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On a rate of convergence of truncated hypersingular integrals associated to Riesz and Bessel potentials☆,☆☆

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ABSTRACT

The notion of a μ-smooth point of a L_p-function φ and a family of truncated hypersingular integrals depending on a parameter ε are introduced. Then the connection between the order of μ-smoothness of the function φ and the rate of convergence of the family of truncated hypersingular integrals to φ, when ε tends to 0, is obtained.

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1. Introduction

The classical Riesz and Bessel potentials have been proved to be powerful tools in Harmonic Analysis and its various application. These potentials are defined in terms of the Fourier transform by

$$(I^\alpha\varphi)^\wedge(x) = |x|^{-\alpha}\varphi^\wedge(x), \quad x\in\mathbb{R}^n, 0<\alpha<n;$$
$$(\mathfrak{J}^\alpha\varphi)^\wedge(x) = (1+|x|^2)^{-\alpha/2}\varphi^\wedge(x), \quad x\in\mathbb{R}^n, 0<\alpha<\infty. \tag{1}$$

Here, φ is a Schwartz function which is orthogonal to all polynomials (the class of these functions is dense in $L_p(\mathbb{R}^n)$ spaces for $1<p<\infty$, see [15]). According to the formulas (1), the potential operators I^α and $\mathfrak{J}^\alpha$, are interpreted as negative (fractional) powers of differential operators $(-\Delta) = -\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ and $(E-\Delta)$, respectively. One of the important trends in potential theory is finding inversion formulas for the potential operators. Hypersingular integral theory has been appeared as a result of investigation of the subject by E. Stein [19], P. Lizorkin [9], R. Wheeden [20], M. Fisher [8], S. Samko [16,15,17], V. Nogin and S. Samko [10], B. Rubin [11–14] and many other mathematicians (see references in [15,17,13] for more detailed information). The wavelet-approach to inversion of the potentials was developed by B. Rubin [13,14], I.A. Aliev, and B. Rubin [4], and I.A. Aliev [1]; see also [5,3,2,18].

In the paper [12] (see also, [13, pp. 222–224]) B. Rubin introduced some family of "truncated" integrals $D_\varepsilon^\alpha f$ and $\mathfrak{D}_\varepsilon^\alpha f$, $(\varepsilon>0)$, generated by the Gauss–Weierstrass semigroup, and proved that under some conditions on function $\varphi\in L_p(\mathbb{R}^n)$ and parameter $\alpha>0$, the expressions $D_\varepsilon^\alpha I^\alpha\varphi$, and $\mathfrak{D}_\varepsilon^\alpha\mathfrak{J}^\alpha\varphi$ converge to φ as $\varepsilon\to 0^+$, pointwise (a.e.) and in the L_p-norm.

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Naturally, the following question arises: Let a function $\varphi \in L_p(\mathbb{R}^n)$ has some smoothness properties at a point $x^0 \in \mathbb{R}^n$. What connection takes place between the "order of smoothness" of a function φ and the "rate of convergence" of the family $D_\varepsilon^\alpha I^\alpha \varphi$, and $\mathfrak{D}_\varepsilon^\alpha \mathfrak{J}^\alpha \varphi$ to $\varphi(x^0)$ as $\varepsilon \to 0$? The aim of this paper is to answer this question as far as possible.

2. Auxiliary definitions and lemmas

The well-known Gauss–Weierstrass semigroup, generated by a function $f(x)$, $x \in \mathbb{R}^n$, is defined by

$$(\mathrm{U}f)(x,t) = \int_{\mathbb{R}^n} W(y,t) f(x-y)\,dy, \quad (t>0), \tag{2}$$

where $W(y,t)$ is the following Gauss–Weierstrass kernel:

$$W(y,t) = (4\pi t)^{-n/2} e^{-|y|^2/4t}, \quad (t>0). \tag{3}$$

For the various properties of $(\mathrm{U}f)(.,t)$, see, for instance, [12] and [13, p. 223]. The modified Gauss–Weierstrass semigroup, $\mathrm{U}_M f$, is defined as

$$(\mathrm{U}_M f)(x,t) = e^{-t}(\mathrm{U}f)(x,t), \quad (t>0, x \in \mathbb{R}^n). \tag{4}$$

For $t=0$ we set $(\mathrm{U}f)(x,0) = (\mathrm{U}_M f)(x,0) = f(x)$.

