



Angular and unrestricted limits of one-parameter semigroups in the unit disk



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ABSTRACT

We study local boundary behaviour of one-parameter semigroups of holomorphic functions in the unit disk. Earlier, under some additional condition (the position of the Denjoy–Wolff point) it was shown in [13] that elements of one-parameter semigroups have *angular limits everywhere on the unit circle and unrestricted limits at all boundary fixed points*. We prove stronger versions of these statements with no assumption on the position of the Denjoy–Wolff point. In contrast to many other problems, in the question of existence for unrestricted limits it appears to be more complicated to deal with the boundary Denjoy–Wolff point (the case not covered in [13]) than with all the other boundary fixed points of the semigroup.

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1. Introduction

One-parameter semigroups in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ are classical objects of study in Complex Analysis and can be defined as *continuous homomorphisms* from the additive semigroup $([0, +\infty), +)$ of non-negative reals to the topological semigroup $\text{Hol}(\mathbb{D}, \mathbb{D})$ consisting of all holomorphic self-maps $\phi : \mathbb{D} \rightarrow \mathbb{D}$ and endowed with operation of composition $(\phi, \psi) \mapsto \psi \circ \phi$ and the topology induced by the locally uniform convergence in \mathbb{D} . In other words, a *one-parameter semigroup in \mathbb{D}* is a family $(\phi_t) \in \text{Hol}(\mathbb{D}, \mathbb{D})$ satisfying the following conditions:

- (i) $\phi_0 = \text{id}_{\mathbb{D}}$;
- (ii) $\phi_{t+s} = \phi_t \circ \phi_s = \phi_s \circ \phi_t$ for any $t, s \geq 0$;
- (iii) $\phi_t(z) \rightarrow z$ as $t \rightarrow +0$ for any $z \in \mathbb{D}$.

Due to the fact that $\text{Hol}(\mathbb{D}, \mathbb{D})$ is a normal family in \mathbb{D} , condition (iii) expresses the continuity of the map $t \mapsto \phi_t$.

In a similar way one can define one-parameter semigroups in other domains,² *e.g.*, in the right half-plane $\mathbb{H}_1 := \{z : \text{Re } z > 0\}$. In what follows we will omit the words “in \mathbb{D} ” and specify the domain only in the rare cases when it is different from \mathbb{D} .

Interest to one-parameter semigroups comes from different areas. In the Iteration Theory in \mathbb{D} they appear as *fractional iterates*, see, *e.g.*, [36,15,23,17]. In Operator Theory, one-parameter semigroups in \mathbb{D} have been extensively investigated in connection to the study of one-parameter semigroups of *composition operators*, see, *e.g.*, [5,35]. The *embedding problem* for time-homogeneous stochastic branching processes is also very much related to one-parameter semigroups, see, *e.g.*, [27,24,25]. Finally, one can consider this notion as a special (autonomous) case of evolution families in \mathbb{D} playing important role in much celebrated Loewner Theory [22,7,2]. It is also worth to mention that one-parameter semigroups lying in a given subsemigroup $S \subset \text{Hol}(\mathbb{D}, \mathbb{D})$ give useful information about the infinitesimal structure of S [22,23]. More detailed discussion on the history, recent developments and applications of one-parameter semigroups can be found, *e.g.*, in the survey paper [26].

Not every element of $\text{Hol}(\mathbb{D}, \mathbb{D})$ can be embedded into a one-parameter semigroup. Elements of one-parameter semigroups enjoy some very specific nice properties. For example, these functions are *univalent* (see, *e.g.*, [1, Proposition (1.4.6)]). But especially brightly this shows up in their boundary behaviour. In this paper we study mainly *local boundary behaviour* of one-parameter semigroups.

1.1. Preliminaries

Here we collect some fundamental results on one-parameter semigroups we use in this paper.

First of all, in spite of the fact that in the definition one requires only continuity of a one-parameter semigroup (ϕ_t) w.r.t. the parameter t , the algebraic semigroup structure enhances regularity in t . In fact, the map $t \mapsto \phi_t(z)$ is smooth. Moreover, any one-parameter semigroup is a semiflow of some holomorphic

² It is worth to mention that one-parameter semigroups in a domain $D \subset \bar{\mathbb{C}}$ constitute a very narrow class of objects unless D is conformally equivalent to \mathbb{D} or to $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$. Passing from \mathbb{D} to another domain can make sense when the geometry of the new domain suits the problem better, *e.g.*, in the case of a boundary attracting fixed point (the boundary Denjoy–Wolff point).

vector field [5, Theorem (1.1)], see also [34, §3.2] or [1, Theorem (1.4.11)]. More rigorously these statements can be formulated in the following form.

Theorem A. *For any one-parameter semigroup (ϕ_t) the limit*

$$G(z) := \lim_{t \rightarrow +0} \frac{\phi_t(z) - z}{t}, \quad z \in \mathbb{D}, \quad (1.1)$$

exists and G is a holomorphic function in \mathbb{D} .

Moreover, for each $z \in \mathbb{D}$, the function $[0, +\infty) \ni t \mapsto w(t) := \phi_t(z) \in \mathbb{D}$ is the unique solution to the initial value problem

$$\frac{dw(t)}{dt} = G(w(t)), \quad t \geq 0, \quad w(0) = z. \quad (1.2)$$

Definition 1.1. The function G in Theorem A is called the *infinitesimal generator* of the one-parameter semigroup (ϕ_t) .

Clearly, not every holomorphic function in \mathbb{D} is a generator of a one-parameter semigroup. Berkson and Porta [5] obtained the following very useful characterization of infinitesimal generators (see also [1, Theorem (1.4.19)]).

Theorem B. *A function $G : \mathbb{D} \rightarrow \mathbb{C}$ is an infinitesimal generator (of some one-parameter semigroup in \mathbb{D}) if and only if it can be represented as*

$$G(z) = (\tau - z)(1 - \bar{\tau}z)p(z), \quad (1.3)$$

where τ is a point of $\bar{\mathbb{D}}$ and $p \in \text{Hol}(\mathbb{D}, \mathbb{C})$ satisfies the condition $\text{Re } p(z) \geq 0$ for all $z \in \mathbb{D}$.

The representation (1.3) is unique unless $G \equiv 0$, in which case (trivially) $p \equiv 0$ and τ is any point in $\bar{\mathbb{D}}$.

Assumption. In what follows we assume that all one-parameter semigroups (ϕ_t) we consider are non-trivial, i.e. at least one of ϕ_t 's is different from $\text{id}_{\mathbb{D}}$. Except for the case of elliptic automorphisms, this condition in fact implies (see, e.g., [1, pp. 108–109]) that $\phi_t \neq \text{id}_{\mathbb{D}}$ for all $t > 0$ and that the infinitesimal generator $G \neq 0$.

It is an immediate consequence of the Schwarz Lemma that a self-map $\phi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\text{id}_{\mathbb{D}}\}$ can have at most one fixed point in \mathbb{D} . However, there can be much more so-called *boundary fixed points*.

Definition 1.2. Let $\phi \in \text{Hol}(\mathbb{D}, \mathbb{D})$ and $\sigma \in \mathbb{T} := \partial\mathbb{D}$. The point σ is called a *contact point* of ϕ if the angular limit $\phi(\sigma) := \angle \lim_{z \rightarrow \sigma} \phi(z)$ exists and belongs to \mathbb{T} . If in addition, $\varphi(\sigma) = \sigma$, then σ is said to be a *boundary fixed point* of ϕ .

It is known that if $\sigma \in \mathbb{T}$ is a contact point of $\phi \in \text{Hol}(\mathbb{D}, \mathbb{D})$, then the angular limit $\phi'(\sigma) := \angle \lim_{z \rightarrow \sigma} (\phi(z) - \phi(\sigma))/(z - \sigma)$, referred to as the *angular derivative* of ϕ at σ , exists, *finite or infinite*, see, e.g., [33, Proposition 4.13].

Definition 1.3. A contact point (resp., boundary fixed point) $\sigma \in \mathbb{T}$ of a self-map $\phi \in \text{Hol}(\mathbb{D}, \mathbb{D})$ is said to be *regular* if $\phi'(\sigma) \neq \infty$.

Remark 1.4. By the classical Julia Lemma (see, e.g., [20, Chapter 1, Exercises 6, 7]) the following two statements are equivalent:

- (a) $\phi \in \text{Hol}(\mathbb{D}, \mathbb{D})$ has a regular contact point at $\sigma \in \mathbb{T}$;
- (b) the dilation

$$\alpha_\phi(\sigma) := \liminf_{\mathbb{D} \ni z \rightarrow \sigma} \frac{1 - |\phi(z)|}{1 - |z|}$$

is finite.

Moreover, if the above conditions are fulfilled, then $\phi'(\sigma) = \bar{\sigma}\phi(\sigma)\alpha_\phi(\sigma)$.

