J. Math. Anal. Appl. 409 (2014) 485–493

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

A continuous method for nonlocal functional differential equations with delayed or advanced arguments

X.Y. Li [a], B.Y. Wu [b,*]

[a] Department of Mathematics, Changshu Institute of Technology, Suzhou, Jiangsu 215500, China
[b] Department of Mathematics, Harbin Institute of Technology, Harbin, Heilongjiang 150001, China

ARTICLE INFO

Article history:
Received 29 December 2012
Available online 23 July 2013
Submitted by M. Laczkovich

Keywords:
Reproducing kernel method
Nonlocal problems
Functional differential equations
Error estimation

ABSTRACT

In the previous works, the authors presented the reproducing kernel method (RKM) for solving various differential equations. However, to the best of our knowledge, there exist no results for functional differential equations. The aim of this paper is to extend the application of reproducing kernel theory to nonlocal functional differential equations with delayed or advanced arguments, and give the error estimation for the present method. Some numerical examples are provided to show the validity of the present method.

1. Introduction

Reproducing kernel theory has important application in numerical analysis, differential equations, probability and statistics, learning theory and so on. Recently, reproducing kernel methods for solving a variety of differential equations were presented by Cui and Geng [4–9], Lin and Zhou [12,13], Yao, Chen and Jiang [3,26], Wang, Li and Wu [11,22], Mohammadi and Mokhtari [15], Akram and Ur Rehman [1], Arqub, Al-Smadi, Momani [2], Özen and Oruçoğlu [17], Wang, Han and Yamamoto [23].

In this paper, we consider the following functional differential equations with linear nonlocal conditions:

$$\begin{cases} u'(x)+a(x)u(x)+b(x)u(\tau(x))=f(x), & x\in I=[0,1],\\ B(u(c),u)=0, \end{cases} \tag{1.1}$$

where $c\in I$, $a(x)$, $b(x)\in C[0,1]$, $\tau(x)\in C^1[0,1]$, and f is given such that (1.1) satisfies the existence and uniqueness of the solutions.

Notice that function $B(u(c),u)=0$ includes several types of boundary conditions: initial conditions, final conditions, periodic conditions, or more general functional conditions, as

$$B(u(c),u)=\lambda u(c)-\int_0^1 u(s)ds,\quad \lambda\in R.$$

Functional differential equations arise in a variety of applications, such as number theory, electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structure, quantum mechanics and cell growth. Therefore, the problems have attracted a great deal of attention. Liu and Li [14] studied the analytical and numerical solutions of the multi-pantograph equations. Sezer [20,21] gave the series solutions of multi-pantograph equations with variable coefficients.

* Corresponding author.
E-mail addresses: lixiuying1112@126.com (X.Y. Li), mathwby@sina.com (B.Y. Wu).

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.07.039

Wang, Qin and Li [25] proposed one-leg h-methods for nonlinear neutral differential equations with proportional delay. In [10,16,19], the authors discussed the series solutions of some functional differential equations. Wang, Shen and Luo [24] obtained the existence of solutions of second-order multi-point functional differential equations by using the method of upper and lower solutions. Rodríguez-López [18] studied the existence of a solution to a nonlocal boundary value problem for a class of second-order functional differential equations with piecewise constant arguments.

However, numerical solutions of nonlocal functional differential equations are seldom discussed. The aim of this paper is to fill this gap.

The rest of the paper is organized as follows. In the next section, the reproducing kernel method for solving (1.1) is introduced. The error estimation is presented in Section 3. Numerical examples are provided in Section 4. Section 5 ends this paper with a brief conclusion.

2. Reproducing kernel method for (1.1)

In this section, we extend the application of reproducing kernel theory to nonlocal functional differential equation (1.1).

To solve (1.1), first, we construct reproducing kernel spaces $W^m[0,1]$, $(m \geq 2)$ in which every function satisfies the boundary condition of (1.1).

