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Extensions and applications of ACF mappings

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ABSTRACT

Using a definition of ASF sequences derived from the definition of asymptotic contractions of the final type of ACF, we give some new fixed point theorem for cyclic mappings and an alternating mapping which extend results from Suzuki (2007) [8, Theorem 2] and Zhang (2007) [10, Theorem 1].

1. Introduction

Many extensions of the well known Banach contraction principle [1] have been proposed in the nonlinear analysis literature. Among them fixed point theorems for Meir–Keeler contraction have been extensively studied [5,4,7] and a final (in some sense) generalization defined as *asymptotic contraction of the final type* (ACF, for short) has been stated by T. Suzuki [8, Theorem 5]. Our aim in this paper is to extend the results of T. Suzuki to more general cases with regard to the mappings. More precisely, we want to be able to use the same framework for proving fixed point theorems for alternating mappings Section 6 or for cyclic mappings Section 4. For that purpose we propose the definition of p-ASF-1 and p-ASF-2 sequences which are defined without references to a mapping and prove some Cauchy properties of such sequences in Theorem 6. In Section 3, we recall the definition of ACF mapping and relate the ACF mapping to p-ASF mappings. When the p-ASF sequences are generated using $\{T^n x\}$ we show that the two definitions coincide (Theorem 9). We give an application to cyclic mappings in Section 4 by providing a fixed point theorem which extends [8, Theorem 2] to continuous p-ASF mappings. In Section 6 we give an application to an alternating mapping through Theorem 22 which extends the results of [10].

2. ACF sequences

In [8] T. Suzuki introduces the definition of an *asymptotic contraction of the final type* (ACF, for short) and proves that if a mapping T is an ACF, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by $x_n \stackrel{\text{def}}{=} T^n x$ is a Cauchy sequence for all $x \in \mathbb{X}$. Since our aim is to extend T. Suzuki results when sequences $\{x_n\}_{n\in\mathbb{N}}$ are generated by more general processes, we introduce a new definition that we call ASF, which stands for *asymptotic sequences of the final type*. The definition characterizes two sequences and not a mapping. The link between the two definitions is the following. Suppose that the mapping T is an ACF and for $x, y \in \mathbb{X}$ define two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ by $x_n \stackrel{\text{def}}{=} T^n x$ and $y_n \stackrel{\text{def}}{=} T^n y$. If for all $n \in \mathbb{N}$ we have $x_n \neq y_n$ then the two sequences are ASF. Properties of ASF sequences are given in Lemma 2 and a proof is given but note that the proof is mostly a simple rephrase of [8, Lemmas 1 and 2]. We first start with the ASF definition.

In the sequel $(\mathbb{X}, d)$ is a complete metric space and p is a given function from $\mathbb{X} \times \mathbb{X}$ into $[0, \infty)$.

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Definition 1. We say that two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ with $x_n, y_n \in \mathbb{X}$ are p-ASF-1 if the following are satisfied:

(C_1) For each $\epsilon > 0$ there exists $\delta > 0$ such that if for $i \in \mathbb{N}$ we have $p(x_i, y_i) < \delta$ then $\limsup_{n\to\infty} p(x_n, y_n) \le \epsilon$.
(C_2) For each $\epsilon > 0$, there exists $\delta > 0$ such that for $i, j \in \mathbb{N}$ with $\epsilon < p(x_i, y_i) < \epsilon + \delta$, there exists $\nu \in \mathbb{N}$ such that $p(x_{\nu+i}, y_{\nu+i}) \le \epsilon$.
(C_3) For each given (x_i, y_i) such that $p(x_i, y_i) \ne 0$ there exists $\nu \in \mathbb{N}$ such that

$$p(x_{\nu+i}, y_{\nu+i}) < p(x_i, y_i).$$

Lemma 2. *Let $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ be two p-ASF-1 sequences. Then $\lim_{n\to\infty} p(x_n, y_n) = 0$.*

Proof. We follow [8, Lemma 2]. If we suppose that there exists $i \in \mathbb{N}$ such that $p(x_i, y_i) = 0$, we conclude directly using (C_1) that $\lim_{n\to\infty} p(x_n, y_n) = 0$. Thus we assume now that $p(x_n, y_n) \ne 0$ for all $n \in \mathbb{N}$. We first prove that if $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ satisfy (C_2) and (C_3) then $\liminf_{n\to\infty} p(x_n, y_n) = 0$. Using the fact that p is nonnegative and repeatedly using Property (C_3) it is possible to build an extracted decreasing subsequence $p(x_{\sigma(n)}, y_{\sigma(n)})$ such that $0 \le p(x_{\sigma(n)}, y_{\sigma(n)}) \le p(x_0, y_0)$ which implies that $\liminf_{n\to\infty} p(x_n, y_n) = \alpha$ exists and is finite. Suppose that $\alpha > 0$. We first show that we must have $\alpha < p(x_n, y_n)$ for all $n \in \mathbb{N}$. Indeed suppose that there exists n_0 such that $p(x_{n_0}, y_{n_0}) \le \alpha$ then repeatedly using (C_3) we can build an extracted decreasing sequence $p(x_{\sigma(n)}, y_{\sigma(n)})$ such that $p(x_{\sigma(n)}, y_{\sigma(n)}) < p(x_{n_0}, y_{n_0}) \le \alpha$. This decreasing sequence will converge to a cluster point of $p(x_n, y_n)$ strictly smaller than α which is contradictory with the definition of α. Thus we have $\alpha < p(x_n, y_n)$ for all $n \in \mathbb{N}$ and $\alpha > 0$. We then consider $\delta(\alpha)$ given by (C_2) for $\epsilon = \alpha$. By definition of α we can find (x_i, y_i) such that $\alpha < p(x_i, y_i) < \alpha + \delta(\alpha)$ and by (C_2) we will obtain $\nu \in \mathbb{N}$ such that $p(x_{\nu+i}, y_{\nu+i}) \le \alpha$ which contradicts $\alpha < p(x_n, y_n)$ for all $n \in \mathbb{N}$. Thus we conclude that $\alpha = 0$.