The finite difference of the function $g(t)$, $(t \in \mathbb{R})$ with order $l \in \mathbb{N}$ and step τ is defined by

$$\Delta_\tau^l[g](t) = \sum_{k=0}^{l} \binom{l}{k} (-1)^k g(t+k\tau). \tag{5}$$

By making use of $(\mathrm{U}f)(x,t)$, $(\mathrm{U}_M f)(x,t)$ and $\Delta_\tau^l[g](t)$, we introduce the following "truncated" integrals (cf. [13, p. 224 and p. 262]):

$$D_\varepsilon^\alpha f(x) = \frac{1}{\chi(\frac{\alpha}{2};l)} \int_\varepsilon^\infty \Delta_\tau^l[(\mathrm{U}f)(x,\cdot)](0) \frac{d\tau}{\tau^{1+\frac{\alpha}{2}}}; \tag{6}$$

$$\mathfrak{D}_\varepsilon^\alpha f(x) = \frac{1}{\chi(\frac{\alpha}{2};l)} \int_\varepsilon^\infty \Delta_\tau^l[(\mathrm{U}_M f)(x,\cdot)](0) \frac{d\tau}{\tau^{1+\frac{\alpha}{2}}}, \tag{7}$$

where the normalized coefficient $\chi(\alpha/2;l)$ is defined by

$$\chi(\alpha/2;l) = \int_0^\infty (1-e^{-t})^l \frac{dt}{t^{1+\alpha/2}}, \quad \left(0 < \frac{\alpha}{2} < l, l \in \mathbb{N}\right).$$

As shown in the following lemma, there is a close connection between the constructions (6)–(7) and the potentials $I^\alpha\varphi$ and $\mathfrak{J}^\alpha\varphi$.

Lemma 2.1 (*Rubin [12], [13, p. 224 and p. 262]*)**.**

(a) *Let $\varphi \in L_p(\mathbb{R}^n)$, $(1 \le p < \infty)$ and $0 < \alpha < n/p$. Then for any $\varepsilon > 0$ and for a.e. $x \in \mathbb{R}^n$,*

$$(D_\varepsilon^\alpha I^\alpha \varphi)(x) = \int_0^\infty K_{\alpha/2}^{(l)}(\eta)(\mathrm{U}\varphi)(x,\varepsilon\eta)\,d\eta; \tag{8}$$

(b) *Let $\varphi \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$ and $0 < \alpha < \infty$. Then for any $\varepsilon > 0$ and for a.e. $x \in \mathbb{R}^n$,*

$$(\mathfrak{D}_\varepsilon^\alpha \mathfrak{J}^\alpha \varphi)(x) = \int_0^\infty K_{\alpha/2}^{(l)}(\eta)(\mathrm{U}_M\varphi)(x,\varepsilon\eta)\,d\eta. \tag{9}$$

Here the function $K_{\alpha/2}^{(l)}(\eta)$ is defined as

$$K_{\alpha/2}^{(l)}(\eta) = \left(\Gamma(1+\alpha/2)\chi(\alpha/2,l)\right)^{-1} \eta^{-1} \Delta_{-1}^l[\eta_+^{\alpha/2}], \tag{10}$$

where

$$\Delta_{-1}^l[\eta_+^{\alpha/2}] = \sum_{k=0}^{l} \binom{l}{k} (-1)^k (\eta-k)_+^{\alpha/2}, \quad \text{and} \quad a_+^{\alpha/2} = \begin{cases} a^{\alpha/2}, & \text{if } a>0, \\ 0, & \text{if } a \le 0, \end{cases}.$$

For the sake of convenience of the reader, we give a proof of the Lemma 2.1.

Proof. The equalities (8) and (9) are based on the formulas (see: [13, p. 223 and p. 262]):

(i) Let $\varphi \in L_p(\mathbb{R}^n)$, $1 \le p < n/\alpha$. Then

$$(\mathrm{U}[I^\alpha \varphi])(x,t) = \left(I_-^{\alpha/2}(\mathrm{U}\varphi)(x,\cdot)\right)(t), \quad x \in \mathbb{R}^n, t > 0;$$

(ii) Let $\varphi \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$. Then

$$(\mathrm{U}_M[\mathfrak{I}^\alpha \varphi])(x,t) = \left(I_-^{\alpha/2}(\mathrm{U}_M\varphi)(x,\cdot)\right)(t), \quad x \in \mathbb{R}^n, t > 0.$$

Here

$$I_-^{\alpha/2}(g)(t) = \frac{1}{\Gamma(\alpha/2)}\int_t^\infty \frac{g(r)dr}{(r-t)^{1-\alpha/2}} = \frac{1}{\Gamma(\alpha/2)}\int_0^\infty \frac{g(r+t)}{r^{1-\alpha/2}}dr, \quad (t>0)$$

is the Riemann–Liouville fractional integral of the function g.

