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On stacked central configurations with n bodies when one body is removed

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ABSTRACT

A stacked central configuration is a central configuration of the n-body problem for which a proper subset of the bodies is already in a central configuration. In this paper we study all the possibilities of stacked central configurations for the n-body problem when one body is removed. The results have simple and analytic proofs.

1. Introduction

In celestial mechanics the classical n-body problem consists in the study of the motion of n point bodies with positive masses $m_1, \ldots, m_n$, interacting among themselves through no other forces than their mutual gravitational attraction according to Newton's gravitational law [17]. Consider such bodies with position vectors $r_1, \ldots, r_n$, where $r_i \in \mathbb{R}^d$, $d = 2, 3$. The equations of motion are given by

$$\ddot{r}_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{m_j}{r_{ij}^3}(r_i - r_j), \tag{1}$$

for $i = 1, 2, \ldots, n$. Here the gravitational constant is taken equal to 1 and $r_{ij} = |r_i - r_j|$ is the Euclidean distance between the bodies at r_i and r_j.

Note that Eq. (1) is not well defined if $r_{ij} = 0$. So we consider configurations $r = (r_1, r_2, \ldots, r_n)$ out of the collision set, assuming $r_{ij} \neq 0$ for all $i \neq j$. We also consider the inertial barycentric system, that is, the origin of the inertial system is the center of mass of the system, which is given by $\mathcal{C}_n = \sum_{j=1}^{n} m_j r_j / \mathcal{M}_n$, where $\mathcal{M}_n = m_1 + \cdots + m_n$ is the total mass.

The n-body problem is not solvable by direct integration via quadratures when $n > 2$. So, the study of particular solutions becomes very important. One interesting case of a particular solution is the homographic one: the initial form of the configuration is preserved in the time evolution. In other words, the motions are given by dilations or rotations (centered at the center of mass).

The first homographic solutions for the three-body problem were found by Euler [5]. In Euler's solutions three bodies move on conical sections, but on a straight line at each fixed instant. Lagrange [11] studied other homographic solutions for

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the three-body problem, in which the bodies are orbiting around the center of mass of the system on conical sections at the vertices of an equilateral triangle in each fixed instant.

At a given instant $t = t_0$ the n bodies are in a *central configuration* if there exists $\lambda \neq 0$ such that $\ddot{r}_i = \lambda r_i$, referred to the inertial barycentric system, for all $i = 1, \ldots, n$. By a simple computation we have $\lambda = -U/I$, where

$$U = \sum_{1 \le i < j \le n} \frac{m_i m_j}{r_{ij}}, \qquad I = \frac{1}{\mathcal{M}_n} \sum_{1 \le i < j \le n} m_i m_j r_{ij}^2 \tag{2}$$

are the Newtonian potential and the moment of inertia of the n bodies, respectively.

Two central configurations $(r_1, r_2, \ldots, r_n)$ and $(\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n)$ of the n bodies are related if we can pass from one to the other through a dilation or a rotation (centered at the center of mass). So we can study classes of central configurations defined by the above equivalence relation.

Taking into account this equivalence there are exactly five classes of central configurations in the three-body problem. The finiteness of the number of central configurations performed by n bodies with positive masses is a question posed by Chazy in [3], Wintner in [23], and reformulated to the planar case by Smale in [22]. This problem has an affirmative answer when $n = 4$. See [9]. Alternatively, see [2] for a proof of the finiteness when $n = 4$ and a partial answer when $n = 5$. This question is open when $n > 5$.

It is important to emphasize that the computation of central configurations is reduced to a resolution of a set of algebraic equations, since by (1) we have

$$\lambda r_i = -\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{r_{ij}^3}(r_i - r_j), \tag{3}$$

for each instant of time and $i = 1, \ldots, n$. The expressions in Eq. (3) are called the *equations of central configurations*.

A powerful tool for studying central configurations in the planar n-body problem is provided by the so called planar Laura–Andoyer equations. Consider n non-collinear bodies with masses $m_1, \ldots, m_n$ and positions $r_1, \ldots, r_n$, $r_i \in \mathbb{R}^2$. Then the planar Laura–Andoyer equations are given by

$$f_{ij} = \sum_{\substack{k=1 \\ k \neq i,j}}^{n} m_k (R_{ik} - R_{jk}) \Delta_{ijk} = 0, \tag{4}$$

for $1 \le i < j \le n$, where $R_{ij} = r_{ij}^{-3}$ and $\Delta_{ijk} = (r_i - r_j) \wedge (r_i - r_k)$ is twice the oriented area defined by the triangle with vertices at r_i, r_j and r_k. The $n(n-1)/2$ Laura–Andoyer equations are equivalent to the equations of central configurations in (3). See [7] or [14], for instance.

