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Global attractivity in a population model with nonlinear death rate and distributed delays

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ABSTRACT

We study the global attractivity of the unique positive equilibrium of a population model with distributed delays and nonlinear death rate. Both delay dependent and delay independent criteria are obtained which generalize, unify and improve known criteria. These results will be applied to some models with bounded and unbounded death functions.

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1. Introduction

In this paper, we investigate the global attractivity of first order differential equations of the form

$$x'(t) = p(t)\left[f\left(\int_{\tau(t)}^{t} h\big(x(s)\big)\, d_s\mu(t,s) \right) - g\big(x(t)\big) \right], \quad t \geqslant 0 \tag{1.1}$$

which model the growth of a single species population with distributed delays in the production (birth) rate f and nonlinear death (mortality) rate g. The presence of the function $p(t)$ makes it possible to model the growth when the environment parameters are changing proportionally with time. We assume that $p \in C[[0,\infty),(0,\infty)]$, $\tau \in C[[0,\infty),R]$ such that $\lim_{t\to\infty} \tau(t) = \infty$, $\tau(t) < t$ and

(C1) $\mu(t,s)$ is nondecreasing in s, continuous with respect to t and is normalized so that $\int_{\tau(t)}^{t} d_s\mu(t,s) = 1$.
(C2) $f \circ h$, $h \in C[(a,\infty),(0,\infty)]$ and g is a positive continuous and increasing function on (a,∞) for some $a \geqslant -\infty$.

With each solution of (1.1) we associate an initial continuous function $\phi : [-\tau, 0] \to (a,\infty)$ where $\tau = -\inf\{\tau(t):\ t \in [0,\infty)\} > 0$. The existence and uniqueness of a solution x associated with certain initial function ϕ can be proved using well known standard techniques; see for example the one used by [6] which in general requires the functions f, h to be Lipschitzian. Also, for some particular cases of (1.1) as the equation

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$$x'(t) = p(t)\left[f\left(\int_{\tau(t)}^{\beta(t)} h\big(x(s)\big)\, d_s\bar{\mu}(t,s) \right) - \alpha x(t) \right], \tag{1.2}$$

where $\tau(t) \leqslant \beta(t) < t$ and all $t \in (0,\infty)$, a method of steps as in [19, Theorem 1.1.3] can be easily used to prove that its initial value problem has a unique solution without further assumptions on f, h other than their continuity. Therefore, we focus in this work only on the global attractivity of (1.1) bearing in mind that we deal only with continuously differentiable solutions on $[a,\infty)$.

Eq. (1.1) contains many important known general models from mathematical biology such as the non-autonomous balance equation

$$x'(t) = p(t)\big[H\big(x(t-\tau)\big) - g\big(x(t)\big)\big]. \tag{1.3}$$

The autonomous version of this equation has been derived by Blythe et al. [5] which in turn includes many important prototypes such as:

$$x'(t) = \frac{\beta_0\theta^n}{\theta^n + x^n(t-\tau)} - \gamma x(t), \tag{1.4}$$

$$x'(t) = \frac{\beta_0\theta^n x(t-\tau)}{\theta^n + x^n(t-\tau)} - \gamma x(t), \tag{1.5}$$

$$x'(t) = px(t-\tau)e^{-ax(t-\tau)} - \delta x(t), \tag{1.6}$$

and

$$x'(t) = pe^{-\gamma x(t-\tau)} - \mu x(t). \tag{1.7}$$

Eqs. (1.4) and (1.5) were used by Mackey and Glass [29] in order to describe some physiological control systems. Eq. (1.6) is the celebrated Nicholson blowflies model which was proposed by Gurney et al. [18]; while Eq. (1.7) was used by Wazewska-Czyzewska and Lasota [33] as a model of an erythropoietin system. For further results on the dynamical characteristics of the models (1.4)–(1.7) the reader is referred to [15,17,19,20,24,25,27,28].

Very little research was done on the global attractivity for equations with nonlinear death function in comparison with equations with linear one. For example; Freedman and Gopalsamy [14] studied the autonomous version of (1.3) which appears also as a special case of the model studied by Ivanov et al. [22]. Kuang [25] gave many global attractivity criteria for the equation

$$x'(t) = f\left(\int_{-\tau}^{-\sigma} x(t+s)\, d\mu(s) \right) - g\big(x(t)\big). \tag{1.8}$$

Recently, Qian [32] improved some of Kuang's results for the non-autonomous model (1.3). Arino et al. [1] and Berezansky et al. [2] studied two prototypes of (1.3) when g is quadratic.

In this work, we consider increasing death functions that can be bounded at infinity. Therefore the obtained results unify, generalize and improve some known global attractivity criteria; particularly our results improve the delay dependent criteria obtained by [6,22,25,32] and contain those of [1,2]. The obtained results will be applied on models with nonlinear death function such as a model mentioned by [3] which has the form

$$x'(t) = qx(t-\tau)e^{-x(t-\tau)} - g(x), \tag{1.9}$$

where g might have the form: $g(x) = \frac{\alpha x}{x+b}$ with $\alpha, b > 0$.

We will also be able to obtain sufficient criteria for the global attractivity of the positive equilibrium; say $\bar{y}$, of models of the form

$$y'(t) = p(t)y(t)\left[f\left(\int_{\tau(t)}^{t} h^*\big(y(s)\big)\, d_s\mu(t,s) \right) - g^*\big(y(t)\big) \right]. \tag{1.10}$$

In fact, using the transformation $y(t) = \bar{y}e^{x(t)}$ which transforms (1.10) to equation of the form (1.1) with $h(x(t)) = h^*(\bar{y}e^{x(t)})$ and $g(x(t)) = g^*(\bar{y}e^{x(t)})$, all our results can be restated to (1.10). This will enable us to obtain new global attractivity criteria for the non-autonomous Mackey–Glass model (see [29])

$$z'(t) = p(t)\left[a - b\frac{z(t)z^n(t-\tau)}{1+z^n(t-\tau)} \right], \quad t \geqslant 0,$$

by transforming it to an equation of the form (1.10).

2. Main results

We consider the following assumptions

(C3) $(H(x)-g(x))(x-\bar{x})<0$ for all $x>a$, $x\neq\bar{x}$ where $H(x):=f(h(x))$ for all $x>a$.
(C4) $H(x)\leqslant M<g(\infty)$ for all $x\in(a,\bar{x})$ and some $M\in(0,\infty)$.
(C5) $g(+a)<H(x)$ for all $x>a$,

where $g(+a)=\lim_{x\to a^+}g(x)$ and $g(\infty)=\lim_{x\to\infty}g(x)\leqslant\infty$.

The negative feedback criterion (C3) implies that $\bar{x}$ is the unique fixed point in (a,∞) of the equation

$$H(x)-g(x)=0.$$

Therefore, (1.1) has a unique equilibrium in (a,∞).

To avoid repetition in our proofs, we refer that (1.1) can be written in the form

$$x'(t)=p(t)\big[H\big(x\big(T(t)\big)\big)-g\big(x(t)\big)\big] \tag{2.1}$$

where $T(t)$ is a certain number in the interval $(\tau(t),t)$ for each $t>0$. The existence of T can be easily proved using the normalized property (C1) and the Intermediate Value Theorem.

Lemma 2.1. *Assume that* (C1)–(C2) *and* (C5) *are satisfied. If x is a solution of* (1.1)*, then $x(t)>a$ for all $t>0$.*

Proof. The lemma is clearly satisfied when $a=-\infty$. For the case $a>-\infty$, we assume the contrary; that is there exists $\bar{a}>0$ such that $x(\bar{a})=a$ with $x(t)>a$ for all $t<\bar{a}$. Therefore, $x'(\bar{a})\leqslant 0$ which yields in view of (1.1) that

$$0\geqslant p(\bar{a})\left[f\left(\int_{\tau(\bar{a})}^{\bar{a}}h\big(x(s)\big)\,d_s\mu(\bar{a},s)\right)-g\big(+x(\bar{a})\big)\right]=p(\bar{a})\big[H\big(x\big(T(\bar{a})\big)\big)-g(+a)\big]>0$$

which is a contradiction. □

Lemma 2.2. *Assume that* (C1)–(C5) *are satisfied. If x is a solution of* (1.1)*, then*

$$a<\liminf_{t\to\infty}x(t)\leqslant\limsup_{t\to\infty}x(t)<\infty.$$

Proof. We use contradiction to prove this result. For this reason, we assume that

$$\limsup_{t\to\infty}x(t)=\infty.$$

Then there exists a sequence $\{t_n\}$, $t_n>0$ for all $n=1,2,\ldots$ and $\lim_{n\to\infty}t_n=\infty$ such that

$$x(t_n)=\max\big\{x(t)\colon t\leqslant t_n\big\},$$

and

$$x'(t_n)\geqslant 0\quad\text{for all } n\geqslant 1.$$

Letting $t=t_n$ in (2.1). It follows that

$$g\big(x(t_n)\big)\leqslant H\big(x\big(T(t_n)\big)\big). \tag{2.2}$$

Using the increasing nature of g, we have

$$g\big(x\big(T(t_n)\big)\big)\leqslant H\big(x\big(T(t_n)\big)\big)$$

which, according to (C3), means that $x(T(t_n))\in(a,\bar{x}]$ and hence (C4), (2.2) yield

$$g\big(x(t_n)\big)\leqslant M<g(\infty).$$

Passing n to ∞, we obtain $g(\infty)\leqslant M<g(\infty)$ which is impossible.