The relation in (i), for "nice" function φ, may be checked in Fourier terms as follows:

$$[\mathrm{U}[I^\alpha\varphi](x,t)]^\wedge(\xi) = e^{-t|\xi|^2}|\xi|^{-\alpha}\varphi^\wedge(\xi), \quad (t>0, \xi \in \mathbb{R}^n),$$

and likewise,

$$\begin{aligned}
\left[(I_-^{\alpha/2}(\mathrm{U}\varphi)(x,\cdot))(t)\right]^\wedge(\xi) &= \frac{1}{\Gamma(\alpha/2)}\left[\int_0^\infty (\mathrm{U}\varphi)(x,r+t)\, r^{\frac{\alpha}{2}-1}dr\right]^\wedge(\xi)\\
&= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty [\,(\mathrm{U}\varphi)(x,r+t)\,]^\wedge(\xi)\, r^{\frac{\alpha}{2}-1}dr\\
&= \frac{1}{\Gamma(\alpha/2)}\varphi^\wedge(\xi)\int_0^\infty e^{-(r+t)|\xi|^2} r^{\frac{\alpha}{2}-1}dr\\
&= \frac{1}{\Gamma(\alpha/2)}\varphi^\wedge(\xi)\, e^{-t|\xi|^2}\int_0^\infty e^{-r|\xi|^2} r^{\frac{\alpha}{2}-1}dr\\
&\quad \text{(changing variable as } r = \tau/|\xi|^2,\ 0<\tau<\infty)\\
&= e^{-t|\xi|^2}|\xi|^{-\alpha}\varphi^\wedge(\xi), \quad (t>0, \xi\in\mathbb{R}^n).
\end{aligned}$$

Now, let us pass to a strict argument. We have

$$\begin{aligned}
\int_0^\infty r^{\frac{\alpha}{2}-1}(\mathrm{U}f)(x,r)dr &= \int_0^\infty r^{\frac{\alpha}{2}-1}\left(\int_{\mathbb{R}^n} W(y,r) f(x-y)\,dy\right)dr\\
&= \int_{\mathbb{R}^n} f(x-y)\left(\int_0^\infty r^{\frac{\alpha}{2}-1} W(y,r)dr\right)dy\\
&\overset{(3)}{=} (4\pi)^{-n/2}\int_{\mathbb{R}^n} f(x-y)\left(\int_0^\infty r^{\frac{\alpha-n}{2}-1} e^{-|y|^2/4r}\,dr\right)dy\\
&\quad \left(\text{we set } r = \frac{1}{4}|y|^2/\tau,\ (0<\tau<\infty)\right)\\
&= \pi^{-n/2}2^{-\alpha}\Gamma\left(\frac{n-\alpha}{2}\right)\int_{\mathbb{R}^n}\frac{f(x-y)}{|y|^{n-\alpha}}dy.
\end{aligned}$$

Using the following integral representation of the Riesz potentials (see, e.g. [13, p. 215])

$$(I^\alpha f)(x) = \pi^{-n/2}2^{-\alpha}\Gamma\left(\frac{n-\alpha}{2}\right)\frac{1}{\Gamma(\alpha/2)}\int_{\mathbb{R}^n}\frac{f(x-y)}{|y|^{n-\alpha}}\,dy,$$

we have

$$\frac{1}{\Gamma(\alpha/2)}\int_0^\infty r^{\frac{\alpha}{2}-1}(\mathrm{U}f)(x,r)\,dr = (I^\alpha f)(x). \tag{11}$$

This formula is known as the Johnson representation of the Riesz potentials ([17, p. 497]; [13, p. 223]).

Since the convolution operators, I^α and U are commutative,

$$\begin{aligned}(\mathrm{U}[I^\alpha\varphi])(x,t) &= I^\alpha[(\mathrm{U}\varphi)(\cdot,t)](x) \overset{(11)}{=} \frac{1}{\Gamma(\alpha/2)}\int_0^\infty r^{\frac{\alpha}{2}-1}\mathrm{U}[(\mathrm{U}\varphi)(\cdot,t)](x,r)\,dr\\ &\quad \text{(we use the semigroup property } \mathrm{U}[(\mathrm{U}\varphi)(\cdot,t)](x,r) = (\mathrm{U}\varphi)(x,t+r))\\ &= \frac{1}{\Gamma(\alpha/2)}\int_0^\infty r^{\frac{\alpha}{2}-1}(\mathrm{U}\varphi)(x,t+r)\,dr \equiv \left(I_-^{\alpha/2}(\mathrm{U}\varphi)(x,\cdot)\right)(t).\end{aligned}$$

This is the relation in (i). The relation in (ii) is proved by the similar way.

