & -(\gamma - 1) e \xi_1 & -(\gamma - 1) e \xi_2 & -(\gamma - 1) e \xi_3 & \lambda - \xi u \end{bmatrix}$$

and

$$B_2 := \begin{bmatrix} \lambda - \xi u & -\rho \xi_1 & -\rho \xi_2 & -\rho \xi_3 \\ -(\gamma - 1) \frac{e}{\rho} \xi_1 & \lambda - \xi u & 0 & 0 \\ -(\gamma - 1) \frac{e}{\rho} \xi_2 & 0 & \lambda - \xi u & 0 \\ -(\gamma - 1) \frac{e}{\rho} \xi_3 & 0 & 0 & \lambda - \xi u \end{bmatrix}.$$

Then we have

$$P_3(\lambda, \xi; \tau) = \lambda^3 \det B_1 - \lambda^2 \frac{\kappa}{\rho \tau} |\xi|^2 \det B_2.$$

By a direct computation it follows that

$$\det B_1 = (\lambda - \xi u)^3 \{(\lambda - \xi u)^2 - \gamma(\gamma - 1)e|\xi|^2\}, \quad (3.2)$$

and

$$\det B_2 = (\lambda - \xi u)^2 \{(\lambda - \xi u)^2 - (\gamma - 1)e|\xi|^2\}, \quad (3.3)$$

hence

$$\begin{aligned} P_3(\lambda, \xi; \tau) &= \lambda^2 (\lambda - \xi u)^2 \left\{ \lambda (\lambda - \xi u) ((\lambda - \xi u)^2 - \gamma(\gamma - 1)e|\xi|^2) - \frac{\kappa}{\rho \tau} |\xi|^2 ((\lambda - \xi u)^2 - (\gamma - 1)e|\xi|^2) \right\} \\ &:= \lambda^2 (\lambda - \xi u)^2 Q(\lambda, \xi; \tau). \end{aligned} \quad (3.4)$$

**Remark 3.1.** In the one- and two-dimensional cases, we have similar expressions for the corresponding characteristic polynomials  $P_d$  ( $d = 1, 2$ ):

$$P_d(\lambda, \xi; \tau) = \lambda^{d-1} (\lambda - \xi u)^{d-1} Q(\lambda, \xi; \tau), \quad (3.5)$$

where  $u, \xi \in \mathbb{R}^{d-1}$ .

It is clear that the polynomial  $P_3(\lambda, \xi; \tau)$  in (3.4) has eight roots  $\lambda_j$  ( $j = 1, 2, \dots, 8$ ) with  $\lambda_1 = \lambda_2 = 0$ ,  $\lambda_3 = \lambda_4 = \xi u$ , and  $\lambda_5, \lambda_6, \lambda_7, \lambda_8$  being the roots to the equation  $Q(\lambda, \xi; \tau) = 0$ . We observe that  $\lambda_1, \lambda_2, \lambda_3, \lambda_4$  are independent of  $\tau$ , while  $\lambda_5, \lambda_6, \lambda_7, \lambda_8$  are not. So we turn to the expansion of eigenvalues  $\lambda_5, \lambda_6, \lambda_7, \lambda_8$  in terms of the parameter  $\tau$  as  $\tau \rightarrow 0$ .

Let

$$F(\lambda, \tau) := \tau Q(\lambda, \xi; \tau) = \tau \lambda (\lambda - \xi u) \{(\lambda - \xi u)^2 - (\gamma - 1)\gamma e\} - \frac{\kappa}{\rho} \{(\lambda - \xi u)^2 - (\gamma - 1)e\}.$$

By letting  $\tau = 0$  and solving the equation

$$F(\lambda, 0) = -\frac{\kappa}{\rho} \{(\lambda - \xi u)^2 - (\gamma - 1)e\} = 0,$$

we obtain the solution  $\lambda_{5,6}^0$  with

$$\lambda_5^0 = \xi u - \sqrt{(\gamma - 1)e} \quad \lambda_6^0 = \xi u + \sqrt{(\gamma - 1)e}.$$

Note that if  $(\xi u)^2 = (\gamma - 1)e$ , one of  $\lambda_5$  and  $\lambda_6$  is 0, which equals  $\lambda_1$  and  $\lambda_2$ . Since

$$\frac{\partial F}{\partial \lambda}(\lambda, 0) = -\frac{2\kappa}{\rho}(\lambda - \xi u),$$

we have

$$\frac{\partial F}{\partial \lambda}(\lambda_{5,6}^0, 0) = \pm \frac{2\kappa}{\rho} \sqrt{(\gamma - 1)e} \neq 0.$$

By the implicit function theorem, we have  $\lambda_{5,6} = \lambda_{5,6}(\tau)$  near  $(\lambda_{5,6}^0, 0)$  from the equation  $F(\lambda, \tau) = 0$ . Differentiating  $F(\lambda(\tau), \tau) = 0$  with respect to  $\tau$ , we get

$$\frac{\partial F}{\partial \lambda} \lambda'(\tau) + \frac{\partial F}{\partial \tau} = 0.$$

Thus, we obtain

$$\lambda'(0) = -\frac{\frac{\partial F}{\partial \tau}(\lambda(0), 0)}{\frac{\partial F}{\partial \lambda}(\lambda(0), 0)}.$$

Since

$$\frac{\partial F}{\partial \tau} = \lambda(\lambda - \xi u) \{(\lambda - \xi u)^2 - (\gamma - 1)\gamma e\}$$

we obtain

$$\frac{\partial F}{\partial \tau}(\lambda_5^0, 0) = (\gamma - 1)^2 e \sqrt{(\gamma - 1)e} (\xi u - \sqrt{(\gamma - 1)e}),$$

and

$$\frac{\partial F}{\partial \tau}(\lambda_6^0, 0) = -(\gamma - 1)^2 e \sqrt{(\gamma - 1)e} (\xi u + \sqrt{(\gamma - 1)e}).$$

Thus,

$$\begin{aligned} \lambda_5'(0) &= -\frac{\rho}{2\kappa} (\gamma - 1)^2 e (\xi u - \sqrt{(\gamma - 1)e}), \\ \lambda_6'(0) &= -\frac{\rho}{2\kappa} (\gamma - 1)^2 e (\xi u + \sqrt{(\gamma - 1)e}). \end{aligned}$$

We conclude

$$\begin{aligned} \lambda_5(\tau) &= \lambda_5^0 + \lambda_5'(0)\tau + O(\tau^2) \\ &= (\xi u - \sqrt{(\gamma - 1)e}) - \frac{\rho}{2\kappa} (\gamma - 1)^2 e (\xi u - \sqrt{(\gamma - 1)e})\tau + O(\tau^2), \\ \lambda_6(\tau) &= \lambda_6^0 + \lambda_6'(0)\tau + O(\tau^2) \\ &= (\xi u + \sqrt{(\gamma - 1)e}) - \frac{\rho}{2\kappa} (\gamma - 1)^2 e (\xi u + \sqrt{(\gamma - 1)e})\tau + O(\tau^2). \end{aligned}$$

To expand  $\lambda_7$  and  $\lambda_8$ , we define, with  $\Lambda := \frac{1}{\lambda}$ ,

$$\begin{aligned} G(\Lambda, \tau) &:= \frac{1}{\lambda^4} F(\lambda, \tau) \\ &= \tau (1 - \xi u \Lambda) \{ (1 - \xi u \Lambda)^2 - (\gamma - 1)\gamma e \Lambda^2 \} - \frac{\kappa}{\rho} \Lambda^2 \{ (1 - \xi u \Lambda)^2 - (\gamma - 1)e \Lambda^2 \}. \end{aligned}$$

Then  $G(0, 0) = 0$  and  $\frac{\partial G}{\partial \Lambda}(0, 0) = 0$ . Also, we have  $\frac{\partial G}{\partial \tau}(0, 0) = 1$ , hence, using the implicit function theorem again, we get  $\tau = \tau(\Lambda)$  near  $(0, 0)$  from the equation  $G(\Lambda, \tau) = 0$ , and we conclude

$$\begin{aligned} \frac{\partial G}{\partial \Lambda} + \frac{\partial G}{\partial \tau} \tau' &= 0, \quad \text{hence } \tau'(0) = 0, \\ \left( \frac{\partial^2 G}{\partial \Lambda^2} + \frac{\partial^2 G}{\partial \Lambda \partial \tau} \tau' \right) + \frac{\partial G}{\partial \tau} \tau'' + \tau' \left( \frac{\partial^2 G}{\partial \tau \partial \Lambda} + \frac{\partial^2 G}{\partial \tau^2} \tau' \right) &= 0. \end{aligned}$$

Since  $\frac{\partial^2 G}{\partial \Lambda^2}(0, 0) = -\frac{2\kappa}{\rho}$ , we have  $\tau''(0) = \frac{2\kappa}{\rho}$ . Hence

$$\tau(\Lambda) = \frac{\kappa}{\rho} \Lambda^2 + O(\Lambda^3),$$

and then

$$\frac{1}{\Lambda^2} = \frac{\kappa}{\rho \tau} (1 + O(\Lambda)),$$

that is,

$$\lambda^2 = \frac{\kappa}{\rho \tau} (1 + O(\sqrt{\tau})),$$

and we have

$$\lambda_7(\tau) = -\sqrt{\frac{\kappa}{\rho \tau}} + O(1), \quad \lambda_8(\tau) = \sqrt{\frac{\kappa}{\rho \tau}} + O(1).$$

Summarizing, we obtain the following lemma on the eigenvalues of the matrix  $\hat{A}(\xi)$ .