Definition 2.1. $W_0^m[0,1] = \{u(x) \mid u^{(m-1)}(x)$ is an absolutely continuous real value function, $u^{(m)}(x) \in L^2[0,1]\}$. The inner product and norm in $W_0^m[0,1]$ are given respectively by

$$(u,v)_m = \sum_{i=0}^{m-1} u^{(i)}(0)v^{(i)}(0) + \int_0^1 u^{(m)}(x)v^{(m)}(x)dx$$

and

$$\|u\|_m = \sqrt{(u,u)_m}, \quad u, v \in W_0^m[0,1].$$

By [6,8], $W_0^m[0,1]$ is a reproducing kernel space and its reproducing kernel $k_0(x,y)$ can be obtained.

Next, we construct reproducing kernel space $W^m[0,1]$ in which every function satisfies $B(u(c),u) = 0$.

Definition 2.2. $W^m[0,1] = \{u(x) \mid u(x) \in W_0^m[0,1], B(u(c),u) = 0\}$.

Clearly, $W^m[0,X]$ is a closed subspace of $W_0^m[0,X]$ and therefore it is also a reproducing kernel space.

Put $Pu(x) = B(u(c),u)$.

Theorem 2.1. *If $P_xP_yk(x,y) \neq 0$, then the reproducing kernel $K(x,y)$ of $W^m[0,1]$ is given by*

$$K(x,y) = k_0(x,y) - \frac{P_xk_0(x,y)P_yk_0(x,y)}{P_xP_yk_0(x,y),} \tag{2.1}$$

where the subscript x by the operator P indicates that the operator P applies to the function of x.

Proof. It is easy to see that $PK(x,y) = 0$, and therefore $K(x,y) \in W^m[0,1]$.

For all $u(y) \in W^m[0,1]$, obviously, $P_yu(y) = 0$, it follows that

$$(u(y), K(x,y))_m = (u(y), k_0(x,y))_m = u(x).$$

That is, $K(x,y)$ is of "reproducing property". Thus, $K(x,y)$ is the reproducing kernel of $W^m[0,1]$ and the proof is complete. □

In [6], Cui and Lin defined reproducing kernel space $W^1[0,1]$ and gave its reproducing kernel

$$\bar{k}(x,y) = \begin{cases} 1+y, & y \leq x, \\ 1+x, & y > x. \end{cases}$$

Put

$$Lu(x) = u'(x) + a(x)u(x) + b(x)u(\tau(x)).$$

Theorem 2.2. *$L : W^m[0,1] \to W^1[0,1]$ is a bounded linear operator.*

Proof. It is easy to see that

$$\begin{aligned} |u(x)| &= |(u(\cdot),\ K(x,\cdot))_m| \\ &\leq \|u(\cdot)\|_m\, \|K(x,\cdot)\|_m. \end{aligned} \tag{2.2}$$

Similarly,

$$|u'(x)| = \left|\left(u(\cdot), \frac{\partial K(x,\cdot)}{\partial x}\right)_m\right| \le \|u(\cdot)\|_m \left\|\frac{\partial K(x,\cdot)}{\partial x}\right\|_m \tag{2.3}$$

and

$$|u''(x)| \le \|u(\cdot)\|_m \left\|\frac{\partial^2 K(x,\cdot)}{\partial x^2}\right\|_m. \tag{2.4}$$

By (2.2) and (2.3), there exists a positive constant M_1 such that

$$\begin{aligned}\|b(x)u(x)\|_1 &\le \|b(x)\|_1\,\|u(x)\|_1 \\ &\le M_0\left[u^2(0) + \int_0^1 (u'(x))^2 dx\right]^{1/2} \\ &\le M_0\left[\|u\|_m^2\,\|K(0,\cdot)\|_m^2 + \|u(\cdot)\|_m^2 \int_0^1 \left\|\frac{\partial K(x,\cdot)}{\partial x}\right\|_m^2 dx\right]^{1/2} \\ &\le M_1\|u\|_m.\end{aligned} \tag{2.5}$$

Similarly, we have

$$\begin{aligned}\|a(x)u(\tau(x))\|_1 &\le \|a(x)\|_1\,\|u(\tau(x))\|_1 \\ &\le M_2\left[u^2(\tau(0)) + \int_0^1 (u'(\tau(x))\tau'(x))^2 dx\right]^{1/2} \\ &\le M_3\left[\|u\|_m^2\,\|K(0,\cdot)\|_m^2 + \|u(\cdot)\|_m^2 \int_0^1 \left\|\frac{\partial K(x,\cdot)}{\partial x}\right\|_m^2 dx\right]^{1/2} \\ &\le M_4\|u\|_m\end{aligned} \tag{2.6}$$

where M_2, M_3 and M_4 are positive constants.