To prove that $\liminf_{t\to\infty}x(t)>a$, we use similar arguments as above. So, assume that

$$\liminf_{t\to\infty}x(t)=a,$$

we have two partial cases:

Case 1: $\lim_{t\to\infty} x(t) = a$ and Case 2: $a = \liminf_{t\to\infty} x(t) < \limsup_{t\to\infty} x(t)$. In any of these cases, one can pick a positive sequence $\{t'_n\}$ and a sufficiently large $t_0 > 0$ such that $\lim_{n\to\infty} t'_n = \infty$, $x(t'_n) = \min\{x(t):\ t \leqslant t'_n\}$, $\lim_{n\to\infty} x(t'_n) = a$ with $x'(t'_n) \leqslant 0$, $t'_n > t_0$ and $x(t) < \bar{x}$ for all $t > t_0$ for the first case while $x(t'_n) < \bar{x}$, $x'(t'_n) = 0$ for the second case where $n = 1, 2, \ldots$.

Now, if the first case holds, then (2.1) yields

$$g\big(x(t'_n)\big) \geqslant H\big(x\big(T\big(t'_n\big)\big)\big) \quad \text{for all } n > 0,$$

which in view of the increasing nature of g and the definition of $x(t'_n)$ yields

$$g\big(x\big(T\big(t'_n\big)\big)\big) \geqslant H\big(x\big(T\big(t'_n\big)\big)\big) \quad \text{for all } n > 0,$$

and hence, using (C3), we obtain

$$x\big(T\big(t'_n\big)\big) \geqslant \bar{x} \quad \text{for all } n > 0,$$

which contradicts the choice of $\{t'_n\}$ for this case.

If the second case holds, then (2.1) yields

$$g\big(x(t'_n)\big) = H\big(x\big(T\big(t'_n\big)\big)\big) \quad \text{for all } n > 0,$$

which implies the previous inequality. Thus $\{x(T(t'_n))\}$ is a bounded sequence and therefore has a convergent subsequence; say to a number $l \geqslant \bar{x}$. Then $g(+a) = H(l)$ which contradicts (C5). □

The following lemma gives us more insight into the boundedness property proved in Lemma 2.2.

Lemma 2.3. *Assume that* (C1)–(C5) *are satisfied. If x is a solution of* (1.1) *that is not attracted to $\bar{x}$, then there exist real numbers $L, S, L_*, S_* > a$ such that $L_*, S_* \in [L, S]$,*

$$\begin{aligned} &\limsup_{t\to\infty} x(t) = S, \qquad \liminf_{t\to\infty} x(t) = L,\\ &g(S) = H(S_*), \qquad g(L) = H(L_*), \end{aligned} \tag{2.3}$$

and

$$L \leqslant S_* < \bar{x} < L_* \leqslant S. \tag{2.4}$$

Moreover, $S \leqslant g^{-1}(M)$ and $L \geqslant g^{-1}(m) > a$ where $m = \min\{H(x):\ x \in [\bar{x}, g^{-1}(M)]\}$.

Proof. From Lemma 2.2, we deduce that $\liminf_{t\to\infty} x(t)$ and $\limsup_{t\to\infty} x(t)$ are finite. Since these two limits are not equal, then $x(t)$ fluctuates between these two values at infinity. Therefore, one can choose two sequences $\{\lambda_n\}$ and $\{\beta_n\}$ with the following properties:

$$\begin{aligned} &\lim_{n\to\infty} \lambda_n = \lim_{n\to\infty} \beta_n = \infty,\\ &x'(\lambda_n) = 0, \qquad \lim_{n\to\infty} x(\lambda_n) = L,\\ &x'(\beta_n) = 0, \qquad \lim_{n\to\infty} x(\beta_n) = S, \end{aligned}$$

and

$$\lim_{n\to\infty} x\big(T(\lambda_n)\big) = L_*, \qquad \lim_{n\to\infty} x\big(T(\beta_n)\big) = S_*.$$

Now calculating (1.1) at $t = \lambda_n$, $t = \beta_n$ and taking limits as $n \to \infty$, we obtain (2.3).

Since $L_*, S_* \in [L, S]$, then (2.3) yields

$$g(S_*) \leqslant H(S_*) \quad \text{and} \quad g(L_*) \geqslant H(L_*).$$

In view (C3), these two inequalities imply that $S_* \leqslant \bar{x} \leqslant L_*$. In fact $S_* \neq \bar{x}$ and $L_* \neq \bar{x}$ since otherwise the relations in (2.3) lead to $L = S = \bar{x}$ which is not possible since $L < S$. Thus (2.4) is proved.

The rest of the lemma follows immediately from (C4), (2.3) and the increasing nature of g. □

Since $g^{-1}(M)$, $g^{-1}(m)$ are independent of any solution of (1.1), then Lemma 2.3 implies that (1.1) is permanent.

When the system has no equilibrium other that a with $a > -\infty$, we assume that $H(a) = g(a)$ and

(C6) $H(x) < g(x)$ for all $x > a$.

Lemma 2.4. *Assume that* (C1)–(C2), (C5) *and* (C6) *are satisfied and*

$$\int^{\infty} p(t)\,dt = \infty. \tag{2.5}$$

If x is a solution of (1.1), *then* $\lim_{t\to\infty} x(t) = a$.

Proof. Starting as in the proof of Lemma 2.2, we obtain (2.2) which, in view of (C6), yields $g(x(t_n)) < g(x(T(t_n)))$. Using the increasing nature of g, we get $x(t_n) < x(T(t_n))$ which contradicts the definition of $x(t_n)$. Thus $\limsup_{t\to\infty} x(t) < \infty$.

If $\limsup_{t\to\infty} x(t) > a$, then we see that $\lim_{t\to\infty} x(t) = l \geqslant a$ since otherwise one may define the sequence $\{\beta_n\}$ and the numbers S, S_* as in the proof of Lemma 2.3 so (2.1) yields

$$0 = \lim_{n\to\infty} H\big(x\big(T(\beta_n)\big)\big) - g\big(x(\beta_n)\big) = H(S_*) - g(S),$$

which is impossible since $S_* \geqslant a$.

Now if $l > a$, then

$$\lim_{t\to\infty} H\big(x\big(T(t)\big)\big) - g\big(x(t)\big) = H(l) - g(l) = -d < 0.$$

Hence, for sufficiently small $\epsilon > 0$ there exists $t^* > 0$ such that

$$H\big(x\big(T(t)\big)\big) - g\big(x(t)\big) < -(d-\epsilon) < 0 \quad \text{for all } t > t^*.$$

Combining this with (2.1), we have

$$x'(t) < -(d-\epsilon)p(t), \quad t > t^*.$$

Integrating from t^* to t and allowing $t \to \infty$, then condition (2.5) yields $\lim_{t\to\infty} x(t) = -\infty$ which is a contradiction with the assumption that $l > a$. □

The following lemma is adapted from [12, Lemma 2.5]. It is the basic tool of proving our main results.

Lemma 2.5. *Assume that $F \in C[I, I]$ for a real interval I and there exists a unique $p \in I$ such that*

$$\big(F(x) - x\big)(x - p) < 0 \quad \textit{for all } x \in I,\ x \neq p.$$

If p is the global attractor of the first order difference equation

$$x_{n+1} = F(x_n), \quad x_0 \in I,\ n = 0, 1, 2, \ldots$$

then there are no points $c \neq d$ in I $(c < p < d)$ satisfying that

$$F(c) \geqslant d, \qquad F(d) \leqslant c.$$

Now, let us define α_1, α_2 and F as follows

$$\alpha_1 = g^{-1}(m), \qquad \alpha_2 = g^{-1}(M), \qquad F = g^{-1} \circ H.$$

Then from (C3), we obtain

$$\big(F(x) - x\big)(x - \bar{x}) < 0 \quad \text{for all } x \in [\alpha_1, \alpha_2],\ x \neq \bar{x}.$$

Moreover, from (C3), (C4) and the definition of m in Lemma 2.3, we have $F : I \to I$ where $I = [\alpha_1, \alpha_2]$. On the other hand (2.3) yields

$$L_* \leqslant S = F(S_*) \quad \text{and} \quad S_* \geqslant L = F(L_*).$$

Therefore $\bar{x}$ cannot be a global attractor of the equation $x_{n+1} = F(x_n)$ (for $x_0 \in I$). This proves the following interesting result.