**Lemma 3.2.** *Let  $\tau > 0$  be sufficiently small. Then the characteristic polynomial  $P_3(\lambda, \xi; \tau)$  of  $\hat{A}(\xi)$  has eight real roots, depending on  $\tau$ , as below:*

$$\lambda_1(\tau) = \lambda_2(\tau) = 0, \tag{3.6}$$

$$\lambda_3(\tau) = \lambda_4(\tau) = \xi u, \tag{3.7}$$

$$\lambda_5(\tau) = \left( \xi u - \sqrt{(\gamma - 1)e} \right) - \frac{\rho}{2\kappa} (\gamma - 1)^2 e \left( \xi u - \sqrt{(\gamma - 1)e} \right) \tau + O(\tau^2), \tag{3.8}$$

$$\lambda_6(\tau) = \left( \xi u + \sqrt{(\gamma - 1)e} \right) - \frac{\rho}{2\kappa} (\gamma - 1)^2 e \left( \xi u + \sqrt{(\gamma - 1)e} \right) \tau + O(\tau^2), \tag{3.9}$$

$$\lambda_7(\tau) = -\sqrt{\frac{\kappa}{\rho \tau}} + O(1), \tag{3.10}$$

$$\lambda_8(\tau) = \sqrt{\frac{\kappa}{\rho \tau}} + O(1). \tag{3.11}$$

### 3.2. The right and left eigenvectors

Next, we compute the eigenvectors corresponding to the above eigenvalues of  $\hat{A}(\xi)$ . First, we determine the *right* eigenvectors  $x = (x_1, \dots, x_8)^T$  satisfying the equation, assuming  $|\xi|^2 = 1$ ,

$$\left( \lambda I - \sum_{j=1}^3 \xi_j A_j \right) x = 0. \tag{3.12}$$

There are two cases.

Case 1.  $\xi u = 0$ .

Then

$$P_3(\lambda, \xi; \tau) = \lambda^4 \left\{ \lambda^2 (\lambda^2 - \gamma(\gamma - 1)e) - \frac{\kappa}{\rho \tau} (\lambda^2 - (\gamma - 1)e) \right\}.$$

The characteristic polynomial and the matrix  $\hat{A}(\xi)$  coincide with those in the thermoelastic situation in [6], in the sense of the following correspondence (the notations in the paper [6] are placed on the left side of “ $\sim$ ”, and the ones for our equations

are placed on the right):

$$\begin{aligned} \operatorname{div} u^p &\sim -\frac{\tilde{\rho}}{\rho}, & u_t^p &\sim \tilde{u}^p, & \theta &\sim \tilde{e}, & q^p &\sim \tilde{q}^p, \\ \alpha^2 &\sim (\gamma - 1)e, & \beta &\sim \gamma - 1, & \delta &\sim (\gamma - 1)e, & \gamma &\sim \frac{1}{\rho}. \end{aligned}$$

Therefore, in order to focus on the new ingredients, we omit the right eigenvectors for this case.

Case 2.  $\xi u \neq 0$ .

Then we have the eigenvalues  $\lambda_1 = \lambda_2 = 0$ ,  $\lambda_3 = \lambda_4 = \xi u$ , and  $\lambda_5, \lambda_6, \lambda_7, \lambda_8$ , which are expressed by (3.8)–(3.11).

Case 2.1. For the multiple eigenvalues  $\lambda_1 = \lambda_2 = 0$ , we have  $x_5 = 0$  by the last three equations of (3.12), and the first four equations of (3.12) read

$$\begin{aligned} -\xi u x_1 & & -\rho \xi_1 x_2 & & -\rho \xi_2 x_3 & & -\rho \xi_3 x_4 & = 0, \\ -(\gamma - 1)e \xi_1 x_1 & & -\rho \xi u x_2 & & & & & = 0, \\ -(\gamma - 1)e \xi_2 x_1 & & & & -\rho \xi u x_3 & & & = 0, \\ -(\gamma - 1)e \xi_3 x_1 & & & & & & -\rho \xi u x_4 & = 0, \end{aligned}$$

hence

$$\{(\gamma - 1)e - (\xi u)^2\}x_1 = 0.$$

Case 2.1.1. If  $(\gamma - 1)e \neq (\xi u)^2$ , then  $x_1 = 0$  and, since  $\xi u \neq 0$ , we have  $x_2 = x_3 = x_4 = 0$ . This implies, by the fifth equation in (3.12), that

$$\xi_1 x_6 + \xi_2 x_7 + \xi_3 x_8 = 0.$$

Thus, denoting by  $\Phi_{\tau_1}, \Phi_{\tau_2}, \xi$  (thinking of  $\xi = \nabla \Phi^0$ ) an orthonormal basis of  $\mathbb{R}^3$ , we have the right eigenvectors for  $\lambda_1 = \lambda_2 = 0$  as follows:

$$\vec{r}_1 = (\vec{0}_{1 \times 5}, \Phi_{\tau_1}^T)^T, \quad \vec{r}_2 = (\vec{0}_{1 \times 5}, \Phi_{\tau_2}^T)^T.$$

Case 2.1.2. If  $(\gamma - 1)e = (\xi u)^2$ , then the multiplicity of the eigenvalue 0 is 3 and we have three eigenvectors

$$\vec{r}_1, \vec{r}_2, \text{ and } \vec{r}_* = \left( \xi u, -(\gamma - 1)\frac{e}{\rho}\xi^T, 0, (\gamma - 1)^2 e^2 \xi^T \right)^T.$$

Case 2.2. Consider the eigenvalues  $\lambda_3 = \lambda_4 = \xi u$ .

Then Eqs. (3.12) imply

$$\begin{aligned} \xi_1 x_2 + \xi_2 x_3 + \xi_3 x_4 &= 0, \\ \frac{e}{\rho} \xi x_1 + \xi x_5 &= 0 \Rightarrow \frac{e}{\rho} x_1 + x_5 = 0, \\ \xi_1 x_6 + \xi_2 x_7 + \xi_3 x_8 &= 0, \\ -\frac{\kappa}{\tau} x_5 \cdot \xi + \xi u \begin{bmatrix} x_6 \\ x_7 \\ x_8 \end{bmatrix} &= 0 \Rightarrow -\frac{\kappa}{\tau} |\xi|^2 x_5 + \xi u \underbrace{(\xi_1 x_6 + \xi_2 x_7 + \xi_3 x_8)}_{=0} = 0, \end{aligned}$$

hence, we have  $x_1 = x_5 = 0$ , and since  $\xi u \neq 0$ ,  $x_6 = x_7 = x_8 = 0$ . Thus, the corresponding right eigenvectors are

$$\vec{r}_3 = (0, \Phi_{\tau_1}^T, \vec{0}_{1 \times 4})^T, \quad \vec{r}_4 = (0, \Phi_{\tau_2}^T, \vec{0}_{1 \times 4})^T.$$

Case 2.3. Consider the last eigenvalues  $\lambda_5, \lambda_6, \lambda_7, \lambda_8$  which satisfy the equation

$$Q(\lambda, \xi; \tau) = \lambda(\lambda - \xi u)\{(\lambda - \xi u)^2 - \gamma(\gamma - 1)e\} - \frac{\kappa}{\rho\tau}\{(\lambda - \xi u)^2 - (\gamma - 1)e\} = 0.$$

The first four equations in (3.12) yield

$$\begin{aligned} \frac{\lambda - \xi u}{\rho} x_1 - \xi_1 x_2 - \xi_2 x_3 - \xi_3 x_4 &= 0, \\ -\frac{(\gamma - 1)}{\rho} |\xi|^2 x_1 + (\lambda - \xi u)(\xi_1 x_2 + \xi_2 x_3 + \xi_3 x_4) - (\gamma - 1)|\xi|^2 \xi_5 &= 0, \end{aligned}$$

implying

$$\frac{1}{\rho}\{(\lambda - \xi u)^2 - (\gamma - 1)e\}x_1 - (\gamma - 1)x_5 = 0. \quad (3.13)$$



The last four equations yield

$$\begin{aligned} -\frac{(\gamma-1)e}{\rho}(\lambda-\xi u)x_1 + (\lambda-\xi u)x_5 - \frac{1}{\rho}(\xi_1x_6 + \xi_2x_7 + \xi_3x_8) &= 0, \\ -\frac{\kappa}{\tau}|\xi|^2x_5 + \lambda(\xi_1x_6 + \xi_2x_7 + \xi_3x_8) &= 0, \end{aligned}$$

implying

$$-\frac{(\gamma-1)e}{\rho}(\lambda-\xi u)x_1 + \left(\lambda-\xi u - \frac{\kappa}{\rho\tau\lambda}\right)x_5 = 0. \quad (3.14)$$

The determinant  $\Gamma$  corresponding to the linear system (3.13), (3.14) is

$$\Gamma = (\lambda-\xi u)\{(\lambda-\xi u)^2 - \gamma(\gamma-1)e\} - \frac{\kappa}{\rho\tau\lambda}\{(\lambda-\xi u)^2 - (\gamma-1)e\} = Q(\lambda, \xi; \tau),$$

which equals 0 for the eigenvalues  $\lambda_5, \lambda_6, \lambda_7, \lambda_8$ .

Letting  $x_1 = (\gamma-1)$ , we have  $x_5 = \frac{1}{\rho}\{(\lambda-\xi u)^2 - (\gamma-1)e\}$ , thus

$$\begin{aligned} x_2 &= \frac{(\gamma-1)}{\rho}(\lambda-\xi u)\xi_1, \\ x_3 &= \frac{(\gamma-1)}{\rho}(\lambda-\xi u)\xi_2, \\ x_4 &= \frac{(\gamma-1)}{\rho}(\lambda-\xi u)\xi_3, \\ x_6 &= \frac{\kappa}{\lambda\tau}\xi_1x_5 = \frac{\kappa}{\rho\tau\lambda}\{(\lambda-\xi u)^2 - (\gamma-1)e\}\xi_1, \\ x_7 &= \frac{\kappa}{\rho\tau\lambda}\{(\lambda-\xi u)^2 - (\gamma-1)e\}\xi_2, \\ x_8 &= \frac{\kappa}{\rho\tau\lambda}\{(\lambda-\xi u)^2 - (\gamma-1)e\}\xi_3. \end{aligned}$$

Hence, the right eigenvectors for  $\lambda_j (j = 5, 6, 7, 8)$  are

$$\vec{r}_j = \left( \gamma-1, \frac{\gamma-1}{\rho}(\lambda_j - \xi u)\xi^T, \frac{1}{\rho}\{(\lambda_j - \xi u)^2 - (\gamma-1)e\}, \frac{\kappa}{\rho\tau\lambda_j}\{(\lambda_j - \xi u)^2 - (\gamma-1)e\}\xi^T \right)^T.$$

One can verify that in case  $(\gamma-1)e = (\xi u)^2$ , one of the above eigenvectors coincides with  $\vec{r}_*$  we obtained already.