Combining (2.3) and (2.4), there exists a positive constant M_5 such that

$$\begin{aligned}\|u'(x)\|_1 &= \left[(u'(0))^2 + \int_0^1 (u''(x))^2 dx\right]^{1/2} \\ &\le M_5\|u\|_m.\end{aligned} \tag{2.7}$$

From (2.5)–(2.7), it follows that

$$\begin{aligned}\|Lu\|_1 &\le (M_1 + M_4 + M_5)\|u\|_m \\ &\le M\|u\|_m,\end{aligned}$$

where M is a positive constant.

Therefore, $L : W^m[0,1] \to W^1[0,1]$ is a bounded linear operator and the proof is complete. □

Put $\varphi_i(x) = \overline{k}(x_i, x)$ and $\psi_i(x) = L^*\varphi_i(x)$ where $\overline{k}(x_i, x)$ is the reproducing kernel of $W^1[0,1]$, L^* is the adjoint operator of L. The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W^m[0,1]$ can be derived from Gram–Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0,\ i = 1, 2, \ldots). \tag{2.8}$$

According to [6,8], we have the following theorem:

Theorem 2.3. *If $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W^m[0,1]$ and $\psi_i(x) = L_s K(x,s)|_{s=x_i}$.*

Theorem 2.4. *If $\{x_i\}_{i=1}^{\infty}$ is dense in $[0,1]$ and the solution of (1.1) is unique, then the solution of (1.1) is*

$$u(x) = \sum_{j=1}^{\infty} A_j \overline{\psi}_j(x), \tag{2.9}$$

where $A_j = \sum_{l=1}^{j} \beta_{jl} f(x_l)$.

Now, the approximate solution $u(x)$ can be obtained by taking finitely many terms in the series representation of $u(x)$ and

$$u_N(x) = \sum_{j=1}^{N} A_j \overline{\psi}_j(x). \tag{2.10}$$

Remark. Since $W^m[0,1]$ is a Hilbert space, it is clear that $\sum_{i=1}^{\infty}(\sum_{k=1}^{i} \beta_{ik} f(x_k))^2 < \infty$. Therefore, the sequence u_N is convergent in the sense of norm $\|\cdot\|_m$.

Lemma 2.1. *If $u(x) \in W^m[0,1]$, then there exists a constant C such that $|u(x)| \le C\|u(x)\|_m$, $|u^{(k)}(x)| \le C\|u(x)\|_m$, $1 \le k \le m-1$.*

Proof. Since

$$|u(x)| = |(u(y), k(x,y))_m| \le \|u(y)\|_m \, \|k(x,y)\|_m,$$

there exists a constant c_0 such that

$$|u(x)| \le c_0\|u\|_m.$$

Note that

$$\begin{aligned} |u^{(i)}(x)| &= \left| \left(u(y), \frac{\partial^i k(x,y)}{\partial x^i} \right)_m \right| \\ &\le \|u\|_m \left\| \frac{\partial^i k(x,y)}{\partial x^i} \right\|_m \\ &\le c_i\|u\|_m, \quad (i = 0, 1, 2, \ldots, m-1), \end{aligned}$$

where c_i are constants.

Putting $C = \max_{0 \le i \le m-1}\{c_i\}$ and the proof of the lemma is complete. □

From the above lemma and convergence of $u_n(x)$ in the sense of norm, it is easy to obtain the following theorem.

Theorem 2.5. *The approximate solution $u_n(x)$ and its derivatives $u_n^{(k)}(x)$, $1 \le k \le m-1$ are all uniformly convergent.*

3. Error estimations

To give the error estimation for the solution of (1.1) it is assumed that $b(x) < 0$, $a(x) + b(x) \ge \alpha > 0$.

