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Low Mach number limit of the full compressible Navier–Stokes–Maxwell system

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ABSTRACT

This paper deals with the low Mach number limit of the full compressible Navier–Stokes–Maxwell system. It is justified rigorously that, for the well-prepared initial data, the solutions of the full compressible Navier–Stokes–Maxwell system converge to that of the incompressible Navier–Stokes–Maxwell system as the Mach number tends to zero.

1. Introduction and main results

In this paper we consider the low Mach number limit of the following full compressible Navier–Stokes–Maxwell system [10,18,19]:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\rho(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) + \nabla p = \operatorname{div}\big(2\mu\mathbf{P} + \mu' \operatorname{div}\mathbf{u}\mathbf{I}_3\big) + \mathbf{J}\times\mathbf{B}, \tag{1.2}$$

$$\rho\frac{\partial e}{\partial\theta}(\theta_t + \mathbf{u}\cdot\nabla\theta) + \theta\frac{\partial p}{\partial\theta}\operatorname{div}\mathbf{u} = \kappa\Delta\theta + \Psi + \mathbf{J}\cdot(\mathbf{E} + \mathbf{u}\times\mathbf{B}), \tag{1.3}$$

$$\nu\mathbf{E}_t - \frac{1}{\mu_0}\operatorname{curl}\mathbf{B} + \mathbf{J} = 0, \tag{1.4}$$

$$\mathbf{B}_t + \operatorname{curl}\mathbf{E} = 0, \quad \operatorname{div}\mathbf{B} = 0. \tag{1.5}$$

Here the unknowns $\rho > 0$, $\mathbf{u} = (u_1, u_2, u_3)$, $\theta > 0$, $\mathbf{E} = (E_1, E_2, E_3)$, and $\mathbf{B} = (B_1, B_2, B_3)$ represent the mass density, the velocity, the absolute temperature, the electric field, and the magnetic flux density, respectively. $\mathbf{I}_3$ is a 3×3 unit vector. $\mathbf{P}$ is the deformation tensor, whose entries are

$$P_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, 2, 3.$$

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The current density $\mathbf{J}$ is given by Ohm's law $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})$.

$$\Psi = 2\mu \sum_{i,j=1}^{3} (P_{ij})^2 + \mu'(\operatorname{div}\mathbf{u})^2$$

is the viscous dissipation function. The pressure $p = p(\rho, \theta)$ and the internal energy $e = e(\rho, \theta)$ are smooth functions of the density ρ and the temperature θ of the flow and satisfy the Gibbs relation

$$\theta \mathrm{d}S = \mathrm{d}e + p\mathrm{d}\left(\frac{1}{\rho}\right),$$

for some smooth function $S = S(\rho, \theta)$ which expresses the first law of the thermodynamics. The viscosity coefficients μ and μ', the heat conductivity coefficient κ, and the electric conductivity coefficient σ are assumed to be constants such that $\mu > 0$, $2\mu + \mu' > 0$, $\kappa > 0$, and $\sigma > 0$. The dielectric constant ν and the magnetic permeability μ_0 are assumed to be positive constants.

The system (1.1)–(1.5) describes the motion of compressible quasi-neutrally ionized fluids under the influence of electromagnetic fields. By taking the dielectric constant $\nu = 0$ in (1.4), then we obtain that $\mathbf{J} = \operatorname{curl}\mathbf{B}/\mu_0$. Thanks to Ohm's law, we can eliminate the electric field $\mathbf{E}$ in (1.2), (1.3), and (1.5), and finally obtain the full compressible magnetohydrodynamic equations (see [11] for more details)

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.6}$$

$$\rho(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + \nabla p = \operatorname{div}\big(2\mu \mathbf{P} + \mu' \operatorname{div}\mathbf{u}\mathbf{I}\big) + \frac{1}{\mu_0} \operatorname{curl}\mathbf{B} \times \mathbf{B}, \tag{1.7}$$

$$\rho \frac{\partial e}{\partial \theta}(\theta_t + \mathbf{u} \cdot \nabla \theta) + \theta \frac{\partial p}{\partial \theta} \operatorname{div}\mathbf{u} = \kappa \Delta \theta + \Psi + \frac{1}{\sigma \mu_0^2} |\operatorname{curl}\mathbf{B}|^2, \tag{1.8}$$

$$\mathbf{B}_t - \operatorname{curl}(\mathbf{u} \times \mathbf{B}) + \frac{1}{\sigma \mu_0} \operatorname{curl}\operatorname{curl}\mathbf{B} = 0, \quad \operatorname{div}\mathbf{B} = 0. \tag{1.9}$$

There is a lot of literature on the full compressible magnetohydrodynamic equations (1.6)–(1.9) due to their physical importance, complexity, rich phenomena, and mathematical challenges, see [2–4,6,14,16] and the references cited therein. In [14], Jiang, Ju, and Li studied the low Mach number limit to the system (1.6)–(1.9) in the framework of the local smooth solutions for small variational density and temperature. Later, the large variational density and temperature case was studied in [16]. For more results on low Mach number limit to the magnetohydrodynamic equations, please see [7,12,13] on isentropic case, [15] on the ideal non-isentropic case, [17] on full magnetohydrodynamics equations in framework of weak solutions, and [22] on magnetohydrodynamics equations with strong stratification.

To the authors' best knowledge, there are no results on low Mach number limit of the full compressible Navier–Stokes–Maxwell system (1.1)–(1.5). In this paper we extend the results in [14] to the system (1.1)–(1.5). We shall focus our study on the fluid obeying the perfect relations $p = \Re\rho\theta$ and $e = c_V\theta$, where the constants $\Re > 0$ and $c_V > 0$ are the gas constant and the heat capacity volume, respectively.

In order to study the low Mach number limit of the system (1.1)–(1.5), similarly to [14], we use its appropriate dimensionless form as follows

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \tag{1.10}$$

$$\rho\big(\mathbf{u}_t + (\mathbf{u} \cdot \nabla \mathbf{u})\mathbf{u}\big) + \frac{\nabla(\rho\theta)}{\epsilon^2} = 2\mu \operatorname{div}\mathbf{P} + \mu' \operatorname{div}(\operatorname{div}\mathbf{u}\mathbf{I}_3) + \mathbf{J} \times \mathbf{B}, \tag{1.11}$$

$$\rho(\theta_t + \mathbf{u} \cdot \nabla \theta) + (\gamma - 1)\rho\theta \operatorname{div}\mathbf{u} = \kappa \Delta \theta + \epsilon^2\big(\Psi + \mathbf{J} \cdot (\mathbf{E} + \mathbf{u} \times \mathbf{B})\big), \tag{1.12}$$

$$\nu \mathbf{E}_t - (1/\mu_0) \operatorname{curl}\mathbf{B} + \mathbf{J} = 0, \quad \mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}), \tag{1.13}$$

$$\mathbf{B}_t + \operatorname{curl}\mathbf{E} = 0, \quad \operatorname{div}\mathbf{B} = 0, \tag{1.14}$$

where ϵ is the Mach number and the coefficients μ, μ', ν, and κ are the scaled parameters. $\gamma = 1 + \Re/c_V$ is the ratio of specific heats. Note that we have used the same notations and assumed that the coefficients μ, μ', ν, and κ are independent of ϵ for simplicity.