Theorem 2.1. *Assume that* (C1)–(C5) *are satisfied. If $\bar{x}$ is the global attractor of the deference equation $x_{n+1} = g^{-1}(H(x_n))$ with respect to I, then $\bar{x}$ is the global attractor of* (1.1) *on* (a, ∞).

When H is nondecreasing on I, we see from (C3) that (2.3) holds only when $L = S = \bar{x}$. This proves the following particular case of Theorem 2.1 which cover the cases of models with quadratic death functions studied in [1,2].

Corollary 2.1. *Assume that* (C1)–(C5) *are satisfied. If H is nondecreasing on I, then $\bar{x}$ is the global attractor of* (1.1) *on* (a, ∞).

Theorem 2.2. *Assume that* (C1)–(C5) *are satisfied. If x is a solution of* (1.1) *which is not attracted to $\bar{x}$ and*

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds \leqslant p_0,$$

then

$$\int_{S_*}^{S} \frac{dx}{\bar{H} - g(x)} \leqslant p_0 \tag{2.6}$$

and

$$\int_{L_*}^{L} \frac{dx}{\underline{H} - g(x)} \leqslant p_0, \tag{2.7}$$

where $\underline{H} = \min\{H(x)\colon x \in [L, S]\}$, $\bar{H} = \max\{H(x)\colon x \in [L, S]\}$ *and* L, L_*, S_*, S *are defined as in Lemma 2.3.*

Proof. Since $\liminf_{t\to\infty} x(t) = L < \limsup_{t\to\infty} x(t) = S$, then for a sufficiently small $\epsilon > 0$, there exists $t_* > 0$ such that $x(t), x(T(t)) \in \Omega_\epsilon = [L - \epsilon, S + \epsilon]$ for all $t \geqslant t_*$. Define two positive increasing sequences $\{R_n\}$, $\{R_n^*\}$ such that $R_1 > t_*$, $T(R_n) > t_*$ for $n \geqslant 1$, $\lim_{n\to\infty} R_n = \lim_{n\to\infty} R_n^* = \infty$, and

$$\lim_{n\to\infty} x(R_n) = S, \quad x(R_n) < S \ \big(\text{where } x'(R_n) \geqslant 0\big) \text{ for all } n > 1.$$

For a suitably small $\lambda > 0$ one can find a positive sequence $\{\epsilon_n^1\}$, $\lim_{n\to\infty} \epsilon_n^1 = \infty$ such that:

$$\lim_{n\to\infty} x\big(R_n - \epsilon_n^1\big) = S - \lambda.$$

Now $\{R_n^*\}$ is defined as follows

$$R_n^* = l_n \in \big[T(R_n), R_n\big) \quad \text{for all } n \geqslant 1 \text{ such that } x(t) \in [S_*, S),$$
$$\text{for all } t \in [l_n, R_n) \text{ and } \lim_{n\to\infty} x(l_n) = S_*.$$

In a similar fashion; two positive increasing sequences $\{r_n\}$, $\{r_n^*\}$ related to L, L_* can be defined such that $r_1 > t_*$, $T(r_n) \geqslant t_*$, $\lim_{n\to\infty} r_n = \lim_{n\to\infty} r_n^* = \infty$ and $\lim_{n\to\infty} x(r_n) = L$ where $x(r_n) \geqslant L$ (here $x'(r_n) \leqslant 0$). It follows that a positive sequence $\{\epsilon_n^2\}$ exists such that $\lim_{n\to\infty} \epsilon_n^2 = \infty$ and $\lim_{n\to\infty} x(r_n - \epsilon_n^2) = L + \lambda$ where λ is defined as before. The sequence $\{r_n^*\}$ is defined as follows:

$$r_n^* = l_n^* \in \big[T(r_n), r_n\big) \quad \text{for all } n \geqslant 1 \text{ such that } x(t) \in (L, L_*],$$
$$\text{for all } t \in \big[l_n^*, r_n\big) \text{ and } \lim_{n\to\infty} x\big(l_n^*\big) = L_*.$$

Now, (2.1) leads to the following two inequalities:

$$x'(t) \leqslant p(t)\big[\bar{H}_\epsilon - g\big(x(t)\big)\big], \quad t \in \big[R_n^*, R_n\big) \text{ for all } n \geqslant 1, \tag{2.8}$$

and

$$x'(t) \geqslant p(t)\big[\underline{H}^\epsilon - g\big(x(t)\big)\big], \quad t \in \big[r_n^*, r_n\big) \text{ for all } n \geqslant 1. \tag{2.9}$$

But

$$g\big(x(t)\big) > g(L) = H(L_*) \geqslant \underline{H}^\epsilon \quad \text{for all } t \in \big[r_n^*, r_n\big)$$

and

$$g\big(x(t)\big) < g(S) = H(S_*) \leqslant \bar{H}_\epsilon \quad \text{for all } t \in \big[R_n^*, R_n\big),$$

where $\underline{H}^\epsilon = \min\{H(x), x \in \Omega_\epsilon\}$ and $\bar{H}_\epsilon = \max\{H(x), x \in \Omega_\epsilon\}$. On the other hand, taking λ to be small enough (so that $L + \lambda < L_*$, $S - \lambda > S_*$) will guarantee the existence of a sufficiently large integer n_0 such that

$$R_n - \epsilon_n^1 > R_n^*, \qquad r_n - \epsilon_n^2 > r_n^* \quad \text{for all } n \geqslant n_0,$$

then (2.8) and (2.9) imply respectively that

$$\int_{x(R_n^*)}^{x(R_n-\epsilon_n^1)} \frac{dx}{\bar{H}_\epsilon - g(x)} \leqslant \int_{R_n^*}^{R_n-\epsilon_n^1} p(s)\,ds \leqslant \int_{\tau(R_n)}^{R_n} p(s)\,ds$$

and

$$\int_{x(r_n^*)}^{x(r_n-\epsilon_n^2)} \frac{dx}{\underline{H}^\epsilon - g(x)} \leqslant \int_{r_n^*}^{r_n-\epsilon_n^2} p(s)\,ds \leqslant \int_{\tau(r_n)}^{r_n} p(s)\,ds,$$

for all $n \geqslant n_0$. Now, let $n \to \infty$. Then, we obtain

$$\int_{S_*}^{S-\lambda} \frac{dx}{\bar{H}_\epsilon - g(x)} \leqslant \limsup_{n\to\infty} \int_{\tau(R_n)}^{R_n} p(s)\,ds \leqslant p_0,$$

and

$$\int_{L_*}^{L+\lambda} \frac{dx}{\underline{H}^\epsilon - g(x)} \leqslant \limsup_{n\to\infty} \int_{\tau(r_n)}^{r_n} p(s)\,ds \leqslant p_0.$$

Letting $\epsilon \to 0$ in the above two integrals, we get

$$\int_{S_*}^{S-\lambda} \frac{dx}{\bar{H} - g(x)} \leqslant p_0,$$

and

$$\int_{L_*}^{L+\lambda} \frac{dx}{\underline{H} - g(x)} \leqslant p_0.$$

Since the above two integrals are decreasing in λ and are bounded, then they exist as $\lambda \to 0$. That is the inequalities (2.6), (2.7) are satisfied. □

The contrapositive of the previous result supplies us with the following interesting global attracting criterion.

Theorem 2.3. *Assume that* (C1)–(C5) *are satisfied. If the only possible solution of* (2.6) *and* (2.7) *is* $L = S = \bar{x}$, *then* $\bar{x}$ *is the global attractor of all solutions of* (1.1).

We give the following particular cases of Theorem 2.3. The proofs of these results are straightforward applications of Theorem 2.3.

Corollary 2.2. *Assume that all assumptions of Theorem 2.3 are satisfied. If* $p_0 = 0$, *then* $\bar{x}$ *is the global attractor of all solutions of* (1.3).

The above result tells us that the delay has a harmless effect on the global attractivity when $p_0 = 0$. For example; if (C1)–(C5) are satisfied, $p(t) = 1/t$ and $\tau(t) = t - \tau$, then the unique positive equilibrium of (1.1) is globally attractive regardless the length of the delay τ.