We prove now that $\liminf_{n\to\infty} p(x_n, y_n) = 0$ and (C_1) imply that $\limsup_{n\to\infty} p(x_n, y_n) = 0$. For $\epsilon > 0$ given, we consider δ given by (C_1). Since $\liminf_{n\to\infty} p(x_n, y_n) = 0$ then we can find $i \in \mathbb{N}$ such that $p(x_i, y_i) < \delta$. Thus by (C_1) we have $\limsup_{n\to\infty} p(x_{n+i}, y_{n+i}) \le \epsilon$ and thus successively $\limsup_{n\to\infty} p(x_n, y_n) \le \epsilon$ and $\limsup_{n\to\infty} p(x_n, y_n) = 0$ and the result follows. □

Definition 3. We say that a sequence $\{x_n\}_{n\in\mathbb{N}}$, with $x_n \in \mathbb{X}$ is p-ASF-2 if we have the following property.

(C_4) For each $\epsilon > 0$, there exist $\delta > 0$ and $\nu \in \mathbb{N}$ such that if for $i,j \in \mathbb{N}$ we have $\epsilon < p(x_i, x_j) < \epsilon + \delta$, then $p(x_{\nu+i}, x_{\nu+j}) \le \epsilon$.

Let q be a given function from $\mathbb{X} \times \mathbb{X}$ into $[0, \infty)$ and $p = G \circ q$ where the mapping G is a nondecreasing right continuous function such that $G(t) > 0$ for $t > 0$. We first show here that when a sequence is $(G \circ q)$-ASF-2 then it is also a q-ASF-2 sequence if (C_5) is satisfied by p. Note that Property (C_4) (resp. (C_5)) is a kind of uniform extension of (C_2) (resp. (C_3)) when only one sequence is involved.

Lemma 4 (*[9, in Theorem 6]*)**.** *Let $\{x_n\}_{n\in\mathbb{N}}$ be a p-ASF-2 sequence and suppose that $p = G \circ q$ where G is a nondecreasing right continuous function such that $G(t) > 0$ for $t > 0$. Suppose that we have the following assumption*

(C_5) *for each given (x_i, x_j) such that $p(x_i, x_j) \ne 0$ there exists $\nu \in \mathbb{N}$ such that*

$$p(x_{\nu+i}, x_{\nu+j}) < p(x_i, x_j),$$

then $\{x_n\}_{n\in\mathbb{N}}$ is a q-ASF-2 sequence.

Proof. The proof is contained in [9, Theorem 6]. Fix $\eta > 0$ and consider $\epsilon = G(\eta)$. Since $G(t) > 0$ for $t > 0$ we have $\epsilon > 0$. Then we can use (C_4) to obtain $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\epsilon < p(x_i, x_j) < \epsilon + \delta$, for some $i, j \in \mathbb{N}$ implies $p(x_{\nu+i}, x_{\nu+j}) \le \epsilon$. Since G is nondecreasing right continuous we can find β such that $G([\eta, \eta + \beta]) \subset [\epsilon, \epsilon + \delta)$. Thus suppose that $\eta < q(x_i, x_j) < \eta + \beta$, we then have $\epsilon \le G(q(x_i, x_j)) < \epsilon + \delta$. Since G is nondecreasing it can be constant and equal to ϵ on a nonempty interval $[\eta, \eta + \overline{\beta}) \subset \eta + \beta$; on the contrary we will have $\epsilon < G(\eta + \gamma)$ for $\gamma \in (0, \beta)$. If we are in the second case then $\epsilon < G(q(x_i, x_j)) < \epsilon + \delta$ and using (C_4) we obtain $G(q(x_{i+\nu}, x_{j+\nu})) \le \epsilon < G(\eta + \gamma)$; we thus have $q(x_{i+\nu}, x_{j+\nu}) < \eta + \gamma$ for all $\gamma \in (0, \beta)$ and consequently $q(x_{i+\nu}, x_{j+\nu}) \le \eta$. In the first case we have $G(q(x_i, x_j)) = \epsilon$ for $\eta < q(x_i, x_j) < \eta + \overline{\beta}$. Using ($C_5$) we can find $\nu \in \mathbb{N}$ such that

$$G(q(x_{i+\nu}, x_{j+\nu})) < G(q(x_i, x_j)) = \epsilon = G(\eta) \tag{1}$$

and thus $q(x_{i+\nu}, x_{j+\nu}) \le \eta$. We thus have proved that (C_4) is satisfied by q. □

We prove now that p-ASF-2 sequences mixed with the convergence properties of the sequence $p(x_n, x_{n+1})$ gives p-Cauchy properties. More precisely we have the following lemma.

Lemma 5. *Let $\{x_n\}_{n\in\mathbb{N}}$, be a p-ASF-2 sequence and suppose that p is such that $p(x, y) \le p(x, z) + r(z, y)$ and $p(x, y) \le r(x, z) + p(z, y)$ for all $x, y, z \in \mathbb{X}$ where the mapping $r : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ satisfies the triangle inequality $r(x, y) \le r(x, z) + r(z, y)$ for all $x, y, z \in \mathbb{X}$. If the sequence $\{x_n\}_{n\in\mathbb{N}}$ is such that $\lim_{n\to\infty} r(x_n, x_{n+1}) = 0$ and $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$ then we have $\lim_{n\to\infty} \sup_{m>n} p(x_n, x_m) = 0$.*