Setting

$$\rho = 1 + \epsilon q, \qquad \theta = 1 + \epsilon \phi,$$

then we can rewrite (1.10)–(1.14) as

$$\partial_t q^\epsilon + \mathbf{u}^\epsilon \cdot \nabla q^\epsilon + \frac{1}{\epsilon}\big(1+\epsilon q^\epsilon\big)\operatorname{div}\mathbf{u}^\epsilon = 0, \tag{1.15}$$

$$\big(1+\epsilon q^\epsilon\big)\big(\partial_t \mathbf{u}^\epsilon + \mathbf{u}^\epsilon \cdot \nabla \mathbf{u}^\epsilon\big) + \frac{1}{\epsilon}\big[\big(1+\epsilon q^\epsilon\big)\cdot\nabla\phi^\epsilon + \big(1+\epsilon\phi^\epsilon\big)\cdot\nabla q^\epsilon\big]$$
$$= 2\mu\operatorname{div}\mathbf{P}^\epsilon + \mu'\operatorname{div}\big(\operatorname{div}\mathbf{u}^\epsilon \mathbf{I}_3\big) + \mathbf{J}^\epsilon \times \mathbf{B}^\epsilon, \tag{1.16}$$

$$\nu\partial_t \mathbf{E}^\epsilon - \frac{1}{\mu_0}\operatorname{curl}\mathbf{B}^\epsilon = -\mathbf{J}^\epsilon, \quad \mathbf{J}^\epsilon = \sigma\big(\mathbf{E}^\epsilon + \mathbf{u}^\epsilon \times \mathbf{B}^\epsilon\big), \tag{1.17}$$

$$\partial_t \mathbf{B}^\epsilon + \operatorname{curl}\mathbf{E}^\epsilon = 0, \quad \operatorname{div}\mathbf{B}^\epsilon = 0, \tag{1.18}$$

$$\big(1+\epsilon q^\epsilon\big)\big(\partial_t\phi^\epsilon + \mathbf{u}^\epsilon\cdot\nabla\phi^\epsilon\big) + \frac{\gamma-1}{\epsilon}\big(1+\epsilon q^\epsilon\big)\big(1+\epsilon\phi^\epsilon\big)\operatorname{div}\mathbf{u}^\epsilon = \epsilon\big(\Psi^\epsilon + \mathbf{J}^\epsilon\cdot\big(\mathbf{E}^\epsilon + \mathbf{u}^\epsilon\times\mathbf{B}^\epsilon\big)\big) + \kappa\Delta\phi^\epsilon, \tag{1.19}$$

with

$$\big(P^\epsilon\big)_{ij} = \frac{1}{2}\left(\frac{\partial u_i^\epsilon}{\partial x_j} + \frac{\partial u_j^\epsilon}{\partial x_i}\right), \qquad \Psi^\epsilon = 2\mu\sum_{i,j=1}^{3}\big(\big(P^\epsilon\big)_{ij}\big)^2 + \mu'\big(\operatorname{div}\mathbf{u}^\epsilon\big)^2.$$

The system (1.15)–(1.19) is equipped with initial data

$$\big(q^\epsilon, \mathbf{u}^\epsilon, \mathbf{E}^\epsilon, \mathbf{B}^\epsilon, \phi^\epsilon\big)\big|_{t=0} = \big(q_0^\epsilon(x), \mathbf{u}_0^\epsilon(x), \mathbf{E}_0^\epsilon(x), \mathbf{B}_0^\epsilon(x), \phi_0^\epsilon(x)\big), \quad x\in\Omega. \tag{1.20}$$

Here $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$, and we have added the superscript ϵ on the unknowns $(q^\epsilon, \mathbf{u}^\epsilon, \mathbf{E}^\epsilon, \mathbf{B}^\epsilon, \phi^\epsilon)$ to stress the dependence of the parameter ϵ.

Formal if we take the limit $\epsilon \to 0$ in (1.15)–(1.19), we obtain the following incompressible Navier–Stokes–Maxwell equations (we suppose that the limits $\mathbf{u}^\epsilon \to \mathbf{w}$, $\mathbf{B}^\epsilon \to \mathbf{H}$, and $\mathbf{E}^\epsilon \to \mathbf{D}$ exist)

$$\partial_t\mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{w} + \nabla\pi = \mu\Delta\mathbf{w} + \mathbf{J}\times\mathbf{H}, \tag{1.21}$$

$$\nu\partial_t\mathbf{D} - \frac{1}{\mu_0}\operatorname{curl}\mathbf{H} + \mathbf{J} = 0, \quad \mathbf{J} = \sigma(\mathbf{D} + \mathbf{w}\times\mathbf{H}), \tag{1.22}$$

$$\partial_t\mathbf{H} + \operatorname{curl}\mathbf{D} = 0, \tag{1.23}$$

$$\operatorname{div}\mathbf{w} = 0, \quad \operatorname{div}\mathbf{H} = 0. \tag{1.24}$$

The system (1.21)–(1.24) is supplemented with initial data

$$(\mathbf{w}, \mathbf{D}, \mathbf{H})\big|_{t=0} = \big(\mathbf{w}_0(x), \mathbf{D}_0(x), \mathbf{H}_0(x)\big), \quad x\in\Omega. \tag{1.25}$$

In this paper we shall establish the above limit rigorously in Ω. Moreover, we shall show that for sufficiently small Mach number, the compressible Navier–Stokes–Maxwell system (1.15)–(1.19) admits a smooth solution on the time interval where the smooth solution of the incompressible Navier–Stokes–Maxwell equations (1.21)–(1.24) exists.

Before stating our main result we recall some known results on the incompressible Navier–Stokes–Maxwell equations (1.21)–(1.24). Masmoudi [21] obtained the existence and uniqueness of global strong solutions (1.21)–(1.24) for the initial data $(\mathbf{w}_0, \mathbf{D}_0, \mathbf{H}_0) \in L^2(\mathbb{R}^2)\times H^s(\mathbb{R}^2)\times H^s(\mathbb{R}^2)$ with $s>0$. Ibrahim and Keraani [8] established the strong solutions to the system (1.21)–(1.24) for the initial belonging to $\dot{B}^{1/2}_{2,1}(\mathbb{R}^3)\times\dot{H}^{1/2}(\mathbb{R}^3)\times\dot{H}^{1/2}(\mathbb{R}^3)$. Very recently, this result was improved by Germain, Ibrahim, and Masmoudi [5] where the initial data lie in $\dot{H}^{1/2}(\mathbb{R}^3)\times\dot{H}^{1/2}(\mathbb{R}^3)\times\dot{H}^{1/2}(\mathbb{R}^3)$. Ibrahim and Yoneda [9] constructed local smooth solution for non-decaying initial data on the torus $\mathbb{T}^3$ and showed the loss of smoothness of solutions. We remark that although there are some results on the Navier–Stokes–Maxwell system (1.21)–(1.24), the global finite energy weak solution (Leray-type solution) to (1.21)–(1.24) for initial data lying in $L^2(\mathbb{R}^d)$ is still an interesting open problem for $d=2,3$.

In this paper, for simplicity, we assume that the problem (1.21)–(1.25) has a local smooth solution as obtained in [9]. The main result of the present paper is the following.