For the left eigenvectors  $y = (y_1, \dots, y_8)$  satisfying  $y \cdot \left( \lambda I - \sum_{j=1}^3 \xi_j A_j \right) = 0$ , we carry out a similar analysis and conclude as follows.

Case 1.  $\xi u = 0$ .

As for the right eigenvectors, we have in this case again the same situation as in [6].

Case 2.  $\xi u \neq 0$ .

Case 2.1. Consider the multiple eigenvalues  $\lambda_1 = \lambda_2 = 0$ .

Case 2.1.1. If  $(\gamma-1)e \neq (\xi u)^2$ , the left eigenvectors are

$$\vec{l}_1 = (\vec{0}_{1 \times 5}, \Phi_{\tau_1}^T), \quad \vec{l}_2 = (\vec{0}_{1 \times 5}, \Phi_{\tau_2}^T).$$

Case 2.1.2. If  $(\gamma-1)e = (\xi u)^2$ , we have three left eigenvectors,

$$\vec{l}_1, \vec{l}_2, \text{ and } \vec{l}_* = \left( \xi u, -\rho\xi^T, 0, (\gamma-1)\frac{\rho\tau}{\kappa}\xi^T \right).$$

Case 2.2. For the multiple eigenvalues  $\lambda_3 = \lambda_4 = \xi u$ , the corresponding left eigenvectors are

$$\vec{l}_3 = (0, \Phi_{\tau_1}^T, \vec{0}_{1 \times 4}), \quad \vec{l}_4 = (0, \Phi_{\tau_2}^T, \vec{0}_{1 \times 4}).$$

Case 2.3. Consider  $\lambda_j (j = 5, 6, 7, 8)$ .

The left eigenvectors are

$$\vec{l}_j = \left( (\gamma-1)e, \rho(\lambda_j - \xi u)\xi^T, \frac{\rho}{(\gamma-1)e}\{(\lambda_j - \xi u)^2 - (\gamma-1)e\}, \frac{1}{(\gamma-1)e\lambda_j}\{(\lambda_j - \xi u)^2 - (\gamma-1)e\}\xi^T \right).$$

Again one can verify that in case  $(\gamma-1)e = (\xi u)^2$ , one of the above left eigenvectors coincides with  $\vec{l}_*$  already obtained previously.

Summarizing, we obtain the following lemma on the eigenvectors for the matrix  $\hat{A}(\xi)$ .

**Lemma 3.3.** Suppose  $\tau > 0$  is sufficiently small such that the eigenvalues of  $\hat{A}(\xi)$  are expressed as in Lemma 3.2. Then the right eigenvectors of  $\hat{A}(\xi)$  are

$$\begin{aligned}\vec{r}_1 &= (0_{1 \times 5}, \Phi_{\tau_1}^T)^T, \\ \vec{r}_2 &= (0_{1 \times 5}, \Phi_{\tau_2}^T)^T, \\ \vec{r}_3 &= (0, \Phi_{\tau_1}^T, 0_{1 \times 4})^T, \\ \vec{r}_4 &= (0, \Phi_{\tau_2}^T, 0_{1 \times 4})^T, \\ \vec{r}_j &= \left( \gamma - 1, \frac{\gamma - 1}{\rho}(\lambda_j - \xi u)\xi^T, \frac{1}{\rho}\{(\lambda_j - \xi u)^2 - (\gamma - 1)e\}, \frac{\kappa}{\rho\tau\lambda_j}\{(\lambda_j - \xi u)^2 - (\gamma - 1)e\}\xi^T \right)^T, \\ &\quad (j = 5, 6, 7, 8),\end{aligned}$$

and the left eigenvectors are

$$\begin{aligned}\vec{l}_1 &= (0_{1 \times 5}, \Phi_{\tau_1}^T), \\ \vec{l}_2 &= (0_{1 \times 5}, \Phi_{\tau_2}^T), \\ \vec{l}_3 &= (0, \Phi_{\tau_1}^T, 0_{1 \times 4}), \\ \vec{l}_4 &= (0, \Phi_{\tau_2}^T, 0_{1 \times 4}), \\ \vec{l}_j &= C_j \left( (\gamma - 1)e, \rho(\lambda_j - \xi u)\xi^T, \frac{\rho}{(\gamma - 1)e}\{(\lambda_j - \xi u)^2 - (\gamma - 1)e\}, \frac{1}{(\gamma - 1)e\lambda_j}\{(\lambda_j - \xi u)^2 - (\gamma - 1)e\}\xi^T \right), \\ &\quad (j = 5, 6, 7, 8),\end{aligned}$$

where  $C_j$  ( $j = 5, 6, 7, 8$ ) are chosen constants such that the following normalization condition holds:

$$\vec{l}_j \vec{r}_k = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases} \quad (j, k = 5, 6, 7, 8).$$

Moreover, by substituting the expansion of  $\lambda_j$  ( $j = 5, 6, 7, 8$ ) in terms of  $\tau$ , we obtain, via a direct computation, that

$$C_5 = \frac{1}{2(\gamma - 1)^2 e} (1 + O(\tau)), \quad (3.15)$$

$$C_6 = \frac{1}{2(\gamma - 1)^2 e} (1 + O(\tau)), \quad (3.16)$$

$$C_7 = \frac{(\gamma - 1)e}{2} \left( \frac{\rho\tau}{\kappa} \right)^2 (1 + O(\sqrt{\tau})), \quad (3.17)$$

$$C_8 = \frac{(\gamma - 1)e}{2} \left( \frac{\rho\tau}{\kappa} \right)^2 (1 + O(\sqrt{\tau})). \quad (3.18)$$

### 3.3. The evolution of the initial singularity

With the help of the matrices of left resp. right eigenvectors with  $\xi = \nabla\Phi^0$ ,

$$L := \begin{bmatrix} \vec{l}_1 \\ \vdots \\ \vec{l}_8 \end{bmatrix} = L(x), \quad R := [\vec{r}_1, \dots, \vec{r}_8] = R(x),$$

satisfying  $LR = I_{8 \times 8}$  and depending on  $x$  (dependence on  $\nabla\Phi^0$ ), we transform the original differential equation (2.10) for  $V$  into an equation for  $W := LV$ ,

$$\partial_t W + \sum_{j=1}^3 (LA_j R) \partial_{x_j} W + \left\{ \sum_{j=1}^3 (LA_j \partial_{x_j} R) + LA_0 R \right\} W = 0.$$

Defining

$$\tilde{A}_0 := LA_0 R + \sum_{j=1}^3 LA_j \partial_{x_j} R$$

and an initial value

$$W^0 := LV^0 = LV|_{t=0},$$

we may rewrite this as

$$\partial_t W + \sum_{j=1}^3 (L_j R) \partial_{x_j} W + \tilde{A}_0 W = 0, \quad W|_{t=0} = W^0. \quad (3.19)$$

In order to describe the evolution of jumps in the initial data, we need the evolution of the initial surface  $\sigma$ , along which jumps are present,  $\sigma \equiv \Sigma_0$ . This is given through the characteristic surfaces,

$$\Sigma_0 := \{(t, x) \mid \Phi_0(t, x) := \Phi^0(x) = 0\}, \quad \text{for the eigenvalues } \lambda_{1,2},$$

$$\Sigma_1 := \{(t, x) \mid \Phi_1(t, x) := -\xi u \cdot t + \Phi^0(x) = 0\}, \quad \text{for the eigenvalues } \lambda_{3,4},$$

$$\Sigma_k := \{(t, x) \mid \Phi_k(t, x) := -\lambda_k \cdot t + \Phi^0(x) = 0\}, \quad \text{for the eigenvalues } \lambda_k (k = 5, \dots, 8).$$

In case that  $\xi u = 0$ , that is,  $\lambda_{1,2} = \lambda_{3,4}$ , the right and left eigenvectors are the same as in the paper [6], therefore the argument for the evolution of the singularities is also the same. In order to concentrate on the new ingredients, we omit the analysis of this part and refer the reader to [6] for details.

Hereafter, we always assume that  $\xi u \neq 0$ . We also assume that  $(\xi u)^2 \neq (\gamma - 1)e$  to avoid that  $\lambda_5$  or  $\lambda_6$  equals 0.

By  $[f]_{\Sigma_j}$  we denote the jump of  $f$  along  $\Sigma_j$ , i.e. the difference of the values of the function  $f$  on both sides of the surface  $\Sigma_j$ ,  $j = 0, 1, 5, 6, 7, 8$ . Then an analogous argument to the one in [6] will lead us to the next two results.

**Lemma 3.4.** *Let  $W$  be a bounded piecewise smooth solution to the Cauchy problem (3.19). Then  $W_{1,2}$  are continuous on  $\Sigma_1 \cup (\bigcup_{k=5}^8 \Sigma_k)$ ,  $W_{3,4}$  are continuous on  $\Sigma_0 \cup (\bigcup_{k=5}^8 \Sigma_k)$ , and  $W_i (i = 5, 6, 7, 8)$  are continuous on  $\Sigma_0 \cup \Sigma_1 \cup (\bigcup_{k=5, k \neq i}^8 \Sigma_k)$ . That is,*

$$[W_i]_{\Sigma_k} = 0, \quad (i = 1, 2, k = 1), \quad \text{or} \quad (i = 3, 4, k = 0)$$

$$[W_i]_{\Sigma_k} = 0, \quad (i = 1, 2, 3, 4, k = 5, 6, 7, 8)$$

$$[W_i]_{\Sigma_k} = 0, \quad (i = 5, 6, 7, 8, k = 0, 1)$$

$$[W_i]_{\Sigma_k} = 0, \quad (i, k = 5, 6, 7, 8, i \neq k).$$

This means that, for any  $i = 1, \dots, 8$ , the singularity of  $W_i$  can only propagate along the characteristic surface corresponding to  $\lambda_i$ .