Proof. We follow [8, Lemma 2] where a similar proof is given when $r = p$. Let $\epsilon > 0$ be fixed and consider δ and ν given by (C_4). There exists $N \in \mathbb{N}$ such that $r(x_n, x_{n+1}) < \delta/\nu$ and $p(x_n, x_{n+1}) < \delta/\nu$ for all $n \geq N$. We first have for $k \leq \nu$

$$r(x_n, x_{n+k}) \leq \sum_{i=0}^{k-1} r(x_{n+i}, x_{n+i+1}) < k\frac{\delta}{\nu} \leq \delta \tag{2}$$

and

$$p(x_n, x_{n+k}) \leq p(x_n, x_{n+1}) + \sum_{i=1}^{k-1} r(x_{n+i}, x_{n+i+1}) < k\frac{\delta}{\nu} \leq \delta. \tag{3}$$

We suppose that $p(x_n, x_{n+\alpha}) < \epsilon + \delta$ is satisfied for $\alpha \in [1, k]$ and we want to prove that the same inequalities are satisfied for $\alpha \in [1, k+1]$. Using (3) we may assume that $k \geq \nu$. Using the mixed triangle inequality satisfied by p we have the two separate inequalities

$$\begin{aligned} p(x_n, x_{n+k+1}) &\leq p(x_n, x_{n+k+1-\nu}) + \sum_{i=1-\nu}^{0} r(x_{n+k+i}, x_{n+k+i+1}) \\ &< p(x_n, x_{n+k+1-\nu}) + \delta \end{aligned} \tag{4}$$

and

$$p(x_n, x_{n+k+1}) \leq r(x_n, x_{n+\nu}) + p(x_{n+\nu}, x_{n+k+1-\nu+\nu}). \tag{5}$$

By hypothesis we have $p(x_n, x_{n+k+1-\nu}) < \epsilon + \delta$. If $p(x_n, x_{n+k+1-\nu}) \leq \epsilon$ then using (4) we obtain $p(x_n, x_{n+k+1}) < \epsilon + \delta$ else we can use (C_4) to first get $p(x_{n+\nu}, x_{n+k+1-\nu+\nu}) \leq \epsilon$ and using (2) and (5) we obtain $p(x_n, x_{n+k+1}) < \epsilon + \delta$. □

In [6], T. Suzuki introduces the definition of a τ-distance. We just recall here two properties which are satisfied by τ-distance: if a function p from $\mathbb{X} \times \mathbb{X}$ into $\mathbb{R}^+$ is a τ-distance it satisfies $p(x, y) \leq p(x, z) + p(z, y)$ for all $x, y, z \in \mathbb{X}$ and if a sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}$ satisfies $\lim_{n\to\infty} \sup_{m>n} p(x_n, x_m) = 0$ then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. We thus have the following theorem.

Theorem 6. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a p-ASF-2 sequence in $\mathbb{X}$ such that $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are p-ASF-1 for $y_n = x_{n+1}$ for all $n \in \mathbb{N}$. If one of the following assumptions holds true*

(i) *$p = q$ and q is a τ-distance;*
(ii) *$p = G(q)$ where q is a τ-distance and where G is a nondecreasing right continuous function such that $G(t) > 0$ for $t > 0$ and (C_5) is satisfied by the sequence $\{x_n\}_{n\in\mathbb{N}}$ (for the mapping $p = G(q)$);*
(iii) *p is a τ-distance such that $p(x, y) \leq p(x, z) + q(z, y)$ and $p(x, y) \leq q(x, z) + p(z, y)$ for all $x, y, z \in \mathbb{X}$ where the mapping $q : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ satisfies the triangle inequality $q(x, y) \leq q(x, z) + q(z, y)$ for all $x, y, z \in \mathbb{X}$ and $\lim_{n\to\infty} q(x_n, x_{n+1}) = 0$;*

then, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Proof. First note that, in the three cases, using Lemma 2 we have that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0.$$

(i) We consider the case $p = q$. Since $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$ we can use Lemma 5 (with $r = p$) to obtain $\lim_{n\to\infty} \sup_{m>n} p(x_n, x_m) = 0$ and since p is a τ-distance we obtain the fact that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence [6, Lemma 1].

(ii) Suppose now that $p = G(q)$; we have $\lim_{n\to\infty} G(q(x_n, x_{n+1})) = 0$. This is only possible if $G(0) = 0$ and we thus also obtain that $\lim_{n\to\infty} q(x_n, x_{n+1}) = 0$. Using Lemma 4 we obtain that $\{x_n\}_{n\in\mathbb{N}}$ is q-ASF-2 and we conclude as in part (i) using now the τ-distance q.

(iii) Here we can use Lemma 5 to obtain $\lim_{n\to\infty} \sup_{m>n} p(x_n, x_m) = 0$ and using the fact that p is a τ-distance the conclusion follows the lines of case (i). □

Remark 7. Note that we have proved during the proof of Theorem 6 that if we have two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ which are p-ASF-1 with $p = G \circ q$ then $G(0) = 0$.

3. Links with ACF sequences

We first recall here the definition of an ACF mapping. Then we give a definition of a T-ASF mapping by defining properties which are to be satisfied by the sequences $\{T^n x\}$ for $x \in \mathbb{X}$. We prove in Theorem 9 that the two definitions are equivalent.

Definition 8 ([8, Definition 1]). Let $(\mathbb{X}, d)$ be a metric space. Then a mapping T on $\mathbb{X}$ is said to be an asymptotic contraction of the final type (ACF, for short) if the following hold:

(D_1) $\lim_{\delta\to 0^+} \sup\{\limsup_{n\to\infty} d(T^n x, T^n y) : d(x, y) < \delta\} = 0$.
(D_2) For each $\epsilon > 0$, there exists $\delta > 0$ such that for $x, y \in \mathbb{X}$ with $\epsilon < d(x, y) < \epsilon + \delta$, there exists $\nu \in N$ such that $d(T^\nu x, T^\nu y) \le \epsilon$.
(D_3) For $x, y \in \mathbb{X}$ with $x \ne y$, there exists $\nu \in N$ such that $d(T^\nu x, T^\nu y) < d(x, y)$.
(D_4) For $x \in \mathbb{X}$ and $\epsilon > 0$, there exist $\delta > 0$ and $\nu \in N$ such that

$$\epsilon < d(T^i x, T^j x) < \epsilon + \delta \text{ implies } d(T^\nu \circ T^i x, T^\nu \circ T^j x) \le \epsilon \tag{6}$$

for all $i, j \in N$.