Further, from (6) we have

$$(D_\varepsilon^\alpha I^\alpha\varphi)(x) = \frac{1}{\chi(\frac{\alpha}{2};l)}\int_\varepsilon^\infty \frac{1}{\tau^{1+\alpha/2}}\left(\sum_{k=0}^{l}\binom{l}{k}(-1)^k(\mathrm{U}[I^\alpha\varphi])(x,k\tau)\right)d\tau.$$

By making use of the formula (i) above, we obtain

$$\begin{aligned}\sum_{k=0}^{l}\binom{l}{k}(-1)^k(\mathrm{U}[I^\alpha\varphi])(x,k\tau) &= \sum_{k=0}^{l}\binom{l}{k}(-1)^k(I_-^{\alpha/2}(\mathrm{U}\varphi)(x,\cdot))(k\tau)\\ &= \frac{1}{\Gamma(\alpha/2)}\sum_{k=0}^{l}\binom{l}{k}(-1)^k\int_{k\tau}^\infty (r-k\tau)^{\frac{\alpha}{2}-1}(\mathrm{U}\varphi)(x,r)dr\\ &\equiv \int_0^\infty h_\tau(r)(\mathrm{U}\varphi)(x,r)dr,\end{aligned}$$

where

$$h_\tau(r) = \frac{1}{\Gamma(\alpha/2)}\sum_{k=0}^{l}\binom{l}{k}(-1)^k(r-k\tau)_+^{\frac{\alpha}{2}-1}, \quad \text{and} \quad (r-k\tau)_+^{\frac{\alpha}{2}-1} = \left\{\begin{matrix}(r-k\tau)^{\frac{\alpha}{2}-1}, & \text{if } r>k\tau,\\ 0, & \text{if } r\le k\tau\end{matrix}\right\}.$$

Therefore,

$$\begin{aligned}(D_\varepsilon^\alpha I^\alpha\varphi)(x) &= \frac{1}{\chi(\frac{\alpha}{2};l)}\int_\varepsilon^\infty \frac{1}{\tau^{1+\alpha/2}}\left(\int_0^\infty h_\tau(r)(\mathrm{U}\varphi)(x,r)dr\right)d\tau\\ &= \frac{1}{\chi(\frac{\alpha}{2};l)}\int_0^\infty (\mathrm{U}\varphi)(x,r)\left(\int_\varepsilon^\infty \frac{1}{\tau^{1+\alpha/2}}h_\tau(r)d\tau\right)dr\\ &\quad \text{(by making change of variable } r=\varepsilon\eta,\ 0<\eta<\infty)\\ &= \frac{\varepsilon}{\chi(\frac{\alpha}{2};l)}\int_0^\infty (\mathrm{U}\varphi)(x,\varepsilon\eta)\left(\int_\varepsilon^\infty \frac{1}{\tau^{1+\alpha/2}}h_\tau(\varepsilon\eta)d\tau\right)d\eta\\ &= \frac{\varepsilon}{\chi(\frac{\alpha}{2};l)}\int_0^\infty (\mathrm{U}\varphi)(x,\varepsilon\eta)\left(\sum_{k=0}^{l}\binom{l}{k}(-1)^k\int_\varepsilon^\infty \frac{1}{\tau^{1+\alpha/2}}(\varepsilon\eta-k\tau)_+^{\frac{\alpha}{2}-1}d\tau\right)d\eta.\end{aligned}$$

We need to evaluate the following integrals

$$i_k = \int_\varepsilon^\infty \tau^{-1-\frac{\alpha}{2}}(\varepsilon\eta-k\tau)_+^{\frac{\alpha}{2}-1}d\tau, \quad (k=0,1,\dots,l).$$

In case of $k\ge 1$

$$i_k = \int_\varepsilon^{\varepsilon\eta/k}\tau^{-1-\frac{\alpha}{2}}(\varepsilon\eta-k\tau)^{\frac{\alpha}{2}-1}d\tau \quad \text{for } \eta>k,$$

and $i_k=0$ for $\eta\le k$. By making a change of variables as

$$\tau = \frac{\varepsilon\eta}{k}\frac{1}{t+1}, \qquad d\tau = -\frac{\varepsilon\eta}{k}\frac{1}{(t+1)^2},$$

after simple calculation we obtain

$$i_k = \frac{2}{\varepsilon\eta\alpha}(\eta-k)_+^{\alpha/2}, \quad (k=1,2,\dots,l).$$

It is easy to see that the same formula is valid also for $k=0$.