Lemma 3.1. *Let $u(x) \in C^1[0,1]$. If $u(0) \ge 0$, $Lu \ge 0$, $\forall x \in I$, then $u \ge 0$, $\forall x \in I$.*

Proof. It can be easily proven by contradiction. Suppose $x^* \in (0,1)$ be such that

$$u(x^*) = \min_{x\in(0,1)} u(x), \quad u(x^*) < 0.$$

Then it is clear that $u'(x^*) = 0$, therefore we have

$$Lu(x)|_{x=x^*} = a(x^*)u(x^*) + b(x^*)u(\tau(x^*)).$$

From $b(x) < 0$ and $a(x) + b(x) \ge \alpha > 0$, it follows that

$$Lu(x)|_{x=x^*} \le a(x^*)u(x^*) + b(x^*)u(x^*) = [a(x^*) + b(x^*)]u(x^*) < 0$$

which is a contradiction. □

Lemma 3.2. *The solution of (1.1) satisfies*

$$|u(x)| \le D \max\left\{|u(0)|, \max_{x\in(0,1)} |Lu|\right\},$$

where D is a positive constant and $D \ge \frac{1}{\alpha}$.

Proof. Define two functions

$$\phi^{\pm}(x) = D \max\left\{|u(0)|, \max_{x\in(0,1)} |Lu|\right\} \pm u(x),$$

where D is a positive constant and $D \ge \frac{1}{\alpha}$.

It is clear that $\phi^{\pm}(0) \geq 0$, and

$$\begin{aligned} L\phi^{\pm}(x) &= D\,[a(x)+b(x)] \max\left\{|u(0)|, \max_{x\in(0,1)} |Lu|\right\} \pm Lu(x) \\ &\geq \max\left\{|u(0)|, \max_{x\in(0,1)} |Lu|\right\} \pm Lu(x) \\ &\geq 0. \end{aligned}$$

By Lemma 3.1, we get

$$|u(x)| \leq D \max\left\{|u(0)|, \max_{x\in(0,1)} |Lu|\right\}. \quad \square$$

Theorem 3.1. *Let $u_N(x)$ be the approximate solution of (1.1) in space $W^3[0,1]$ and $u(x)$ be the exact solution of (1.1). If $0 = x_1 < x_2 < \cdots < x_N = 1$, $a(x), b(x), \tau(x), f(x) \in C^2[0,1]$, and if $|u(0)| \leq |Lu(0)|$, then*

$$\|u(x) - u_N(x)\|_\infty \leq d_1 h^2,$$

where $\|u(x)\|_\infty = \max_{x\in[0,1]} |u(x)|$, d_1 is a positive constant, $h = \max_{1\leq i\leq N-1} |x_{i+1} - x_i|$.

Proof. Note here that

$$Lu_N(x) = \sum_{i=1}^{N} A_i L\overline{\psi}_i(x)$$

and

$$(Lu_N)(x_n) = \sum_{i=1}^{N} A_i (L\overline{\psi}_i, \varphi_n) = \sum_{i=1}^{N} A_i (\overline{\psi}_i, L^*\varphi_n) = \sum_{i=1}^{N} A_i (\overline{\psi}_i, \psi_n).$$

Therefore,

$$\sum_{j=1}^{n} \beta_{nj} (Lu_N)(x_j) = \sum_{i=1}^{N} A_i \left(\overline{\psi}_i, \sum_{j=1}^{n} \beta_{nj} \psi_j\right) = \sum_{i=1}^{N} A_i (\overline{\psi}_i, \overline{\psi}_n) = A_n. \tag{3.1}$$

If $n = 1$, then $(Lu_N)(x_1) = f(x_1)$.

If $n = 2$, then $\beta_{21}(Lu_N)(x_1) + \beta_{22}(Lu_N)(x_2) = \beta_{21} f(x_1) + \beta_{22} f(x_2)$.

It is clear that $(Lu_N)(x_2) = f(x_2)$.