Theorem 1.1. *Let $s>3/2+2$. Suppose that the initial data $(q_0^\epsilon(x), \mathbf{u}_0^\epsilon(x), \mathbf{E}_0^\epsilon(x), \mathbf{B}_0^\epsilon(x), \phi_0^\epsilon(x))$ belong to H^s and satisfy*

$$\big\|\big(q_0^\epsilon(x), \mathbf{u}_0^\epsilon(x) - \mathbf{w}_0(x), \mathbf{E}_0^\epsilon(x) - \mathbf{D}_0(x), \mathbf{B}_0^\epsilon(x) - \mathbf{H}_0(x)\big)\big\|_s = O(\epsilon) \tag{1.26}$$

for some smooth functions $(\mathbf{w}_0(x), \mathbf{D}_0(x), \mathbf{H}_0(x))$. Let $(\mathbf{w}, \mathbf{D}, \mathbf{H}, \pi)$ be a smooth solution to the system (1.21)–(1.24) with initial data $(\mathbf{w}_0(x), \mathbf{D}_0(x), \mathbf{H}_0(x))$ on $[0,T^*]\times\Omega$ for $T^*>0$ finite, then there exists a constant $\epsilon_0>0$ such that, for all $\epsilon\leqslant\epsilon_0$, the problem (1.15)–(1.20) has a unique smooth solution $(q^\epsilon, \mathbf{u}^\epsilon, \mathbf{E}^\epsilon, \mathbf{B}^\epsilon, \phi^\epsilon)\in C([0,T^*], H^s)$. Moreover, there exists a positive constant $K>0$, independent of ϵ, such that, for all $\epsilon\leqslant\epsilon_0$,

$$\sup_{t\in[0,T^*]}\left\|\left(q^\epsilon - \frac{\epsilon}{2}\pi, \mathbf{u}^\epsilon - \mathbf{w}, \mathbf{E}^\epsilon - \mathbf{D}, \mathbf{B}^\epsilon - \mathbf{H}, \phi^\epsilon - \frac{\epsilon}{2}\pi\right)\right\|_s \leqslant K\epsilon. \tag{1.27}$$

The proof of Theorem 1.1 is based on the method developed in [14]. The key point is to obtain the uniform estimates of the error system and apply convergence-stability lemma [1,24] and Gronwall-type inequality. Compared with the full compressible magnetohydrodynamic equations (1.6)–(1.9), the full compressible Navier–Stokes–Maxwell system (1.15)–(1.19) is more complicated and more refined analysis is needed in our arguments, see the details presented in the next section.

Before ending the introduction, we give the notations used throughout the current paper. We use the letter C to denote various positive constants independent of ϵ. For convenience, we denote by $H^l \equiv H^l(\Omega)$ $(l \in \mathbb{R})$ the standard Sobolev spaces and write $\|\cdot\|_l$ for the standard norm of H^l and $\|\cdot\|$ for $\|\cdot\|_0$. The remainder of this paper is devoted to the proof of Theorem 1.1.

2. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. Denoting $U^\epsilon = (q^\epsilon, \mathbf{u}^\epsilon, \mathbf{E}^\epsilon, \mathbf{B}^\epsilon, \phi^\epsilon)^\top$, we rewrite the system (1.15)–(1.19) in the vector form

$$A_0(U^\epsilon)\partial_t U^\epsilon + \sum_{j=1}^{3} A_j(U^\epsilon)\partial_j U^\epsilon = Q(U^\epsilon), \tag{2.1}$$

where the matrices $A_j(U^\epsilon)$ $(0 \leqslant j \leqslant 3)$ are given by

$$A_0(U^\epsilon) = \operatorname{diag}(1, 1+\epsilon q^\epsilon, 1+\epsilon q^\epsilon, 1+\epsilon q^\epsilon, \nu, \nu, \nu, 1, 1, 1, 1+\epsilon q^\epsilon),$$

$$A_1(U^\epsilon) = \begin{pmatrix}
u_1^\epsilon & \frac{1+\epsilon q^\epsilon}{\epsilon} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1+\epsilon \phi^\epsilon}{\epsilon} & u_1^\epsilon(1+\epsilon q^\epsilon) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+\epsilon q^\epsilon}{\epsilon} \\
0 & 0 & u_1^\epsilon(1+\epsilon q^\epsilon) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & u_1^\epsilon(1+\epsilon q^\epsilon) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\mu_0} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu_0} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{(\gamma-1)(1+\epsilon q^\epsilon)(1+\epsilon\phi^\epsilon)}{\epsilon} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_1^\epsilon(1+\epsilon q^\epsilon)
\end{pmatrix},$$

$$A_2(U^\epsilon) = \begin{pmatrix}
u_2^\epsilon & \frac{1+\epsilon q^\epsilon}{\epsilon} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & u_2^\epsilon(1+\epsilon q^\epsilon) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1+\epsilon \phi^\epsilon}{\epsilon} & 0 & u_2^\epsilon(1+\epsilon q^\epsilon) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+\epsilon q^\epsilon}{\epsilon} \\
0 & 0 & 0 & u_2^\epsilon(1+\epsilon q^\epsilon) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu_0} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu_0} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{(\gamma-1)(1+\epsilon q^\epsilon)(1+\epsilon\phi^\epsilon)}{\epsilon} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_2^\epsilon(1+\epsilon q^\epsilon)
\end{pmatrix},$$

$$A_3(U^\epsilon) = \begin{pmatrix}
u_3^\epsilon & 0 & 0 & \frac{1+\epsilon q^\epsilon}{\epsilon} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & u_3^\epsilon(1+\epsilon q^\epsilon) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & u_3^\epsilon(1+\epsilon q^\epsilon) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1+\epsilon \phi^\epsilon}{\epsilon} & 0 & 0 & u_3^\epsilon(1+\epsilon q^\epsilon) & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1+\epsilon q^\epsilon}{\epsilon} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\mu_0} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\mu_0} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{(\gamma-1)(1+\epsilon q^\epsilon)(1+\epsilon\phi^\epsilon)}{\epsilon} & 0 & 0 & 0 & 0 & 0 & 0 & u_3^\epsilon(1+\epsilon q^\epsilon)
\end{pmatrix},$$

and

$$Q\left(U^\epsilon\right) := \left(0, L\left(U^\epsilon\right), -\mathbf{J}^\epsilon, 0, 0, 0, \kappa\Delta\phi^\epsilon + \epsilon\left(\Psi^\epsilon + \mathbf{J}^\epsilon \cdot \left(\mathbf{E}^\epsilon + \mathbf{u}^\epsilon \times \mathbf{B}^\epsilon\right)\right)\right)^\top$$

with

$$\begin{aligned}
\mathbf{J}^\epsilon &= \sigma\left(\mathbf{E}^\epsilon + \mathbf{u} \times \mathbf{B}^\epsilon\right),\\
L\left(U^\epsilon\right) &= 2\mu \operatorname{div}\mathbf{P}^\epsilon + \mu' \operatorname{div}\left(\operatorname{div}\mathbf{u}^\epsilon \mathbf{I}_3\right) + \mathbf{J}^\epsilon \times \mathbf{B}^\epsilon\\
&= \mu\Delta\mathbf{u}^\epsilon + \left(\mu + \mu'\right)\nabla \operatorname{div}\mathbf{u}^\epsilon + \mathbf{J}^\epsilon \times \mathbf{B}^\epsilon.
\end{aligned}$$

It is easy to see that the matrices $A_j(U^\epsilon)$ $(0 \leqslant j \leqslant 3)$ can be symmetrized by choosing

$$\hat{A}_0\left(U^\epsilon\right) = \operatorname{diag}\left(\frac{1+\epsilon\phi^\epsilon}{1+\epsilon q^\epsilon}, 1, 1, 1, \mu_0, \mu_0, \mu_0, 1, 1, 1, \frac{1}{(\gamma-1)(1+\epsilon\phi^\epsilon)}\right).$$

Moreover, for $U^\epsilon \in \bar{G}_1 \Subset G$ with G being the state space for the system (2.1), $\tilde{A}_0(U^\epsilon)$ is a positive definite symmetric matrix for sufficiently small ϵ.

Assume that the initial data

$$U^\epsilon(0,x) = U_0^\epsilon(x) := \left(q_0^\epsilon(x), \mathbf{u}_0^\epsilon(x), \mathbf{E}_0^\epsilon(x)\mathbf{B}_0^\epsilon(x), \phi_0^\epsilon(x)\right)^\top \in H^s$$

and $U_0^\epsilon(x) \in G_0$, $\bar{G}_0 \Subset G$.