As a result, we have

$$(D_\varepsilon^\alpha I^\alpha \varphi)(x) = \frac{1}{\frac{\alpha}{2}\Gamma(\frac{\alpha}{2})\chi(\frac{\alpha}{2};l)} \int_0^\infty (U\varphi)(x,\varepsilon\eta)\left(\sum_{k=0}^{l}\binom{l}{k}(-1)^k(\eta-k)_+^{\frac{\alpha}{2}}\right)\frac{d\eta}{\eta}$$
$$= \int_0^\infty (U\varphi)(x,\varepsilon\eta)K_{\alpha/2}^{(l)}(\eta)\,d\eta,$$

where $K_{\alpha/2}^{(l)}(\eta)$ is defined by (10).

Thus, (8) is proved. (The proof of (9) is similar to (8) and based on the formula (ii).) □

The next lemma gives some properties of the function $K_{\alpha/2}^{(l)}(\eta)$, that will be used later.

Lemma 2.2 (*See [17, p. 125] and [13, p. 158]*)**.**

(a) $K_{\alpha/2}^{(l)}(\eta) \in L_1(0,\infty)$ *and* $\int_0^\infty K_{\alpha/2}^{(l)}(\eta)\,d\eta = 1$.

(b) $K_{\alpha/2}^{(l)}(\eta) = \begin{Bmatrix} O(\eta^{\frac{\alpha}{2}-1}), & \text{if } \eta \longrightarrow 0^+; \\ O(\eta^{\frac{\alpha}{2}-l-1}), & \text{if } \eta \longrightarrow \infty \end{Bmatrix}$.

Definition 1. Let $0 < \rho < 1$ and a function $\mu(r)$, $(0 \le r \le \rho)$ be a modulus of continuity, which means that, μ is continuous on $[0,\rho]$, subadditive, non-decreasing, positive on $(0,\rho]$, and $\mu(0) = 0$. We say that a function $\varphi \in L_1^{loc}(\mathbb{R}^n)$ has μ-smoothness property at a point $x^0 \in \mathbb{R}^n$ if

$$M_\mu(x^0) \overset{def}{=} \sup_{0<r\le\rho} \frac{1}{r^n\mu(r)} \int_{|x|\le r} |\varphi(x^0 - x) - \varphi(x^0)|dx < \infty. \tag{12}$$

Here, as usual, $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $dx = dx_1 \ldots dx_n$.

From now on it will be assumed that $\mu(t) \ge at$, $(0 \le t \le \rho)$ for some $a > 0$ and $\mu(r) = \mu(\rho)$ for $\rho \le r < \infty$. It is well known that $\mu(\lambda t) \le (\lambda+1)\mu(t)$ for $\lambda \ge 0$ (see, for instance [7, p. 41]).

The following lemma is playing a crucial role in the sequel.

Lemma 2.3. *Let function* $\varphi \in L_1^{loc}(\mathbb{R}^n)$ *has the* μ*-smoothness property at a point* $x^0 \in \mathbb{R}^n$ *and the function* g *be defined as*

$$g(x) = \left|\varphi(x^0 - x) - \varphi(x^0)\right|, \qquad |x| \le \rho < 1.$$

Further, let the function $\psi(r)$, $(0 \le r \le \rho)$ *be decreasing, nonnegative and continuously differentiable on* $[0,\rho]$. *Then*

$$\int_{|x|\le\rho} g(x)\psi(|x|)dx \le M_\mu(x^0)\left[\int_0^\rho r^n\mu(r)(-\psi'(r))dr + \rho^n\mu(\rho)\psi(\rho)\right], \tag{13}$$

where $M_\mu(x^0)$ *is defined as in* (12).