The spatial Laura–Andoyer equations are defined analogously. Consider n non-planar bodies with masses $m_1, \ldots, m_n$, and positions $r_1, \ldots, r_n$, with $r_i \in \mathbb{R}^3$. Then the spatial Laura–Andoyer equations are given by

$$f_{ijl} = \sum_{\substack{k=1 \\ k \neq i,j,l}}^{n} m_k (R_{ik} - R_{jk}) \Delta_{ijlk} = 0, \tag{5}$$

for $i < j$, $l \neq i$, $l \neq j$, $i, j, l = 1, \ldots, n$, where $R_{ij} = r_{ij}^{-3}$ and $\Delta_{ijlk} = (r_i - r_j) \wedge (r_i - r_l) \cdot (r_i - r_k)$ is six times the oriented volume defined by the tetrahedron with vertices at r_i, r_j, r_l and r_k.

The knowledge of central configurations allows us to compute homographic solutions [16]; there is a relation between central configurations and the bifurcations of the hypersurfaces of constant energy and angular momentum [21]; if the n bodies are moving towards a simultaneous collision then the bodies tend to the set of central configurations [18]. See also the Refs. [7,19,23].

Hampton [8] provides a new family of planar central configurations for the five-body problem with an interesting property: two bodies can be removed and the remaining three bodies are already in a central configuration. Such configurations are called *stacked central configurations*.

Here we adopt the following nomenclature: a central configuration is called (n, k)-*stacked* when the n bodies are in a central configuration and we can remove $0 < k < n$ bodies such that the remaining $n - k$ bodies are already in a central configuration. Without loss of generality, the k bodies to be removed are those of position vectors $r_{n-k+1}, r_{n-k+2}, \ldots, r_n$.

Some $(5, 2)$-stacked planar central configurations were studied by Llibre and Mello [12], and by Llibre, Mello and Perez-Chavela [13]. Stacked central configurations in the spatial five-body and seven-body problems were studied by Santos [20] and Hampton and Santoprete [10], respectively.

Fernandes and Mello [6] studied $(5, 1)$-stacked planar central configurations. The authors conclude that the only non-collinear $(5, 1)$-stacked planar central configuration is formed by four bodies in a co-circular central configuration and one body of arbitrary mass at the center of the circle.

In this paper we extend the results of [6] studying the general case of $(n, 1)$-stacked central configurations. The two main results are the following.

Theorem 1. *Consider the planar non-collinear n-body problem with $n \geq 4$. Then the only $(n, 1)$-stacked central configurations are formed by $n-1$ bodies in a co-circular central configuration and one body (to be removed) of arbitrary mass at the center of the circle.*

Theorem 2. *Consider the spatial non-planar n-body problem with $n \geq 4$. Then the only $(n, 1)$-stacked central configurations are formed by $n-1$ bodies in a co-spherical central configuration and one body of arbitrary mass at the center of the sphere.*

The proofs of Theorems 1 and 2 are given in Sections 2 and 3, respectively.

The $(n, 1)$-stacked central configurations have the following fundamental property: they are the only non-collinear central configurations in which we can vary the value of one mass of the configuration keeping the positions and other masses fixed and still have a central configuration.

2. Proof of Theorem 1

The proof of Theorem 1 is divided into two lemmas.

Lemma 3. *In order to have an $(n, 1)$-stacked central configuration it is necessary that the central configuration of the n bodies does not depend on the value of the mass of the body to be removed, that is m_n.*

Proof. The planar Laura–Andoyer equations $f_{ij} = 0$ must be satisfied for the n bodies. So, consider the Laura–Andoyer equations with $i \neq n$ and $j \neq n$. These equations can be written as

$$f_{ij} = \sum_{k \neq i,j,n} m_k \left(R_{ik} - R_{jk}\right) \Delta_{ijk} + m_n \left(R_{in} - R_{jn}\right) \Delta_{ijn} = 0, \tag{6}$$

for all indices i and j such that $0 < i < j < n$. Note that in (6) the parts under summation are exactly the Laura–Andoyer equations for $n-1$ bodies, which must vanish too. So, as we consider $m_n > 0$, the following equations are necessary conditions in order to have an $(n, 1)$-stacked central configuration:

$$\left(R_{in} - R_{jn}\right) \Delta_{ijn} = 0, \tag{7}$$

for all indices i and j such that $0 < i < j < n$. Remember that in equations $f_{in} = 0$ the masses m_n do not appear. Thus the lemma is proved. $\square$

The next lemma says that the remaining $n-1$ bodies must be on a circle with center at r_n.