Corollary 2.3. *Assume that all assumptions of Theorem 2.3 are satisfied. If the inequalities*

$$\int_{\bar{x}}^{S} \frac{dx}{\bar{H} - g(x)} \leqslant p_0 \tag{2.10}$$

and

$$\int_{\bar{x}}^{L} \frac{dx}{\underline{H} - g(x)} \leqslant p_0, \tag{2.11}$$

are satisfied only if $S = L = \bar{x}$, then $\bar{x}$ is the global attractor of all solutions of (1.1).

If H is enveloped by a decreasing function E, that is $(E(x) - H(x))(x - \bar{x}) \leqslant 0$ for $x \in I - \{\bar{x}\}$, then (2.3) yields $S_* \leqslant E^{-1}(g(S))$, $L_* \geqslant E^{-1}(g(L))$, $\bar{E} = E(L)$ and $\underline{E} = E(S)$. This leads to the following result.

Corollary 2.4. *Assume that all assumptions of Theorem 2.3 are satisfied where H is enveloped by a decreasing function E on I. If the inequalities*

$$\int_{E^{-1}(g(S))}^{S} \frac{dx}{E(L) - g(x)} \leqslant p_0, \tag{2.12}$$

and

$$\int_{E^{-1}(g(L))}^{L} \frac{dx}{E(S) - g(x)} \leqslant p_0 \tag{2.13}$$

are satisfied only if $S = L = \bar{x}$, then $\bar{x}$ is the global attractor of all solutions of (1.1).

Remark 2.1. For the first glance, the inequalities (2.6), (2.7) may look vague. But the integrals in these inequalities are increasing in S, L_* and decreasing in L, S_*. Moreover, the negative feedback assumption (C3) and (2.3) lead to the existence of the numbers $L \leqslant \eta < \bar{x} < \zeta \leqslant S$ such that $\bar{H} = H(\eta)$ and $\underline{H} = H(\zeta)$. Solving (2.6) and (2.7) in S and L respectively, a function $\Gamma(p_0, x, y)$ could be found such that

$$S \leqslant \Gamma(p_0, S_*, \eta) \quad \text{and} \quad L \leqslant \Gamma(p_0, L_*, \zeta).$$

If it is possible to substitute S_* and L_* in these inequalities, using (2.3), (2.4) in terms may be of $\bar{x}$ or S and L, then we come into the following inequalities

$$S \leqslant W(\eta) \quad \text{and} \quad L \geqslant W(\zeta)$$

for some function W and hence Lemma 2.5 can be used to obtain explicit global attractivity criteria. This idea will be applied in the next section; see for example the proofs of Theorems 3.3 and 3.4.

Corollary 2.5. *Assume that all assumptions of Theorem 2.3 are satisfied. If there exists a function $\bar{g} : [\alpha_1, \alpha_2] \to [\alpha_1, \alpha_2]$ such that $(\bar{g}(x) - g(x))(x - \bar{x}) < 0$ for all $x \in [\alpha_1, \alpha_2]$, $x \neq \bar{x}$, then $\bar{x}$ is the global attractor of all solutions of (1.1) if all assumptions of Corollary 2.3 are satisfied with g replaced by $\bar{g}$.*

The following two corollaries are direct applications of Corollary 2.3 and Corollary 2.4 respectively when $g(x) = \alpha x$ for all $x > a$.

Corollary 2.6. *Assume that* (C1)–(C5) *hold with $g(x) := \alpha x$ for all $x > a$ and some constant $\alpha > 0$. Let*

$$K(x) = \bar{x}e^{-\alpha p_0} + \frac{1}{\alpha}\left(1 - e^{-\alpha p_0}\right)H(x) \quad \text{for all } x \in [\alpha_1, \alpha_2].$$

Then

(i) $(K(x) - x)(x - \bar{x}) < 0$ *for all $x \in [\alpha_1, \alpha_2]$, $x \neq \bar{x}$ and*

$$S \leqslant K(\eta), \qquad L \geqslant K(\zeta),$$

where $S_* \leqslant \eta < \bar{x} < \zeta \leqslant L_*$.

(ii) *If $\bar{x}$ is the global attractor of K on $[\alpha_1, \alpha_2]$ then $\bar{x}$ is the global attractor of (1.1) on (a, ∞).*

Corollary 2.7. *Assume that* (C1)–(C5) *hold with* $g(x) := \alpha x$ *for some* $\alpha > 0$ *and* H *is enveloped by a decreasing function* E *on* I. *Let*

$$\rho(x) = \sigma^{-1}\left(\frac{1}{\alpha}\left(1 - e^{-\alpha p_0}\right)E(x)\right),$$

where σ^{-1} *is the inverse function of a function* σ *defined by* $\sigma(x) = x - e^{-\alpha p_0}E^{-1}(g(x))$ *on* I. *Then*

(i) $(\rho(x) - x)(x - \bar{x}) < 0$ *for all* $x \in I$, $x \neq \bar{x}$ *and*

$$S \leqslant \rho(\eta), \qquad L \geqslant \rho(\zeta),$$

where $E^{-1}(g(S)) \leqslant \eta < \bar{x} < \zeta \leqslant E^{-1}(g(L))$.

(ii) *If* $\bar{x}$ *is the global attractor of* ρ *on* I, *then* $\bar{x}$ *is the global attractor of* (1.1) *on* (a, ∞).

We give Corollary 2.5 to increase the computability of our conditions. Another way for doing this is to use the increasing nature of g on $[L, S]$. In this case; since

$$\int_{S_*}^{S} \frac{dx}{H(\eta) - g(x)} \geqslant \frac{S - S_*}{H(\eta) - g(S_*)}$$

and

$$\int_{L_*}^{L} \frac{dx}{H(\zeta) - g(x)} \geqslant \frac{L - L_*}{H(\zeta) - g(L_*)},$$

then the inequalities (2.6) and (2.7) imply that

$$\begin{cases} S \leqslant S_* + p_0\big(H(\eta) - g(S_*)\big) \\ L \geqslant L_* + p_0\big(H(\zeta) - g(L_*)\big), \end{cases} \tag{2.14}$$

and similar inequalities corresponding to (2.10)–(2.13) can be obtained. Therefore, the following results can be easily derived from Theorem 2.3, Corollary 2.3 and Corollary 2.4.

Corollary 2.8. *Assume that* (C1)–(C5) *are satisfied and* α_1 *and* α_2 *are defined as in Theorem 2.1. Then* $L, S \in I$ *and in addition to* (2.14), *either one of the pairs*:

$$\begin{cases} S < \bar{x} + p_0\big(H(\eta) - g(\bar{x})\big), \\ L > \bar{x} + p_0\big(H(\zeta) - g(\bar{x})\big), \end{cases} \tag{2.15}$$

or

$$\begin{cases} S - E^{-1}\big(g(S)\big) + p_0 g\big(E^{-1}\big(g(S)\big)\big) \leqslant p_0 E(L), \\ L - E^{-1}\big(g(L)\big) + p_0 g\big(E^{-1}\big(g(L)\big)\big) \geqslant p_0 E(S), \end{cases} \tag{2.16}$$

is satisfied where E *is a decreasing function enveloping* H *on* I. *Moreover*; *if any of* (2.14), (2.15) *or* (2.16) *implies that* $L = S$, *then* $\bar{x}$ *is the global attractor of all solutions of* (1.1).

The following result is an application of the inequality (2.15) of the above corollary.

Corollary 2.9. *Assume that* (C1)–(C5) *are satisfied if* H *is Lipschitzian on* $[\alpha_1, \alpha_2]$ *with Lipschitz constant* D *and*

$$p_0 D \leqslant 1. \tag{2.17}$$

Then $\bar{x}$ *is the global attractor of all solutions of* (1.1).

Proof. From (2.15), we have

$$\begin{aligned} S - L &< p_0\big(H(\eta) - H(\zeta)\big) \\ &= p_0\big|H(\eta) - H(\zeta)\big| \leqslant p_0 D|\eta - \zeta| \\ &\leqslant S - L \end{aligned}$$

which is impossible. Thus $L = S = \bar{x}$.

Remark 2.2. The above results either improve or extend most of the known results for (1.1) and its particular cases. For example, Corollary 2.6 improves [20, Corollary 17]. Corollary 2.9 improves the mean result of [32] as well as [6, Theorem 7]. Also, we refer that [27,28] obtained sharp global stability conditions for the functional differential equation

$$x'(t) = -\delta x(t) + f(t, x_t), \quad x_t = x(t+s),\ s \in [\tau, 0].$$

These criteria are not contained in our results although they are applicable to (1.4)–(1.7) and equations with distributed delays as well.