Theorem 9. *Let $(\mathbb{X}, d)$ be a metric space. A mapping T on $\mathbb{X}$ is said to be p-ASF if for all $x, y \in \mathbb{X}$ the sequences $\{T^n x\}$ and $\{T^n y\}$ are p-ASF-1 and $\{T^n x\}$ is p-ASF-2. Then, T is an ACF mapping is equivalent to T is a d-ASF mapping.*

Proof. Suppose that the mapping T is an *ACF*. For each $x, y \in \mathbb{X}$ it is very easy to check and left to the reader that $\{T^n x\}$ and $\{T^n y\}$ are d-ASF-1 and $\{T^n x\}$ is d-ASF-2. Thus, T is d-ASF.

If T is d-ASF, using Lemma 2 we obtain $\lim_{n\to\infty} d(T^n x, T^n y) = 0$. If we consider the special case $y = Tx$ and the sequence $x_n = T^n x$ we obtain using Theorem 6 that $\{T^n x\}$ is a Cauchy sequence. Then using [8, Theorem 6][1] we obtain that the mapping T is an ACF. □

The existence and uniqueness of fixed points of p-ASF mappings is now obtained. Note that, in the special case where the mapping p is equal to d (i.e. when we use the τ-distance $p = d$ in (i)) the next theorem gives the same results as [8, Theorem 5].

Theorem 10. *Let $(\mathbb{X}, d)$ be a complete metric space, T be a p-ASF mapping which is such that T^l is continuous for some $l \in \mathbb{N}$ $(l > 0)$. We suppose that the function q is a τ-distance and one of the following holds true for the mapping p:*

(i) $p = q$.
(ii) $p = G(q)$ where G is a nondecreasing right continuous function such that $G(t) > 0$ for $t > 0$ and (C_5) is satisfied by the sequence $\{x_n\}_{n\in\mathbb{N}}$ (for the mapping $p = G(q)$)

then, there exists a fixed point $z \in \mathbb{X}$ of T. Moreover, if for every sequence $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ $\lim_{n\to\infty} p(x_n, y_n) = 0$ implies that $\lim_{n\to\infty} d(x_n, y_n) = 0$ then the fixed point is unique and $\lim_{n\to\infty} T^n x = z$ holds true for every $x \in \mathbb{X}$.

Proof. For every $x \in \mathbb{X}$ using Theorem 6 we know that $\{T^n x\}$ is a Cauchy sequence. By Lemma 2 we know that $\lim_{n\to\infty} p(T^n x, T^n y) = 0$. We then have all the ingredients of [8, Theorem 4 and Lemma 3] to conclude the proof. □

4. An application to *ACF* cyclic mappings

We suppose here that $\mathbb{X}$ is a uniformly convex Banach space and thus $d(x, y) \stackrel{\text{def}}{=} \|x-y\|$. We consider A and B two nonempty subsets of $\mathbb{X}$, A being convex, and a cyclic mapping $T : A \cup B \to A \cup B$. We recall that T is a cyclic mapping if $T(A) \subset B$ and $T(B) \subset A$. We define a mapping $p : \mathbb{X} \times \mathbb{X} \to \mathbb{R}^+$ by $p(x, y) \stackrel{\text{def}}{=} d(x, y) - d(A, B)$ where $d(A, B) \stackrel{\text{def}}{=} \inf\{d(x, y) \mid x \in A, y \in B\}$.

Then, using the previous results we can give a short proof of a theorem which extends [2, Theorem 1].

Theorem 11. *Suppose that the mapping T is p-ASF, then the sequence $\{T^{2n} x\}$ for $x \in A$ is a d-Cauchy sequence.*

Proof. For a given $x \in \mathbb{X}$, we consider the sequence $x_n = T^n x$. Since T is p-ASF we have by Lemma 2 that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Using the definition of p we immediately also have $\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B)$. Using Lemma [2, Lemma 4] we obtain that $\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = 0$ (convexity of A and uniform convexity of $\mathbb{X}$ are used here). We now consider the sequence $\{T^{2n} x\}$ taking values in A. We have $\lim_{n\to\infty} p(x_{2n}, x_{2n+2}) = 0$ and as it was already shown $\lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = 0$. If the sequence $x_n = T^n x$ is p-ASF-2 then it is the same for the sequence $\{T^{2n} x\}$. The distance d satisfies the triangle inequality and it is straightforward to see that we have the two mixed triangle inequalities $p(x, y) \le p(x, z) + d(z, y)$ and $p(x, y) \le d(x, y) + p(z, y)$ for all $x, y, z \in \mathbb{X}$. We can thus apply Lemma 5 to the sequence $\{T^{2n} x\}$ with $x \in A$ to obtain that it is a p-Cauchy sequence. It is now easy to see by contradiction that a p-Cauchy sequence is a d-Cauchy sequence [2, Proof of Theorem 2]. The key argument being again the use of [2, Lemma 4]. □

[1] We first recall from [8, Theorem 6] that for a mapping T on a metric space $(\mathbb{X}, d)$ the following are equivalent:

(i) T is an ACF.
(ii) $\lim_{n\to\infty} d(T^n x, T^n y) = 0$ holds true and $\{T^n x\}$ is a Cauchy sequence for all $x, y \in \mathbb{X}$.

We extend now [2, Theorem 2] which was stated for continuous cyclic Meir–Keeler contractions to continuous p-ASF mappings.