Moreover, it is easy to see by induction that

$$(Lu_N)(x_j) = f(x_j), \quad j = 1, 2, \ldots, N. \tag{3.2}$$

Put $R_N(x) = f(x) - Lu_N(x)$. Obviously, $R_N(x) \in C^2[0,1]$ and $R_N(x_j) = 0,\ j = 1, 2, \ldots, N$. Suppose that $l(x)$ is a polynomial of degree $= 1$ that interpolates the function $R_N(x)$ at x_i, x_{i+1}. It is clear that $l(x) = 0$. Also, for $\forall x \in [x_i, x_{i+1}]$,

$$R_N(x) = R_N(x) - l(x) = \frac{R_N''(\xi_i)}{2!}(x - x_i)(x - x_{i+1}), \quad \xi_i \in [x_i, x_{i+1}]. \tag{3.3}$$

Hence, for $\forall x \in [x_i, x_{i+1}]$,

$$|R_N(x)| \leq \frac{|R_N''(\xi_i)|}{8} h_i^2 = c_i h_i^2, \quad c_i = \frac{|R_N''(\xi_i)|}{8}, \; h_i = |x_{i+1} - x_i|.$$

Putting $cc = \max_{1\leq i\leq N-1} c_i$ and $h = \max_{1\leq i\leq N-1} h_i$, we have

$$\|R_N(x)\|_\infty = \max_{x\in[0,1]} |R_N(x)| \leq cc\, h^2.$$

Obviously, $u(x) - u_N(x)$ is the solution of $Lv(x) = R_N(x)$.

According to Lemma 3.2, there exists a positive constant d_1 such that

$$\|u(x) - u_N(x)\|_\infty \leq d_1 h^2. \quad \square$$

Theorem 3.2. *Let $u_N(x)$ be the approximate solution of (1.1) in space $W^4[0,1]$ and $u(x)$ be the exact solution of (1.1). If $0 = x_1 < x_2 < \cdots < x_N = 1$, $a(x), b(x), \tau(x), f(x) \in C^4[0,1]$, and if $|u(0)| \le |Lu(0)|$, then*

$$\|u(x) - u_N(x)\|_\infty \le d_2\, h^4,$$

where d_2 is a positive constant, $h = \max_{1\le i\le N-1} |x_{i+1} - x_i|$.

Proof. From the proof of Theorem 3.1, we have

$$Lu_N(x_j) = f(x_j), \quad j = 1, 2, \ldots, N.$$

Put

$$R_N(x) = f(x) - Lu_N(x).$$

Obviously,

$$R_N(x) \in C^4[0,1], \quad R_N(x_j) = 0, \quad j = 1, 2, \ldots, N.$$

On interval $[x_i, x_{i+1}]$, the application of Roll's theorem to $R_N(x)$ yields

$$R'_N(y_i) = 0, \quad y_i \in (x_i, x_{i+1}),\ i = 1, 2, \ldots, N-1.$$

On interval $[y_i, y_{i+1}]$, the application of Roll's theorem to $R'_N(x)$ yields

$$R''_N(z_i) = 0, \quad z_i \in (y_i, y_{i+1}),\ i = 1, 2, \ldots, N-2.$$

Putting

$$h = \max_{1\le i\le N-1}\{|x_{i+1} - x_i|\}, \qquad h_y = \max_{1\le i\le N-2}\{|y_{i+1} - y_i|\}, \qquad h_z = \max_{1\le i\le N-3}\{|z_{i+1} - z_i|\},$$

clearly,

$$h_y \le 2h, \qquad h_z \le 4h.$$

Suppose that $l_1(x)$ is a polynomial of degree $= 1$ that interpolates the function $R''_N(x)$ at z_1, z_2. It is clear that $l_1(x) = 0$. Also, for $\forall x \in [x_1, z_2]$, there exist $\eta_1 \in [x_1, z_2]$ and a positive constant b_1 such that

$$R''_N(x) = R''_N(x) - l_1(x) = \frac{R_N^{(4)}(\eta_1)}{2!}(x - z_1)(x - z_2) \le b_1\, h^2.$$

In a similar way, there exist positive constants c_i, b_2 such that

$$R''_N(x) \le c_i\, h^2, \quad x \in [z_i, z_{i+1}],\ i = 2, 3, \ldots, N-3,$$

and

$$R''_N(x) \le b_2\, h^2, \quad x \in [z_{N-2}, x_N].$$

Hence, there exists a positive constant d_2 such that

$$\|R''_N(x)\|_\infty \le d_2\, h^2.$$

On interval $[x_i, x_{i+1}]$ $i = 1, 2, \ldots, N-1$, noting that

$$R'_N(x) = \int_{y_i}^{x} R''_N(s)ds,$$

there exist constants $\bar{a}_i$

$$|R'_N(x)| \le \|R''_N(x)\|_\infty |x - y_i| \le \bar{a}_i\, h^3.$$

It turns out that

$$\|R'_N(x)\|_\infty \le a_0\, h^3, \quad x \in [x_1, x_N] = [0,1] \tag{3.4}$$

where a_0 is a positive constant.