**Lemma 3.5.**  *$W_{1,2}$  are continuous on  $\Sigma_0$ , so are  $W_{3,4}$  on  $\Sigma_1$ . That is,*

$$[W_i]_{\Sigma_0} = 0, \quad i = 1, 2,$$

$$[W_i]_{\Sigma_1} = 0, \quad i = 3, 4.$$

**Remark 3.6.** To obtain this lemma, we exploit the property that  $\nabla \times \tilde{u}^p \equiv \nabla \times \tilde{q}^p \equiv 0$ , which comes from the decomposition  $\tilde{u} = \tilde{u}^p + \tilde{u}^s$  and  $\tilde{q} = \tilde{q}^p + \tilde{q}^s$ . Therefore, we present the proof.

**Proof.** By definition, we have

$$W_1 = \tilde{q}^p \cdot \Phi_{\tau_1}, \quad W_2 = \tilde{q}^p \cdot \Phi_{\tau_2}, \quad W_3 = \tilde{u}^p \cdot \Phi_{\tau_1}, \quad W_4 = \tilde{u}^p \cdot \Phi_{\tau_2},$$

where  $\Phi_{\tau_1}$ ,  $\Phi_{\tau_2}$  and  $\nabla \Phi^0$  are unit vectors perpendicular to each other.

We only prove the continuity of  $W_{1,2}$  across  $\Sigma_0$ :

$$[W_i]_{\Sigma_0} = 0, \quad i = 1, 2$$

the continuity of  $W_{3,4}$  across  $\Sigma_1$  can be analogously verified.

We have, via the decomposition  $\tilde{q} = \tilde{q}^p + \tilde{q}^s$ , that

$$\nabla \times \tilde{q}^p \equiv 0,$$

in the sense of distribution, and also classically away from  $\Sigma_0$ . Therefore, for a bounded domain  $G \subset \mathbb{R}^+ \times \mathbb{R}^3$  with  $G \cap \Sigma_0 \neq \emptyset$  and any  $\varphi \in (C_c^\infty(\Omega))^3$ , we have

$$\int_G \tilde{q}^p \cdot (\nabla \times \varphi) = 0,$$

which implies

$$\int_{G \cap \Sigma_0} ([\tilde{q}^p]_{\Sigma_0} \times (\nabla \Phi^0)) \cdot \varphi = 0.$$

Hence, since  $\varphi$  is arbitrary, we obtain that

$$[\tilde{q}^p]_{\Sigma_0} \times (\nabla \Phi^0) = 0,$$

which yields that

$$[W_i]_{\Sigma_0} = 0, \quad i = 1, 2. \quad \square$$

In view of Lemmas 3.4 and 3.5, the remaining problem is to discuss the propagation of the jumps of  $W_k$  on  $\Sigma_k$  for  $k = 5, 6, 7, 8$ .

The  $k$ -th equation ( $k = 5, 6, 7, 8$ ) in (3.19) can be written as

$$\partial_t W_k + \sum_{m=1}^8 \sum_{j=1}^3 (LA_j R)_{km} \partial_{x_j} W_m = - \sum_{m=1}^8 (\tilde{A}_0)_{km} W_m.$$

Since  $\lambda_k - \left( \sum_{j=1}^3 (LA_j R)_{kk} \partial_{x_j} \Phi^0 \right)_{kk} = 0$ , we have that the operator  $\partial_t + \sum_{j=1}^3 (LA_j R)_{kk} \partial_{x_j}$  is tangential to  $\Sigma_k = \{-\lambda_k \cdot t + \Phi^0(x) = 0\}$ . Analogously, since all entries in the  $k$ -th line of the matrix  $\lambda_k I - \sum_{j=1}^3 (LA_j R)_{kj} \partial_{x_j} \Phi^0 = \lambda_k I - \tilde{\lambda}$  vanish, we obtain that, when  $m \neq k$ , the vector  $(0, (LA_1 R)_{km}, (LA_2 R)_{km}, (LA_3 R)_{km})$  is orthogonal to the normal direction  $(-\lambda_k, (\nabla \Phi^0)^T)$  of  $\Sigma_k$ , thus the operator  $\sum_{j=1}^3 (LA_j R)_{km} \partial_{x_j}$  is also tangential to  $\Sigma_k$  when  $m \neq k$ . Therefore, by applying Lemmas 3.4 and 3.5, we obtain that the jumps  $[W_k]_{\Sigma_k}$  ( $k = 5, 6, 7, 8$ ) are governed by the following transport equations:

$$\left( \partial_t + \sum_{j=1}^3 (LA_j R)_{kk} \partial_{x_j} + (\tilde{A}_0)_{kk} \right) [W_k]_{\Sigma_k} = 0, \quad (3.20)$$

with initial conditions:  $[W_k]_{\Sigma_k}|_{t=0} = [W_k^0]_{\{\Phi^0(x)=0\}}$ .

Thus, to determine the behavior of  $[W_k]_{\Sigma_k}$ , it is essential to study  $(\tilde{A}_0)_{kk}$  with  $\tilde{A}_0 = LA_0 R + \sum_{j=1}^3 LA_j \partial_{x_j} R$ .

We first look at the part  $LA_0 R$ .

For  $k = 5, \dots, 8$ , we have

$$(LA_0 R)_{kk} = \frac{1}{\tau} \sum_{j=6}^8 l_{kj} r_{jk} = \frac{1}{\tau} C_k \frac{1}{(\gamma - 1)e\lambda_k} \{(\lambda_k - \xi u)^2 - (\gamma - 1)e\}^2 \frac{\kappa}{\rho\tau\lambda_k}.$$

Thus, by (3.8)–(3.11) and (3.15)–(3.18), we have

$$\begin{aligned} (LA_0 R)_{55} &= \frac{1}{2(\gamma - 1)^2 e} (1 + O(\tau)) \frac{\kappa}{(\gamma - 1)e\rho\tau^2\lambda_5^2} \cdot \{(\lambda_5 - \xi u + \sqrt{(\gamma - 1)e})(\lambda_5 - \xi u - \sqrt{(\gamma - 1)e})\}^2 \\ &= \frac{1 + O(\tau)}{2(\gamma - 1)^2 e} \frac{\kappa}{(\gamma - 1)e\rho\tau^2\lambda_5^2} \cdot \left\{ \left( -\frac{\rho}{2\kappa} (\gamma - 1)^2 e (\xi u - \sqrt{(\gamma - 1)e}) \tau \right) (-2\sqrt{(\gamma - 1)e})(1 + O(\tau)) \right\}^2 \\ &= \frac{\rho}{2\kappa} (\gamma - 1)^2 e (1 + O(\tau)), \end{aligned}$$

and similarly

$$(LA_0 R)_{66} = \frac{\rho}{2\kappa} (\gamma - 1)^2 e (1 + O(\tau)).$$

Also

$$\begin{aligned} (LA_0 R)_{77} &= \frac{(\gamma - 1)e}{2} \left( \frac{\rho\tau}{\kappa} \right)^2 (1 + O(\sqrt{\tau})) \frac{\kappa}{(\gamma - 1)e\rho\tau^2\lambda_7^2} \left\{ \left( -\sqrt{\frac{\kappa}{\rho\tau}} + O(1) \right)^2 - (\gamma - 1)^2 e \right\}^2 \\ &= \frac{1}{2} \frac{\rho^2\tau(1 + O(\sqrt{\tau}))}{\kappa^2} \left\{ \frac{\kappa}{\rho\tau} (1 + O(\sqrt{\tau})) + (\gamma - 1)e \right\}^2 \\ &= \frac{1}{2\tau} (1 + O(\sqrt{\tau})), \end{aligned}$$

and similarly

$$(LA_0R)_{88} = \frac{1}{2\tau}(1 + O(\sqrt{\tau})).$$

Now we turn to the second part  $\left(\sum_{j=1}^3 (LA_j \partial_{x_j} R)\right)_{kk}$  for  $k = 5, 6, 7, 8$ .

Direct computations follow that

$$\begin{aligned}\bar{l}_k A_1 \partial_{x_1} \bar{r}_k &= (\rho l_{k1} + u_1 l_{k2} + (\gamma - 1)el_{k5}) \partial_{x_1} r_{2k} + u_1 l_{k3} \partial_{x_1} r_{3k} + u_1 l_{k4} \partial_{x_1} r_{4k} \\ &\quad + \left((\gamma - 1)l_{k2} + u_1 l_{k5} + \frac{\kappa}{\tau} l_{k6}\right) \partial_{x_1} r_{5k} + \frac{1}{\rho} l_{k5} \partial_{x_1} r_{6k}, \\ \bar{l}_k A_2 \partial_{x_2} \bar{r}_k &= u_2 l_{k2} \partial_{x_2} r_{2k} + (\rho l_{k1} + u_2 l_{k3} + (\gamma - 1)el_{k5}) \partial_{x_2} r_{3k} + u_2 l_{k4} \partial_{x_2} r_{4k} \\ &\quad + \left((\gamma - 1)l_{k3} + u_2 l_{k5} + \frac{k}{\tau} l_{k7}\right) \partial_{x_2} r_{5k} + \frac{1}{\rho} l_{k5} \partial_{x_2} r_{7k}, \\ \bar{l}_k A_3 \partial_{x_3} \bar{r}_k &= u_3 l_{k2} \partial_{x_3} r_{2k} + u_3 l_{k3} \partial_{x_3} r_{3k} + (\rho l_{k1} + u_3 l_{k4} + (\gamma - 1)el_{k5}) \partial_{x_3} r_{4k} \\ &\quad + \left((\gamma - 1)l_{k4} + u_3 l_{k5} + \frac{\kappa}{\tau} l_{k8}\right) \partial_{x_3} r_{5k} + \frac{1}{\rho} l_{k5} \partial_{x_3} r_{8k}.\end{aligned}$$