As we have seen in the last results; we need to study the global attractivity of an accompanied first order difference equation; say $x_{n+1} = \bar{h}(x_n)$. This type of studies have been developed by many authors. For example [8,21,30] (and some papers cited therein) studied this problem using a well known result due to Coppel [7] which relates the global attractivity of the above equation to the absence of period two points of $\bar{h}$. Recently, Cull [9] introduced the enveloping-method to study this problem. In addition to the celebrated Lyapunov method, it is very common to use the Schwarzian derivative. This derivative for a map $\bar{h} : I \to I$, with a real interval I, is denoted by $(S\bar{h})(x)$ and is defined when $\bar{h}$ is a C^3-map as follows:

$$(S\bar{h})(x) = \frac{\bar{h}'''}{\bar{h}'} - \frac{3}{2}\left(\frac{\bar{h}''}{\bar{h}'}\right)^2, \quad x \in I.$$

When $(S\bar{h})(x) < 0$ for all $x \in I$, it is found that the local asymptotic stability implies the global one if $\bar{h}$ is decreasing on I or has at most one hump (see [11,26]). Recently [12] proved that this result for one hump functions still true when $(S\bar{h})(x) < 0$ for all $x \in (x^*, \infty) \cap I$ where x^* is the unique critical point (a maximum) of $\bar{h}(x)$.

Lemma 2.6. *Let $f : J \to J$ be a continuous map on a real interval J with a critical point x^* which is the right most among all absolute maximum points of f in J. Assume also that p is a fixed point of f with the negative feedback property*

$$\big(f(x) - x\big)(x - p) < 0 \quad \text{for all } x \in J - \{p\}.$$

Then p is the global attractor of f if one of the following conditions is satisfied:

(i) $f(x^*) = p$ *or* $p < f(x^*)$ *and* f *is nondecreasing on* $[p, f(x^*)]$.
(ii) $x^* < p < f(x^*)$, $f \in C^3[x^*, f(x^*)]$, $f'(p) \geqslant -1$, $f'(x) < 0$ *and* $(Sf)(x) < 0$ *for all* $x \in (x^*, f(x^*))$.

Proof. Consider the difference equation $x_{n+1} = f(x_n)$, $x_0 \in J$ where x_n does not converge to p. Then

$$x_{n+1} \leqslant f\big(x^*\big), \quad n \geqslant 0. \tag{2.18}$$

Thus there exist a, b, A, B such that $a, b \in [A, B] \subseteq J$ with $A = \liminf_{n\to\infty} x_n$, $B = \limsup_{n\to\infty} x_n$ and

$$A = f(b), \qquad B = f(a). \tag{2.19}$$

Therefore,

$$A \leqslant a < p < b \leqslant B. \tag{2.20}$$

If (i) holds, then (2.19) yields $B \leqslant p$ or $A \geqslant p$ which is impossible according to (2.20). Thus $\lim_{n\to\infty} x_n = p$ for this case.

If (ii) holds, we see from (2.19) that

$$a \geqslant A \geqslant f(B) = f^2(a).$$

On the other hand (ii) implies that p is locally asymptotically stable (see [10, pp. 28]) which according to [4, pp. 103], leads to the existence of a neighborhood of p (say O_p) such that

$$f^2(x) > x \quad \text{for } x < p,\ x \in O_p.$$

Thus f has a period two point which can be easily proved to lie in $[x^*, f(x^*)]$. If $u < p < v$ are the conjugate pair of this period, then

$$f^2(u) = u, \qquad f^2(p) = p \quad \text{and} \quad f^2(v) = v.$$

Hence there exist $u^* \in (u, \bar{x})$ and $v^* \in (\bar{x}, v)$ such that $\frac{df^2(x)}{dx^2}|_{x=u^*} = \frac{df^2(x)}{dx^2}|_{x=v^*} = 1$. Since $(Sf)(x) < 0$ and $\frac{df^2(x)}{dx^2}|_{x=p} = (f'(p))^2 \leqslant 1$, then we arrive at a contradiction due to Lemma 2.6 of Singer [31]. □

Remark 2.3. The above result can also be proved using [12, Cor. 2.9].

The following result is derived using Lemma 2.6 with Corollary 2.8.

Corollary 2.10. *Assume that* (C1)–(C5) *are satisfied and*

$$f(x) := \bar{x} + p_0\big(H(x) - g(\bar{x})\big) \quad \text{for } x \in [\alpha_1, \alpha_2]$$

satisfies the assumptions of Lemma 2.6 (for $p = \bar{x}$) with either (i) *or* (ii) *hold. Then $\bar{x}$ is the global attractor of all solutions of (1.1).*

Note that the assumption $-1 \leqslant f'(\bar{x}) < 0$ of Lemma 2.6 is equivalent to saying that

$$-1 \leqslant p_0 H'(\bar{x}) \leqslant 0 \tag{2.21}$$

which certainly improves (2.17) when $(SH)(x) < 0$ on $[\alpha_1, \alpha_2]$ in particular for one-hump functions.

Remark 2.4. The idea of relating the global attractivity of differential equations to first order difference equations is not new; see, for example, [19,20,22,23] and the references cited therein. One of the advantages of our results is the relaxation of the restriction (2.5) which is common in the related literatures (e.g. [6,13,32]).

3. Applications

Consider (1.1) with $g(x) = \frac{\alpha x}{b+x}$ and $H(x) = qxe^{-x}$ for all $x > 0$ where $\alpha, b, q > 0$. In this case Eq. (1.1) contains a model mentioned by [3, Section 4(7)] and (1.6) as a special case as well as the following models;

$$x'(t) = p(t)\left[q\int_{t-\tau}^{t} K(t,s)x(s)e^{-x(s)}\,ds - \frac{\alpha x(t)}{b+x(t)}\right],$$

$$x'(t) = p(t)\left[\sum_{i=1}^{n} q_i x(t-\tau_i)e^{-x(t-\tau_i)}\,ds - \frac{\alpha x(t)}{b+x(t)}\right],$$

where $\int_{t-\tau}^{t} K(t,s)\,ds = 1$ for all $t > 0$ and $\sum_{i=1}^{n} q_i = q$, $\tau = \max\{\tau_i:\ 1 \leqslant i \leqslant n\}$.

For an easy reference, we write our equation in the form

$$x'(t) = p(t)\left[\int_{\tau(t)}^{t} qx(s)e^{-x(s)}\,d_s\mu(t,s) - \frac{\alpha x(t)}{b+x(t)}\right], \tag{3.1}$$

where H is defined as above, and $\mu(t,s)$ satisfies (C1).

Lemma 3.1. *There exists a unique positive equilibrium $\bar{x}$ of (3.1) such that* (C3) *holds if and only if*

$$bq - \alpha \geqslant 0, \quad \text{with strict inequality for } b \geqslant 1 \tag{3.2}$$

Proof. The positive equilibrium $\bar{x}$ is the positive solution of the equation

$$qxe^{-x} - \frac{\alpha x}{b+x} = 0.$$

So it is also the positive zero of the function

$$c(x) := q(x+b)e^{-x} - \alpha.$$

Note that $c(0) = bq - \alpha$, $c(\infty) = -\alpha < 0$, and

$$c'(x) = qe^{-x}(1 - b - x), \quad x > 0.$$

For $b \geqslant 1$, we see that $c'(x) < 0$ and $c(0)c(\infty) < 0$. Then $c(x)$ has a unique zero. When $b < 1$, the function $c(x)$ becomes of one hump at $x = 1 - b$. Since $c(0) \geqslant 0$, then $c(1-b) > 0$ and $c(x)$ has also a unique positive zero in this case.

If (3.2) does not hold, then $c(0) \leqslant 0$ and therefore, $c(x) < 0$ for $x > 0$ if $b \geqslant 1$. For $b < 1$, we see that $c(x) < 0$ when $qe^{b-1} < \alpha$, $c(x) = 0$ only at $x = 1 - b$ when $qe^{b-1} = \alpha$, and $c(x) = 0$ for exactly two values when $qe^{b-1} > \alpha$. So we see that (3.1) has a unique positive equilibrium only when $qe^{b-1} = \alpha$ but in this case $H(x) < g(x)$ for other positive values of x which means that (C3) does not hold. □

Notice that; since $g(\infty)=\alpha$ and $H(x)=qxe^{-x}\leqslant \frac{q}{e}$ for all $x\geqslant 0$. Then (C4) holds if

$$M=\frac{q}{e}<\alpha. \tag{3.3}$$

Moreover, in our case we take $a=0$ and hence (C5) holds trivially since $qxe^{-x}>0$ for all $x>0$.

Now we give our first global attractivity result which is of absolute type, or by other words *delay independent*.

Theorem 3.1. *Assume that (3.2) and (3.3) are satisfied. If*

$$(\bar{x}_+ + b)qe^{-\bar{x}_+}-\alpha\leqslant 0 \quad \text{where } \bar{x}_+=\tfrac{1-b+\sqrt{b^2+6b+1}}{2}, \tag{3.4}$$

then $\bar{x}$ is the global attractor of (3.1).