Proof. Putting $x = r\theta$, where $r = |x|$, we have

$$I \equiv \int_{|x|\le\rho} g(x)\psi(|x|)dx = \int_0^\rho r^{n-1}\psi(r)\left(\int_{|\theta|=1} g(r\theta)d\sigma(\theta)\right)dr.$$

By introducing the functions

$$\lambda(r) = \int_{|\theta|=1} g(r\theta)d\sigma(\theta) \quad \text{and} \quad \Omega(r) = \int_0^r \lambda(t)t^{n-1}dt,$$

we get

$$I = \int_0^\rho \psi(r)\lambda(r)r^{n-1}dr = \int_0^\rho \psi(r)d\Omega(r) = \psi(r)\Omega(r)\Big|_0^\rho - \int_0^\rho \Omega(r)\psi'(r)dr$$
$$= \psi(\rho)\Omega(\rho) + \int_0^\rho \Omega(r)(-\psi'(r))dr.$$

It follows from (12) that

$$\Omega(r) = \int_0^r \lambda(t)t^{n-1}dt = \int_{|x|\le r} g(x)\,dx$$
$$= \int_{|x|\le r} \left|\varphi(x^0 - x) - \varphi(x^0)\right|dx \le r^n\mu(r)M_\mu(x^0).$$

Hence,

$$I \le M_\mu(x^0)\left\{\rho^n\mu(\rho)\psi(\rho)+\int_0^\rho r^n\mu(r)(-\psi'(r))dr\right\}. \quad \square$$

Corollary 2.4. *Let $\psi(|x|)\equiv\psi_\varepsilon(|x|)$ be the Gauss–Weierstrass kernel with parameter $\varepsilon>0$ (cf. (3)):*

$$\psi_\varepsilon(|x|)=(4\pi\varepsilon)^{-n/2}e^{-|x|^2/4\varepsilon}, \quad (x\in\mathbb{R}^n).$$

Then

$$\int_{|x|\le r} g(x)\psi_\varepsilon(|x|)\le c\mu(\sqrt{\varepsilon}), \tag{14}$$

where $c>0$ does not depend on $\varepsilon\ll 1$.

Proof. Since $\psi_\varepsilon(r)=(4\pi\varepsilon)^{-n/2}e^{-r^2/4\varepsilon}$, after simple calculation we get

$$-\psi_\varepsilon'(r)=c_1 r\varepsilon^{-1-(n/2)}e^{-r^2/4\varepsilon}; \qquad c_1=2^{-(n+1)}\pi^{-n/2}.$$

Putting this value of $(-\psi_\varepsilon'(r))$ into (13), we have for $\rho<1$

$$\int_{|x|\le\rho} g(x)\psi_\varepsilon(|x|)dx\le c_2M_\mu(x^0)\varepsilon^{-1-(n/2)}\left\{\int_0^\rho r^{n+1}\mu(r)e^{-r^2/4\varepsilon}dr+\varepsilon^{-n/2}e^{-\rho^2/4\varepsilon}\right\}.$$

By changing variables as $r=\sqrt{\varepsilon}t$, we get

$$\int_{|x|\le\rho} g(x)\psi_\varepsilon(|x|)dx\le c_3\left\{\int_0^{\rho/\sqrt{\varepsilon}} t^{n+1}\mu(\sqrt{\varepsilon}t)e^{-t^2/4}dt+\varepsilon^{-n/2}e^{-\rho^2/4\varepsilon}\right\}. \tag{15}$$

From the inequality $\mu(\sqrt{\varepsilon}t)\le(1+t)\mu(\sqrt{\varepsilon})$ it follows that

$$\int_0^{\rho/\sqrt{\varepsilon}} t^{n+1}\mu(\sqrt{\varepsilon}t)e^{-t^2/4}dt\le\mu(\sqrt{\varepsilon})\int_0^\infty t^{n+1}(1+t)e^{-t^2/4}dt\le c_4\mu(\sqrt{\varepsilon}).$$

On the other hand, since for any $k>0$, $\lim_{\varepsilon\to0}\varepsilon^{-k}e^{-\rho^2/4\varepsilon}=0$, we have $\varepsilon^{-n/2}e^{-\rho^2/4\varepsilon}\le c_5\sqrt{\varepsilon}$ for some $c_5>0$. By making use of these in (15), we have

$$\int_{|x|\le\rho} g(x)\psi_\varepsilon(|x|)dx\le c_6(\mu(\sqrt{\varepsilon})+\sqrt{\varepsilon}).$$

The condition $\mu(t)\ge$ at, $(0\le t\le\rho, a>0)$ yields $(\mu(\sqrt{\varepsilon})+\sqrt{\varepsilon})\le c_7\mu(\sqrt{\varepsilon})$, and the desired result

$$\int_{|x|\le\rho} g(x)\psi_\varepsilon(|x|)dx\le c\mu(\sqrt{\varepsilon})$$

follows. $\square$

Example 2.5. For $0<\beta\le1$ and $\gamma\ge0$ set $\mu(t)=t^\beta(\ln\frac{1}{t})^\gamma$, $(0<t\le\rho<1)$. It is easy to see that $\mu(t)$ is modulus of continuity (simple calculation shows that $(\mu(t)/t)'<0$, and therefore, $\mu(t)/t$ is decreasing, which guaranties that $\mu(t)$ is a modulus of continuity, see [7, p. 41]).