Theorem 12. *Suppose in addition that A is closed, T is p-ASF and T^l is continuous for some $l \in \mathbb{N}$ ($l > 0$) then there exists a unique best proximity point $z \in A$ (i.e. $d(z, Tz) = d(A, B)$). Moreover $\lim_{n\to\infty} T^{2n}x = z$ for each $x \in A$.*

Proof. Using Theorem 11, the sequence $\{T^{2n}x\}$ for each $x \in A$ is a d-Cauchy sequence. Using Lemma 2, we have $\lim_{n\to\infty} p(T^n x, T^n y) = 0$ for each $x, y \in A$; hence for (x, Tx) it gives $\lim_{n\to\infty} p(T^{2n}x, T^{2n+1}x) = 0$ and for (Tx, y) it gives $\lim_{n\to\infty} p(T^{2n+1}x, T^{2n}y) = 0$. Using again [2, Lemma 4] we obtain $\lim d(T^{2n}x, T^{2n}y) = 0$ and we can use [8, Theorem 4 and Lemma 3] to conclude the proof. □

5. ASMK sequences

We introduce in this section the definition of ASMK sequences. It is an adaptation to sequences of the ACMK (asymptotic contraction of Meir–Keeler type) definition used for mappings [7]. It is proved in [8, Theorem 3] that an ACMK mapping on a metric space is an ACF mapping. We will prove in this section similar results which relate ASMK sequences to ASF sequences. These results will be used in the next section for studying sequences of alternating mappings.

Definition 13. We say that two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ with $x_n, y_n \in \mathbb{X}$ are p-ASMK-1 if there exists a sequence $\{\psi_n\}$ of functions from $[0, \infty)$ into itself satisfying $\psi_n(0) = 0$[2] for all $n \in \mathbb{N}$ and the following:

(C_6) $\limsup_n \psi_n(\epsilon) < \epsilon$ for all $\epsilon > 0$.
(C_7) For each $\epsilon > 0$, there exists $\delta > 0$ such that for each $t \in [\epsilon, \epsilon + \delta]$ there exists $\nu \in \mathbb{N}$ such that $\psi_\nu(t) < \epsilon$.
(C_8) $F(p(x_{n+i}, y_{n+i})) \le \psi_n(F(p(x_i, y_i)))$ for all $n, i \in \mathbb{N}$. F is a given right continuous nondecreasing mapping such that $F(t) > 0$ for $t \ne 0$.

Lemma 14. *Suppose that the two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ are p-ASMK-1 then they are p-ASF-1.*

Proof. (C_1): For all $n, i \in \mathbb{N}$ we have by (C_8) and (C_6) when $F(p(x_i, y_i)) \ne 0$ that

$$\begin{aligned} F(p(x_{n+i}, y_{n+i})) &\le \psi_n(F(p(x_i, y_i))) \\ &\le \limsup_{n\to\infty} \psi_n(F(p(x_i, y_i))) \\ &< F(p(x_i, y_i)). \end{aligned}$$

Since F is nondecreasing and the inequality is strict we obtain for all $n \in \mathbb{N}$

$$p(x_{n+i}, y_{n+i}) < p(x_i, y_i),$$

and thus

$$\limsup_{n\to\infty} p(x_{n+i}, y_{n+i}) \le p(x_i, y_i).$$

Then (C_1) follows easily when $F(p(x_i, y_i)) \ne 0$. When $F(p(x_i, y_i)) = 0$, we have by (C_8) $F(p(x_{n+i}, y_{n+i})) \le 0$ for all $n \in \mathbb{N}$. Since F is a right continuous mapping such that $F(t) > 0$ we must have $F(0) \ge 0$. Thus we have that $F(p(x_{n+i}, y_{n+i})) = 0$ for all $n \in \mathbb{N}$ and thus $p(x_{n+i}, y_{n+i}) = 0$ for all $n \in \mathbb{N}$ and the same conclusion holds. (C_2): for $\epsilon > 0$ we know that $F(\epsilon) > 0$ and we can use (C_7) to find $\delta > 0$ such that for each $t \in [F(\epsilon), F(\epsilon) + \delta]$ we can find $\nu \in \mathbb{N}$ such that $\psi_\nu(t) < F(\epsilon)$. Since F is right continuous and nondecreasing we can find δ' such that $F([\epsilon, \epsilon + \delta']) \subset [F(\epsilon), F(\epsilon) + \delta]$. Thus, taking $i \in \mathbb{N}$ such that $\epsilon < p(x_i, y_i) < \epsilon + \delta'$, we can find ν such that $\psi_\nu(F(p(x_i, y_i))) < F(\epsilon)$. And we conclude using (C_8) that

$$F(p(x_{\nu+i}, y_{\nu+i})) \le \psi_\nu(F(p(x_i, y_i))) < F(\epsilon) \le F(p(x_i, y_i)). \tag{7}$$

Thus we have $F(p(x_{\nu+i}, y_{\nu+i})) < F(p(x_i, y_i))$ and since F is nondecreasing and the inequality is strict we obtain (C_2).

(C_3): Let i be given such that $p(x_i, y_i) \ne 0$ and start as in the previous paragraph using $\epsilon = p(x_i, y_i)$. We can find $\nu \in \mathbb{N}$ such that $\psi_\nu(F(p(x_i, y_i))) < F(\epsilon)$ which combined with (C_8) gives

$$F(p(x_{\nu+i}, y_{\nu+i})) \le \psi_\nu(F(p(x_i, y_i))) < F(\epsilon) = F(p(x_i, y_i)). \tag{8}$$

Since F is nondecreasing and the inequality is strict the result follows. □

Definition 15. We say that two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ with $x_n, y_n \in \mathbb{X}$ are p-ASMK-2 when (C_8) is replaced by

(C_9) $F\left(p(x_{n+i}, y_{n+j})\right) \le \psi_n\left(F\left(p(x_i, y_j)\right)\right)$ for all $n, p, i, j \in \mathbb{N}$.