First, following the proof of the local existence theory for the initial value problem of symmetrizable hyperbolic–parabolic systems by Volpert and Hudjaev [23], we obtain that there exists a time interval $[0,T]$ with $T > 0$, so that the system (2.1) with initial data $U_0^\epsilon(x)$ has a unique classical solution $U^\epsilon(t,x) \in C([0,T], H^s)$ and $U^\epsilon(t,x) \in G_2$ with $\bar{G}_2 \Subset G$. We remark that the crucial step in the proof of local existence result is to prove the uniform boundedness of the solutions.

Now we define

$$T_\epsilon = \sup\left\{T > 0\colon\ U^\epsilon(t,x) \in C\left([0,T], H^s\right),\ U^\epsilon(t,x) \in G_2,\ \forall(t,x) \in [0,T] \times \Omega\right\}.$$

Note that T_ϵ depends on ϵ and may tend to zero as ϵ goes to 0.

To show that $\liminf_{\epsilon\to 0} T_\epsilon > 0$, we shall make use of the convergence-stability lemma which was established in [1,24] for hyperbolic systems of balance laws. Similarly to [14], for the hyperbolic–parabolic system (2.1), we have the following convergence-stability lemma.

Lemma 2.1. *Let $s > 3/2 + 2$. Suppose that $U_0^\epsilon(x) \in G_0$, $\bar{G}_0 \Subset G$, and $U_0^\epsilon(x) \in H^s$, and the following convergence assumption* (A) *holds.*

(A) *For each ϵ, there exist $T_\star > 0$ and $U_\epsilon \in L^\infty(0, T_\star; H^s)$ satisfying*

$$\bigcup_{x,t,\epsilon}\left\{U_\epsilon(t,x)\right\} \Subset G,$$

such that, for $t \in [0, \min\{T_\star, T_\epsilon\})$,

$$\sup_{x,t}\left|U^\epsilon(t,x) - U_\epsilon(t,x)\right| = o(1), \qquad \sup_t\left\|U^\epsilon(t,x) - U_\epsilon(t,x)\right\|_s = O(1), \quad \text{as } \epsilon \to 0.$$

Then there exists an $\bar{\epsilon} > 0$ such that, for all $\epsilon \in (0, \bar{\epsilon}]$, it holds that

$$T_\epsilon > T_\star.$$

To apply Lemma 2.1, we construct the approximation $U_\epsilon = (q_\epsilon, \mathbf{v}_\epsilon, \mathbf{D}_\epsilon, \mathbf{H}_\epsilon, \phi_\epsilon)^\top$ to the original system (2.1) with $q_\epsilon = \epsilon\pi/2$, $\mathbf{v}_\epsilon = \mathbf{w}$, $\mathbf{D}_\epsilon = \mathbf{D}$, $\mathbf{H}_\epsilon = \mathbf{H}$, and $\phi_\epsilon = \epsilon\pi/2$, where $(\mathbf{w}, \mathbf{D}, \mathbf{H}, \pi)$ is the smooth solution to the problem (1.21)–(1.25). It is easy to verify that U_ϵ satisfies

$$\partial_t q_\epsilon + \mathbf{v}_\epsilon \cdot \nabla q_\epsilon + \frac{1}{\epsilon}(1+\epsilon q_\epsilon)\operatorname{div}\mathbf{v}_\epsilon = \frac{\epsilon}{2}(\pi_t + \mathbf{w}\cdot\nabla\pi), \tag{2.2}$$

$$\begin{aligned}
&(1+\epsilon q_\epsilon)(\partial_t\mathbf{v}_\epsilon + \mathbf{v}_\epsilon\cdot\nabla\mathbf{v}_\epsilon) + \frac{1}{\epsilon}\left[(1+\epsilon q_\epsilon)\nabla\phi_\epsilon + (1+\epsilon\phi_\epsilon)\nabla q_\epsilon\right]\\
&\quad = \mu\Delta\mathbf{v}_\epsilon + \frac{\epsilon^2}{2}\pi(\mathbf{w}_t + \mathbf{w}\cdot\nabla\mathbf{w} + \nabla\pi) + \mathbf{J}_\epsilon \times \mathbf{H}_\epsilon,
\end{aligned} \tag{2.3}$$

$$\nu\partial_t\mathbf{D}_\epsilon - \frac{1}{\mu_0}\operatorname{curl}\mathbf{H}_\epsilon = -\mathbf{J}_\epsilon, \quad \mathbf{J}_\epsilon = \sigma(\mathbf{D}_\epsilon + \mathbf{w}_\epsilon \times \mathbf{H}_\epsilon), \tag{2.4}$$

$$\partial_t\mathbf{H}_\epsilon + \operatorname{curl}\mathbf{D}_\epsilon = 0, \quad \operatorname{div}\mathbf{H}_\epsilon = 0, \tag{2.5}$$

$$\begin{aligned}
&(1+\epsilon q_\epsilon)(\partial_t\phi_\epsilon + \mathbf{v}_\epsilon\cdot\nabla\phi_\epsilon) + \frac{\gamma-1}{\epsilon}(1+\epsilon q_\epsilon)(1+\epsilon\phi_\epsilon)\operatorname{div}\mathbf{v}_\epsilon\\
&\quad = \left(\frac{\epsilon}{2} + \frac{\epsilon^3}{4}\pi\right)(\pi_t + \mathbf{w}\cdot\nabla\pi).
\end{aligned} \tag{2.6}$$

We rewrite the system (2.2)–(2.6) in the following vector form

$$A_0(U_\epsilon)\partial_t U_\epsilon + \sum_{j=1}^{3} A_j(U_\epsilon)\partial_j U_\epsilon = S(U_\epsilon) + R \tag{2.7}$$

with $S(U_\epsilon) = (0, \mu\Delta\mathbf{v}_\epsilon + \mathbf{J}_\epsilon \times \mathbf{H}_\epsilon, -\mathbf{J}_\epsilon, 0, 0, 0, 0)^\top$ and

$$R = \begin{pmatrix} \frac{\epsilon}{2}(\pi_t + \mathbf{w}\cdot\nabla\pi) \\ \frac{\epsilon^2}{2}\pi(\mathbf{w}_t + \mathbf{w}\cdot\nabla\mathbf{w} + \nabla\pi) \\ \mathbf{0} \\ \mathbf{0} \\ \left(\frac{\epsilon}{2} + \frac{\epsilon^3}{4}\pi\right)(\pi_t + \mathbf{w}\cdot\nabla\pi) \end{pmatrix}.$$

Due to the regularity assumptions on $(\mathbf{w}, \pi)$ in Theorem 1.1, we have

$$\max_{t\in[0,T^*]} \left\| R(t) \right\|_s \leqslant C\epsilon. \tag{2.8}$$

In order to prove Theorem 1.1, thanks to Lemma 2.1, it suffices to establish the error estimate in (1.27) for $t \in [0, \min\{T^*, T_\epsilon\})$. To this end, introducing

$$F = U^\epsilon - U_\epsilon \quad \text{and} \quad \mathcal{A}_j(U) = A_0^{-1}(U)A_j(U),$$

and utilizing (2.1) and (2.7), we arrive at

$$F_t + \sum_{j=1}^{3} \mathcal{A}_j\left(U^\epsilon\right)F_{x_j} = \sum_{j=1}^{3}\left(\mathcal{A}_j(U_\epsilon) - \mathcal{A}_j\left(U^\epsilon\right)\right)U_{\epsilon x_j} + A_0^{-1}\left(U^\epsilon\right)Q\left(U^\epsilon\right) - A_0^{-1}(U_\epsilon)\left(S(U_\epsilon) + R\right). \tag{2.9}$$