The following statement is fundamental for the study of $\text{Hol}(\mathbb{D}, \mathbb{D})$, see, e.g., [34, §§1.3, 1.4].

Theorem C (*Denjoy–Wolff Theorem*). *Let $\phi \in \text{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\text{id}_{\mathbb{D}}\}$. Then there exists a unique (boundary) fixed point $\tau \in \bar{\mathbb{D}}$ such that³ $|\phi'(\tau)| \leq 1$. Moreover, if ϕ is not an elliptic automorphism of \mathbb{D} , then the iterates $\phi^{\circ n} \rightarrow \tau$ locally uniformly in \mathbb{D} as $n \rightarrow +\infty$.*

The point τ in the above theorem is called the *Denjoy–Wolff point* of ϕ (abbreviated, “DW-point”). For any boundary fixed point σ different from the DW-point we have $\phi'(\sigma) \in (1, +\infty)$ or $\phi'(\sigma) = \infty$. Accordingly, σ is said to be *repulsive* in the former case and *super-repulsive* in the latter case.

It is known (see, e.g., [1, Corollary (1.4.18), Theorem (1.4.19)]) that for a one-parameter semigroup (ϕ_t) the functions ϕ_t , $t > 0$, share the same DW-point, which coincides with the point τ in the Berkson–Porta formula (1.3) for the infinitesimal generator G of (ϕ_t) . This point is called the *Denjoy–Wolff point of the one-parameter semigroup* (ϕ_t) . Similarly, the following theorem allows to define boundary (super-)repulsive fixed points of one-parameter semigroups.

Theorem D. (See [13, Theorems 1 and 5]; [10, Lemmas 1 and 3].) *Let (ϕ_t) be a one-parameter semigroup in \mathbb{D} and $\sigma \in \mathbb{T}$. Then:*

- (i) σ is a fixed point of ϕ_t for some $t > 0$ if and only if it is a fixed point of ϕ_t for all $t > 0$;
- (ii) σ is a boundary regular fixed point of ϕ_t for some $t > 0$ if and only if it is a boundary regular fixed point of ϕ_t for all $t > 0$, with $\phi'_t(\sigma) = e^{\lambda t}$ for some $\lambda \in \mathbb{R}$ and all $t \geq 0$.

Remark 1.5. The Denjoy–Wolff Theorem implies easily, see, e.g., [1, Theorem (1.4.17)], that similar to the case of discrete iteration, any one-parameter semigroup (ϕ_t) such that ϕ_t ’s are not elliptic automorphisms of \mathbb{D} for $t > 0$, converges locally uniformly in \mathbb{D} to its Denjoy–Wolff point as $t \rightarrow +\infty$.

Besides infinitesimal representation given by Theorem A, one-parameter semigroups can be represented by means of the so-called *linearization models*, see, e.g., [15, 18, 35]. For a one-parameter semigroup (ϕ_t) a linearization model can be defined as a three-tuple (h, Ω, \mathcal{T}) , where $\mathcal{T} = (\mathcal{L}_t)$ is a one-parameter semigroup in $\bar{\mathbb{C}}$ consisting of Möbius transformations and h is a conformal mapping of \mathbb{D} onto a domain $\Omega \subset \bar{\mathbb{C}}$ such that $\mathcal{L}_t(\Omega) \subset \Omega$ and $\mathcal{L}_t \circ h = h \circ \phi_t$ for all $t \geq 0$. The choice of a “standard” linearization model depends on whether the DW-point lies in \mathbb{D} or on its boundary.

Theorem E. (See, e.g., [1, Theorems (1.4.22), (1.4.23)].) *Let (ϕ_t) be a one-parameter semigroup in \mathbb{D} and τ its Denjoy–Wolff point. Then:*

³ If $\tau \in \partial\mathbb{D}$, then $\phi'(\tau)$ stands, of course, for the angular derivative at τ .

- (A) If $\tau \in \mathbb{D}$, then there exists a univalent holomorphic function $h : \mathbb{D} \rightarrow \mathbb{C}$ with $h(\tau) = 0$ satisfying the Schröder functional equation

$$h \circ \phi_t = \phi'_t(\tau)h \quad \text{for all } t \geq 0.$$

Such a function h is unique up to multiplication by a complex constant $h \mapsto ch$, where $c \in \mathbb{C}^*$.

- (B) If $\tau \in \mathbb{T}$, then there exists a univalent holomorphic function $h : \mathbb{D} \rightarrow \mathbb{C}$ satisfying the Abel functional equation

$$h \circ \phi_t = h + t \quad \text{for all } t \geq 0. \quad (1.4)$$

Such a function h is unique up to a translation $h \mapsto h + c$, where $c \in \mathbb{C}$.

The function h in the above theorem is called the *Kœnigs function* of the one-parameter semigroup (ϕ_t) . Usually, to fix the unique solution to the Schröder and Abel equations, one assumes that $h'(\tau) = 1$ or $h(0) = 0$ in the former and latter cases, respectively. However, for our purposes it will be more convenient not to impose this normalization in the case of boundary DW-point.

1.2. Main results

Although one-parameter semigroups in \mathbb{D} constitute a classical topic and the study of holomorphic self-maps of \mathbb{D} suggests looking for angular limits on the boundary, it was not realized before [13] that *elements of one-parameter semigroups have angular limits everywhere on the unit circle*. This remarkable fact does not seem to be widely known: in that paper it was stated only in a proof (the proof of Theorem 5) and only for the case of boundary DW-point. We note that it is, in fact, enough to consider this case, as one can see using the idea from [8, Proof of Theorem 3.3]. The corresponding auxiliary statement is proved in Section 2, which allows us to concentrate in what follows on the case of boundary DW-point.

First of our main results, Theorem 3.1 is a “uniform version” of the fact stated above. As usual we denote by $\phi_t(\sigma)$, $\sigma \in \mathbb{T}$, the angular limit of ϕ_t at σ . We show that for each Stolz angle S with vertex $\sigma \in \mathbb{T}$ the convergence $\phi_t(z) \rightarrow \phi_t(\sigma)$ as $S \ni z \rightarrow \sigma$ is *locally uniform in t* .

Further in Proposition 3.2 we show that a one-parameter semigroup (ϕ_t) being considered as a family of maps $[0, +\infty) \ni t \mapsto \phi_t(z) \in \mathbb{D}$ parameterized by $z \in \mathbb{D}$, is uniformly continuous. In particular, *for each $\sigma \in \mathbb{T}$ the trajectory $t \mapsto \phi_t(\sigma)$ is continuous*. Moreover, as a byproduct, in Section 5 we will see (Remark 5.1) that *if the DW-point τ belongs to $\partial\mathbb{D}$, then $\sigma \in \mathbb{T}$ is either a boundary fixed point of (ϕ_t) , or $\phi_t(\sigma) \rightarrow \tau$ as $t \rightarrow +\infty$* . The analogous statement for $\tau \in \mathbb{D}$ follows readily from [13, Theorem 4].

Despite of the above remarkable facts, the extension of ϕ_t by angular limits is not necessarily continuous on \mathbb{T} . In other words, the unrestricted limits do not need to exist everywhere on \mathbb{T} . As a “compensation”, the *unrestricted limits still do exist at all (regular and non-regular) boundary fixed points*. For the first time this was proved in [13, Corollary 3] for the case of interior DW-point. We prove a “uniform version” of this statement for the boundary DW-point, see Theorem 4.1, which automatically extends to the interior case due to Proposition 2.1. For all repulsive (and super-repulsive) fixed points on \mathbb{T} we could employ essentially the same idea as in [13]: the key point is to use the translational invariance of $\Omega := h(\mathbb{D})$ in order to prove that the Kœnigs function h has unrestricted limits at all boundary (super-)repulsive fixed points. However, the analogous statement for the DW-point does not hold, and we had to give an independent proof for this distinguished fixed point. This aspect is really new in the boundary DW-point case.

The proof of Theorem 4.1 involves several more technical results, e.g., Propositions 3.7, 4.6, 4.16, and 4.17, which might be of some interest for specialists.

In Section 5 we consider three examples. The first two of them are related to the local dynamical behaviour of $\phi_t : \mathbb{D} \rightarrow \mathbb{D}$ in a neighbourhood of the boundary DW-point. The third example shows that Theorem 4.1 cannot be extended to the *contact points* of one-parameter semigroups.

Remark 1.6. One might ask whether at every boundary fixed point there exists also the unrestricted limit of the derivative ϕ'_t and/or the “unrestricted derivative”:

$$\lim_{\mathbb{D} \ni z \rightarrow \sigma} \phi'_t(z), \quad \lim_{\mathbb{D} \ni z \rightarrow \sigma} \frac{\phi_t(z) - \phi_t(\sigma)}{z - \sigma}.$$

The answer is negative. Both unrestricted limits above fail to exist if our boundary regular fixed point σ is not isolated, *i.e.* if it is a limit of a sequence of boundary fixed points different from σ . For repulsive fixed points such examples can be obtained by modifying the construction given in [11, p. 260]. Non-isolated DW-points of hyperbolic and parabolic types appear in Example 1 and in [6, Example 3.5], respectively.