[2] Note that this assumption can be removed when $F(0) > 0$.

Corollary 16. *If two sequences* $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ *with* $y_n = x_{n+1}$ *are p-ASMK-2 then they are p-ASF-1 and the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is p-ASF-2. Moreover, assumption* (C_5) *holds true for p.*

Proof. It is obvious to see that if two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ are p-ASMK-2 then they are p-ASMK-1. Thus by Lemma 14 they are p-ASF-1. Proving that (C_4) holds true is similar to the proof that (C_2) holds true in Lemma 14 and proving that (C_5) holds true follows the same steps as the proof that (C_3) holds true in Lemma 14. □

6. A sequence of alternating mappings

In this section p is a given function from $\mathbb{X}\times\mathbb{X}$ into $[0,\infty)$ such that $p(x,y)\le p(x,z)+p(z,y)$ for all $x,y,z\in\mathbb{X}$ and $p(x,y)=p(y,x)$ for all $x,y\in\mathbb{X}$.

Definition 17. We will say that the pair (T,S) satisfies the (F,ψ)-contraction property if we can find two functions F and ψ such that

$$F(p(Tx,Sy))\le\psi(F(M(x,y))) \tag{9}$$

where

$$M(x,y)\overset{\text{def}}{=}\max\left\{p(x,y),p(Tx,x),p(Sy,y),\frac{1}{2}\{p(Tx,y)+p(Sy,x)\}\right\}. \tag{10}$$

The function $F:\mathbb{R}^+\to\mathbb{R}^+$ is a given right continuous nondecreasing mapping such that $F(t)>0$ for $t\neq 0$. The function $\psi:\mathbb{R}^+\to\mathbb{R}^+$ is a given nondecreasing upper semicontinuous function satisfying $\psi(t)<t$ for each $t>0$ and $\psi(0)=0$.

We first start with a technical lemma.

Lemma 18. *Let the pair of mappings* (T,S) *be an* (F,ψ)*-contraction. Suppose that* $x=S\alpha$ *and* $p(x,Tx)\neq 0$ *then we have*

$$F(p(x,Tx))\le\psi(F(p(S\alpha,\alpha))). \tag{11}$$

Suppose that $y=T\alpha$ *and* $p(y,Sy)\neq 0$ *then we have*

$$F(p(Sy,y))\le\psi(F(p(\alpha,T\alpha))). \tag{12}$$

Proof. We prove the first inequality (11). We suppose that $x=S\alpha$ then we have

$$F(p(x,Tx))=F(p(Tx,x))=F(p(Tx,S\alpha))\le\psi(F(M(x,\alpha)))$$

and we have

$$\begin{aligned}M(x,\alpha)&=\max\left\{p(x,\alpha),p(Tx,x),p(S\alpha,\alpha),\frac{1}{2}\{p(Tx,\alpha)+p(S\alpha,x)\}\right\}\\&=\max\left\{p(x,\alpha),p(Tx,x),\frac{1}{2}p(Tx,\alpha)\right\}\\&=\max\{p(x,\alpha),p(Tx,x)\}.\end{aligned} \tag{13}$$

We show now that the maximum cannot be achieved by $p(Tx,x)$. Indeed, suppose that $M(x,\alpha)=p(Tx,x)$ then we would have

$$F(p(x,Tx))\le\psi(F(p(x,Tx)))$$

which is not possible since $p(x,Tx)\neq 0$ and there does not exist $x>0$ such that $F(x)\le\psi(F(x))$ (since for $x>0$ we have that $F(x)\le\psi(F(x))<F(x)$). The proof for the second inequality is very similar and thus omitted. □

We now introduce the alternating sequence of mappings $\{\Gamma_n\}_{n\in\mathbb{N}}$ defined by

$$\Gamma_n\overset{\text{def}}{=}\begin{cases}T, & \text{if } n \text{ is even}\\ S, & \text{if } n \text{ is odd.}\end{cases} \tag{14}$$

Then, we consider the two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ defined by

$$x_{n+1}=\Gamma_n x_n \quad\text{and}\quad y_{n+1}=\Gamma_{n+1}y_n. \tag{15}$$

It is very easy to check that when the two sequences are initiated with $(x_0,y_0)=(Sx,x)$ for a given $x\in\mathbb{X}$ they are related by $y_{n+1}=x_n$ and that only the two following cases can occur:

$$(x_{n+1},y_{n+1})=\begin{cases}(Sx_n,x_n) & \text{with } x_n=Tx_{n-1} \text{ and } y_n=x_{n-1}\\ (Tx_n,x_n) & \text{with } x_n=Sx_{n-1} \text{ and } y_n=x_{n-1}.\end{cases} \tag{16}$$

If we are in the first (resp. the second) case we use (12) (resp. (11)) to obtain the inequality

$$F(p(x_{n+1}, y_{n+1})) \leq \psi(F(p(x_n, y_n))). \tag{17}$$

We thus have the following easy lemma.