Thus

$$\begin{aligned}\sum_{j=1}^3 (LA_j \partial_{x_j} R)_{kk} &= \bar{l}_k A_1 \partial_{x_1} \bar{r}_k + \bar{l}_k A_2 \partial_{x_2} \bar{r}_k + \bar{l}_k A_3 \partial_{x_3} \bar{r}_k \\ &= (\rho l_{k1} + (\gamma - 1)el_{k5}) (\partial_{x_1} r_{2k} + \partial_{x_2} r_{3k} + \partial_{x_3} r_{4k}) + \frac{1}{\rho} l_{k5} (\partial_{x_1} r_{6k} + \partial_{x_2} r_{7k} + \partial_{x_3} r_{8k}) \\ &\quad + u_1 (l_{k2} \partial_{x_1} r_{2k} + l_{k3} \partial_{x_1} r_{3k} + l_{k4} \partial_{x_1} r_{4k}) + u_2 (l_{k2} \partial_{x_2} r_{2k} + l_{k3} \partial_{x_2} r_{3k} + l_{k4} \partial_{x_2} r_{4k}) \\ &\quad + u_3 (l_{k2} \partial_{x_3} r_{2k} + l_{k3} \partial_{x_3} r_{3k} + l_{k4} \partial_{x_3} r_{4k}) + (\gamma - 1) (l_{k2} \partial_{x_1} r_{5k} + l_{k3} \partial_{x_2} r_{5k} + l_{k4} \partial_{x_3} r_{5k}) \\ &\quad + l_{k5} (u_1 \partial_{x_1} r_{5k} + u_2 \partial_{x_2} r_{5k} + u_3 \partial_{x_3} r_{5k}) + \frac{\kappa}{\tau} (l_{k6} \partial_{x_1} r_{5k} + l_{k7} \partial_{x_2} r_{5k} + l_{k8} \partial_{x_3} r_{5k}).\end{aligned}$$

To get more details on the expressions involving derivatives above, we recall that, for  $k, j = 5, 6, 7, 8$ , we have

$$\begin{aligned}\begin{bmatrix} r_{2k} \\ r_{3k} \\ r_{4k} \end{bmatrix} &= \frac{\gamma - 1}{\rho} (\lambda_k - (\nabla \Phi^0)u) \nabla \Phi^0, \\ r_{5k} &= \frac{1}{\rho} \{(\lambda_k - (\nabla \Phi^0)u)^2 - (\gamma - 1)e\}, \\ \begin{bmatrix} r_{6k} \\ r_{7k} \\ r_{8k} \end{bmatrix} &= \frac{k}{\rho \tau \lambda_k} \{(\lambda_k - (\nabla \Phi^0)u)^2 - (\gamma - 1)e\} \nabla \Phi^0\end{aligned}$$

and

$$\bar{l}_j = C_j \left( (\gamma - 1)e, \rho(\lambda_j - \xi u) \xi^T, \frac{\rho}{(\gamma - 1)e} \{(\lambda_j - \xi u)^2 - (\gamma - 1)e\}, \frac{1}{(\gamma - 1)e \lambda_j} \{(\lambda_j - \xi u)^2 - (\gamma - 1)e\} \xi^T \right).$$

We consider first  $k = 5$ :

$$\lambda_5 - \nabla \Phi^0 u = -\sqrt{(\gamma - 1)e} - \frac{\rho}{2\kappa} (\gamma - 1)^2 e (\nabla \Phi^0 u - \sqrt{(\gamma - 1)e} \tau) + O(\tau^2),$$

and, for  $i, j = 1, 2, 3$ ,

$$\begin{aligned}\partial_{x_i} r_{(j+1)5} &= \partial_{x_i} \left\{ \frac{\gamma - 1}{\rho} \left( -\sqrt{(\gamma - 1)e} - \frac{\rho}{2\kappa} (\gamma - 1)^2 e (\nabla \Phi^0 u - \sqrt{(\gamma - 1)e} \tau) + O(\tau^2) \right) \partial_{x_j} \Phi^0 \right\} \\ &= -\frac{\gamma - 1}{\rho} \sqrt{(\gamma - 1)e} \partial_{x_i x_j} \Phi^0 (1 + O(\tau)),\end{aligned}$$

for  $i = 1, 2, 3$ ,

$$\begin{aligned}\partial_{x_i} r_{55} &= \partial_{x_i} \left\{ \frac{1}{\rho} \left( \left( -\sqrt{(\gamma - 1)e} - \frac{\rho}{2\kappa} (\gamma - 1)^2 e (\nabla \Phi^0 u - \sqrt{(\gamma - 1)e} \tau) + O(\tau^2) \right)^2 - (\gamma - 1)e \right) \right\} \\ &= \frac{1}{\kappa} \sqrt{(\gamma - 1)e} (\gamma - 1)^2 e (\nabla (\partial_{x_i} \Phi^0) u) \tau (1 + O(\tau)) \\ &= O(\tau)\end{aligned}$$

moreover, for  $i, j = 1, 2, 3$ ,

$$\begin{aligned}
 \partial_{x_i} r_{(j+5)} &= \partial_{x_i} \left\{ \frac{\kappa}{\rho \tau \lambda_5} ((\lambda_5 - \nabla \Phi^0 u)^2 - (\gamma - 1)e) \cdot \partial_{x_j} \Phi^0 \right\} \\
 &= \frac{\kappa}{\rho \tau \lambda_5} \left\{ (\lambda_5 - \nabla \Phi^0 u)^2 - (\gamma - 1)e \right\} \partial_{x_i x_j} \Phi^0 - \frac{\kappa \partial_{x_i} \lambda_5}{\rho \tau \lambda_5^2} \left\{ (\lambda_5 - \nabla \Phi^0 u)^2 - (\gamma - 1)e \right\} \partial_{x_j} \Phi^0 \\
 &\quad \times \frac{2\kappa}{\rho \tau \lambda_5} (\lambda_5 - \nabla \Phi^0 u) (\partial_{x_i} \lambda_5 - \nabla + (\partial_{x_i} \Phi^0) u) \partial_{x_j} \Phi^0 \\
 &= \sqrt{(\gamma - 1)e} (\gamma - 1)^2 e \partial_{x_i x_j} \Phi^0 (1 + O(\tau)) \\
 &\quad - \sqrt{(\gamma - 1)e} (\gamma - 1)^2 e \frac{\nabla (\partial_{x_i} \Phi^0) u}{\nabla \Phi^0 u - \sqrt{(\gamma - 1)e}} \partial_{x_j} \Phi^0 (1 + O(\tau)) \\
 &\quad + \sqrt{(\gamma - 1)e} (\gamma - 1)^2 e \frac{\nabla (\partial_{x_i} \Phi^0) u}{\nabla \Phi^0 u - \sqrt{(\gamma - 1)e}} \partial_{x_j} \Phi^0 (1 + O(\tau)) \\
 &= \sqrt{(\gamma - 1)e} (\gamma - 1)^2 e \partial_{x_i x_j} \Phi^0 (1 + O(\tau)).
 \end{aligned}$$

Thus, with the expression (3.15),

$$\begin{aligned}
 (\rho l_{k1} + (\gamma - 1)e l_{k5})(\partial_{x_1} r_{2k} + \partial_{x_2} r_{3k} + \partial_{x_3} r_{4k}) &= C_5 (\rho(\gamma - 1)e + \rho\{(\lambda_5 - \nabla \Phi^0 u)^2 - (\gamma - 1)e\}) \\
 &\quad \cdot \left( -\frac{\gamma - 1}{\rho} \sqrt{(\gamma - 1)e} \Delta \Phi^0 \right) (1 + O(\tau)) \\
 &= -\frac{\sqrt{(\gamma - 1)e}}{2} \Delta \Phi^0 (1 + O(\tau)), \\
 \frac{1}{\rho} l_{k5} (\partial_{x_1} r_{6k} + \partial_{x_2} r_{7k} + \partial_{x_3} r_{8k}) &= C_5 \frac{1}{\rho} \cdot \frac{\rho}{(\gamma - 1)e} \{(\lambda_5 - \nabla \Phi^0 u)^2 - (\gamma - 1)e\} \\
 &\quad \times \left( \sqrt{(\gamma - 1)e} (\gamma - 1)^2 e \Delta \Phi^0 \right) (1 + O(\tau)) \\
 &= O(\tau), \\
 \sum_{j=1}^3 u_j (l_{k2} \partial_{x_j} r_{2k} + l_{k3} \partial_{x_j} r_{3k} + l_{k4} \partial_{x_j} r_{4k}) &= C_5 \rho (\lambda_5 - \nabla \Phi^0 u) \left( -\frac{\gamma - 1}{\rho} \sqrt{(\gamma - 1)e} \right) \\
 &\quad \times \sum_{i=1}^3 \partial_{x_i} \Phi^0 (u_1 \partial_{x_1 x_i} \Phi^0 + u_2 \partial_{x_2 x_i} \Phi^0 + u_3 \partial_{x_3 x_i} \Phi^0) (1 + O(\tau)) \\
 &= \frac{1}{2} (\nabla \Phi^0)^T (D^2 \Phi^0) u + O(\tau) \\
 &= O(\tau),
 \end{aligned}$$

where the last equality holds because  $|\nabla \Phi^0| \equiv 1$  and

$$(\nabla \Phi^0)^T (D^2 \Phi^0) = \left( \nabla \left( \frac{1}{2} |\nabla \Phi^0|^2 \right) \right)^T = \vec{0}.$$

Finally,

$$\begin{aligned}
 (\gamma - 1)(l_{k2} \partial_{x_1} r_{5k} + l_{k3} \partial_{x_2} r_{5k} + l_{k4} \partial_{x_3} r_{5k}) &= C_5 (\gamma - 1) \rho (\lambda_5 - \nabla \Phi^0 u) \nabla \Phi^0 O(\tau) \\
 &= O(\tau), \\
 l_{k5} (u_1 \partial_{x_1} r_{5k} + u_2 \partial_{x_2} r_{5k} + u_3 \partial_{x_3} r_{5k}) &= C_5 \frac{\rho}{(\gamma - 1)e} \{(\lambda_5 - \nabla \Phi^0 u)^2 - (\gamma - 1)e\} u O(\tau) \\
 &= O(\tau), \\
 \frac{\kappa}{\tau} (l_{k6} \partial_{x_1} r_{5k} + l_{k7} \partial_{x_2} r_{5k} + l_{k8} \partial_{x_3} r_{5k}) &= C_5 \frac{\kappa}{\tau} \frac{1}{(\gamma - 1)e \lambda_5} \{(\lambda_5 - \nabla \Phi^0 u)^2 - (\gamma - 1)e\} \nabla \Phi^0 O(\tau) \\
 &= O(\tau).
 \end{aligned}$$