The last section of this paper, Section 6, is devoted to a question concerning boundary behaviour of the non-autonomous generalization of one-parameter semigroups, the so-called *evolution families* in \mathbb{D} . It is known that any univalent $\varphi \in \text{Hol}(\mathbb{D}, \mathbb{D})$ can be embedded into an evolution family. Therefore, we cannot expect any results for evolution families similar to the above results for one-parameter semigroups. However, there is still the question whether the algebraic structure of evolution family affects the relationships between various analytic properties, in particular those of regularity on the boundary. We prove (Proposition 6.3) that if all the elements of an evolution family $(\varphi_{s,t})$ are continuous in $\overline{\mathbb{D}}$, then the map $t \mapsto \varphi_{s,t}$ is continuous w.r.t. the supremum norm for any fixed $s \geq 0$. The proof is based on an extended version of the No-Koebe-Arcs Theorem, see, *e.g.*, [32, Theorem 9.2].

2. Lifting one-parameter semigroups with the interior DW-point

Obviously, using Möbius transformations of \mathbb{D} one can assume that the DW-point of a given one-parameter semigroup is either $\tau = 0$ or $\tau = 1$. In fact, we can further reduce, up to some extend, the case of interior DW-point ($\tau = 0$) to the case of boundary DW-point ($\tau = 1$). This is the meaning of the following elementary proposition. In what follows for $a \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, we denote

$$\mathbb{H}_a := \{z \in \mathbb{C} : \text{Re}(\bar{a}z) > 0\}.$$

Proposition 2.1. *Let (ϕ_t) be a non-trivial one-parameter semigroup in $\text{Hol}(\mathbb{D}, \mathbb{D})$ with the DW-point $\tau = 0$. Let h and G be the Kœnigs function and the infinitesimal generator of (ϕ_t) , respectively. Then there exists a unique one-parameter semigroup $(\tilde{\phi}_t)$ in $\text{Hol}(\mathbb{H}_1, \mathbb{H}_1)$ with the DW-point $\tilde{\tau} = \infty$ such that for all $t \geq 0$ and all $\tilde{z} \in \mathbb{H}_1$ we have*

$$\exp(-\tilde{\phi}_t(\tilde{z})) = \phi_t(\exp(-\tilde{z})). \quad (2.1)$$

Moreover, the Kœnigs function \tilde{h}_0 and the infinitesimal generator \tilde{G} of the one-parameter semigroup $(\tilde{\phi}_t)$ are given by

$$\tilde{h}_0(\tilde{z}) = -\frac{\tilde{h}(\tilde{z})}{G'(0)}, \quad \tilde{G}(\tilde{z}) = -\frac{G(\exp(-\tilde{z}))}{\exp(-\tilde{z})} \quad \text{for all } \tilde{z} \in \mathbb{H}_1, \quad (2.2)$$

where $\tilde{h} : \mathbb{H}_1 \rightarrow \mathbb{C}$ is a holomorphic lifting of $\mathbb{H}_1 \ni \tilde{z} \mapsto h(\exp(-\tilde{z})) \in \mathbb{C}^*$ w.r.t. the covering map $\mathbb{C} \ni \tilde{w} \mapsto \exp(-\tilde{w}) \in \mathbb{C}^*$.

Proof. For the proof of the uniqueness and of formulas (2.2), we first assume that there exists a one-parameter semigroup $(\tilde{\phi}_t)$ satisfying (2.1). Since h is univalent and $h(0) = 0$, we have $h(\mathbb{D}^*) \subset \mathbb{C}^*$. According to the Monodromy Theorem there exists a holomorphic lifting $\tilde{h} : \mathbb{H}_1 \rightarrow \mathbb{C}$ of $\mathbb{H}_1 \ni \tilde{z} \mapsto h(\exp(-\tilde{z})) \in \mathbb{C}^*$ w.r.t. the covering map $\mathbb{C} \ni \tilde{w} \mapsto \exp(-\tilde{w}) \in \mathbb{C}^*$. This means that

$$\exp(-\tilde{h}(\tilde{z})) = h(\exp(-\tilde{z})) \quad \text{for all } \tilde{z} \in \mathbb{H}_1. \quad (2.3)$$

The Koenigs function h of (ϕ_t) satisfies the Schröder functional equation

$$h(\phi_t(z)) = e^{\lambda t} h(z), \quad \text{for all } t \geq 0 \text{ and all } z \in \mathbb{D}, \quad (2.4)$$

where $\lambda := G'(0)$. Combining the latter two equalities with (2.1) one easily obtains

$$\exp[-\tilde{h}(\tilde{\phi}_t(\tilde{z}))] = \exp[\lambda t - \tilde{h}(\tilde{z})] \quad \text{for all } \tilde{z} \in \mathbb{H}_1 \text{ and all } t \geq 0.$$

Taking into account that for any fixed $\tilde{z} \in \mathbb{H}_1$, $[0, +\infty) \ni t \mapsto \tilde{\phi}_t(\tilde{z})$ is continuous and equals \tilde{z} when $t = 0$, we conclude from the above equality that for all $t \geq 0$ the function $\tilde{h}_0 := -\tilde{h}/\lambda$ satisfies

$$\tilde{h}_0 \circ \tilde{\phi}_t = t + \tilde{h}_0. \quad (2.5)$$

Differentiating (2.5) w.r.t. t one obtains $\tilde{h}'_0 = 1/\tilde{G}$, while from (2.4) it follows in a similar way that $h'/h = \lambda/G$. Now combining these two equalities and (2.3), we deduce the second formula in (2.2). In particular, this proves the uniqueness of the one-parameter semigroup $(\tilde{\phi}_t)$, because it is defined uniquely by its infinitesimal generator.

Furthermore, according to the Berkson–Porta formula (1.3), G is of the form $G(z) = -zp(z)$, where $p \in \text{Hol}(\mathbb{D}, \mathbb{C})$ with $\text{Re } p(z) \geq 0$ for all $z \in \mathbb{D}$. Therefore,

$$\text{Re } \tilde{G}(\tilde{z}) \geq 0 \quad \text{for all } \tilde{z} \in \mathbb{H}_1. \quad (2.6)$$

Then, taking into account that $\tilde{h}'_0 = 1/\tilde{G}$ we may conclude with the help of the Noshiro–Warschawski Theorem (see, e.g., [16, p. 47]) that \tilde{h}_0 is univalent in \mathbb{H}_1 . Therefore, this function is a Koenigs function of the semigroup $(\tilde{\phi}_t)$. This proves the first formula in (2.2). Besides that, it follows from (2.6), again with the help of the Berkson–Porta formula, that any one-parameter semigroup $(\tilde{\phi}_t)$ in the half-plane \mathbb{H}_1 satisfying (2.1) must have the DW-point at ∞ .

It remains to prove that such one-parameter semigroup $(\tilde{\phi}_t)$ indeed exists.

As the above argument shows, the fact that G is a generator of a one-parameter semigroup in \mathbb{D} with the DW-point at $\tau = 0$ implies, according to the Berkson–Porta formula, that the function $\tilde{G} : \mathbb{H}_1 \rightarrow \mathbb{C}$ defined by the second formula in (2.2) is a generator of some one-parameter semigroup $(\tilde{\phi}_t)$ in \mathbb{H}_1 with the DW-point at ∞ . We claim that this semigroup satisfies (2.1). Indeed, for any $\tilde{z} \in \mathbb{H}_1$,

$$\begin{aligned} \frac{d}{dt} \exp(-\tilde{\phi}_t(\tilde{z})) &= -\exp(-\tilde{\phi}_t(\tilde{z})) \tilde{G}(\phi_t(\tilde{z})) = G(\exp(-\tilde{\phi}_t(\tilde{z}))), \quad t \geq 0, \\ \exp(-\tilde{\phi}_t(\tilde{z}))|_{t=0} &= z := \exp(-\tilde{z}), \end{aligned}$$

and therefore, by the uniqueness of the solution to the initial value problem for $dw/dt = G(w)$, we have $\exp(-\tilde{\phi}_t(\tilde{z})) = \phi_t(z)$ for all $t \geq 0$, which proves (2.1). \square

3. Angular limits of one-parameter semigroups

3.1. Statement of results

First of all we would like to formulate some general statements on the boundary behaviour of one-parameter semigroups. Possibility to embed a holomorphic self-map φ of \mathbb{D} into a one-parameter semigroup is a quite strong condition. For example, it is well-known that the elements of one-parameter semigroups are univalent in \mathbb{D} . Another, less elementary fact is that these functions must have angular limits at all points on \mathbb{T} . This was proved in [13, p. 479, proof of Theorem 5] for the case of a semigroup with the boundary DW-point. Here we prove a bit stronger statement both for the interior and boundary DW-point.