Lemma 19. *Let the pair of mappings (T, S) be an (F, ψ)-contraction and $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences defined by (15). If the two sequences are initiated by $(x_0, y_0) = (Sx, x)$ we have $y_{n+1} = x_n$ and*

$$F\left(p(x_{n+1}, y_{n+1})\right) \leq \psi\left(F\left(p(x_n, y_n)\right)\right). \tag{18}$$

If $x_{n+1} = Tx_n$ (resp. $x_{n+1} = Sx_n$) and $y_{k+1} = Sy_k$ (resp. $y_{k+1} = Ty_k$) we have

$$F\left(p(x_{n+1}, y_{k+1})\right) \leq \psi\left(F\left(p(x_n, y_k) + \max(p(x_n, x_{n+1}), p(x_k, x_{k+1}))\right)\right). \tag{19}$$

Proof. Since inequation (18) was proved by (17), it just remains to prove inequality (19). Suppose that $x_{n+1} = Tx_n$ and $y_{k+1} = Sy_k$ then we have

$$\begin{aligned} M(x_n, y_k) &= \max\left\{p(x_n, y_k), p(Tx_n, x_n), p(Sy_k, y_k), \frac{1}{2}\{p(Tx_n, y_k) + p(Sy_k, x_n)\}\right\} \\ &= \max\left\{p(x_n, y_k), p(x_{n+1}, x_n), p(y_{k+1}, y_k), \frac{1}{2}\{p(x_{n+1}, y_k) + p(y_{k+1}, x_n)\}\right\} \\ &\leq \max\{p(x_n, y_k), p(x_{n+1}, x_n), p(y_{k+1}, y_k), p(x_n, y_k) + \max(p(x_{n+1}, x_n), p(y_{k+1}, y_k))\} \\ &\leq p(x_n, y_k) + \max(p(x_{n+1}, x_n), p(y_{k+1}, y_k)). \end{aligned} \tag{20}$$

We thus have

$$\begin{aligned} F\left(p(x_{n+1}, y_{k+1})\right) &\leq \psi\left(F\left(M(x_n, y_k)\right)\right) \\ &\leq \psi\left(F\left(p(x_n, y_k) + \max(p(x_{n+1}, x_n), p(y_{k+1}, y_k))\right)\right). \end{aligned} \tag{21}$$

If the opposite situation is $x_{n+1} = Sx_n$ and $y_{k+1} = Ty_k$ we obtain the same result by the same arguments. □

We make here a direct proof of the fact that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence when $\lim_{n\to\infty} p(x_{n+1}, x_n) = 0$ is assumed. This last property will be derived from p-ASMK-1 properties as proved in Theorem 22.

Lemma 20. *Let the pair of mappings (T, S) be an (F, ψ)-contraction. Suppose that*

$$\lim_{n\to\infty} p(x_{n+1}, x_n) = 0,$$

then the sequence $\{x_n\}_{n\in\mathbb{N}}$ given by (15) is a Cauchy sequence.

Proof. We follow here [10] to prove the result by contradiction.

If the sequence is not a Cauchy sequence we can find two subsequences $\sigma(n)$ and $\rho(n)$ such that for all $n \in \mathbb{N}$ $p(x_{\sigma(n)}, x_{\rho(n)}) \geq 2\epsilon$ and $\sigma(n) < \rho(n)$. Since the sequence $\{p(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ converges to zero we can choose N such that $p(x_n, x_{n+1}) < \epsilon$ for all $n \geq N$. Using the triangle inequality

$$p(x_{\sigma(n)}, x_{\rho(n)+1}) \geq p(x_{\sigma(n)}, x_{\rho(n)}) - p(x_{\rho(n)+1}, x_{\rho(n)})$$

we obtain that $p(x_{\sigma(n)}, x_{\rho(n)+1}) > \epsilon$ for large n. Thus, we can always change the subsequence $\rho(n)$ in such a way that the parity between $\sigma(n)$ and $\rho(n)$ conforms to the one we need for applying inequality (21) and such that for all $n \in \mathbb{N}$ $p(x_{\sigma(n)}, x_{\rho(n)}) > \epsilon$.

We now define $k(n)$ as follows:

$$k(n) \stackrel{\text{def}}{=} \min\left\{k > \sigma(n) \mid p(x_{\sigma(n)}, x_k) > \epsilon \text{ with same parity as } \rho(n)\right\}. \tag{22}$$

$k(n)$ is well defined and by construction $\sigma(n) < k(n) \leq \rho(n)$. We now have that

$$\epsilon < p(x_{\sigma(n)}, x_{k(n)}) \leq p(x_{\sigma(n)}, x_{k(n)-2}) + p(x_{k(n)-2}, x_{k(n)}) \leq \epsilon + p(x_{k(n)-2}, x_{k(n)}). \tag{23}$$

The sequence $\{p(x_{k(n)-2}, x_{k(n)})\}_{n\in\mathbb{N}}$ converges to zero since we have

$$p(x_{k(n)-2}, x_{k(n)}) \leq p(x_{k(n)-2}, x_{k(n)-1}) + p(x_{k(n)-1}, x_{k(n)})$$

and thus $p(x_{\sigma(n)}, x_{k(n)}) \to \epsilon^+$ when n goes to infinity. We also obtain that $p(x_{\sigma(n)-1}, x_{k(n)-1}) \to \epsilon$ when n goes to infinity since

$$|p(x_{\sigma(n)}, x_{k(n)}) - p(x_{\sigma(n)-1}, x_{k(n)-1})| \leq p(x_{k(n)}, x_{k(n)-1}) + p(x_{\sigma(n)}, x_{\sigma(n)-1}). \tag{24}$$

We now use inequality (21) to obtain

$$F\left(p(x_{\sigma(n)}, x_{k(n)})\right) \leq \psi\left(F\left(p(x_{\sigma(n)-1}, x_{k(n)-1}) + \delta_n\right)\right) \tag{25}$$

where $\delta_n \stackrel{\text{def}}{=} \max\left(p(x_{\sigma(n)-1}, x_{\sigma(n)}), p(x_{k(n)-1}, x_{k(n)})\right)$. When n goes to infinity, using the facts that F is right continuous and nondecreasing and $\psi \circ F$ is upper semicontinuous we obtain that $F(\epsilon) \leq \psi\left(F(\epsilon)\right)$ which is a contradiction. □

Remark 21. The proof remains valid is we assume as in [10] that the function F is nondecreasing and continuous with $F(0) = 0$ and $F(t) > 0$ for $t > 0$ and that the function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is assumed to be nondecreasing and right upper semicontinuous and satisfies $\psi(t) < t$ for each $t > 0$. The idea is to build the sequences choosing the parity so as to use (21) in the reverse situation where

$$F\left(p(x_{\sigma(n)+1}, x_{k(n)+1})\right) \leq \psi\left(F\left(p(x_{\sigma(n)}, x_{k(n)}) + \delta'_n\right)\right),$$

and where $\delta'_n \stackrel{\text{def}}{=} \max\left(p(x_{\sigma(n)+1}, x_{\sigma(n)}), p(x_{k(n)+1}, x_{k(n)})\right)$.