Proof. According to our assumptions, all requirements of Theorem 2.1 are satisfied. So, $\bar{x}$ is the global attractor of (3.1) provided that it is the global attractor the equation $x_{n+1}=g^{-1}(H(x_n))$, $n=0,1,2,\ldots,n$ with respect to $[\alpha_1,\alpha_2]$. Note that $g^{-1}(H(x))$ is of one hump at $x^*=1$. So, in view of Lemma 2.6, the proof follows when $\bar{x}\leqslant 1$ as in this case we have either $g^{-1}(H(x^*))=\bar{x}$ or $g^{-1}(H(x^*))>\bar{x}$ and $g^{-1}(H(x))$ is increasing on $[\bar{x},g^{-1}(H(x^*))]$.

On the other hand, it is easy to see that

$$S\big(g^{-1}(x)\big)=0 \quad \text{and} \quad S\big(H(x)\big)=-\frac{(x-2)^2+2}{2(1-x)^2}<0,$$

for all $x>0$. Hence Singer [31] yields

$$S\big(g^{-1}\big(H(x)\big)\big)<0 \quad \text{for all } x>0.$$

Now for $\bar{x}>1$; Theorem 2.1 and Lemma 2.6 imply that $\bar{x}$ is the global attractor when

$$\frac{H'(\bar{x})}{g'(\bar{x})}\geqslant -1. \tag{3.5}$$

This inequality is equivalent to saying that

$$\bar{x}^2+(b-1)\bar{x}-2b\leqslant 0$$

which, in turn, holds provided that $\bar{x}\leqslant\bar{x}_+$. Thus one can see that (3.5) is equivalent to the inequality $c(\bar{x}_+)\leqslant 0$ when (3.4) holds where $c(x)$ is defined as in the proof of Lemma 3.1. □

The following result provide the first delay dependent criterion.

Theorem 3.2. *Assume that (3.2) and (3.3) are satisfied. If*

$$p_0D\leqslant 1,$$

where $D=\max\{H'(\alpha_1),\frac{q}{e^2}\}$, then $\bar{x}$ is the global attractor of all solutions of (3.1).

Proof. Since

$$\big|H'(x)\big|=qe^{-x}|1-x|.$$

So, for $x\in[\alpha_1,1]$, we have $H'(\bar{x})\leqslant H'(\alpha_1)$. On the other hand for $x>1$, we have $|H'(x)|=qe^{-x}(x-1)$ which has the absolute maximum $|H'(2)|=\frac{q}{e^2}$. Then

$$\big|H'(x)\big|\leqslant D \quad \text{for all } x>0.$$

The proof now follows by applying Corollary 2.9. □

Theorem 3.3. *Assume that (3.2) and (3.3) are satisfied. Let*

$$\big(1-e^{-\gamma}\big)\frac{H'(\bar{x})}{g'(\bar{x})}\geqslant -1, \tag{3.6}$$

where $\gamma=\frac{p_0(\beta-\alpha)^2-(\alpha_2-\bar{x})(H(\bar{x})-\alpha)}{\alpha b}$, and $\beta=\min\{h(\alpha_1),h(\alpha_2)\}$. Then $\bar{x}$ is the global attractor of all solutions of (3.1).

Proof. If x is a solution of (3.1) that is not attracted to $\bar{x}$, then (2.10), (2.11) are satisfied. So by direct integration, we obtain

$$\frac{S-\bar{x}}{\bar{H}-\alpha}-\frac{\alpha b}{(\bar{H}-\alpha)^2}\ln\frac{b\bar{H}+(\bar{H}-\alpha)S}{b\bar{H}+(\bar{H}-\alpha)\bar{x}}\leqslant p_0, \tag{3.7}$$

and

$$\frac{L-\bar{x}}{\underline{H}-\alpha}-\frac{\alpha b}{(\underline{H}-\alpha)^2}\ln\frac{b\underline{H}+(\underline{H}-\alpha)L}{b\underline{H}+(\underline{H}-\alpha)\bar{x}}\leqslant p_0. \tag{3.8}$$

Since $L, S<\alpha_2$, $0\geqslant \bar{H}-\alpha>\underline{H}-\alpha>\beta-\alpha$ and $\underline{H}<H(\bar{x})<\bar{H}$. Then (3.7) and (3.8) yield respectively that

$$\begin{aligned}\ln\frac{b\bar{H}+(\bar{H}-\alpha)S}{b\bar{H}+(\bar{H}-\alpha)\bar{x}}&\geqslant-\frac{p_0}{\alpha b}(\bar{H}-\alpha)^2+\frac{(S-\bar{x})(\bar{H}-\alpha)}{\alpha b}\\&>-\gamma,\end{aligned}$$

and

$$\begin{aligned}\ln\frac{b\underline{H}+(\underline{H}-\alpha)L}{b\underline{H}+(\underline{H}-\alpha)\bar{x}}&\geqslant-\frac{p_0}{\alpha b}(\underline{H}-\alpha)^2+\frac{(L-\bar{x})(\underline{H}-\alpha)}{\alpha b}\\&>-\gamma.\end{aligned}$$

Rearranging and using the notation $\bar{H}=H(\zeta)$, $\underline{H}=H(\eta)$ and the facts that $g^{-1}(z)=\frac{bz}{\alpha-z}$ and $b\underline{H}+(\underline{H}-\alpha)\bar{x}<b\bar{H}+(\bar{H}-\alpha)\bar{x}$, then

$$S<\bar{x}e^{-\gamma}+\left(1-e^{-\gamma}\right)g^{-1}\left(H(\zeta)\right)$$

and

$$L>\bar{x}e^{-\gamma}+\left(1-e^{-\gamma}\right)g^{-1}\left(H(\eta)\right).$$

Let $f(x):=\bar{x}e^{-\gamma}+(1-e^{-\gamma})g^{-1}(H(x))$ for $x\in[\alpha_1,\alpha_2]$. It is easy to see that $x=1$ is the unique critical point of f and $(f(x)-x)(x-\bar{x})<0$ for $x\in[\alpha_1,\alpha_2]$ and $x\neq\bar{x}$. As in the proof of Theorem 3.1, one can deduce that $\bar{x}$ is the global attractor of all solutions of the equation $x_{n+1}=f(x_n)$ on $[\alpha_1,\alpha_2]$ when $\bar{x}\leqslant 1$. For $\bar{x}>1$, we have

$$S\left(f(x)\right)=S\left(g^{-1}\left(H(x)\right)\right)<0\quad\text{for all } x\in[\alpha_1,\alpha_2].$$

But (3.6) is equivalent to the condition $f'(\bar{x})\geqslant -1$, then Lemma 2.6 implies that $\bar{x}$ is the global attractor of all solutions of equation $x_{n+1}=f(x_n)$ and this completes the proof. □

Our second model is the equation

$$z'(t)=a-b\frac{z(t)z^n(t-\tau)}{1+z^n(t-\tau)},\quad t\geqslant 0, \tag{3.9}$$

where $a,b,\tau,n\in(0,\infty)$. This equation was proposed by Mackey and Glass [29] for studying a "dynamical disease" involving respiratory disorders. Various qualitative analysis has been carried out on this model by [16,25,29,32]; see also the references cited therein.

According to [16]; the initial value problem of (3.9) corresponding to positive initial function produces positive solution. We consider only those positive solutions for Eq. (3.9). Let Z be one such solutions on $[0,\infty)$. Then the substitution $y(t)=\frac{1}{z}$ reduces (3.9) to the equation

$$y'(t)=y(t)\left[\frac{b}{y^n(t-\tau)+1}-ay(t)\right],\quad t\geqslant 0$$

which is a particular case of (1.10). It is easy to check that this equation has a unique positive equilibrium $y(t)\equiv\bar{y}$ which is the positive solution of the algebraic equation

$$\frac{b}{\bar{y}^n+1}=a\bar{y}.$$

As we have mentioned in the introduction that the substitution $y(t)=\bar{y}e^{x(t)}$ implies that

$$x'(t)=H\left(x(t-\tau)\right)-g\left(x(t)\right) \tag{3.10}$$

where $H(x)=\frac{b}{\bar{y}^n e^{nx}+1}$, $g(x)=a\bar{y}e^x$ for all $x\in R$ which is a special case of (1.1) so we are now ready to obtain the global attractivity criteria for (3.9) using Eq. (3.10) as if $x=0$ is a global attractor of all solutions of (3.10), then $\bar{z}=\frac{1}{\bar{y}}$ is the global attractor of all positive solutions of (3.9). Notice that, in this case, $p_0=\tau$ in (2.10) and (2.11).