Theorem 3.1. *Let (ϕ_t) be a one-parameter semigroup. Then for any $t \geq 0$ and any $\sigma \in \mathbb{T}$ there exists the angular limit $\phi_t(\sigma) := \angle \lim_{z \rightarrow \sigma} \phi_t(z)$. Moreover, for each $\sigma \in \mathbb{T}$ and each Stolz angle S with vertex at σ the convergence $\phi_t(z) \rightarrow \phi_t(\sigma)$ as $S \ni z \rightarrow \sigma$ is locally uniform in $t \in [0, +\infty)$.*

Using the above theorem, we extend elements of the semigroup to the unit circle \mathbb{T} . Suppressing the language in the same manner as in the statement of Theorem 3.1, we will denote this extension again by ϕ_t . Of course ϕ_t 's do not need to be continuous w.r.t. z on \mathbb{T} . However, we will show that $t \mapsto \phi_t(z)$ is continuous in t for any $z \in \bar{\mathbb{D}}$.

Proposition 3.2. *Let (ϕ_t) be a one-parameter semigroup. The family of functions*

$$([0, +\infty) \ni t \mapsto \phi_t(z))_{z \in \bar{\mathbb{D}}}$$

is uniformly equicontinuous.

The proofs are given below.

3.2. Boundary behaviour of the Kœnigs function. Proof of Proposition 3.2 and Theorem 3.1

In what follows we will make use of one general statement concerning conformal mappings of the disk. Denote by $\text{diam}_{\mathbb{C}} E$ and $\text{diam}_{\bar{\mathbb{C}}} E$ the diameter of a set E w.r.t. the Euclidean distance in \mathbb{C} and the spherical distance $\chi(z_1, z_2) = |z_2 - z_1| / \sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}$ in $\bar{\mathbb{C}}$, respectively. Let D be any domain of $\bar{\mathbb{C}}$. For $w_1, w_2 \in D$ denote

$$d_D(w_1, w_2) := \inf\{\text{diam}_{\bar{\mathbb{C}}} \Gamma : \Gamma \subset D \text{ is a Jordan arc joining } w_1 \text{ and } w_2\}. \quad (3.1)$$

It is easy to see that d_D is a distance function in D .

Proposition 3.3. *Let $f : \mathbb{D} \rightarrow D$ be a conformal mapping onto a domain $D \subset \bar{\mathbb{C}}$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $K \subset D$ is a pathwise connected set and $\text{diam}_{\bar{\mathbb{C}}} K < \delta$, then $\text{diam}_{\mathbb{C}} f^{-1}(K) < \varepsilon$. In particular, the inverse mapping $f^{-1} : D \rightarrow \mathbb{D}$ is uniformly continuous w.r.t. the distance d_D in D and the Euclidean distance in \mathbb{D} .*

This proposition follows easily from Bagemihl–Seidel's version of the No-Koebe-Arcs Theorem (see, e.g., [32, Corollary 9.1 on p. 267]), taken into account that any conformal map is a normal function (see, e.g., [32, Lemma 9.3 on p. 262]).

Now we turn to the proof of Theorem 3.1. To this end we have to study first the boundary behaviour of the Kœnigs function.

It is known, see, e.g., [33, §3.6], that star-like functions have angular limits,⁴ finite or infinite, at every point on the unit circle. This is also the case for the more general class of spiral-like functions [28, Theorem 3.2], which serve as Koenigs functions of one-parameter semigroups with the interior DW-point. Relations between star-like and fan-like functions, see, e.g., [34, §§5.5–5.6], allows us to extend this result immediately to the case of the boundary DW-point under the additional condition that the image of the Koenigs function is contained in a strip parallel to \mathbb{R} . In all the cases, the reason for the existence of the angular limits is the well-known integral representation of holomorphic functions with positive real part, which appear in the Berkson–Porta formula. Using the Nevanlinna representation (see, e.g., [3, Vol. II, p. 7]) and Remark 3.5 below, it is not difficult to prove the following proposition, which covers the boundary case.

Proposition 3.4. *Let (ϕ_t) be a one-parameter semigroup in \mathbb{D} with the DW-point $\tau = 1$ and h its Koenigs function. Then*

$$\forall \sigma \in \mathbb{T} \quad \exists \angle \lim_{z \rightarrow \sigma} h(z) \in \overline{\mathbb{C}}.$$

Moreover, if $\sigma \in \mathbb{T} \setminus \{\tau = 1\}$, then the radial limit $\lim_{r \rightarrow 1-0} \operatorname{Im} h(r\sigma)$ exists finitely and

$$\limsup_{z \rightarrow \sigma} \operatorname{Re} h(z) < +\infty. \quad (3.2)$$

Remark 3.5. In the notation of the above proposition, let $\sigma \in \mathbb{T} \setminus \{\tau = 1\}$. The function $F_\sigma(z) := (1+z)/(1-z) - (1+\sigma)/(1-\sigma)$ is the conformal mapping of \mathbb{D} onto \mathbb{H}_1 sending the DW-point τ and the point σ to ∞ and to the origin, respectively. The family $(\Phi_t)_{t \geq 0}$ defined by $\Phi_t := F_\sigma \circ \phi_t \circ F_\sigma^{-1}$ for all $t > 0$ is a one-parameter semigroup in \mathbb{H}_1 with the DW-point at ∞ and its Koenigs function is given by $H := h \circ F_\sigma^{-1}$. Differentiating the Abel equation (1.4) w.r.t. t at $t = 0$ and using the Berkson–Porta formula (1.3), one can show that $\operatorname{Re} H'(z) \geq 0$ for all $z \in \mathbb{H}_1$. In fact, it is easy to see that $\operatorname{Re} H'(z) > 0$ for all $z \in \mathbb{H}$ unless h is a linear-fractional mapping and all ϕ_t 's are automorphisms of \mathbb{D} .

Remark 3.6. Using the correspondence between the boundary accessible points of $h(\mathbb{D})$ and the unit circle induced by h (see, e.g., [21, Theorem 1 in Chapter 2, §3]) it is easy to see that $h(r\sigma) \rightarrow \infty$ as $r \rightarrow 1-0$, where $\sigma \in \mathbb{T}$, if and only if σ is a boundary fixed point of (ϕ_t) , which in principle can coincide with the DW-point. We are able to prove a bit more:

Proposition 3.7. *Let (ϕ_t) be a one-parameter semigroup in \mathbb{D} with the DW-point $\tau \in \mathbb{T}$. Let h be the Koenigs function of (ϕ_t) and let $\sigma \in \mathbb{T}$. Then the unrestricted limit $\lim_{\mathbb{D} \ni z \rightarrow \sigma} \operatorname{Im} h(z)$ exists finitely if and only if σ is not a boundary regular fixed point of (ϕ_t) .*

Moreover, the following statements are equivalent:

- (i) σ is a boundary fixed point of (ϕ_t) other than its DW-point;
- (ii) $\operatorname{Re} h(r\sigma) \rightarrow -\infty$ as $r \rightarrow 1-0$;
- (iii) $\lim_{\mathbb{D} \ni z \rightarrow \sigma} \operatorname{Re} h(z) = -\infty$.

We do not use Proposition 3.7 in this section. Its proof will be given in Section 4.2.

Proof of Theorem 3.1. First of all, using Proposition 3.3 together with the forerunning comment, we may assume that the DW-point of (ϕ_t) is $\tau = 1$.

⁴ Note that for univalent functions, and more generally, for all normal functions in \mathbb{D} , the angular limit at a given point on $\mathbb{T} = \partial\mathbb{D}$ exists if and only if the radial limit at this point exists, see e.g. [32, §9.1, Lemma 9.3, Theorem 9.3].

Fix any $\sigma \in \mathbb{T}$. A fundamental family of Stolz angles with vertices at σ is given by

$$S_{\sigma,\alpha} := \{z \in \mathbb{D}: |\arg(1 - \bar{\sigma}z)| < \alpha, |1 - \bar{\sigma}z| < (\cos \alpha)/2\}, \quad \alpha \in (0, \pi/2).$$

Fix any Stolz angle S in this family. Then S is a domain in \mathbb{D} , with $\partial S \cap \partial \mathbb{D} = \{\sigma\}$. That is why it follows from Proposition 3.4 that $h|_S$ admits a continuous extension to \bar{S} (as a mapping into $\bar{\mathbb{C}}$). Taking into account that S is convex, it follows that $h|_S$ is uniformly continuous as a mapping from S endowed with the Euclidean distance into the domain $\Omega := h(\mathbb{D})$ endowed with the distance d_Ω ,⁵ which has been introduced in Section 3.2.