For any multi-index α satisfying $|\alpha| \leqslant s$, we apply the operator D^α to (2.9) to obtain that

$$\partial_t D^\alpha F + \sum_{j=1}^{3} \mathcal{A}_j\left(U^\epsilon\right)\partial_{x_j} D^\alpha F = P_1^\alpha + P_2^\alpha + Q^\alpha + R^\alpha \tag{2.10}$$

with

$$P_1^\alpha = \sum_{j=1}^{3}\left\{\mathcal{A}_j\left(U^\epsilon\right)\partial_{x_j} D^\alpha F - D^\alpha\left(\mathcal{A}_j\left(U^\epsilon\right)\partial_{x_j} F\right)\right\},$$

$$P_2^\alpha = \sum_{j=1}^{3} D^\alpha\left\{\left(\mathcal{A}_j(U_\epsilon) - \mathcal{A}_j\left(U^\epsilon\right)\right)U_{\epsilon x_j}\right\},$$

$$Y^\alpha = D^\alpha\left\{A_0^{-1}\left(U^\epsilon\right)Q\left(U^\epsilon\right) - A_0^{-1}(U_\epsilon)S(U_\epsilon)\right\},$$

$$R^\alpha = D^\alpha\left\{A_0^{-1}(U_\epsilon)R\right\}.$$

As in [14], we define the canonical energy by

$$\|F\|_{\mathrm{e}}^2 := \int \left\langle \tilde{A}_0\left(U^\epsilon\right)F, F\right\rangle dx.$$

Define

$$\tilde{A}_0\left(U^\epsilon\right) = \operatorname{diag}\left(\frac{1+\epsilon\phi^\epsilon}{(1+\epsilon q^\epsilon)^2}, 1, 1, 1, \nu\mu_0, \nu\mu_0, \nu\mu_0, 1, 1, 1, \frac{1}{(\gamma-1)(1+\epsilon\phi^\epsilon)}\right).$$

Note that $\tilde{A}_0(U^\epsilon)$ is a positive definite symmetric matrix and $\tilde{A}_0(U^\epsilon)\mathcal{A}_j(U^\epsilon)$ is symmetric. Now, if we multiply (2.10) by $\tilde{A}_0(U^\epsilon)$ and take the inner product between the resulting system and $D^\alpha F$, we arrive at

$$\frac{d}{dt}\left\|D^\alpha F\right\|_{\mathrm{e}}^2 = \int \left\langle \Gamma D^\alpha F, D^\alpha F\right\rangle dx + 2\int \left(D^\alpha F\right)^T \tilde{A}_0\left(U^\epsilon\right)\left(P_1^\alpha + P_2^\alpha + Y^\alpha + R^\alpha\right) dx, \tag{2.11}$$

where

$$\Gamma = (\partial_t, \nabla)\cdot\left(\tilde{A}_0, \tilde{A}_0\left(U^\epsilon\right)\mathcal{A}_1\left(U^\epsilon\right), \tilde{A}_0\left(U^\epsilon\right)\mathcal{A}_2\left(U^\epsilon\right), \tilde{A}_0\left(U^\epsilon\right)\mathcal{A}_3\left(U^\epsilon\right)\right).$$

Next, we estimate every term on the right-hand side of (2.10). Note that our estimates only need to be done for $t \in [0, \min\{T^*, T_\epsilon\})$ where both U^ϵ and U_ϵ are regular enough and take values in a convex compact subset of the state space. Thus, we have

$$C^{-1}\int \left|D^\alpha F\right|^2 dx \leqslant \left\|D^\alpha F\right\|_{\mathrm{e}}^2 \leqslant C\int \left|D^\alpha F\right|^2 dx \tag{2.12}$$

for some $C > 0$ and

$$\left|\left(D^\alpha F\right)^\top \tilde{A}_0\left(U^\epsilon\right)\left(P_1^\alpha + P_2^\alpha + R^\alpha\right)\right| \leqslant C\left(\left|D^\alpha F\right|^2 + \left|P_1^\alpha\right|^2 + \left|P_2^\alpha\right|^2 + \left|R^\alpha\right|^2\right). \tag{2.13}$$

To estimate Γ, we rewrite $\mathcal{A}_j(U^\epsilon) = \mathcal{B}_j + \bar{\mathcal{A}}_j(U^\epsilon)$ with

$$\mathcal{B}_j = \operatorname{diag}\left(u_j^\epsilon, u_j^\epsilon, u_j^\epsilon, u_j^\epsilon, 0, 0, 0, 0, 0, 0, u_j^\epsilon\right).$$

Notice that $\bar{\mathcal{A}}_j(U^\epsilon)$ depends only on q^ϵ and ϕ^ϵ. Thus we have

$$\begin{aligned}
\Gamma &= \frac{\partial}{\partial t}\tilde{A}_0 + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\left(\tilde{A}_0\mathcal{B}_j + \tilde{A}_0\bar{\mathcal{A}}_j\right) \\
&= \tilde{A}_0 \operatorname{div}\mathbf{u}^\epsilon + \sum_{j=1}^{3}\frac{\partial}{\partial x_j}\tilde{A}_0\bar{\mathcal{A}}_j - \tilde{A}'_{0\eta_1}\left(1+\epsilon q^\epsilon\right)\operatorname{div}\mathbf{u}^\epsilon - \tilde{A}'_{0\eta_2}\left\{(\gamma-1)\left(1+\epsilon\phi^\epsilon\right)\operatorname{div}\mathbf{u}^\epsilon - \kappa\epsilon\left(1+\epsilon q^\epsilon\right)^{-1}\Delta\phi^\epsilon\right. \\
&\quad \left. - \epsilon^2\left(1+\epsilon q^\epsilon\right)^{-1}\left(\Psi^\epsilon + \mathbf{J}^\epsilon\cdot\left(\mathbf{E}^\epsilon + \mathbf{u}^\epsilon\times\mathbf{B}^\epsilon\right)\right)\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\Gamma| &\leqslant C + C\left(\left|\nabla q^\epsilon\right| + \left|\nabla\phi^\epsilon\right| + \left|\Delta\phi^\epsilon\right| + \left|\nabla\mathbf{u}^\epsilon\right|^2 + \left|\mathbf{E}^\epsilon\right|^2 + \left|\mathbf{u}^\epsilon\right|^4 + \left|\mathbf{B}^\epsilon\right|^4\right) \\
&\leqslant C + C\left(|\nabla F| + |\nabla F|^2\right) + C\left|\Delta\left(\phi^\epsilon - \phi_\epsilon\right)\right| + C\left(|F|^2 + |F|^4\right) + C\left(|\nabla\mathbf{v}_\epsilon| + |\nabla\mathbf{v}_\epsilon|^2 + 1\right) \\
&\leqslant C + C\left(\|F\|_s + \|F\|_s^4\right) \\
&\leqslant C\left(1 + \|F\|_s^4\right).
\end{aligned} \tag{2.14}$$

Here we have used Sobolev's embedding theorem and the fact that $s > 3/2 + 2$, and the symbols $\tilde{A}'_{0\eta_1}$ and $\tilde{A}'_{0\eta_2}$ denote the differentiation of $\tilde{A}_0$ with respect to ρ^ϵ and θ^ϵ, respectively.