Hence, concluding the above computation, we obtain, for  $k = 5$ , that

$$\sum_{j=1}^3 (L A_j \partial_{x_j} R)_{55} = -\frac{\sqrt{(\gamma - 1)e}}{2} \Delta \Phi^0 + O(\tau). \quad (3.21)$$

For  $k = 6$ , with the expression (3.16), we obtain similarly that, for  $i, j = 1, 2, 3$ ,

$$\begin{aligned}\lambda_6 - \nabla \Phi^0 u &= \sqrt{(\gamma - 1)e} - \frac{\rho}{2\kappa} (\gamma - 1)^2 e (\nabla \Phi^0 u + \sqrt{(\gamma - 1)e} \tau) + O(\tau^2), \\ \partial_{x_i} r_{(j+1)6} &= \frac{\gamma - 1}{\rho} \sqrt{(\gamma - 1)e} \partial_{x_i x_j} \Phi^0 (1 + O(\tau)), \\ \partial_{x_i} r_{56} &= -\frac{(\gamma - 1)^2 e \sqrt{(\gamma - 1)e}}{\kappa} \nabla (\partial_{x_i} \Phi^0) u \tau (1 + O(\tau)), \\ \partial_{x_i} r_{(j+5)6} &= -(\gamma - 1)^2 \sqrt{(\gamma - 1)e} \partial_{x_i x_j} \Phi^0 (1 + O(\tau))\end{aligned}$$

and

$$\sum_{j=1}^3 (L A_j \partial_{x_j} R)_{66} = \frac{\sqrt{(\gamma - 1)e}}{2} \Delta \Phi^0 + O(\tau). \quad (3.22)$$

For  $k = 7$ , we have, for  $i, j = 1, 2, 3$ ,

$$\begin{aligned}\lambda_7 &= -\sqrt{\frac{\kappa}{\rho\tau}} + O(1), \\ \partial_{x_i} r_{(j+1)7} &= \partial_{x_i} \left\{ \frac{\gamma - 1}{\rho} (\lambda_7 - \nabla \Phi^0 u) \partial_{x_j} \Phi^0 \right\} \\ &= \partial_{x_i} \left\{ -\sqrt{\frac{\kappa}{\rho\tau}} \frac{\gamma - 1}{\rho} \partial_{x_j} \Phi^0 (1 + O(\sqrt{\tau})) \right\} \\ &= -\sqrt{\frac{\kappa}{\rho\tau}} \frac{(\gamma - 1)}{\rho} \partial_{x_i x_j} \Phi^0 (1 + O(\sqrt{\tau})), \\ \partial_{x_i} r_{57} &= \partial_{x_i} \left\{ \frac{1}{\rho} ((\lambda_7 - \nabla \Phi^0 u)^2 - (\gamma - 1)e) \right\} \\ &= \partial_{x_i} \left\{ \frac{1}{\rho} \frac{\kappa}{\rho\tau} (1 + O(\sqrt{\tau})) \right\} \\ &= O\left(\frac{1}{\sqrt{\tau}}\right), \\ \partial_{x_i} r_{(5+j)7} &= \partial_{x_i} \left\{ \frac{\kappa}{\rho\tau\lambda_7} ((\lambda_7 - \nabla \Phi^0 u)^2 - (\gamma - 1)e) \partial_{x_j} \Phi^0 \right\} \\ &= \partial_{x_i} \left\{ \frac{\kappa}{\rho\tau} \left(-\sqrt{\frac{\rho\tau}{\kappa}}\right) (1 + O(\sqrt{\tau})) \frac{\kappa}{\rho\tau} (1 + O(\sqrt{\tau})) \partial_{x_j} \Phi^0 \right\} \\ &= -\left(\frac{\kappa}{\rho\tau}\right)^{\frac{3}{2}} \partial_{x_i x_j} \Phi^0 (1 + O(\sqrt{\tau})).\end{aligned}$$

Thus, with the expression (3.17),

$$\begin{aligned}(\rho l_{k1} + (\gamma - 1)e l_{k5})(\partial_{x_1} r_{2k} + \partial_{x_2} r_{3k} + \partial_{x_3} r_{4k}) &= C_7 (\rho(\gamma - 1)e + \rho\{(\lambda_7 - \nabla \Phi^0 u)^2 - (\gamma - 1)e\}) \\ &\quad \cdot \left(-\frac{\gamma - 1}{\rho} \sqrt{\frac{\kappa}{\rho\tau}} \Delta \Phi^0\right) (1 + O(\sqrt{\tau})) \\ &= -\frac{(\gamma - 1)^2 e}{2} \sqrt{\frac{\rho\tau}{\kappa}} \Delta \Phi^0 (1 + O(\sqrt{\tau})), \\ \frac{1}{\rho} l_{k5}(\partial_{x_1} r_{6k} + \partial_{x_2} r_{7k} + \partial_{x_3} r_{8k}) &= C_7 \frac{1}{\rho} \cdot \frac{\rho}{(\gamma - 1)e} \{(\lambda_7 - \nabla \Phi^0 u)^2 - (\gamma - 1)e\} \\ &\quad \times \left((-1) \left(\frac{\kappa}{\rho\tau}\right)^{\frac{3}{2}} \Delta \Phi^0\right) (1 + O(\sqrt{\tau})) \\ &= -\frac{1}{2} \left(\frac{\kappa}{\rho\tau}\right)^{\frac{1}{2}} \Delta \Phi^0 (1 + O(\sqrt{\tau})), \\ \sum_{j=1}^3 u_j (l_{k2} \partial_{x_j} r_{2k} + l_{k3} \partial_{x_j} r_{3k} + l_{k4} \partial_{x_j} r_{4k}) &= C_7 \rho (\lambda_7 - \nabla \Phi^0 u) \sum_{i=1}^3 \left(-\frac{\gamma - 1}{\rho} \sqrt{\frac{\kappa}{\rho\tau}}\right) \\ &\quad \times \partial_{x_i} \Phi^0 (u \cdot \nabla_{x'}) (\partial_{x_i} \Phi^0) (1 + O(\sqrt{\tau})) \\ &= O(\tau).\end{aligned}$$

Finally,

$$\begin{aligned}
 (\gamma - 1)(l_{k2}\partial_{x_1}r_{5k} + l_{k3}\partial_{x_2}r_{5k} + l_{k4}\partial_{x_3}r_{5k}) &= C_7(\gamma - 1)\rho(\lambda_7 - \nabla\Phi^0u)\nabla\Phi^0O\left(\frac{1}{\sqrt{\tau}}\right) \\
 &= O(\tau), \\
 l_{k5}(u_1\partial_{x_1}r_{5k} + u_2\partial_{x_2}r_{5k} + u_3\partial_{x_3}r_{5k}) &= C_7\frac{\rho}{(\gamma - 1)e}\{(\lambda_7 - \nabla\Phi^0u)^2 - (\gamma - 1)e\}uO\left(\frac{1}{\sqrt{\tau}}\right) \\
 &= O(\sqrt{\tau}), \\
 \frac{\kappa}{\tau}(l_{k6}\partial_{x_1}r_{5k} + l_{k7}\partial_{x_2}r_{5k} + l_{k8}\partial_{x_3}r_{5k}) &= C_7\frac{\kappa}{\tau}(\gamma - 1)e\lambda_7\{(\lambda_7 - \nabla\Phi^0u)^2 - (\gamma - 1)e\}\nabla\Phi^0O\left(\frac{1}{\sqrt{\tau}}\right) \\
 &= O(1).
 \end{aligned}$$

Hence, concluding the above computation, we obtain, for  $k = 7$ , that

$$\sum_{j=1}^3 (LA_j\partial_{x_j}R)_{77} = -\frac{1}{2}\left(\frac{\kappa}{\rho\tau}\right)^{\frac{1}{2}}\Delta\Phi^0(1 + O(\sqrt{\tau})). \quad (3.23)$$

For  $k = 8$ , with the expression (3.18), we obtain similarly that, for  $i, j = 1, 2, 3$ ,

$$\begin{aligned}
 \lambda_8 &= \sqrt{\frac{\kappa}{\rho\tau}} + O(1), \\
 \partial_{x_i}r_{(j+1)8} &= \sqrt{\frac{\kappa}{\rho\tau}}\frac{(\gamma - 1)}{\rho}\partial_{x_ix_j}\Phi^0(1 + O(\sqrt{\tau})), \\
 \partial_{x_i}r_{58} &= O\left(\frac{1}{\sqrt{\tau}}\right), \\
 \partial_{x_i}r_{(5+j)8} &= \left(\frac{\kappa}{\rho\tau}\right)^{\frac{3}{2}}\partial_{x_ix_j}\Phi^0(1 + O(\sqrt{\tau}))
 \end{aligned}$$

and

$$\sum_{j=1}^3 (LA_j\partial_{x_j}R)_{88} = \frac{1}{2}\left(\frac{\kappa}{\rho\tau}\right)^{\frac{1}{2}}\Delta\Phi^0(1 + O(\sqrt{\tau})). \quad (3.24)$$

Summarizing, we have obtained the asymptotic expansions

$$(\tilde{A}_0)_{kk} = \begin{cases} \frac{\rho}{2\kappa}(\gamma - 1)^2e \mp \frac{\sqrt{(\gamma - 1)e}}{2}\Delta\Phi^0 + O(\tau), & \text{for } k = 5, 6, \\ \frac{1}{2\tau} \mp \frac{1}{2}\left(\frac{\kappa}{\rho\tau}\right)^{\frac{1}{2}}\Delta\Phi^0(1 + O(\sqrt{\tau})), & \text{for } k = 7, 8. \end{cases} \quad (3.25)$$