Fix now $T > 0$. The family of translations $\mathcal{T} := (\bar{\mathbb{C}} \ni w \mapsto w + t)_{t \in [0, T]}$, where as usually we set $\infty + t = \infty$, is uniformly equicontinuous in $\bar{\mathbb{C}}$ w.r.t. the spherical distance. Since Ω is invariant w.r.t. elements of \mathcal{T} , it is easy to see that \mathcal{T} is a uniformly equicontinuous family of self-maps of Ω endowed with the distance d_Ω .

Finally, recall that h is univalent. By Proposition 3.3, h^{-1} is uniformly continuous in Ω w.r.t. the distance d_Ω .

Now combining the above facts, it is easy to conclude that the composite family

$$(S \ni z \mapsto \phi_t(z) = h^{-1}(h(z) + t) \in \mathbb{D})_{t \in [0, T]}$$

is uniformly equicontinuous in S . Hence it admits uniformly equicontinuous extension to \bar{S} . The statement of Theorem 3.1 follows now immediately. \square

Proof of Proposition 3.2. Let τ be the DW-point of (ϕ_t) . Clearly, it is sufficient to consider cases $\tau = 0$ and $\tau = 1$.

Case 1: $\tau = 1$.

In this case the Koenigs function $h : \mathbb{D} \rightarrow \mathbb{C}$ of (ϕ_t) satisfies the Abel functional equation $h(\phi_t(z)) = h(z) + t$ for all $t \geq 0$ and all $z \in \mathbb{D}$. It follows that the line segment $[h(\phi_{t_1}(z)), h(\phi_{t_2}(z))]$ lies entirely in $\Omega := h(\mathbb{D})$ and hence

$$\begin{aligned} d_\Omega(h(\phi_{t_1}(z)), h(\phi_{t_2}(z))) &\leq \text{diam}_{\bar{\mathbb{C}}}[h(\phi_{t_1}(z)), h(\phi_{t_2}(z))] \\ &\leq |h(\phi_{t_2}(z)) - h(\phi_{t_1}(z))| = |t_1 - t_2| \end{aligned}$$

for any $z \in \mathbb{D}$ and any $t_1, t_2 \in [0, +\infty)$.

Recall that h is univalent. Therefore, the statement of the proposition for z ranging in \mathbb{D} follows from Proposition 3.3. It is also true for the closed unit disk, because $\phi_t(z)$ is continuous in z on each radius $[0, \sigma]$, where $\sigma \in \mathbb{T}$. Thus for $\tau = 1$ the proof is finished.

Case 2: $\tau = 0$.

Fix any $t_1 \geq 0$ and $t_2 \geq t_1$. Using the Maximum Principle for holomorphic functions we see that

$$\sup_{z \in \mathbb{D}} |\phi_{t_2}(z) - \phi_{t_1}(z)| \leq \sup_{z \in \mathbb{D}} |\phi_{t_2-t_1}(z) - z| = \sup_{2/3 < |z| < 1} |\phi_{t_2-t_1}(z) - z| =: \mathcal{Y}(t_2 - t_1).$$

It is enough to show that $\mathcal{Y}(t) \rightarrow 0$ as $t \rightarrow +0$.

⁵ Given two points $z_1, z_2 \in S$, consider $h([z_1, z_2])$ as a candidate for Γ in (3.1).

Taking advantage of [Proposition 2.1](#), consider the one-parameter semigroup formed by the functions $\psi_t := p_0^{-1} \circ \tilde{\phi}_t \circ p_0$, $t \geq 0$, where $p_0(\zeta) := (1 + \zeta)/(1 - \zeta)$ is the Cayley map. Then $\phi_t(e^{-p_0(\zeta)}) = e^{-p_0(\psi_t(\zeta))}$ for all $z \in \mathbb{D}$. Hence

$$\Upsilon(t) = \sup_{\zeta \in \Pi} |e^{-p_0(\psi_t(\zeta))} - e^{-p_0(\zeta)}|, \quad \text{where } \Pi := p_0^{-1}(\{\tilde{x} + i\tilde{y}: 0 < \tilde{x} \leq \log 3/2, |\tilde{y}| \leq \pi\}).$$

By Case 1, for all $\zeta \in \Pi$,

$$\psi_t(\zeta) \in \tilde{\Pi} := p_0^{-1}(\{\tilde{x} + i\tilde{y}: 0 < \tilde{x} \leq \log 2, |\tilde{y}| \leq 2\pi\})$$

provided $t > 0$ is small enough. Since the derivative of the map $\zeta \mapsto e^{-p_0(\zeta)}$ is bounded on the convex hull of $\tilde{\Pi}$, there exists $M > 0$ such that $|e^{-p_0(\psi_t(\zeta))} - e^{-p_0(\zeta)}| \leq M|\psi_t(\zeta) - \zeta|$ for all $\zeta \in \Pi$ and all $t > 0$ small enough. Now by Case 1, it follows that $\Upsilon(t) \rightarrow 0$ as $t \rightarrow +0$. The proof is now complete. \square

4. Unrestricted limits at boundary fixed points

4.1. Main theorem

Now we formulate the main result of this paper. In [\[13, Corollary 3\]](#) it was proved that elements of a one-parameter semigroup with the interior DW-point can be extended continuously to the boundary fixed points. This statement can also be proved for the case when the DW-point $\tau \in \partial\mathbb{D}$. For repulsive and super-repulsive boundary fixed points σ the reason why ϕ_t has the unrestricted limit at σ is essentially the same in both cases: the Koenigs function has the unrestricted limit at all (super-)repulsive boundary fixed points. However, for $\sigma = \tau$ this is not true any more. The Koenigs function does not need to have the unrestricted limit at the boundary DW-point τ , see, e.g., [Example 1](#) in [Section 5](#) below.

Theorem 4.1. *Let (ϕ_t) be a one-parameter semigroup in \mathbb{D} . For each $t \geq 0$ and each $\sigma \in \mathbb{T}$ denote $\phi_t(\sigma) := \angle \lim_{z \rightarrow \sigma} \phi_t(z)$. Then for any $T > 0$ the family of mappings*

$$\Phi_T := (\overline{\mathbb{D}} \ni z \mapsto \phi_t(z) \in \overline{\mathbb{D}})_{t \in [0, T]}$$

is equicontinuous at every boundary fixed point of (ϕ_t) . Moreover, if the semigroup (ϕ_t) is of hyperbolic type,⁶ then the family

$$\Phi := (\overline{\mathbb{D}} \ni z \mapsto \phi_t(z) \in \overline{\mathbb{D}})_{t \geq 0}$$

is equicontinuous at the DW-point τ of (ϕ_t) .

The proof of this theorem is given in [Section 4.3](#).

Remark 4.2. By the theorem above, the unrestricted limit $\lim_{\mathbb{D} \ni z \rightarrow \sigma} \phi_t(z)$ exists at any boundary fixed point $\sigma \in \mathbb{T}$ of (ϕ_t) . One might ask if the same holds for the contact points of (ϕ_t) . The answer is: “not necessarily”, see [Example 3](#) in [Section 5](#).

⁶ A one-parameter semigroup (ϕ_t) is said to be of *hyperbolic type* if its DW-point $\tau \in \mathbb{T}$ and $\phi'_t(\tau) < 1$ for some (and hence for all) $t > 0$. See, e.g., [\[18\]](#) or [\[19\]](#) for more details.

4.2. Kœnigs function of a one-parameter semigroup with the boundary DW-point

In this subsection we prove some auxiliary statements concerning Kœnigs functions of one-parameter semigroups with the boundary DW-point, and obtain from them some consequences characterizing the semigroup itself. Throughout the subsection we will assume that (ϕ_t) is a one-parameter semigroup with the DW-point $\tau = 1$. By h we denote its Kœnigs function, and let $\Omega := h(\mathbb{D})$.

First of all we need to make some general remarks and recall some definitions.

Definition 4.3. By a *slit* (or an *end-cut*) γ in a domain $D \subset \bar{\mathbb{C}}$ we mean the image of $[0, 1)$ under an injective continuous mapping $\varphi : [0, 1] \rightarrow \bar{D}$ with $\varphi([0, 1)) \subset D$ and $\varphi(1) \in \partial D$. The point $\varphi(1)$ is said to be the *root* or the *landing point* of the slit γ .

Remark 4.4. Two slits γ_1, γ_2 in a domain D are called *equivalent* if they share the same root $\omega_0 \in \partial D$ and any neighbourhood of ω_0 contains a curve $\Gamma \subset D$ that joins γ_1 and γ_2 . It is known (see, e.g., [21, Theorem 1 in Chapter 2, §3]) that if $D = f(\mathbb{D})$, where $f : \mathbb{D} \rightarrow \bar{\mathbb{C}}$ is a conformal mapping, then the preimage $f^{-1}(\gamma)$ of any slit γ in D is a slit in the unit disk \mathbb{D} . Moreover, in this case two slits γ_1, γ_2 in D are equivalent if and only if their preimages $f^{-1}(\gamma_1)$ and $f^{-1}(\gamma_2)$ land at the same point on \mathbb{T} . A *cross-cut* in D can be defined as a union of two non-equivalent slits intersecting only at their common end-point in D . It follows that the preimage $f^{-1}(C)$ of any cross-cut C in D is a cross-cut in \mathbb{D} .