3. Formulation and proof of main results

Theorem 3.1. *Let $\varphi\in L_p(\mathbb{R}^n)$, $(1\le p<\infty)$ has the μ-smoothness property at a point $x^0\in\mathbb{R}^n$, i.e.*

$$M_\mu(x^0)\equiv\sup_{0<r\le\rho<1}\frac{1}{r^n\mu(r)}\int_{|x|\le r}|\varphi(x^0-x)-\varphi(x^0)|\,dx<\infty.$$

Further, let $\mu(r)$ be a modulus of continuity which satisfies the inequality $\mu(r)\ge ar$, $(0\le r\le\rho)$ for some $a>0$. If the constructions $D_\varepsilon^\alpha I^\alpha\varphi$, $(0<\alpha<n/p)$ and $\mathfrak{D}_\varepsilon^\alpha\mathfrak{I}^\alpha\varphi$, $(0<\alpha<\infty)$ are defined as in (8) and (9), and the parameter $l\in\mathbb{N}$ satisfies the condition $l>\frac{\alpha}{2}+1$, then

$$\left|(D_\varepsilon^\alpha I^\alpha\varphi)(x^0)-\varphi(x^0)\right|\le c\mu(\sqrt{\varepsilon}), \quad \text{as } \varepsilon\longrightarrow0, \tag{16}$$

and

$$\left|(\mathfrak{D}_{\varepsilon}^{\alpha}\mathfrak{J}^{\alpha}\varphi)(x^0)-\varphi(x^0)\right|\le c\mu(\sqrt{\varepsilon}),\quad \text{as } \varepsilon\longrightarrow 0. \tag{17}$$

Proof. We will prove (17) only. The proof of (16) follows in the similar way. Using Lemmas 2.1(b) and 2.2(a), we have

$$\left|(\mathfrak{D}_{\varepsilon}^{\alpha}\mathfrak{J}^{\alpha}\varphi)(x^0)-\varphi(x^0)\right|\le\int_0^{\infty}\left|K_{\alpha/2}^{(l)}(\eta)\right|\left|(\mathrm{U}_M\varphi)(x^0,\varepsilon\eta)-\varphi(x^0)\right|d\eta. \tag{18}$$

Let us estimate $i_0=\left|(\mathrm{U}_M\varphi)(x^0,\varepsilon\eta)-\varphi(x^0)\right|$. We have

$$i_0\le(1-e^{-\varepsilon\eta})\left|(\mathrm{U}\varphi)(x^0,\varepsilon\eta)\right|+\left|(\mathrm{U}\varphi)(x^0,\varepsilon\eta)-\varphi(x^0)\right|\equiv i_1+i_2, \tag{19}$$

where $\mathrm{U}\varphi$ is the Gauss–Weierstrass integral (see (2)). Since

$$(1-e^{-\varepsilon\eta})\le\varepsilon\eta\quad\text{and}\quad\sup_{\tau>0}\left|(\mathrm{U}\varphi)(x^0,\tau)\right|\le c_1\mathtt{M}(\varphi)(x^0),$$

where $\mathtt{M}(\varphi)$ is the Hardy–Littlewood maximal function of φ, we get $i_1\le c_2\varepsilon\eta$.

Now let us estimate i_2. It follows that

$$\begin{aligned}i_2&=\left|(\mathrm{U}\varphi)(x^0,\varepsilon\eta)-\varphi(x^0)\right|=\left|\int_{\mathbb{R}^n}W(y,\varepsilon\eta)[\varphi(x^0-y)-\varphi(x^0)]dy\right|\\&\le\int_{|y|\le\rho}W(y,\varepsilon\eta)|\varphi(x^0-y)-\varphi(x^0)|dy+\int_{|y|>\rho}W(y,\varepsilon\eta)|\varphi(x^0-y)-\varphi(x^0)|dy\\&\equiv i_3+i_4.\end{aligned}$$

In order to estimate i_3, we use (14). Since x^0 is the μ-smoothness point of function φ, by (14), we have

$$i_3\le c_3\mu(\sqrt{\varepsilon\eta})\le c_3(1+\sqrt{\eta})\mu(\sqrt{\varepsilon}).$$