Theorem 3.4. *Assume that*

$$2 - b\tau + 2a\tau\bar{y} \geqslant 0, \tag{3.11}$$

and

$$bn\tau \leqslant \frac{(1+\bar{y}^n)^2}{\bar{y}^n} + b\tau\frac{1+\bar{y}^n}{2\bar{y}^n}. \tag{3.12}$$

Then the positive equilibrium of (3.9) is globally attractive.

Proof. Corollary 2.3 will be used to prove this result. First, we assume that there exist $L < 0 < S$ such that (2.10) and (2.11) are satisfied. That is

$$\tau \geqslant \int_0^S \frac{dx}{\bar{H} - a\bar{y}e^x} = \int_0^S \frac{e^{-x}}{\bar{H}e^{-x} - a\bar{y}}\,dx = \frac{1}{\bar{H}}\ln\frac{\bar{H} - a\bar{y}}{\bar{H}e^{-S} - a\bar{y}}, \tag{3.13}$$

and

$$\tau \geqslant \int_0^L \frac{dx}{\underline{H} - a\bar{y}e^x} = \int_0^L \frac{e^{-x}}{\underline{H}e^{-x} - a\bar{y}}\,dx = \frac{1}{\underline{H}}\ln\frac{\underline{H} - a\bar{y}}{\underline{H}e^{-L} - a\bar{y}}. \tag{3.14}$$

Since $\frac{\bar{H}-a\bar{y}}{\bar{H}e^{-S}-a\bar{y}} > 1$ and $\frac{\underline{H}-a\bar{y}}{\underline{H}e^{-L}-a\bar{y}} > 1$, then applying the inequality

$$\ln y > \frac{2(y-1)}{y+1} \quad \text{for } y > 1 \tag{3.15}$$

to (3.13), we obtain

$$\tau\bar{H} > 2\left(\frac{\bar{H} - a\bar{y}}{\bar{H}e^{-S} - a\bar{y}} - 1\right)\Big/\left(\frac{\bar{H} - a\bar{y}}{\bar{H}e^{-S} - a\bar{y}} + 1\right) = \frac{2\bar{H}(1-e^{-S})}{\bar{H} + \bar{H}e^{-S} - 2a\bar{y}}.$$

Thus

$$\frac{2(1-e^{-S})}{\bar{H} + \bar{H}e^{-S} - 2a\bar{y}} < \tau.$$

Rearranging and substituting $\bar{H} = H(\eta)$ in the resulting inequality, it follows that

$$S < \ln\frac{2 + \tau H(\eta)}{2 + 2\tau a\bar{y} - \tau H(\eta)}.$$

Similarly, using (3.15) and replacing $\underline{H}$ by $H(\zeta)$, inequality (3.14) yields

$$L > \ln\frac{2 + \tau H(\zeta)}{2 + 2\tau a\bar{y} - \tau H(\zeta)}.$$

Now, to apply Lemma 2.6, we need to prove that $x = 0$ is the global attractor of the equation $x_{n+1} = f(x_n)$ where

$$f(x) := \ln\frac{2 + \tau H(x)}{2 + 2\tau a\bar{y} - \tau H(x)}, \quad x \in I = [\alpha_1, \alpha_2].$$

Direct computations show that

$$f'(x) = \frac{\tau(4 + 2\tau a\bar{y})H'(x)}{(2 + \tau H(x))(2 + 2\tau a\bar{y} - \tau H(x))} < 0, \quad x \in I.$$

Also,

$$\begin{aligned} S\big(f(x)\big) = {} & -n^2\big(16\bar{y}^{4n}e^{4nx}(1 + a\tau\bar{y})^2 + 16\bar{y}^{3n}e^{3nx}(1 + a\tau\bar{y})\big(4 + 4a\tau\bar{y} + ab\tau^2\bar{y}\big) \\ & + 24\bar{y}^{2n}e^{2nx}(2 + b\tau)(1 + a\tau\bar{y})(2 - b\tau + 2a\tau\bar{y}) \\ & + 4\bar{y}^n e^{nx}(2 + b\tau)(2 - b\tau + 2a\tau\bar{y})\big(4 + 4a\tau\bar{y} + ab\tau^2\bar{y}\big) \\ & + (2 + b\tau)^2(2 - b\tau + 2a\tau\bar{y})^2\big) \\ & \div \big(2\big(2 + 2\bar{y}^n e^{nx} + b\tau\big)^2\big(2 - b\tau + 2a\tau\bar{y} + 2\bar{y}^n e^{nx}(1 + a\tau\bar{y})\big)^2\big). \end{aligned}$$

Then $S(f(x)) < 0$ for all $x \in I$ due to (3.11). Moreover, since $f(0) = 0$ and $f(x)$ is decreasing on I, then $xf(x) < 0$ for $x \in I - \{0\}$ which is the negative feedback condition of Lemma 2.6.

Finally, simple calculations imply that

$$f'(0) = \frac{-2bn\tau\bar{y}^n}{(1+\bar{y}^n)^2(2+a\tau\bar{y})}.$$

Then $f'(0) \geqslant -1$ if and only if

$$\begin{aligned} bn\tau &\leqslant (2+a\tau\bar{y})\frac{(1+\bar{y}^n)^2}{2\bar{y}^n} \\ &= \frac{(1+\bar{y}^n)^2}{\bar{y}^n} + a\tau\bar{y}\frac{(1+\bar{y}^n)^2}{2\bar{y}^n} \\ &= \frac{(1+\bar{y}^n)^2}{\bar{y}^n} + b\tau\frac{(1+\bar{y}^n)}{2\bar{y}^n}, \end{aligned}$$

which is our condition (3.12). Thus all requirements of Lemma 2.6 are satisfied which implies that $x = 0$ is the global attractor of all solutions of the first order difference equation $x_{n+1} = f(x_n)$ and hence Lemma 2.5 implies that $L = S = 0$. □

The following result provides another delay-dependent criterion which is a particular case of (3.12) and relaxes the restriction (3.11).

Theorem 3.5. *Assume that*

$$bn\tau \leqslant \frac{(1+\bar{y}^n)^2}{\bar{y}^n}. \tag{3.16}$$

Then the positive equilibrium of (3.9) is globally attractive.

Proof. Here, we apply Corollary 2.10. In this context, we see that

$$f(x) := \tau\big(H(x) - a\bar{y}\big), \quad x \in I.$$

This function satisfies

$$f'(x) = \tau H'(x) < 0,$$

and

$$S\big(f(x)\big) = -\frac{n^2}{2}$$

on I. So all assumptions of Lemma 2.6 are satisfied when $f'(0) \geqslant -1$ which is equivalent to the inequality (3.16). Thus $x = 0$ is the global attractor of the equation $x_{n+1} = f(x_n)$. The proof follows by applying Lemma 2.5. □

We note that if $f(x) = g^{-1}(H(x))$, then $xf(x) < 0$ for all $x \in I - \{0\}$,

$$\begin{aligned} f'(x) &= \frac{H'(x)}{g'f((x))} < 0, \quad x \in I, \\ f'(0) &= -n\frac{\bar{y}^n}{\bar{y}^n+1} \geqslant -1, \quad \text{when } n \leqslant \frac{\bar{y}^n+1}{\bar{y}^n}, \end{aligned}$$

and

$$S\big(f(x)\big) = -n^2\frac{1+2\bar{y}^n e^{nx}}{2(1+\bar{y}^n e^{nx})^2} < 0, \quad x \in I.$$

Therefore using similar reasoning as in the previous proofs, we obtain the following delay-dependent global attractivity criterion which was established also by [22, Theorem 5].

Theorem 3.6. *Assume that*

$$n \leqslant \frac{\bar{y}^n+1}{\bar{y}^n}, \tag{3.17}$$

then the positive equilibrium of (3.9) is globally attractive.

The last three result have equivalent forms that contain only some or all the parameters a, b, n, τ using the function $\nu(t) := \frac{b}{1+x^n} - ax$ which satisfies that $\nu(x) = 0$ only at $\bar{y}$ and therefore

$$\nu(x)(x-\bar{y}) < 0, \quad \text{for all } x > 0 \text{ and } x \neq \bar{y}. \tag{3.18}$$

To find these formulas we have to solve the inequalities in these results in $\bar{y}$ and then use (3.18) to find the desired criteria. For example; Eq. (3.17) is equivalent to $n \leqslant 1$ or

$$n > 1 \quad \text{and} \quad \bar{y} \leqslant \frac{1}{(n-1)^{\frac{1}{n}}}.$$

So $\nu(\frac{1}{(n-1)^{\frac{1}{n}}}) \leqslant 0$ which holds if and only if

$$\frac{a}{b} \geqslant \frac{(n-1)^{1+\frac{1}{n}}}{n}. \tag{3.19}$$

Thus, Theorem 3.6 can be read as follows: *Assume that either $n \leqslant 1$ or $n > 1$ and (3.19) holds. Then the positive equilibrium of (3.9) is globally attractive.*

Remarks 3.1. 1. It should be noticed that the above results hold also for the non-autonomous version of (3.9);

$$z'(t) = p(t)\left[a - b\frac{z(t)z^n(t-\tau)}{1+z^n(t-\tau)}\right], \quad t \geqslant 0 \tag{3.20}$$

with τ replaced by p_0 and considering only the positive solutions. The results obtained for this case will be completely new.