Remark 4.5. Let $w_0 \in \Omega$. The Abel equation (1.4) implies that $R_0 := \{w_0 + t : t > 0\} \subset \Omega$. Moreover, R_0 is a slit in Ω , whose preimage $h^{-1}(R_0)$ is a slit in \mathbb{D} landing at the DW-point $\tau = 1$, because $h^{-1}(w_0 + t) = \phi_t(h^{-1}(w_0)) \rightarrow \tau$ as $t \rightarrow +\infty$.

First we prove that continuity of h at a boundary point implies continuity of $z \mapsto \phi_t(z)$ at that point locally uniformly w.r.t. t .

Proposition 4.6. Assume that h has the unrestricted limit, finite or infinite, at a point $\sigma \in \mathbb{T}$. Then the functions ϕ_t also have unrestricted limits at σ and the convergence $\phi_t(z) \rightarrow \phi_t(\sigma)$ as $\mathbb{D} \ni z \rightarrow \sigma$ is locally uniform w.r.t. $t \geq 0$.

Proof. As an elementary argument of *reductio ad absurdum* shows, it is sufficient to prove that given any slit γ in \mathbb{D} landing at σ , the functions $\phi_t(z)$ tend to $\phi_t(\sigma)$ locally uniformly w.r.t. t as z tends to σ along γ .

We use essentially the same idea as in the proof of Theorem 3.1. The restriction $h|_\gamma$ as a mapping from γ endowed with the Euclidean distance to the domain Ω endowed with the distance d_Ω , is uniformly continuous. For each $T > 0$ the family of self-maps $(\Omega \ni w \mapsto w + t \in \Omega)_{t \in [0, T]}$ is uniformly equicontinuous in Ω w.r.t. the distance d_Ω . Using Proposition 3.3 for $f := h$, we conclude that for any $T > 0$ the family

$$(\gamma \ni z \mapsto \phi_t(z) = h^{-1}(h(z) + t))_{t \in [0, T]}$$

is uniformly equicontinuous. This means that $\phi_t|_\gamma$ has the limit at σ locally uniformly in $t \geq 0$. According to Lemma 9.3 and Theorem 9.3 in [32, pp. 262–268], all the asymptotic values of a univalent holomorphic function in \mathbb{D} at a given point of $\partial\mathbb{D}$, if any exists, coincide with the angular limit at this point. Hence, $\lim_{\gamma \ni z \rightarrow \sigma} \phi_t(z) = \phi_t(\sigma)$. The proof is now complete. \square

Now we concentrate on the study of h near the boundary fixed points of (ϕ_t) . Note that the angular limit of h , which exists according to Proposition 3.4, equals ∞ at each boundary fixed point. Our arguments use extensively the theory of boundary correspondence under conformal mappings of simply connected domains. We refer the reader to [9, Chapter 9] or [33, §§2.4–2.5] for the basic theory and definitions.

By ∂E we will denote the boundary of a set $E \subset \overline{\mathbb{C}}$. If $E \subset \mathbb{C}$, we will write $\partial_{\mathbb{C}} E$ for $\partial E \setminus \{\infty\}$. For a prime end P of a simply connected domain $\Omega \subset \overline{\mathbb{C}}$, we will denote by $I(P)$ its impression. A prime end is said to be *trivial* if its impression is a singleton.

Let (C_n) be a null-chain of cross-cuts in Ω representing some prime end P . For each $n \in \mathbb{N}$ denote by D_n the connected component of $\Omega \setminus C_n$ that contains C_{n+1} . We recall now one definition from the theory of prime ends.

Definition 4.7. A sequence $(w_k) \subset \Omega$ is said to *converge to the prime end P* , if for the null-chain (C_n) representing the prime end P (and hence for all such null-chains) the following statement holds: for every $n \in \mathbb{N}$, there exists k_0 such that $w_k \in D_n$ whenever $k > k_0$. In a similar way one defines convergence to a prime end for slits in Ω and more general continuous families $(w_x \in \Omega)_{x \in J}$, $J \subset \mathbb{R}$.

Definition 4.8. We say that (C_n) *converges to a point $w_0 \in \partial\Omega$* if for any neighbourhood \mathcal{O} of w_0 all but a finite number of C_n 's lie in \mathcal{O} . Taking into account that $\text{diam}_{\overline{\mathbb{C}}}(C_n) \rightarrow 0$ as $n \rightarrow +\infty$ by the very definition of a null-chain, the equivalent condition is that there exists a sequence $(w_n) \subset \Omega$ converging to w_0 such that $w_n \in C_n$ for all $n \in \mathbb{N}$.

Now we are going to prove that h is continuous at every (super-)repulsive boundary fixed point of (ϕ_t) . The proof is based on the following lemma.

Lemma 4.9. *Let $x_0 \in \mathbb{R}$ and let $y_0 : (-\infty, x_0] \rightarrow \mathbb{R}$ be a continuous function. Suppose that the graph $\Gamma := \{x + iy_0(x) : x \leq x_0\}$ lies entirely in $\Omega := h(\mathbb{D})$ and that there exist $w_1, w_2 \in \mathbb{C} \setminus \Omega$ such that $\text{Re } w_1 = \text{Re } w_2 = x_0$ and $\text{Im } w_1 < y_0(x_0) < \text{Im } w_2$. Then $x \mapsto x + iy_0(x)$ converges, as $x \rightarrow -\infty$, to a trivial prime end of Ω .*

Proof. For each $x \leq x_0$, define $Y(x)$ to be the connected component of $\{y \in \mathbb{R} : x + iy \in \Omega\}$ that contains the point $y_0(x)$. Since Ω is invariant under translations $w \mapsto w + t$, $t > 0$, it follows that $Y(x)$ is bounded for all $x \leq x_0$ and that $Y(x') \subset Y(x)$ whenever $x' \leq x \leq x_0$. (To check the latter statement one has to take into account that y_0 is continuous.) We claim that the intervals

$$C_n := \{x + iy : x = x_n, y \in Y(x_n)\}, \quad \text{where } x_n := x_0 - n,$$

form a null-chain in Ω . Indeed, each C_n is a cross-cut in Ω , the closures of C_n 's are pairwise disjoint, $\text{diam}_{\overline{\mathbb{C}}}(C_n) \rightarrow 0$ as $n \rightarrow +\infty$. Moreover, recall that since C_n is a cross-cut, $\Omega \setminus C_n$ has exactly two components for each $n \in \mathbb{N}$. Consider the set

$$G_n := \{x + iy : x < x_n, y \in Y(x_n)\}.$$

It is open and $\partial G_n \cap (\Omega \setminus C_n) = \emptyset$. Therefore, $\Omega \setminus C_n = D_n \cup D'_n$, where $D_n := (\Omega \setminus C_n) \cap G_n = \Omega \cap G_n$ and $D'_n := \Omega \setminus \overline{G_n}$ are both open and non-empty. (Indeed, note that $\Gamma \cap G_n \neq \emptyset$, but $\Gamma \not\subset \overline{G_n}$.) Hence, D_n and D'_n are the two connected components of $\Omega \setminus C_n$. Clearly, $C_{n+1} \subset D_n$ and $C_{n-1} \subset D'_n$. Thus (C_n) is a null-chain and the impression of the prime end P defined by this null-chain is $I(P) = \bigcap_n \overline{D_n} \subset \bigcap_n \overline{G_n} = \{\infty\}$.