Let us now estimate i_4. By Hölder inequality, we have

$$i_4\le|\varphi(x^0)|\int_{|y|>\rho}W(y,\varepsilon\eta)dy+\|\varphi\|_p\left(\int_{|y|>\rho}(W(y,\varepsilon\eta))^{p'}dy\right)^{1/p'}\equiv i_5+i_6.$$

$$\begin{aligned}i_5&=|\varphi(x^0)|\int_{|y|>\rho}W(y,\varepsilon\eta)dy=c_4(\varepsilon\eta)^{-n/2}\int_{\rho}^{\infty}r^{n-1}e^{-r^2/4\varepsilon\eta}dr\\&=c_5\int_{\rho/2\sqrt{\varepsilon\eta}}^{\infty}t^{n-1}e^{-t^2}dt=c_5\int_{\rho/2\sqrt{\varepsilon\eta}}^{\infty}t^{n-1}e^{-t^2/2}e^{-t^2/2}dt\le c_6e^{-\rho^2/8\varepsilon\eta}.\end{aligned}$$

The equality $\inf_{\tau>0}(\tau e^{\delta/\tau})=e\delta$ yields that $e^{-\delta/\tau}\le\frac{1}{e\delta}\tau$, and therefore $i_5\le c_7\varepsilon\eta$, where c_7 does not depend on ε and η.

Similarly,

$$\begin{aligned}i_6&=\|\varphi\|_p\left(\int_{|y|>\rho}(W(y,\varepsilon\eta))^{p'}dy\right)^{1/p'}=c_8(\varepsilon\eta)^{-n/2}\left(\int_{\rho}^{\infty}r^{n-1}e^{-r^2p'/4\varepsilon\eta}dr\right)^{1/p'}\\&=c_9(\varepsilon\eta)^{-n/2p}\left(\int_{\rho/2\sqrt{\varepsilon\eta}}^{\infty}r^{n-1}e^{-r^2p'}dr\right)^{1/p'}\\&\le c_9(\varepsilon\eta)^{-n/2p}e^{-\rho^2/8\varepsilon\eta}\left(\int_0^{\infty}r^{n-1}e^{-\frac{1}{2}r^2p'}dr\right)^{1/p'}\\&=c_{10}(\varepsilon\eta)^{-n/2p}e^{-\rho^2/8\varepsilon\eta}.\end{aligned}$$

Since $\inf_{\tau>0}(\tau\tau^{\frac{n}{2p}}e^{\delta/\tau})=(\frac{2e\delta p}{n+2p})^{\frac{n+2p}{2p}}$, then $\tau^{-n/2p}e^{-\delta/\tau}\le c_{11}\tau$, where $\delta=\rho^2/8$ and the coefficient $c_{11}=\left(\frac{2e\delta p}{n+2p}\right)^{-\frac{n+2p}{2p}}$ does not depend on $\tau>0$. Therefore, $i_6\le c_{12}\varepsilon\eta$. By taking into account of all these in (19) and using the condition $\mu(r)\ge ar$, $(0\le r\le\rho)$, we have, for small $\varepsilon>0$,

$$i_0\le c_{13}\big(\varepsilon\eta+(1+\sqrt{\eta})\mu(\sqrt{\varepsilon})\big)\le c_{13}(\sqrt{\varepsilon}\eta+(1+\sqrt{\eta})\mu(\sqrt{\varepsilon}))\le c_{14}(1+\sqrt{\eta}+\eta)\mu(\sqrt{\varepsilon}).$$

Using this in (18) we have,

$$\left|(\mathfrak{D}_{\varepsilon}^{\alpha}\mathfrak{J}^{\alpha}\varphi)(x^0)-\varphi(x^0)\right|\le c_{13}\mu(\sqrt{\varepsilon})\int_0^{\infty}\left|K_{\alpha/2}^{(l)}(\eta)\right|\left(1+\eta+\sqrt{\eta}\right)d\eta. \tag{20}$$

Finally, using the condition $l > \frac{\alpha}{2} + 1$ and taking in mind the asymptotic behavior of the function $K^{(l)}_{\alpha/2}(\eta)$ as $\eta \to \infty$ (see Lemma 2.2(b)) we get that the integral at the right hand side of (20) converges, that is,

$$\left|(\mathfrak{D}^{\alpha}_{\varepsilon}\mathfrak{J}^{\alpha}\varphi)(x^0) - \varphi(x^0)\right| \le c\mu(\sqrt{\varepsilon}), \quad \text{as } \varepsilon \to 0. \quad \square$$

Remark. For the special case $\mu(\varepsilon) = \varepsilon^{\alpha}$, $(0 < \alpha \le 1)$ the analogous theorem has been proved in [6].

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