2. Theorem 3.4 may be improved further if one could complete the proof without the need to the inequality (3.15) or any similar ones.

3. In Qian [32] the equilibrium of (3.9) is proved to be globally attractive if $\tau nb < 4$. It is easy to see that (3.16) is a substantial improvement of Qian's criterion since $\frac{(1+\bar{y})^2}{\bar{y}} > 4$.

4. Our main results can be easily extended to more general equations than (1.1). For example, Theorem 2.1 holds without any changes for the equation

$$x'(t) = p(t)D\big(x(t)\big)\left[f\left(\int_{\tau(t)}^{t} h\big(x(s)\big)\, d_s\mu(t,s)\right) - g\big(x(t)\big)\right]$$

which generalizes (1.10) where p, h, f, τ, μ, and g are defined as before while $D \in C((a,\infty),(0,\infty))$. But Theorems 2.2, 2.3 still true only after dividing the integrands in (2.6) and (2.7) by $D(x)$.

Open problems and conjectures.

1. We see that better criteria for (3.20) could be proved if Corollary 2.4 was applied. In this regard we believe that the following conjecture is true.

   Conjecture 3.1. *Assume that*

   $$e^{-ap_0\bar{y}} \geqslant \frac{c^2-c}{c^2+1}, \quad \text{where } c = -\frac{H'(0)}{a\bar{y}}. \tag{3.21}$$

   Then the positive equilibrium of (3.20) is globally attractive.

   Note that (3.21) implies (3.17) when $c \leqslant 1$. The main challenge in proving this conjecture using Corollary 2.4 is that the targeted map W of Remark 2.1 is implicitly defined and therefore its Schwarzian derivative is very hard to compute. Ivanov el al. [22] also faced this complexity but were able to obtain good criterion for (3.9) (see [22, Corollary 4]), although they used a particular case of Corollary 2.4. To deal with this situation and to reveal the full bower of our results, this author suggests that a theory need to be developed for the attractivity behavior of the equilibrium points of implicit difference equations of the form $T(x_{n+1}, x_n) = 0$ where T is a certain map. We also, stress on the need to obtain more practical global attractivity criteria for the explicit equation $x_{n+1} = f(x_n)$ that avoid the use of Schwarzian derivative.

2. It is very interesting to develop a technique for studying the global attractivity for the model

$$x'(t) = p(t)D\big(x(t)\big)\left[f\left(\int_{\tau(t)}^{t} h\big(x(s)\big)\, d_s\mu(t,s) \right) - g\big(x\big(\sigma(t)\big)\big) \right]$$

where $\sigma(t)$ is a certain delay function. Care should be taken in this case as our reasoning seems to be not applicable for this equation due to the appearance of the delay in the death rate.

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References

[1] J. Arino, L. Wang, G.S.K. Wolkowicz, An alternative formulation for a delayed logistic equation, J. Theoret. Biol. 241 (2006) 109–119.
[2] L. Berezansky, J. Baštinec, J. Diblik, Z. Šmarda, On a delay population model with quadratic nonlinearity, Adv. Difference Equ. 201 (2012) 230.
[3] L. Berezansky, E. Braverman, L. Idels, Nicholson's blowflies differential equations revisited: Main results and open problems, Appl. Math. Model. 34 (2010) 1405–1417.
[4] L.S. Block, W.A. Coppel, Dynamics in One Dimension, Lecture Notes in Math., vol. 1513, Springer, Berlin, 1992.
[5] S.P. Blythe, R.M. Nisbet, W.S.C. Gurney, Instability and complex dynamic behaviour in population models with long time delays, Theor. Popul. Biol. 22 (1982) 147–176.
[6] E. Braverman, S. Zhukovskiy, Absolute and delay-dependent stability of equations with a distributed delay, Discrete Contin. Dyn. Syst. Ser. A 32 (2012) 2041–2061.
[7] W.A. Coppel, The solution of equations by iteration, Proc. Cambridge Philos. Soc. 51 (1955) 41–43.
[8] P. Cull, Stability of discrete one-dimensional population models, Bull. Math. Biol. 50 (1988) 67–75.
[9] P. Cull, Stability in one-dimensional models, Sci. Math. Jpn. 8 (2003) 349–357.
[10] S. Elaydi, An Introduction to Difference Equations, 3rd ed., Springer Verlag, New York, 2005.
[11] H.A. El-Morshedy, The global attractivity of difference equations of nonincreasing nonlinearities with applications, Comput. Math. Appl. 45 (2003) 749–758.
[12] H.A. El-Morshedy, V.J. López, Global attractors for difference equations dominated by one-dimensional maps, J. Difference Equ. Appl. 14 (2008) 391–410.
[13] T. Faria, Asymptotic stability for delayed logistic type equations, Math. Comput. Modelling 43 (2006) 433–445.
[14] H.I. Freedman, K. Gopalsamy, Global stability in time-delayed single-species dynamics, Bull. Math. Biol. 48 (1986) 485–492.
[15] K. Gopalsamy, Stability and Oscillation in Delay Differential Equations of Population Dynamics, Kluwer Academic Publishers, Dordrecht, 1992.
[16] K. Gopalsamy, M.R.S. Kulenovic, G. Ladas, Oscillations and global attractivity in respiratory dynamics, Dyn. Stab. Syst. 4 (1989) 131–139.
[17] K. Gopalsamy, M.R. Kulenovic, G. Ladas, Oscillations and global attractivity in model of hematopoiesis, J. Dynam. Differential Equations 2 (1990) 117–132.
[18] W.S.C. Gurney, S.P. Blythe, R.M. Nisbet, Nicholson's blowflies revisited, Nature 287 (1980) 17–21.
[19] I. Györi, G. Ladas, Oscillation Theory of Delay Differential Equations with Applications, Clarendon Press, Oxford, 1991.
[20] I. Györi, S.I. Trofimchuk, Global attractivity in $x'(t) = -\delta x(t) + pf(x(t-\tau))$, Dynam. Systems Appl. 8 (1999) 197–210.
[21] Y.N. Huang, A note on global stability for discrete one dimensional population models, Math. Biosci. 102 (1990) 121–124.
[22] A. Ivanov, E. Liz, S. Trofimchuk, Global stability of a class of scalar nonlinear delay differential equations, Differ. Equ. Dyn. Syst. 11 (2003) 33–54.
[23] A.F. Ivanov, A.N. Sharkovsky, Oscillations in Singularly Perturbed Delay Equations, Dynamics Reported (New Series), vol. 1, Springer Verlag, 1991, pp. 165–224.
[24] G. Karakostas, Ch.G. Philos, Y.G. Sficas, Stable steady state of some population models, J. Dynam. Differential Equations 4 (1992) 161–190.
[25] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, Boston, MA, 1993.
[26] E. Liz, Local stability implies global stability in some one-dimensional discrete single-species models, Discrete Contin. Dyn. Syst. Ser. B 7 (2007) 191–199.
[27] E. Liz, M. Pinto, V. Tkachenko, S. Trofimchuk, A global stability criterion for a family of delayed population models, Quart. Appl. Math. LXIII (2005) 56–70.
[28] E. Liz, V. Tkachenko, S. Trofimchuk, A global stability criterion for scalar functional differential equations, SIAM J. Math. Anal. 35 (2003) 596–622.
[29] M.C. Mackey, L. Glass, Oscillation and chaos in physiological control systems, Science 197 (1977) 287–289.
[30] G. Rosenkranz, On global stability of discrete population models, Math. Biosci. 64 (1983) 227–231.
[31] D. Singer, Stable orbits and bifurcation of maps of the interval, SIAM J. Appl. Math. 35 (1978) 260–267.
[32] C. Qian, Global attractivity in a nonlinear delay differential equation with applications, Nonlinear Anal. 71 (2009) 1893–1900.
[33] M. Wazewska-Czyzewska, A. Lasota, Mathematical problems of the red-blood cell system, Ann. Polish Math. Soc. Ser. III, Appl. Math. 6 (1976) 23–40.