Lemma 4. *In order to have an $(n, 1)$-stacked central configuration it is necessary that the remaining $n-1$ bodies must be in a co-circular central configuration with center at r_n.*

Proof. By hypothesis the configuration must be non-collinear, so in Eq. (7) at least one Δ_{ijn} is different from zero. Without loss of generality, consider $\Delta_{12n} \neq 0$. From

$$(R_{1n} - R_{2n}) \Delta_{12n} = 0$$

we have

$$R_{1n} - R_{2n} = 0,$$

which implies that $r_{1n} = r_{2n} = d > 0$. Thus, r_1 and r_2 belong to a circle with radius d and center at r_n.
We can classify the other indices into two sets

$$\mathfrak{C}_1 = \left\{j : \Delta_{1jn} = 0\right\}$$

and

$$\mathfrak{C}_2 = \left\{j : \Delta_{1jn} \neq 0\right\},$$

that is $\mathfrak{C}_1$ contains the indices of the bodies whose vector positions are collinear with r_1 and r_n, while $\mathfrak{C}_2$ contains the indices of the bodies whose vector positions are not collinear with r_1 and r_n. For $j \in \mathfrak{C}_2$ and from

$$\left(R_{1n} - R_{jn}\right) \Delta_{1jn} = 0$$

we have

$$R_{1n} - R_{jn} = 0.$$

Thus, $r_{jn} = r_{1n} = d > 0$ for all $j \in \mathfrak{C}_2$. Then r_1, r_2 and r_j belong to the circle of radius d and center at r_n for all $j \in \mathfrak{C}_2$.
To complete the proof of the lemma we need to show that $\mathfrak{C}_1$ has at most one element. Suppose, by contradiction, that there exist two indices $b, c \in \mathfrak{C}_1$. So $\Delta_{1bn} = 0$, which implies that $\Delta_{2bn} \neq 0$. From

$$(R_{2n} - R_{bn}) \Delta_{2bn} = 0,$$

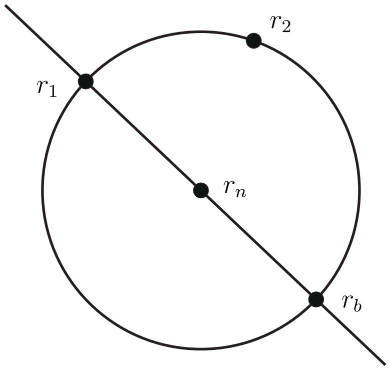

Fig. 1. There is no position for r_c out of the collision set.

we have

$$R_{2n} - R_{bn} = 0,$$

which implies that $r_{bn} = r_{2n} = d > 0$. Thus r_b belongs to the circle of radius d and center at r_n. As the central configurations are out of the collision set, r_b must be diametrically opposite to r_1. Now consider the index $c \in \mathfrak{C}_1$. So $\Delta_{1cn} = 0$, which implies that $\Delta_{2cn} \neq 0$. From

$$(R_{2n} - R_{cn})\,\Delta_{2cn} = 0$$

we have

$$R_{2n} - R_{cn} = 0,$$

which implies that $r_{cn} = r_{2n} = d > 0$. Here we have a contradiction, since, in this case, r_c coincides with either r_1 or r_b. See Fig. 1. The lemma is proved. □

Lemma 4 has the following corollary.

Corollary 5. *In a non-collinear* $(n, 1)$*-stacked planar central configuration the polygon formed by the position vectors of the remaining* $n - 1$ *bodies is convex.*

The proof of Theorem 1 follows from Lemmas 3 and 4.

3. Proof of Theorem 2

The proof of Theorem 2 is divided into two lemmas.

Lemma 6. *In order to have an* $(n, 1)$*-stacked spatial central configuration it is necessary that the central configuration of the* n *bodies does not depend on the value of the mass of the body to be removed, that is* m_n*.*

The proof of Lemma 6 is similar to the proof of Lemma 3. The main difference is the use of spatial Laura–Andoyer equations. In other words, the following equations are necessary to an $(n, 1)$-stacked spatial central configuration:

$$\left(R_{in} - R_{jn}\right)\Delta_{ijkn} = 0, \tag{8}$$

for all $1 \le i < j \le n$ and $k \neq i, j, n$.

The equations in (8) have some geometrical implications for the central configuration.