In a similar way, there exists a positive constant a_1 such that

$$\|R_N(x)\|_\infty \le a_1\, h^4, \quad x \in [0,1].$$

Obviously, $u(x) - u_N(x)$ is the solution of $Lv(x) = R_N(x)$.

According to Lemma 3.2, it is easy to see that

$$\|u(x) - u_N(x)\|_\infty \le d_2\, h^4,$$

where d_2 is a positive constant. □

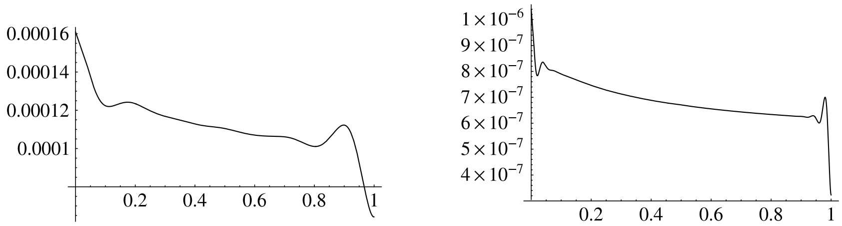

Fig. 1. Absolute errors of $u_{11}(x)$ and $u_{51}(x)$ in W^3 for Example 4.1.

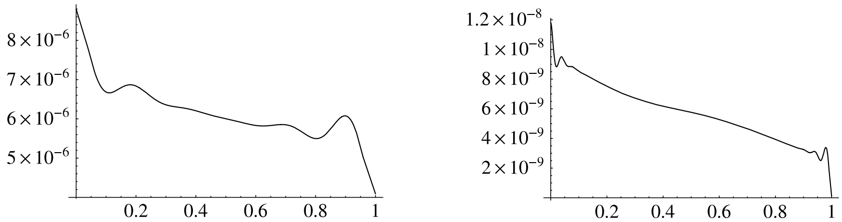

Fig. 2. Absolute errors of $u_{11}(x)$ and $u_{51}(x)$ in W^4 for Example 4.1.

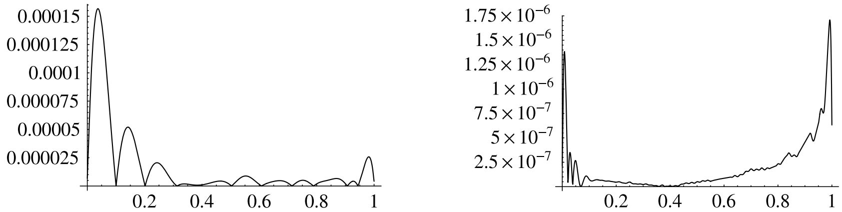

Fig. 3. Absolute errors of $u_{11}(x)$ and $u_{51}(x)$ in W^3 for Example 4.2.

4. Numerical examples

Example 4.1. Consider the following multi-point functional differential equation

$$\begin{cases} u'(x) + u(x) - \sin\left(\sqrt{x}\right) u\left(\dfrac{x^2}{2}\right) = f(x), \quad 0 < x < 1, \\ u(0) - u\left(\dfrac{1}{8}\right) - u\left(\dfrac{1}{2}\right) + a_0 u\left(\dfrac{1}{8}\right) = 0, \end{cases}$$

where $a_0 = \cosh(\frac{7}{8})[\sinh(\frac{1}{8}) - \sinh(\frac{1}{2})]$, $f(x)$ is given such that the exact solution is $u(x) = \sinh(x)$.

Using the method presented in Section 2, taking $x_i = (i-1)/(N-1),\ i = 1, 2, \dots, N, N = 11,\ 51$, the numerical results of $u_N(x)$ in different reproducing kernel spaces are shown in Figs. 1–2.