For any  $x^0 \in \sigma = \{x \in \mathbb{R}^3 \mid \Phi^0(x) = 0\}$ , let

$$t \mapsto (t, x^{(k)}(t; 0, x^0)) = (t, x_1^{(k)}(t; 0, x^0), x_2^{(k)}(t; 0, x^0), x_3^{(k)}(t; 0, x^0))$$

describe the characteristic line of the operator  $\partial_t + \sum_{j=1}^3 (LA_jR)_{kk}\partial_{x_j}$  passing through  $(0, x^0)$ , which lies on the characteristic surface  $\Sigma_k$  for  $k = 5, 6, 7, 8$ :

$$\begin{cases} \frac{\partial x_j^{(k)}(t; 0, x^0)}{\partial t} = (LA_jR)_{kk}(t, x_1, x_2, x_3), \\ x_j^{(k)}(0; 0, x^0) = x_j^0, \quad j = 1, 2, 3, \end{cases}$$

and define

$$[W_k]_{\Sigma_k(t)} := [W_k]_{\Sigma_k}(t, x^{(k)}(t; 0, x^0)).$$

For  $k = 5, 6, 7, 8$ , let

$$D_k(t; \tau) = \int_0^t (\tilde{A}_0)_{kk}(x^{(k)}(s; 0, x^0))ds,$$



that is, in view of (3.25),

$$D_5(t; \tau) = \frac{(\gamma - 1)^2}{2\kappa} \rho e t - \frac{\sqrt{(\gamma - 1)e}}{2} \int_0^t (\Delta \Phi^0(x^{(5)}(t; 0, x^0)) + O(\tau)) ds, \quad (3.26)$$

$$D_6(t; \tau) = \frac{(\gamma - 1)^2}{2\kappa} \rho e t + \frac{\sqrt{(\gamma - 1)e}}{2} \int_0^t (\Delta \Phi^0(x^{(6)}(t; 0, x^0)) + O(\tau)) ds, \quad (3.27)$$

$$D_7(t; \tau) = \frac{1}{2\tau} t - \frac{1}{2} \left( \frac{\kappa}{\rho\tau} \right)^{\frac{1}{2}} \int_0^t (\Delta \Phi^0(x^{(7)}(t; 0, x^0)) + O(\sqrt{\tau})) ds, \quad (3.28)$$

$$D_8(t; \tau) = \frac{1}{2\tau} t + \frac{1}{2} \left( \frac{\kappa}{\rho\tau} \right)^{\frac{1}{2}} \int_0^t (\Delta \Phi^0(x^{(8)}(t; 0, x^0)) + O(\sqrt{\tau})) ds. \quad (3.29)$$

Then we conclude from (3.20) and (3.25) that, for  $k = 5, 6, 7, 8$ ,

$$[W_k]_{\Sigma_k(t)} = [W_k^0]_{\sigma} \exp \left\{ - \int_0^t (\tilde{A}_0)_{kk}(\tilde{x}(s; 0, x^0)) ds \right\} = [W_k^0]_{\sigma} \cdot e^{-D_k(t; \tau)}. \quad (3.30)$$

Remembering the relation  $V = RW$ , we compute  $V = (\tilde{\rho}, \tilde{u}^p, \tilde{e}, \tilde{q}^p)^T$  as

$$\tilde{\rho} = (\gamma - 1) \sum_{j=5}^8 W_j,$$

$$\tilde{u}^p = \Phi_{\tau_1} \cdot W_3 + \Phi_{\tau_2} \cdot W_4 + \nabla \Phi^0 \cdot \sum_{j=5}^8 \frac{\gamma - 1}{\rho} (\lambda_j - \nabla \Phi^0 u) W_j,$$

$$\tilde{e} = \sum_{j=5}^8 \frac{1}{\rho} \{ (\lambda_j - \nabla \Phi^0 u)^2 - (\gamma - 1)e \} W_j,$$

$$\tilde{q}^p = \Phi_{\tau_1} \cdot W_1 + \Phi_{\tau_2} \cdot W_2 + \nabla \Phi^0 \cdot \sum_{j=5}^8 \frac{\kappa}{\rho\tau\lambda_j} \{ (\lambda_j - \nabla \Phi^0 u)^2 - (\gamma - 1)e \} W_j.$$

Therefore, in view of Lemma 3.4, we have

$$\begin{aligned} [\tilde{\rho}]_{\Sigma_k} &= (\gamma - 1)[W_k]_{\Sigma_k}, \quad \text{for } k = 5, 6, 7, 8, \\ [\tilde{u}^p]_{\Sigma_k} &= \begin{cases} \nabla \Phi^0 \frac{\gamma - 1}{\rho} (\mp \sqrt{(\gamma - 1)e})(1 + O(\tau))[W_k]_{\Sigma_k}, & \text{for } k = 5, 6, \\ \nabla \Phi^0 \frac{\gamma - 1}{\rho} \left( \mp \sqrt{\frac{\kappa}{\rho\tau}} \right) (1 + O(\sqrt{\tau}))[W_k]_{\Sigma_k}, & \text{for } k = 7, 8, \end{cases} \\ [\tilde{e}]_{\Sigma_k} &= \begin{cases} \pm \frac{(\gamma - 1)^2 e \sqrt{(\gamma - 1)e}}{\kappa} (\nabla \Phi^0 u \mp \sqrt{(\gamma - 1)e}) \tau (1 + O(\tau))[W_k]_{\Sigma_k}, & \text{for } k = 5, 6 \\ \frac{\kappa}{\rho^2 \tau} (1 + O(\sqrt{\tau}))[W_k]_{\Sigma_k}, & \text{for } k = 7, 8, \end{cases} \\ [\tilde{q}^p]_{\Sigma_k} &= \begin{cases} \nabla \Phi^0 \left( \pm (\gamma - 1)^2 \sqrt{(\gamma - 1)e} \right) (1 + O(\tau))[W_k]_{\Sigma_k} & \text{for } k = 5, 6, \\ \nabla \Phi^0 \left( \mp \left( \frac{\kappa}{\rho\tau} \right)^{\frac{3}{2}} \right) (1 + O(\sqrt{\tau}))[W_k]_{\Sigma_k} & \text{for } k = 7, 8. \end{cases} \end{aligned}$$

Since  $W = LV$ , we have

$$[W_k^0]_0 = [LV^0]_0, \quad W_k = l_{k1}\tilde{\rho} + \sum_{j=2}^8 l_{kj}\tilde{u}_j + l_{k5}\tilde{e} + \sum_{j=8}^8 l_{kj}\tilde{q}_j.$$

Thus we compute, for  $k = 5, 6$ :

$$\begin{aligned} W_k &= \frac{1}{2(\gamma - 1)^2 e} (1 + O(\tau)) \left\{ (\gamma - 1)e\tilde{\rho} + \rho(\lambda_k - \nabla \Phi^0 u)(\nabla \Phi^0 \tilde{u}^p) + \frac{\rho}{(\gamma - 1)e} \right. \\ &\quad \cdot \{ (\lambda_k - \nabla \Phi^0 u)^2 - (\gamma - 1)e \} \tilde{e} + \frac{1}{(\gamma - 1)e\lambda_k} \{ (\lambda_k - \nabla \Phi^0 u)^2 - (\gamma - 1)e \} (\nabla \Phi^0 \tilde{q}^p) \left. \right\} \end{aligned}$$

$$\begin{aligned}
&= \left( \frac{1}{2(\gamma-1)} \tilde{\rho} \mp \frac{\rho \sqrt{(\gamma-1)e}}{2(\gamma-1)^2 e} \nabla \Phi^0 \tilde{u}^p \right) (1 + O(\tau)) + O(\tau) \tilde{e} + O(\tau) (\nabla \Phi^0 \tilde{q}^p) \\
&= \frac{1}{2(\gamma-1)} \left( \tilde{\rho} \mp \frac{\rho}{\sqrt{(\gamma-1)e}} \nabla \Phi^0 \cdot \tilde{u}^p \right) (1 + O(\tau)),
\end{aligned}$$

and for  $k = 7, 8$ :

$$\begin{aligned}
W_k &= \frac{(\gamma-1)e}{2} \left( \frac{\rho\tau}{\kappa} \right)^2 (1 + O(\sqrt{\tau})) \left\{ (\gamma-1)e\tilde{\rho} + \rho(\lambda_k - \nabla \Phi^0 u) (\nabla \Phi^0 \tilde{u}^p) + \frac{\rho}{(\gamma-1)e} \right. \\
&\quad \cdot \{ (\lambda_k - \nabla \Phi^0 u)^2 - (\gamma-1)e\tilde{e} + \frac{1}{(\gamma-1)e\lambda_k} \{ (\lambda_k - \nabla \Phi^0 u)^2 - (\gamma-1)e \} (\nabla \Phi^0 \tilde{q}^p) \} \\
&= \frac{(\gamma-1)^2 e^2}{2} \left( \frac{\rho\tau}{\kappa} \right)^2 \tilde{\rho} (1 + O(\sqrt{\tau})) \mp \frac{(\gamma-1)e\rho}{2} \left( \frac{\rho\tau}{\kappa} \right)^{\frac{3}{2}} \nabla \Phi^0 \tilde{u}^p (1 + O(\sqrt{\tau})) \\
&\quad + \frac{\rho}{2} \left( \frac{\rho\tau}{\kappa} \right) \tilde{e} (1 + O(\sqrt{\tau})) \mp \frac{1}{2} \left( \frac{\rho\tau}{\kappa} \right)^{\frac{3}{2}} \nabla \Phi^0 \tilde{q}^p (1 + O(\sqrt{\tau})) \\
&= \left( \frac{1}{2} \frac{\rho^2 \tau}{\kappa} \tilde{e} \mp \frac{1}{2} \left( \frac{\rho\tau}{\kappa} \right)^{\frac{3}{2}} \nabla \Phi^0 \cdot ((\gamma-1)\rho e \tilde{u}^p + \tilde{q}^p) + \frac{1}{2} (\gamma-1)^2 e^2 \left( \frac{\rho\tau}{\kappa} \right)^2 \right) (1 + O(\sqrt{\tau})) \\
&= \frac{1}{2} \frac{\rho^2 \tau}{\kappa} \tilde{e} (1 + O(\sqrt{\tau})).
\end{aligned}$$