Finally by construction, D_n contains $x + iy_0(x)$ for all $x < x_n$. Therefore, $x \mapsto x + iy_0(x)$ converges to the prime end P as $x \rightarrow -\infty$. This completes the proof. \square

Corollary 4.10. *At every boundary (super-)repulsive fixed point σ of the semigroup (ϕ_t) the function h has unrestricted limit ∞ , with $\text{Re } h(z) \rightarrow -\infty$ as $z \rightarrow \sigma$.*

Proof. Using Möbius transformations of \mathbb{D} fixing $\tau = 1$, we may assume that $\sigma = -1$. Consider the function $S(r) := \text{Re } h(-r)$, $r \in [0, 1)$. From the Berkson–Porta formula (1.3) and the Abel equation (1.4)

differentiated w.r.t. t at $t = 0$, it follows that $S'(r) = -\operatorname{Re}[(1+r)^2 p(-r)]^{-1} \leq 0$. Moreover, $S(r) \rightarrow -\infty$ as $r \rightarrow 1 - 0$, because, as we mentioned in Remark 3.6, $h(-r) \rightarrow \infty$ as $r \rightarrow 1 - 0$, while $\operatorname{Im} h(-r)$ has a finite limit by Proposition 3.4. Therefore, $J := S([0, 1)) = (-\infty, S(0)]$ and the curve $\Gamma := h((-1, 0]) \subset \Omega$ is the graph of the function

$$J \ni x \mapsto y_0(x) := \operatorname{Im} h(-S^{-1}(x)).$$

Note that there exist $w_1, w_2 \in \mathbb{C} \setminus \Omega$ with $\operatorname{Im} w_1 < y_0(x_0) < \operatorname{Im} w_2$ and $\operatorname{Re} w_1 = \operatorname{Re} w_2 = x_0$ for some $x_0 \leq S(0)$. Otherwise, Ω would contain a half-plane $H \supset \Gamma$ with ∂H parallel to \mathbb{R} and consequently Γ , as a slit in Ω , would be equivalent to $R_0 := \{t + w_0 : t \geq 0\}$ for arbitrary fixed $w_0 \in H$. But the landing point of $h^{-1}(R_0)$ is the DW-point $\tau = 1$, see Remark 4.5, while the landing point of $h^{-1}(\Gamma)$ is $\sigma = -1$ by construction.

Thus we can apply Lemma 4.9 to conclude that $h(-r)$ converges, as $[0, 1) \ni r \rightarrow 1$, to a trivial prime end P of Ω . This means (see, e.g., [9, Chapter 9, §4]) that the point $\sigma = -1$ corresponds under h to this trivial prime end, and hence the limit set of $h(z)$ as $\mathbb{D} \ni z \rightarrow -1$ is the singleton $I(P) = \{\infty\}$. Since for the null-chain (C_n) defining the prime end P that we have constructed in the proof of Lemma 4.9 one has $\sup_{w \in D_n} \operatorname{Re} w \rightarrow -\infty$ as $n \rightarrow +\infty$, we may conclude that the unrestricted limit of $\operatorname{Re} h$ at σ also exists and equals $-\infty$. \square

Lemma 4.11. *Let P be a non-trivial prime end of $\Omega := h(\mathbb{D})$. Then P corresponds under the mapping h to the DW-point τ if and only if*

$$\sup\{\operatorname{Re} w : w \in I(P) \cap \mathbb{C}\} = +\infty. \quad (4.1)$$

Proof. Denote by σ the unique point on \mathbb{T} that corresponds to the prime end P under the mapping h .

The fact that if (4.1) holds, then $\sigma = \tau$, follows readily from Proposition 3.4, because $I(P)$ is the limit set of $h(z)$ as $\mathbb{D} \ni z \rightarrow \sigma$, see e.g. [9, Theorem 9.4, p. 173].

Now we prove the converse statement. So let us assume that $\sigma = \tau$. We have to show that (4.1) takes place. Choose any point $w_0 \in I(P) \cap \mathbb{C}$. Then there exists a sequence $(z_n) \subset \mathbb{D}$ converging to τ such that $w_n := h(z_n)$ converges to w_0 . Consider another sequence $(\zeta_n) \subset \mathbb{D}$ converging to τ defined by $\zeta_n := \phi_n(0)$ for all $n \in \mathbb{N}$. Then the segments $[z_n, \zeta_n]$ also converge to τ and hence the limit set of the sequence $(\Gamma_n) := (h([z_n, \zeta_n]))$ is a subset of $I(P)$. On the one hand, $w_n \in \Gamma_n$ for all $n \in \mathbb{N}$ and tends to w_0 as $n \rightarrow +\infty$. On the other hand, $h(0) + n = h(\zeta_n) \in \Gamma_n$ for all $n \in \mathbb{N}$. It follows that for any $x > \operatorname{Re} w_0$ there exists a sequence $\omega_n \in \Gamma_n$ such that $\operatorname{Re} \omega_n \rightarrow x$. Let ξ_x be any limit point of (ω_n) . Then $\xi_x \in I(P)$ and $\operatorname{Re} \xi_x = x$. This implies (4.1) and thus the proof is complete. \square

Remark 4.12. Consider the domain $\mathcal{D} := \{w \in \mathbb{C} : \exists \omega \in \Omega \text{ s.t. } \operatorname{Im}(w - \omega) = 0\}$ and let $K \subset \mathcal{D}$ be a compact subset. Since Ω is invariant w.r.t. the translations $w \mapsto w + t$ for $t \geq 0$, given any $w_0 \in K$ there exists $t(w_0) \geq 0$ such that $w_0 + t \in \Omega$ for all $t \geq t(w_0)$. Taking into account that Ω is open and K is compact, it follows that $K + t_K \subset \Omega$ for some $t_K \geq 0$. In other words, Ω is absorbing for \mathcal{D} w.r.t. the action of the translation semigroup $(w \mapsto w + t)_{t \geq 0} \subset \operatorname{Aut}(\mathcal{D})$.

Remark 4.13. Note that the domain \mathcal{D} defined in the previous remark is either a horizontal strip, or a half-plane whose boundary is parallel to \mathbb{R} , or the whole plane \mathbb{C} . Note that in the first case, (ϕ_t) is of hyperbolic type, while in the second and the third cases it is of parabolic type, see [11, pp. 256–257].

Proposition 4.14. *The following statements hold.*

- (A) *Let P be any prime end of $\Omega := h(\mathbb{D})$. Then $I(P) \cap \mathbb{C}$ is contained in the union of two straight lines parallel to the real axis.*

(B) If in addition,

$$\sup\{\operatorname{Re} w: w \in I(P) \cap \mathbb{C}\} < +\infty, \quad (4.2)$$

then $I(P) \cap \mathbb{C}$ is contained on one straight line parallel to the real axis.

Proof. Suppose on the contrary to (A) that there exist $w_j \in I(P) \cap \mathbb{C}$, $j = 1, 2, 3$, such that $\operatorname{Im} w_1 < \operatorname{Im} w_2 < \operatorname{Im} w_3$. Denote by σ the unique point on \mathbb{T} that corresponds to the prime end P under the mapping h . We proceed by constructing two slits in Ω in the following way.

Fix any $y_1 \in (\operatorname{Im} w_1, \operatorname{Im} w_2)$. The line $L_{y_1} := \{w \in \mathbb{C}: \operatorname{Im} w = y_1\}$ intersects Ω , because Ω is connected and $w_1, w_2 \in \partial_{\mathbb{C}} \Omega$. Recall also that Ω is invariant w.r.t. the translations $w \mapsto w + t$, $t > 0$. It follows that, either $L_{y_1} \subset \Omega$, and in this case we set $w_{y_1} := \infty$, $L_{y_1}^+ := L_{y_1}$, or there exists a point $w_{y_1} \in L_{y_1} \cap \partial \Omega$ such that

$$L_{y_1}^+ := \{w \in L_{y_1}: \operatorname{Re} w > \operatorname{Re} w_{y_1}\} \subset \Omega \quad \text{and} \quad L_{y_1}^- := L_{y_1} \setminus L_{y_1}^+ \subset \mathbb{C} \setminus \Omega.$$

Now fix $y_2 \in (\operatorname{Im} w_2, \operatorname{Im} w_3)$ and construct in the same way w_{y_2} and L_{y_2} . By Remark 4.12 there exists $x_0 \in \mathbb{R}$ such that the segment $\Gamma_0 := [x_0 + iy_1, x_0 + iy_2]$ lies in Ω . Let $\Gamma_j := L_{y_j}^+ \cap \{w: \operatorname{Re} w < x_0\}$, $j = 1, 2$. The union

$$\Gamma := \bigcup_{j=0,1,2} \Gamma_j$$

is a cross-cut in Ω . Indeed, it is easy to see that the limits

$$\sigma_j := \lim_{L_{y_j}^+ \ni w \rightarrow w_j} h^{-1}(w), \quad j = 1, 2,$$

are different, $\sigma_1 \neq \sigma_2$, because the slits Γ_1 and Γ_2 are not equivalent: otherwise we would have $w_{y_1} = w_{y_2} = \infty$ and $\{w \in \mathbb{C}: \operatorname{Im} w \in [y_1, y_2]\} \subset \Omega$, which contradicts the construction.

Moreover, we may assume $\sigma \notin \{\sigma_1, \sigma_2\}$. Indeed, fix $j = 1$ or $j = 2$. If $w_{y_j} = \infty$, then by Lemma 4.9, the slit Γ_j converges to a trivial prime end, but $I(P)$ by the hypothesis contains more than one point. Hence in this case $\sigma_j \neq \sigma$. If $w_{y_j} \neq \infty$ we *a priori* may have the situation that $\sigma_j = \sigma$. This would mean that the slit Γ_j converges to P . If this happens choose in the above argument another value of $y_j \in (\operatorname{Im} w_j, \operatorname{Im} w_{j+1})$. Then the landing point w_{y_j} will change and, since no two slits with different landing points can converge to the same prime end [9, Theorem 9.7, p. 177], we now meet the desired condition $\sigma_j \neq \sigma$.