Example 4.2. Consider the following functional differential equation with integral condition

$$\begin{cases} u'(x) + 2000u(x) - 500e^x u\left(\sqrt{x}\right) = f(x), \quad 0 < x < 1, \\ u(1) = 5\displaystyle\int_0^1 su(s)ds, \end{cases}$$

where $f(x)$ is given such that the exact solution is $u(x) = x^3$.

Using the method presented in Section 2, taking $x_i = (i-1)/(N-1),\ i = 1, 2, \dots, N, N = 11, 51$, the numerical results of $u_N(x)$ in different reproducing kernel spaces are shown in Figs. 3–4.

Example 4.3. For comparison, we consider the pantograph equation [16,20]

$$\begin{cases} u'(x) + u(x) - \dfrac{1}{10} u\left(\dfrac{x}{5}\right) = -\dfrac{1}{10} e^{-0.2x}, \quad 0 < x < 1, \\ u(0) = 1. \end{cases}$$

It is easy to see that the exact solution is $u(x) = e^{-x}$.

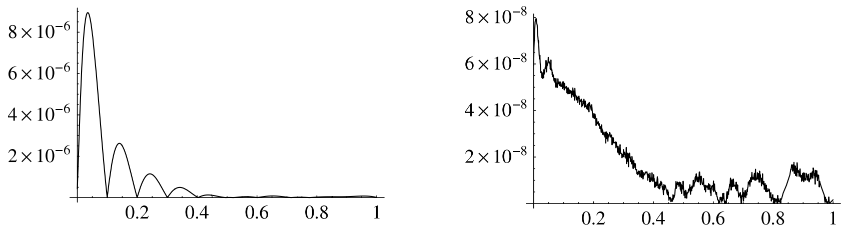

Fig. 4. Absolute errors of $u_{11}(x)$ and $u_{51}(x)$ in W^4 for Example 4.2.

Table 1
Comparison of absolute error for Example 4.3.

x	Muroya [16]	Taylor method [20]	Present method ($N = 11$)	Present method ($N = 151$)
2^{-1}	2.19×10^{-5}	1.24×10^{-10}	1.59×10^{-6}	3.33×10^{-11}
2^{-2}	1.08×10^{-5}	9.74×10^{-11}	1.87×10^{-6}	4.13×10^{-11}
2^{-3}	3.81×10^{-5}	7.00×10^{-11}	2.71×10^{-6}	4.62×10^{-11}
2^{-4}	1.26×10^{-5}	9.14×10^{-11}	2.41×10^{-6}	4.89×10^{-11}
2^{-5}	4.09×10^{-5}	5.28×10^{-11}	1.00×10^{-6}	5.05×10^{-11}
2^{-6}	1.20×10^{-5}	1.95×10^{-11}	3.51×10^{-7}	4.74×10^{-11}

In Table 1, the absolute errors of the present method for $N = 11, 81$ are compared with Muroya method [16] and Taylor method of [20].

5. Conclusion

In this paper, based on reproducing kernel theory, a continuous method is proposed for solving nonlocal first order functional differential equations with delayed or advanced arguments. Also, the error estimation of the present method is developed. The results of numerical examples show that the present method can provide accurate approximate solutions. In addition, the present method is also effective for linear functional differential equations with large coefficients.

Acknowledgments

The author would like to express thanks to the unknown referees for their careful reading and helpful comments. This work is supported by the National Natural Science Foundation of China (No. 11126222, 11271100).