Hence,

$$[W_k^0]_\sigma = \begin{cases} \frac{1}{2(\gamma-1)} \left( [\tilde{\rho}_0]_\sigma \mp \frac{\rho}{\sqrt{(\gamma-1)e}} [\nabla \Phi^0 \cdot \tilde{u}_0^p]_\sigma \right) (1 + O(\tau)), & k = 5, 6 \\ \frac{1}{2} \frac{\rho^2 \tau}{\kappa} [\tilde{e}_0]_\sigma (1 + O(\sqrt{\tau})), & k = 7, 8. \end{cases}$$

Thus we have

$$\begin{aligned}
[\tilde{\rho}]_{\Sigma_k} &= (\gamma-1)[W_k]_{\Sigma_k} = (\gamma-1)[W_k^0]_\sigma \cdot e^{-D_k(t;\tau)} \\
&= \begin{cases} \frac{1}{2} \left( [\tilde{\rho}_0]_\sigma \mp \frac{\rho}{\sqrt{(\gamma-1)e}} [\nabla \Phi^0 \cdot \tilde{u}_0^p]_\sigma \right) (1 + O(\tau)) \cdot e^{-D_k(t;\tau)}, & k = 5, 6, \\ \frac{1}{2} \frac{\rho^2 \tau}{\kappa} [\tilde{e}_0]_\sigma (1 + O(\sqrt{\tau})) \cdot e^{-D_k(t;\tau)}, & k = 7, 8 \end{cases} \\
[\tilde{u}^p]_{\Sigma_k} &= \begin{cases} \nabla \Phi^0 \frac{\gamma-1}{\rho} (\mp \sqrt{(\gamma-1)e}) (1 + O(\tau)) [W_k]_{\Sigma_k}, & k = 5, 6, \\ \nabla \Phi^0 \frac{\gamma-1}{\rho} \left( \mp \sqrt{\frac{\kappa}{\rho\tau}} \right) (1 + O(\sqrt{\tau})) [W_k]_{\Sigma_k}, & k = 7, 8, \end{cases} \\
&= \begin{cases} \nabla \Phi^0 \cdot \frac{\sqrt{(\gamma-1)e}}{2\rho} \left( \mp [\tilde{\rho}_0]_\sigma + \frac{\rho}{\sqrt{(\gamma-1)e}} [\nabla \Phi^0 \cdot \tilde{u}_0^p]_\sigma \right) (1 + O(\tau)) \cdot e^{-D_k(t;\tau)}, & k = 5, 6, \\ \nabla \Phi^0 \cdot \left( \mp \frac{\gamma-1}{2} \sqrt{\frac{\rho\tau}{\kappa}} \right) [\tilde{e}_0]_\sigma (1 + O(\sqrt{\tau})) \cdot e^{-D_k(t;\tau)}, & k = 7, 8, \end{cases} \\
[\tilde{e}]_{\Sigma_k} &= \begin{cases} \pm \frac{(\gamma-1)^2 e \sqrt{(\gamma-1)e}}{\kappa} \left( \nabla \Phi^0 u \mp \sqrt{(\gamma-1)e} \right) \tau (1 + O(\tau)) [W_k]_{\Sigma_k}, & k = 5, 6, \\ \frac{\kappa}{\rho^2 \tau} (1 + O(\sqrt{\tau})) [W_k]_{\Sigma_k}, & k = 7, 8, \end{cases} \\
&= \begin{cases} O(\tau) \cdot e^{-D_k(t;\tau)}, & k = 5, 6, \\ \frac{1}{2} [\tilde{e}_0]_\sigma (1 + O(\sqrt{\tau})) \cdot e^{-D_k(t;\tau)}, & k = 7, 8 \end{cases}
\end{aligned}$$

$$\begin{aligned}
[\tilde{q}^p]_{\Sigma_k} &= \begin{cases} \nabla \Phi^0 \left( \pm(\gamma-1)^2 \sqrt{(\gamma-1)e} \right) (1+O(\tau)) [W_k]_{\Sigma_k}, & k=5, 6, \\ \nabla \Phi^0 \left( \mp \left( \frac{\kappa}{\rho\tau} \right)^{\frac{3}{2}} \right) (1+O(\sqrt{\tau})) [W_k]_{\Sigma_k}, & k=7, 8, \end{cases} \\
&= \begin{cases} \nabla \Phi^0 \cdot \frac{(\gamma-1)\sqrt{(\gamma-1)e}}{2} \left( \pm[\tilde{\rho}_0]_{\sigma} - \frac{\rho}{\sqrt{(\gamma-1)e}} [\nabla \Phi^0 \tilde{u}_0^p]_{\sigma} \right) (1+O(\tau)) \cdot e^{-D_k(t;\tau)} & k=5, 6, \\ \nabla \Phi^0 \cdot \left( \mp \frac{\rho}{2} \sqrt{\frac{\kappa}{\rho\tau}} \right) [\tilde{e}_0]_{\sigma} (1+O(\sqrt{\tau})) \cdot e^{-D_k(t;\tau)}, & k=7, 8. \end{cases}
\end{aligned}$$

Summarizing we obtain the following characteristic behavior of the evolving jumps along  $\Sigma_k$ .

**Theorem 3.7.** Suppose that the initial data of  $V_0 = (\tilde{\rho}_0, \tilde{u}_0^p, \tilde{e}_0, \tilde{q}_0^p)^T$  for the system (2.10) may have jumps on  $\sigma = \{\Phi^0(x) = 0\}$  with  $|\nabla \Phi^0(x)| = 1$ . Then the jumps will propagate along the characteristic surfaces  $\Sigma_k (k = 5, 6, 7, 8)$  and the propagation is given by

$$\begin{aligned}
[\tilde{\rho}]_{\Sigma_k} &= \begin{cases} \frac{1}{2} \left( [\tilde{\rho}_0]_{\sigma} \mp \frac{\rho}{\sqrt{(\gamma-1)e}} [\nabla \Phi^0 \cdot \tilde{u}_0^p]_{\sigma} \right) (1+O(\tau)) \cdot e^{-D_k(t;\tau)}, & k=5, 6, \\ O(\tau) \cdot e^{-D_k(t;\tau)}, & k=7, 8 \end{cases} \\
[\tilde{u}^p]_{\Sigma_k} &= \begin{cases} \nabla \Phi^0 \cdot \frac{\sqrt{(\gamma-1)e}}{2\rho} \left( \mp[\tilde{\rho}_0]_{\sigma} + \frac{\rho}{\sqrt{(\gamma-1)e}} [\nabla \Phi^0 \cdot \tilde{u}_0^p]_{\sigma} \right) (1+O(\tau)) \cdot e^{-D_k(t;\tau)}, & k=5, 6, \\ \nabla \Phi^0 \cdot O(\sqrt{\tau}) \cdot e^{-D_k(t;\tau)}, & k=7, 8 \end{cases} \\
[\tilde{e}]_{\Sigma_k} &= \begin{cases} O(\tau) \cdot e^{-D_k(t;\tau)}, & k=5, 6, \\ \frac{1}{2} [\tilde{e}_0]_{\sigma} (1+O(\sqrt{\tau})) \cdot e^{-D_k(t;\tau)}, & k=7, 8 \end{cases} \\
[\tilde{q}^p]_{\Sigma_k} &= \begin{cases} \nabla \Phi^0 \cdot \frac{(\gamma-1)\sqrt{(\gamma-1)e}}{2} \left( \pm[\tilde{\rho}_0]_{\sigma} - \frac{\rho}{\sqrt{(\gamma-1)e}} [\nabla \Phi^0 \tilde{u}_0^p]_{\sigma} \right) (1+O(\tau)) \cdot e^{-D_k(t;\tau)}, & k=5, 6, \\ \nabla \Phi^0 \cdot \left( \mp \frac{\rho}{2} \sqrt{\frac{\kappa}{\rho\tau}} \right) [\tilde{e}_0]_{\sigma} (1+O(\sqrt{\tau})) \cdot e^{-D_k(t;\tau)}, & k=7, 8 \end{cases}
\end{aligned}$$

where  $D_k(t; \tau)$  with  $k = 5, 6, 7, 8$  is given as in (3.26)–(3.29). That is, as  $t \rightarrow \infty$  or  $\tau \rightarrow 0$ , the propagation of the jumps of  $V = (\tilde{\rho}, \tilde{u}^p, \tilde{e}, \tilde{q}^p)^T$  depends on the parameters of the coefficients of Eqs. (2.10) and the mean curvature

$$H = \frac{\Delta \Phi^0}{2}$$

of the initial surface  $\sigma$ .

1. On the characteristic surfaces  $\Sigma_5$  and  $\Sigma_6$ , as  $\tau \rightarrow 0$ , the jumps of  $\tilde{\rho}$ ,  $\tilde{u}^p$ ,  $\tilde{q}^p$  will remain while the jumps of  $\tilde{e}$  will vanish of order  $O(\tau)$ , which shows a smoothing effect in the system (2.10) when  $\tau \rightarrow 0$ .
2. On the characteristic surfaces  $\Sigma_5$  (or  $\Sigma_6$  resp.), as  $t \rightarrow \infty$ , the jumps of  $\tilde{\rho}$ ,  $\tilde{u}^p$ ,  $\tilde{q}^p$  will decay exponentially as long as  $\frac{(\gamma-1)^2 \rho e}{\kappa} - \sqrt{(\gamma-1)e} \Delta \Phi^0$  (or  $\frac{(\gamma-1)^2 \rho e}{\kappa} + \sqrt{(\gamma-1)e} \Delta \Phi^0$  resp.) are positive.
3. On the characteristic surfaces  $\Sigma_7$  and  $\Sigma_8$ , the jumps of  $V$  will decay exponentially as  $\tau \rightarrow 0$  or  $t \rightarrow \infty$  for a fixed small  $\tau > 0$ .

**Remark 3.8.** The key terms describing the asymptotic behaviors in Theorem 3.7 are similar to the ones for the equations of thermoelasticity with second sound in [6]. Hence, it would be possible to add nonhomogeneous terms or semilinear terms on the right-hand side of Eqs. (2.1)–(2.8) and carry out a similar analysis as in [6].

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