Since

$$\begin{aligned}
P_1^\alpha &= \sum_{j=1}^{3}\mathcal{A}_j\left(U^\epsilon\right)\partial_{x_j}D^\alpha F - D^\alpha\left(\mathcal{A}_j\left(U^\epsilon\right)\partial_{x_j}F\right) \\
&= -\sum_{j=1}^{3}\sum_{0<\beta\leqslant\alpha}\binom{\alpha}{\beta}\partial^\beta\mathcal{A}_j\left(U^\epsilon\right)\partial^{\alpha-\beta}F_{x_j} \\
&= -\sum_{j=1}^{3}\sum_{0<\beta\leqslant\alpha}\binom{\alpha}{\beta}\partial^\beta\left[\mathcal{B}_j + \bar{\mathcal{A}}_j\left(U^\epsilon\right)\right]\partial^{\alpha-\beta}F_{x_j},
\end{aligned}$$

we obtain, with the help of the Moser-type calculus inequalities in Sobolev spaces [20], that

$$\begin{aligned}
\left\|\partial^\beta\mathcal{B}_j\left(U^\epsilon\right)\partial^{\alpha-\beta}F_{x_j}\right\| &\leqslant C\left\|\mathbf{u}^\epsilon\right\|_s\|F_{x_j}\|_{|\alpha|-1} \leqslant C\left\|\mathbf{u}^\epsilon\right\|_s\|F\|_{|\alpha|}, \\
\left\|\partial^\beta\bar{\mathcal{A}}_j\left(U^\epsilon\right)\partial^{\alpha-\beta}F_{x_j}\right\| &= \left\|\epsilon^{-1}\partial^\beta f\left(q^\epsilon,\phi^\epsilon\right)\partial^{\alpha-\beta}F_{x_j}\right\| \\
&\leqslant C\epsilon^{-1}\left(\left\|\nabla\rho^\epsilon\right\|_{s-1} + \left\|\nabla\theta^\epsilon\right\|_{s-1} + \left\|\nabla\rho^\epsilon\right\|_{s-1}^s + \left\|\nabla\theta^\epsilon\right\|_{s-1}^s\right)\|F\|_\alpha \\
&\leqslant C\left(\left\|\left(q^\epsilon,\phi^\epsilon\right)\right\|_s + \left\|\left(q^\epsilon,\phi^\epsilon\right)\right\|_s^s\right)\|F\|_\alpha \\
&\leqslant C\left(1 + \|F\|_s^s\right)\|F\|_\alpha
\end{aligned}$$

where $f(q^\epsilon,\phi^\epsilon) := (1+\epsilon q^\epsilon) + (\gamma-1)(1+\epsilon\phi^\epsilon) + (1+\epsilon q^\epsilon)^{-1}(1+\epsilon\phi^\epsilon)$. Thus, we have

$$\left\|P_1^\alpha\right\| \leqslant C\left(1 + \|F\|_s^s\right)\|F\|_{|\alpha|}. \tag{2.15}$$

Similarly, by the uniform boundedness of $\|U_\epsilon\|_{s+1}$, the term P_2^α can be bounded as follows.

$$\begin{aligned}
\|P_2^\alpha\| &\leqslant C\|U_{\epsilon x_j}\|_s\big\|\mathcal{A}_j(U_\epsilon)-\mathcal{A}_j\big(U^\epsilon\big)\big\|_{|\alpha|} \\
&\leqslant C\big\|\big(\mathbf{u}_j^\epsilon-\mathbf{v}_{\epsilon j}\big)\big\|_{|\alpha|}+\big\|\bar{\mathcal{A}}_j\big(U^\epsilon\big)-\bar{\mathcal{A}}_j(U_\epsilon)\big\|_{|\alpha|} \\
&\leqslant C\big(1+\big\|\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big\|_{|\alpha|}\big)+C\big\|\epsilon^{-1}\big(f\big(q^\epsilon,\phi^\epsilon\big)-f(q_\epsilon,\phi_\epsilon)\big)\big\|_{|\alpha|} \\
&\leqslant C\big(1+\big\|q_\epsilon+\eta_3\big(q^\epsilon-q_\epsilon\big)+\phi_\epsilon+\eta_4\big(\phi^\epsilon-\phi_\epsilon\big)\big\|_s^s\big)\|F\|_{|\alpha|} \\
&\leqslant C\big(1+\|F\|_s^s\big)\|F\|_{|\alpha|},
\end{aligned} \tag{2.16}$$

where $0\leqslant\eta_3,\eta_4\leqslant 1$ are constants.

The estimate of the term $\int(D^\alpha E)^\top\tilde{A}_0(U^\epsilon)Y^\alpha dx$ is more complex and delicate, and we give the details below. Since

$$Y^\alpha=\begin{pmatrix} 0 \\ D^\alpha\big(\frac{L(U^\epsilon)}{1+\epsilon q^\epsilon}-\frac{\mu\Delta\mathbf{v}_\epsilon+\mathbf{J}_\epsilon\times\mathbf{H}_\epsilon}{1+\epsilon q^\epsilon}\big) \\ D^\alpha\big(\frac{\mathbf{J}_\epsilon-\mathbf{J}^\epsilon}{\nu}\big) \\ \mathbf{0} \\ D^\alpha\big(\frac{\kappa\Delta\phi^\epsilon+\epsilon(\Psi^\epsilon+\mathbf{J}^\epsilon(\mathbf{E}^\epsilon+\mathbf{u}^\epsilon\times\mathbf{H}^\epsilon))}{1+\epsilon q^\epsilon}\big)\end{pmatrix},$$

we can rewrite $\int(D^\alpha E)^\top\tilde{A}_0(U^\epsilon)Y^\alpha dx$ as

$$\begin{aligned}
&\int\big(D^\alpha E\big)^\top\tilde{A}_0\big(U^\epsilon\big)Y^\alpha\,dx \\
&=\int D^\alpha\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big)D^\alpha\bigg(\frac{L(U^\epsilon)}{1+\epsilon q^\epsilon}-\frac{\mu\Delta\mathbf{v}_\epsilon+\mathbf{J}_\epsilon\times\mathbf{H}_\epsilon}{1+\epsilon q^\epsilon}\bigg)dx+\int\mu_0D^\alpha\big(\mathbf{E}^\epsilon-\mathbf{D}_\epsilon\big)D^\alpha\big(\mathbf{J}_\epsilon-\mathbf{J}^\epsilon\big)\,dx \\
&\quad+\int\frac{1}{(\gamma-1)(1+\epsilon\phi^\epsilon)}D^\alpha\big(\phi^\epsilon-\phi_\epsilon\big)D^\alpha\bigg(\frac{\kappa\Delta\phi^\epsilon+\epsilon(\Psi^\epsilon+\mathbf{J}^\epsilon\cdot(\mathbf{E}^\epsilon+\mathbf{u}^\epsilon\times\mathbf{H}^\epsilon))}{1+\epsilon q^\epsilon}\bigg)dx \\
&=\int D^\alpha\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big)D^\alpha\bigg(\frac{\mu\Delta\mathbf{u}^\epsilon+(\mu+\mu')\nabla\operatorname{div}\mathbf{u}^\epsilon}{1+\epsilon q^\epsilon}-\frac{\mu\Delta\mathbf{v}_\epsilon}{1+\epsilon q_\epsilon}\bigg)dx \\
&\quad+\int D^\alpha\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big)D^\alpha\bigg(\frac{\mathbf{J}^\epsilon\times\mathbf{B}^\epsilon}{1+\epsilon q^\epsilon}-\frac{\mathbf{J}_\epsilon\times\mathbf{H}_\epsilon}{1+\epsilon q_\epsilon}\bigg)dx+\int\mu_0D^\alpha\big(\mathbf{E}^\epsilon-\mathbf{D}_\epsilon\big)D^\alpha\big(\mathbf{J}_\epsilon-\mathbf{J}^\epsilon\big)\,dx \\
&\quad+\int\frac{1}{(\gamma-1)(1+\epsilon\phi^\epsilon)}D^\alpha\big(\phi^\epsilon-\phi_\epsilon\big)D^\alpha\bigg(\frac{\kappa\Delta\phi^\epsilon}{1+\epsilon q^\epsilon}\bigg)dx \\
&\quad+\epsilon\int\frac{1}{(\gamma-1)(1+\epsilon\phi^\epsilon)}D^\alpha\big(\phi^\epsilon-\phi_\epsilon\big)D^\alpha\bigg(\frac{\Psi^\epsilon}{1+\epsilon q^\epsilon}\bigg)dx \\
&\quad+\epsilon\int\frac{1}{(\gamma-1)(1+\epsilon\phi^\epsilon)}D^\alpha\big(\phi^\epsilon-\phi_\epsilon\big)D^\alpha\bigg(\frac{\mathbf{J}^\epsilon(\mathbf{E}^\epsilon+\mathbf{u}^\epsilon\times\mathbf{B}^\epsilon)}{1+\epsilon q^\epsilon}\bigg)dx \\
&:=\sum_{i=1}^{6}\mathcal{I}_i.
\end{aligned} \tag{2.17}$$