Lemma 7. *In order to have an* $(n, 1)$*-stacked spatial central configuration it is necessary that the remaining* $n - 1$ *bodies must be co-spherical.*

Proof. By hypothesis the configuration must be non-coplanar, so in Eq. (8) at least one Δ_{ijkn} is different from zero. Without loss of generality, consider $\Delta_{123n} \neq 0$. From

$$(R_{1n} - R_{2n})\,\Delta_{123n} = 0$$

we have

$$R_{1n} - R_{2n} = 0,$$

which implies that $r_{1n} = r_{2n} = d > 0$. Thus, r_1 and r_2 belong to a sphere with radius d and center at r_n.

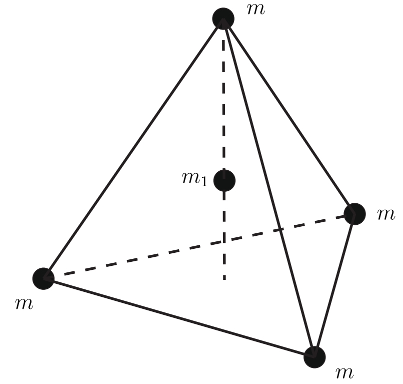

Fig. 2. Example of a (5, 1)-stacked spatial central configuration.

We can classify the other indices into two sets

$$\mathfrak{D}_1 = \left\{ j : \Delta_{12jn} = 0 \right\}$$

and

$$\mathfrak{D}_2 = \left\{ j : \Delta_{12jn} \neq 0 \right\},$$

that is $\mathfrak{D}_1$ contains the indices of the bodies whose vector positions are coplanar with r_1, r_2 and r_n, while $\mathfrak{D}_2$ contains the indices of the bodies whose vector positions are not coplanar with r_1, r_2 and r_n. For $j \in \mathfrak{D}_2$, we have

$$\left(R_{1n} - R_{jn} \right) \Delta_{12jn} = 0,$$

which implies that

$$R_{1n} - R_{jn} = 0.$$

Thus, $r_{jn} = r_{1n} = d > 0$ for all $j \in \mathfrak{D}_2$. Then r_1, r_2 and r_j belong to the sphere of radius d and center at r_n for all $j \in \mathfrak{D}_2$. To complete the proof of the lemma note that the index 3 does not belong to $\mathfrak{D}_1$; thus $r_{3n} = d$. Take now the equations in (8) for which the first index is 3 and the second belongs to $\mathfrak{D}_1$. So

$$\left(R_{3n} - R_{jn} \right) \Delta_{3j1n} = 0$$

and thus

$$R_{3n} - R_{jn} = 0,$$

which implies that $r_{jn} = r_{3n} = d > 0$ for all $j \in \mathfrak{D}_1$, that is $r_{jn} = d > 0$ for all j such that $1 \leq j < n$. The lemma is proved. $\square$

In short, Theorem 2 is proved from Lemmas 6 and 7.

The only possibility of a central configuration with n bodies for which a proper subset of $n-1$ bodies also form a central configuration is given by $n-1$ bodies in a central configuration on a sphere S and one body of arbitrary mass at the center of S.

Little is known about co-spherical central configurations. See [15] for some results on co-spherical kite central configurations. Platonic polyhedrons with bodies of equal masses at the vertices and one body of arbitrary mass at their geometrical barycenters are examples of $(n, 1)$-stacked spatial central configurations. See Fig. 2.

Pyramidal central configurations are also examples of $(n, 1)$-stacked spatial central configurations. See [1]. In particular, these central configurations are the only possible $(n, 1)$-stacked spatial central configurations with a proper planar subset of the bodies which form a planar central configuration. See Fig. 3.

The study of pyramidal central configurations requires a better understanding of co-circular central configurations. See the work of Roberts and Cors [4] for the explanation of the case $n = 4$. The general case still remains open.

4. Concluding remarks

In Theorem 1 (Theorem 2, respectively) we have characterized the $(n, 1)$-stacked central configuration for the planar non-collinear (spatial non-planar, respectively) n-body problem, $n \geq 4$. On the basis of the proofs of Theorems 1 and 2 the following problem arises: What can be said about the center of mass of an $(n, 1)$-stacked central configuration?

This problem will be investigated in a future work.

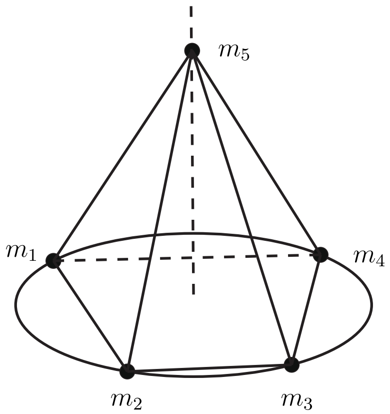

Fig. 3. Example of an $(n, 1)$-stacked spatial central configuration with a planar subset of bodies that also form a central configuration.

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