Theorem 22. *Consider two mappings $T : \mathbb{X} \to \mathbb{X}$ and $S : \mathbb{X} \to \mathbb{X}$ and suppose that the pair (T, S) has the (F, ψ)-contraction property. Let the sequence of function $\{\psi_n\}_{n\in\mathbb{N}}$ be defined by $\psi_n \stackrel{\text{def}}{=} \overbrace{\psi \circ \psi \circ \cdots \psi}^{n}$ and assume that (C_6) and (C_7) are satisfied, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by (15) and initialized by $x_0 = Sx$ is a Cauchy sequence.*

Proof. The only point to prove is that assumption (C_8) is satisfied. We consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ and the sequence $\{y_n\}_{n\in\mathbb{N}}$ defined by (15) and initialized by $y_0 = x$. Using the fact that ψ is nondecreasing, we repeatedly use Eq. (18) in Lemma 19 to obtain assumption (C_8) and conclude that the two sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are p-ASMK-1 and then by Lemmas 14 and 2 we obtain that $\limsup_{n\to\infty} p(x_n, x_{n+1}) = 0$. Using Lemma 20 we conclude that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. □

We make a link here with the result of [10] where it is assumed that $F(0) = 0$ and $F(t) > 0$ for $t > 0$ and F is supposed to be nondecreasing and continuous. The function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is assumed to be nondecreasing and right upper semicontinuous and satisfies $\psi(t) < t$ for each $t > 0$ and $\lim_{n\to\infty} \psi_n(t) = 0$. It is proved in [10] that $F(x) \leq \psi(F(x))$ implies $x = 0$. We prove in the next lemma that these properties of functions F and ψ imply properties (C_6) and (C_7).

Lemma 23. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing, right upper semicontinuous function satisfying $\psi(t) < t$ for each $t > 0$. Then the sequence of functions $\{\psi_n\}_{n\in\mathbb{N}}$ defined by $\psi_n \stackrel{\text{def}}{=} \overbrace{\psi \circ \psi \circ \cdots \psi}^{n}$ satisfies (C_6) and (C_7).*

Proof. (C_6): for $t > 0$, since ψ is nondecreasing and $\psi(t) < t$ we have $\psi_n(t) \leq \psi(t) < t$ and thus (C_6) follows. (C_7): Using [3, Theorem 2] we can find a right continuous function $\overline{\psi} : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(t) \leq \overline{\psi}(t) < t$ for $t > 0$. Thus we easily have (C_7), since proving (C_7) (using $\nu = 1$) for a right continuous function is easy. □

In [10] it is proved that T and S have a common fixed point when $\mathbb{X}$ is a complete metric space and $p = d$. The proof follows the following steps. Since $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence it converges to $\bar{x} \in \mathbb{X}$. Using the definition of M one easily checks that $M(x_{2n}, \bar{x}) \to d(S\bar{x}, \bar{x})$ and $M(x_{2n}, \bar{x}) \geq d(S\bar{x}, \bar{x})$. Moreover $Tx_{2n} = x_{2n+1}$ also converges to $\bar{x}$. We therefore have

$$F\left(d(Tx_{2n}, S\bar{x})\right) \leq \psi\left(F\left(M(x_{2n}, \bar{x})\right)\right). \tag{26}$$

Using next Lemma 24 we obtain that $\bar{x} = S\bar{x}$. Then proving that $\bar{x}$ is also a fixed point of T is given in [10, Theorem 1]. We therefore conclude that in order to obtain convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ to the unique fixed point of T and S requires us to add continuity of F in the hypothesis of Theorem 22.

Lemma 24. *Suppose that F is a continuous nondecreasing function, ψ is a right upper semicontinuous function satisfying one of the following properties:*

(E_1) *$\psi(t) < t$ for all $t > 0$;*
(E_2) *ψ is nondecreasing and for each $t > 0$, there exists $\nu \in \mathbb{N}$, $\nu \geq 1$ such that $\psi_\nu(t) < t$.*

Suppose that we have two sequences $\{\alpha_n\}_{n\in\mathbb{N}}$ and $\{\beta_n\}_{n\in\mathbb{N}}$ such that

$$F(\alpha_n) \leq \psi\left(F(\beta_n)\right). \tag{27}$$

If $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \gamma$ and $\beta_n \geq \gamma$ for all $n \in \mathbb{N}$ then we must have $\gamma = 0$.

Proof. We have

$$F(\gamma) = \lim_{n\to\infty} F(\alpha_n) \leq \limsup_{n\to\infty} \psi\left(F(\beta_n)\right) \leq \psi\left(\limsup_{n\to\infty} F(\beta_n)\right) \leq \psi\left(F(\gamma)\right). \tag{28}$$

If $\gamma \neq 0$ and $\psi(\gamma) < \gamma$ we conclude that $F(\gamma) < F(\gamma)$ which is a contradiction. If ψ is nondecreasing we consider the value of ν associated with γ to obtain $F(\gamma) \leq \psi_\nu\left(F(\gamma)\right) < F(\gamma)$ and conclude again by contradiction. □

Remark 25. Note that using [3, Theorem 2] we obtain that a right upper semicontinuous function ψ satisfying $\psi(t) < t$ for all $t > 0$ satisfies Property (C_7) with $\nu = 1$.

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