By the Moser-type calculus inequalities, Sobolev's embedding theorem, Holder's inequality, and the regularity of $(q_\epsilon,\mathbf{v}_\epsilon,\mathbf{D}_\epsilon,\mathbf{H}_\epsilon,\phi_\epsilon)$, the first four terms $\mathcal{I}_i$ $(i=1,2,3,4)$ can be bounded as follows.

$$\begin{aligned}
\mathcal{I}_1&=\int D^\alpha\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big)D^\alpha\bigg(\frac{\mu\Delta\mathbf{u}^\epsilon+(\mu+\mu')\nabla\operatorname{div}\mathbf{u}^\epsilon}{1+\epsilon q^\epsilon}-\frac{\mu\Delta\mathbf{v}_\epsilon}{1+\epsilon q_\epsilon}\bigg)dx \\
&=\int D^\alpha\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big)D^\alpha\big\{\big(1+\epsilon q^\epsilon\big)^{-1}\mu\Delta\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big)+\big(\mu+\mu'\big)\nabla\operatorname{div}\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big)\big\}\,dx \\
&\quad+\mu\int D^\alpha\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big)D^\alpha\big\{\big[\big(1+\epsilon q^\epsilon\big)^{-1}-(1+\epsilon q_\epsilon)^{-1}\big]\Delta\mathbf{v}_\epsilon\big\}\,dx \\
&\leqslant-\int\frac{\mu}{1+\epsilon q^\epsilon}\big|D^\alpha\nabla\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big)\big|^2\,dx-\int\frac{\mu+\mu'}{1+\epsilon q^\epsilon}\big|D^\alpha\operatorname{div}\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big)\big|^2\,dx \\
&\quad+\bigg|\int D^\alpha\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big)\cdot\sum_{0<\beta\leqslant\alpha}D^\beta\big[\big(1+\epsilon q^\epsilon\big)^{-1}\big]D^{\alpha-\beta}\big\{\mu\Delta\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big) \\
&\quad+\big(\mu+\mu'\big)\nabla\operatorname{div}\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big)\big\}\,dx\bigg|+C\|F\|_{|\alpha|}^2+C\|F\|_s^4+C\epsilon\big\|D^\alpha\nabla\big(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon\big)\big\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leqslant -C\int \mu\big|D^\alpha\nabla(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)\big|^2\,dx - C\int(\mu+\mu')\big|D^\alpha\operatorname{div}(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)\big|^2 + C\|F\|_{|\alpha|}^2\,dx\\
&\quad + C\epsilon\big\|D^\alpha\nabla(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)\big\|^2 + C\|F\|_s^4 + C\|F\|_s^2\|F\|_{|\alpha|}^2 + \bigg|\int D^\alpha(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)\\
&\quad\cdot\sum_{1<\beta\leqslant\alpha} D^\beta\big[(1+\epsilon q^\epsilon)^{-1}\big]D^{\alpha-\beta}\big\{\mu\Delta(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)+(\mu+\mu')\nabla\operatorname{div}(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)\big\}\,dx\bigg|\\
&\leqslant -C\int \mu\big|D^\alpha\nabla(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)\big|^2\,dx - C\int(\mu+\mu')\big|D^\alpha\operatorname{div}(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)\big|^2\,dx\\
&\quad + C\epsilon\big\|D^\alpha\nabla(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)\big\|^2 + C\|F\|_s^4 + C\|F\|_{|\alpha|}^2 + C\|F\|_{|\alpha|}^2\|F\|_s^s,
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
|\mathcal{I}_2| &= \bigg|\int D^\alpha(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)D^\alpha\bigg(\frac{\mathbf{J}^\epsilon\times\mathbf{B}^\epsilon}{1+\epsilon q^\epsilon}-\frac{\mathbf{J}_\epsilon\times\mathbf{H}_\epsilon}{1+\epsilon q_\epsilon}\bigg)\,dx\bigg|\\
&\leqslant \|F\|_{|\alpha|}\bigg\|\frac{\mathbf{J}^\epsilon\times\mathbf{B}^\epsilon}{1+\epsilon q^\epsilon}-\frac{\mathbf{J}_\epsilon\times\mathbf{H}_\epsilon}{1+\epsilon q_\epsilon}\bigg\|_{|\alpha|}\\
&\leqslant \|F\|_{|\alpha|}\bigg(\bigg\|\frac{1}{1+\epsilon q^\epsilon}\bigg\|_s\big\|(\mathbf{J}^\epsilon-\mathbf{J}_\epsilon)\times\mathbf{B}^\epsilon\big\|_{|\alpha|}+\bigg\|\frac{1}{1+\epsilon q^\epsilon}\bigg\|_s\big\|\mathbf{J}_\epsilon\times(\mathbf{B}^\epsilon-\mathbf{H}_\epsilon)\big\|_{|\alpha|}\\
&\quad + \bigg\|\frac{1}{1+\epsilon q^\epsilon}-\frac{1}{1+\epsilon q_\epsilon}\bigg\|_s\|\mathbf{J}_\epsilon\times\mathbf{H}_\epsilon\|_{|\alpha|}\bigg)\\
&\leqslant C\|F\|_{|\alpha|}\big(1+\|F\|_s^{s+2}\big)\|F\|_s,
\end{aligned}
\tag{2.19}
$$