References

[1] G. Akram, H. Ur Rehman, Numerical solution of eighth order boundary value problems in reproducing Kernel space, Numer. Algorithms 62 (2013) 527–540.
[2] O.A. Arqub, M. Al-Smadi, S. Momani, Application of reproducing kernel method for solving nonlinear Fredholm–Volterra integro-differential equations, Abstr. Appl. Anal. 2012 (2012) 1–16.
[3] Z. Chen, W. Jiang, The exact solution of a class of Volterra integral equation with weakly singular kernel, Appl. Math. Comput. 217 (2011) 7515–7519.
[4] M.G. Cui, F.Z. Geng, Solving singular two-point boundary value problem in reproducing kernel space, J. Comput. Appl. Math. 205 (2007) 6–15.
[5] M.G. Cui, F.Z. Geng, A computational method for solving third-order singularly perturbed boundary-value problems, Appl. Math. Comput. 198 (2008) 896–903.
[6] M.G. Cui, Y.Z. Lin, Nonlinear Numerical Analysis in Reproducing Kernel Space, Nova Science Pub Inc., Hauppauge, 2009.
[7] F.Z. Geng, New method based on the HPM and RKHSM for solving forced Duffing equations with integral boundary conditions, J. Comput. Appl. Math. 233 (2009) 165–172.
[8] F.Z. Geng, M.G. Cui, Solving a nonlinear system of second order boundary value problems, J. Math. Anal. Appl. 327 (2007) 1167–1181.
[9] F.Z. Geng, M.G. Cui, A reproducing kernel method for solving nonlocal fractional boundary value problems, Appl. Math. Lett. 25 (2012) 818–823.
[10] H. Kocak, A. Yildirim, Series solution for a delay differential equation arising in electrodynamics, Commun. Numer. Methods. Eng. 25 (2009) 1084–1096.
[11] X.Y. Li, B.Y. Wu, A novel method for nonlinear singular fourth order four-point boundary value problems, Comput. Math. Appl. 62 (2011) 27–31.
[12] Y.Z. Lin, M.G. Cui, Y.F. Zhou, Numerical algorithm for parabolic problems with non-classical conditions, J. Comput. Appl. Math. 230 (2009) 770–780.
[13] Y.Z. Lin, Y.F. Zhou, Solving the reaction–diffusion equation with nonlocal boundary conditions based on reproducing kernel space, Numer. Methods Partial Differential Equations 25 (2009) 1468–1481.
[14] M.Z. Liu, D.S. Li, Properties of analytic solution and numerical solution of multi-pantograph equation, Appl. Math. Comput. 155 (2004) 853–871.
[15] M. Mohammadi, R. Mokhtari, Solving the generalized regularized long wave equation on the basis of a reproducing kernel space, J. Comput. Appl. Math. 235 (2011) 4003–4014.
[16] Y. Muroya, E. Ishiwata, H. Brunner, On the attainable order of collocation methods for pantograph integro-differential equations, J. Comput. Appl. Math. 152 (2003) 347–366.
[17] K. Özen, K. Oruçoğlu, Investigation of numerical solution for fourth-order nonlocal problem by the reproducing kernel method, AIP Conf. Proc. 1389 (2011) 1164–1167.
[18] R. Rodríguez-López, Nonlocal boundary value problems for second-order functional differential equations, Nonlinear Anal. 74 (2011) 7226–7239.
[19] A. Saadatmandi, M. Dehghan, Variational iteration method for solving a generalized pantograph equation, Comput. Math. Appl. 58 (2009) 2190–2196.
[20] M. Sezer, S. Yalçinbaş, M. Gülsu, A taylor polynomial approch for solving generalized pantograph equations with nonhomogeneous term, Int. J. Comput. Math. 85 (2008) 1055–1063.

[21] M. Sezer, S. Yalçinbaş, N. Şahin, Approximate solution of multi-pantograph equation with variable coefficients, J. Comput. Appl. Math. 214 (2008) 406–416.
[22] Y.L. Wang, X.J. Cao, X.N. Li, A new method for solving singular fourth-order boundary value problems with mixed boundary conditions, Appl. Math. Comput. 217 (2011) 7385–7390.
[23] W.Y. Wang, B. Han, M. Yamamoto, Inverse heat problem of determining time-dependent source parameter in reproducing kernel space, Nonlinear Anal. RWA 14 (2013) 875–887.
[24] W.B. Wang, J.H. Shen, Z.G. Luo, Multi-point boundary value problems for second-order functional differential equations, Comput. Math. Appl. 56 (2008) 2065–2072.
[25] W.S. Wang, T. Qin, S.F. Li, Stability of one-leg h-methods for nonlinear neutral differential equations with proportional delay, Appl. Math. Comput. 213 (2009) 177–183.
[26] H.M. Yao, Y.Z. Lin, Solving singular boundary-value problems of higher even-order, J. Comput. Appl. Math. 223 (2009) 703–713.