$$
\begin{aligned}
|\mathcal{I}_3| &= \bigg|\int \mu_0 D^\alpha(\mathbf{E}^\epsilon-\mathbf{D}_\epsilon)D^\alpha(\mathbf{J}_\epsilon-\mathbf{J}^\epsilon)\,dx\bigg|\\
&\leqslant \mu_0\|\mathbf{E}^\epsilon-\mathbf{D}_\epsilon\|_{|\alpha|}\|\mathbf{J}^\epsilon-\mathbf{J}_\epsilon\|_{|\alpha|}\\
&\leqslant C\big(1+\|F\|_s\big)\|F\|_{|\alpha|}^2,
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
|\mathcal{I}_4| &= \bigg|\int\frac{1}{(\gamma-1)(1+\epsilon\phi^\epsilon)}D^\alpha(\phi^\epsilon-\phi_\epsilon)D^\alpha\bigg(\frac{\kappa\Delta\phi^\epsilon}{1+\epsilon q^\epsilon}\bigg)\,dx\bigg|\\
&\leqslant \frac{\kappa}{\gamma-1}\bigg|\int\frac{1}{1+\epsilon\phi^\epsilon}D^\alpha(\phi^\epsilon-\phi_\epsilon)D^\alpha\bigg(\frac{\Delta(\phi^\epsilon-\phi_\epsilon)}{1+\epsilon q^\epsilon}\bigg)\,dx\bigg| + \frac{\kappa}{\gamma-1}\bigg|\int\frac{1}{1+\epsilon\phi^\epsilon}D^\alpha(\phi^\epsilon-\phi_\epsilon)D^\alpha\bigg(\frac{\Delta\phi_\epsilon}{1+\epsilon q^\epsilon}\bigg)\,dx\bigg|\\
&\leqslant \frac{\kappa}{\gamma-1}\bigg|\int\frac{1}{1+\epsilon\phi^\epsilon}D^\alpha(\phi^\epsilon-\phi_\epsilon)\frac{1}{1+\epsilon q^\epsilon}D^\alpha\Delta(\phi^\epsilon-\phi_\epsilon)\,dx\bigg|\\
&\quad + \frac{\kappa}{\gamma-1}\sum_{0<\beta\leqslant\alpha}\bigg|\int D^\alpha(\phi^\epsilon-\phi_\epsilon)D^\beta\big[(1+\epsilon q^\epsilon)^{-1}\big]D^{\alpha-\beta}\Delta(\phi^\epsilon-\phi_\epsilon)\,dx\bigg|\\
&\quad + \frac{\kappa}{(\gamma-1)}\bigg|\int\frac{1}{1+\epsilon\phi^\epsilon}D^\alpha(\phi^\epsilon-\phi_\epsilon)D^\alpha\frac{\Delta\phi_\epsilon}{1+\epsilon q^\epsilon}\,dx\bigg|\\
&\leqslant -C\kappa\int\big|\nabla D^\alpha(\phi^\epsilon-\phi_\epsilon)\big|^2\,dx + C\epsilon\big\|\nabla D^\alpha(\phi^\epsilon-\phi_\epsilon)\big\|^2 + C\|F\|_{|\alpha|}^2 + C\|F\|_s^4 + C\|F\|_{|\alpha|}^2\|F\|_s^s + C\epsilon^2.
\end{aligned}
\tag{2.21}
$$

Similarly, we have

$$
\begin{aligned}
|\mathcal{I}_5| &= \epsilon\bigg|\int\frac{1}{(\gamma-1)(1+\epsilon\phi^\epsilon)}D^\alpha(\phi^\epsilon-\phi_\epsilon)D^\alpha\bigg(\frac{\Psi^\epsilon}{1+\epsilon q^\epsilon}\bigg)\,dx\bigg|\\
&\leqslant C\epsilon\big\|D^\alpha\nabla(\phi^\epsilon-\phi_\epsilon)\big\|^2 + C\|F\|_{|\alpha|}^2 + C\|F\|_s^4 + C\|F\|_{|\alpha|}^2\|F\|_s^s + C\epsilon^2,
\end{aligned}
\tag{2.22}
$$

$$
\begin{aligned}
|\mathcal{I}_6| &= \epsilon\bigg|\int\frac{1}{(\gamma-1)(1+\epsilon\phi^\epsilon)}D^\alpha(\phi^\epsilon-\phi_\epsilon)D^\alpha\bigg(\frac{\mathbf{J}^\epsilon\cdot(\mathbf{E}^\epsilon+\mathbf{u}^\epsilon\times\mathbf{B}^\epsilon)}{1+\epsilon q^\epsilon}\bigg)\,dx\bigg|\\
&\leqslant \frac{\epsilon}{\gamma-1}\bigg|\int\frac{1}{1+\epsilon\phi^\epsilon}\frac{1}{1+\epsilon q^\epsilon}D^\alpha(\phi^\epsilon-\phi_\epsilon)D^\alpha\big(\mathbf{J}^\epsilon\cdot(\mathbf{E}^\epsilon+\mathbf{u}^\epsilon\times\mathbf{B}^\epsilon)\big)\,dx\bigg|\\
&\quad + \frac{\epsilon}{\gamma-1}\sum_{0<\beta\leqslant\alpha}\bigg|\int D^\alpha(\phi^\epsilon-\phi_\epsilon)D^\beta\frac{1}{1+\epsilon q^\epsilon}D^{\alpha-\beta}\big(\mathbf{J}^\epsilon\cdot(\mathbf{E}^\epsilon+\mathbf{u}^\epsilon\times\mathbf{B}^\epsilon)\big)\,dx\bigg|\\
&\leqslant C\|F\|_s^2 + C\|F\|_s^8 + C\|F\|_\alpha^2\big(1+\|F\|_s^{2s}\big) + C\epsilon^2.
\end{aligned}
\tag{2.23}
$$

Putting the estimates (2.8) and (2.13)–(2.23) into (2.11) and taking ϵ small enough, we obtain that

$$\begin{aligned}&\frac{d}{dt}\|D^\alpha F\|_e^2+\xi\int\left|D^\alpha\nabla(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)\right|^2dx+\kappa\int\left|D^\alpha\nabla(\phi^\epsilon-\phi_\epsilon)\right|^2dx\\&\quad\leqslant C\left\|R^\alpha\right\|^2+C\left(1+\|F\|_s^{2s}\right)\|F\|_{|\alpha|}^2+\left(1+\|F\|_s^{s+2}\right)\|F\|_s\|F\|_\alpha+\|F\|_s^2+\|F\|_s^8+C\epsilon^2,\end{aligned}\tag{2.24}$$

where we have used the following estimate

$$\mu\int\left|D^\alpha\nabla(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)\right|^2+(\mu+\mu')\int\left|D^\alpha\operatorname{div}(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)\right|^2\geqslant\xi\int\left|D^\alpha\nabla(\mathbf{u}^\epsilon-\mathbf{v}_\epsilon)\right|^2$$

for some positive constant $\xi>0$.

Thanks to (2.12), we can integrate the inequality (2.24) over $(0,t)$ with $t<\min\{T_\epsilon,T^*\}$ to obtain that

$$\begin{aligned}\left\|D^\alpha F(t)\right\|^2&\leqslant C\left\|D^\alpha F(0)\right\|^2+C\int_0^t\left\|R^\alpha(\tau)\right\|^2d\tau\\&\quad+C\int_0^t\left\{\left(1+\|F\|_s^{2s}\right)\|F\|_{|\alpha|}^2+\|F\|_s^2+\left(1+\|F\|_s^{s+2}\right)\|F\|_s\|F\|_\alpha+\|F\|_s^8+C\epsilon^2\right\}(\tau)\,d\tau.\end{aligned}$$

Summing up this inequality for all α with $|\alpha|\leqslant s$, we get

$$\left\|F(t)\right\|_s^2\leqslant C\left\|F(0)\right\|_s^2+C\int_0^{T^*}\left\|R(\tau)\right\|_s^2d\tau+C\int_0^t\left\{\left(1+\|F\|_s^{2s+2}\right)\|F\|_s^2\right\}(\tau)\,d\tau.$$

With the help of Gronwall's lemma and the fact that

$$\left\|F(0)\right\|_s^2+\int_0^{T^*}\left\|R(t)\right\|_s^2dt=O\left(\epsilon^2\right),$$

we conclude that

$$\left\|F(t)\right\|_s^2\leqslant C\epsilon^2\exp\left\{C\int_0^t\left(1+\left\|F(\tau)\right\|_s^{2s+2}\right)d\tau\right\}\equiv\Phi(t).$$

It is easy to see that $\Phi(t)$ satisfies

$$\Phi'(t)=C\left(1+\left\|F(t)\right\|_s^{2s+2}\right)\Phi(t)\leqslant C\Phi(t)+C\Phi^{s+2}(t).$$

Thus, by employing the nonlinear Gronwall-type inequality, we conclude that there exists a constant K, independent of ϵ, such that

$$\left\|F(t)\right\|_s\leqslant K\epsilon$$

for all $t\in[0,\min\{T_\epsilon,T^*\})$ provided $\Phi(0)=C\epsilon^2<\exp\{-CT^*\}$. Hence the proof of Theorem 1.1 is complete.

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