Using the argument from the proof of Lemma 4.9, one can conclude that the connected components of $\Omega \setminus \Gamma$ are $\Omega_1 := \Omega \setminus \overline{G}$ and $\Omega_2 := \Omega \cap G$, where

$$G := \{w \in \mathbb{C}: \operatorname{Re} w < x_0, y_1 < \operatorname{Im} w < y_2\}.$$

Clearly, $w_1 \in \partial \Omega_1 \setminus \partial \Omega_2$ and $w_2 \in \partial \Omega_2 \setminus \partial \Omega_1$. Recall that both w_1 and w_2 belong to $I(P)$, *i.e.*, to the limit set of $h(z)$ as $\mathbb{D} \ni z \rightarrow \sigma$. Therefore, $\sigma \in \partial h^{-1}(\Omega_1) \cap \partial h^{-1}(\Omega_2)$. The latter means that $\sigma \in \{\sigma_1, \sigma_2\}$. This contradicts the construction and thus proves part (A) of the proposition.

To prove (B) we use similar ideas. Assume on the contrary that the statement does not hold. Then there exist $w_1, w_2 \in I(P)$ with $\operatorname{Im} w_1 < \operatorname{Im} w_2$. Fix any $y_1 \in (\operatorname{Im} w_1, \operatorname{Im} w_2)$ and let $L_{y_1}^+$ be constructed as above. Its preimage $h^{-1}(L_{y_1}^+)$ lands on the unit circle at two points, the DW-point τ and another point $\varsigma \in \mathbb{T} \setminus \{\tau\}$. As above, we may assume that $\sigma \neq \varsigma$. Moreover, by Lemma 4.11, $\sigma \neq \tau$. This leads to a contradiction in a similar way as in the proof of (A) and thus shows that statement (B) is also true. \square

Remark 4.15. In fact we can also prove that if the prime end P corresponds to the DW-point and $I(P) \cap \mathbb{C}$ is not contained in one line, then for both lines, L_1 and L_2 whose union contains E , we have $\sup\{\operatorname{Re} w : w \in I(P) \cap L_j\} = +\infty$.

Proof of Proposition 3.7. We divide the proof into several steps.

Step 1. We first prove the equivalence of (i), (ii) and (iii).

Trivially (iii) \Rightarrow (ii). Moreover, by [Corollary 4.10](#), (i) implies (iii). It remains to show that (ii) implies (i). As we have already mentioned, if $h(r\sigma) \rightarrow \infty$ as $r \rightarrow 1-0$, then σ is a boundary fixed point. Indeed, in this case for each $t \geq 0$, $\Gamma := h([0, \sigma))$ and $\Gamma + t$ are two equivalent slits in Ω . Therefore, by [\[21, Theorem 1 in Chapter 2, §3\]](#), $h^{-1}(\Gamma + t)$ is a slit in \mathbb{D} landing at σ , i.e. $\phi_t(\sigma) = \lim_{r \rightarrow 1-0} h^{-1}(h(r\sigma) + t) = \sigma$. So we only have to show that if additionally $\operatorname{Re} h(r\sigma) \rightarrow -\infty$ as $r \rightarrow 1-0$, then $\sigma \neq \tau$.

Suppose on the contrary that (ii) holds and that σ coincides with the DW-point τ of (ϕ_t) . According to [Remark 3.5](#) applied with $\sigma_0 := -1$ substituted for σ , this means that the function $H \in \operatorname{Hol}(\mathbb{H}_1, \mathbb{C})$ defined in [Remark 3.5](#) satisfy $H(x) \rightarrow -\infty$ as $\mathbb{R} \ni x \rightarrow +\infty$. But this is not possible, because $\operatorname{Re} H' > 0$ in \mathbb{H}_1 . Thus, $\sigma \neq \tau$.

Step 2. Let us now pass to the proof of the statement concerning the limits of $\operatorname{Im} h$. Assume first that σ is not a boundary fixed point (and in particular does not coincide with the DW-point τ). We will show that in this case the unrestricted limit of $\operatorname{Im} h$ at σ exists finitely.

If the impression $I(P)$ of the prime end P that corresponds under the map h to the point σ , does not contain the point ∞ , then from [Proposition 4.14](#) and [Lemma 4.11](#) it follows that I is a closed interval on a straight line parallel to \mathbb{R} and hence the unrestricted limit $\lim_{\mathbb{D} \ni z \rightarrow \sigma} \operatorname{Im} h(z)$ exists finitely.

Now let us consider the case when the impression $I(P)$ of the prime end P contains ∞ . Since by assumption, σ is not a boundary fixed point, the argument of Step 1 shows that the angular limit $h(\sigma) := \angle \lim_{z \rightarrow \sigma} h(z)$ is finite and hence P is not trivial, i.e., $I(P) \neq \{\infty\}$. Using again [Proposition 4.14](#) and [Lemma 4.11](#), we conclude that the impression is of the form $I(P) = \{w_0 - x : x \geq 0\}$, where w_0 is some point⁷ on $\partial_{\mathbb{C}}\Omega$. By [\[9, Theorem 9.8\]](#) there exists a null-chain (C_n) belonging to the prime end P and converging⁸ to $w_1 := h(\sigma) \in I(P) \cap \mathbb{C}$. Moreover, by the proof of [\[9, Theorem 9.3\]](#), we may assume that each C_n is an arc of the circle $\tilde{C}_n := \{w : |w - w_1| = r_n\}$ for some r_n positive and going to 0 as $n \rightarrow +\infty$. Denote by w_n^j , $j = 1, 2$, $\operatorname{Im} w_n^1 \leq \operatorname{Im} w_n^2$, the end-points of C_n and by D_n the connected component of $\Omega \setminus C_n$ that contains C_{n+1} . By invariance of Ω w.r.t. the translations $w \mapsto w + t$, $t \geq 0$, the rays $C_n^j := \{w_n^j - x : x \geq 0\}$ do not intersect Ω . Fix any point $w_* \in \Omega$ with $\operatorname{Re} w_* > A := \operatorname{Re} w_1 + \max\{r_n : n \in \mathbb{N}\}$. Since $D_{n+1} \subset D_n$ for all $n \in \mathbb{N}$ and $\bigcap_{n \in \mathbb{N}} D_n = \emptyset$, dropping a finite number of C_n 's we may assume that $w_* \notin D_n$ for all $n \in \mathbb{N}$. Then we have $\operatorname{Im} w_n^1 \neq \operatorname{Im} w_n^2$, because otherwise D_n would be a subset of the disk bounded by \tilde{C}_n , which is not possible since $I(P) \subset \overline{D_n}$.

Therefore the set $\Gamma_n := C_n \cup C_n^1 \cup C_n^2 \cup \{\infty\}$ is a Jordan curve. Clearly, one of the two Jordan domains bounded by Γ_n contains the half-plane $\{w : \operatorname{Re} w > A\}$. We denote by G_n the other Jordan domain of Γ_n . It is easy to see that $D_n \subset G_n$ for all $n \in \mathbb{N}$, because $w_* \notin D_n$. It follows that $|\operatorname{Im} w - \operatorname{Im} w_1| < r_n$ for any $w \in D_n$ and all $n \in \mathbb{N}$. Recall that there exists a fundamental system (U_n) of neighbourhoods of σ such that $h(U_n \cap \mathbb{D}) = D_n$ for all $n \in \mathbb{N}$. Thus $\operatorname{Im} h(z) \rightarrow \operatorname{Im} w_1$ as $\mathbb{D} \ni z \rightarrow \sigma$.

⁷ In fact, we could show that $h(\sigma) = w_0$, but for our purposes it is enough to know that $h(\sigma) \neq \infty$.

⁸ See [Definition 4.8](#).

Step 3. Now we assume that σ is a boundary non-regular fixed point of (ϕ_t) . Again we have to show that the unrestricted limit of $\operatorname{Im} h$ at σ exists finitely.

We use the arguments from the proofs of Lemma 4.9 and Corollary 4.10. In the notation introduced in the proof of Lemma 4.9, it is sufficient to show that $\operatorname{mes}_{\mathbb{R}} Y(x) \rightarrow 0$ as $x \rightarrow -\infty$, where $\operatorname{mes}_{\mathbb{R}} \cdot$ stands for the length measure on \mathbb{R} . Then it would follow that $\operatorname{Im} h(z)$ tends to the unique point in the intersection $\bigcap_{x < x_0} Y(x)$. Suppose on the contrary that $\operatorname{mes}_{\mathbb{R}} Y(x) > a$ for some constant $a > 0$ and all $x < x_0$. Recall that $Y(x) \subset Y(x')$ if $x > x'$. Therefore, there exists a non-empty interval Y_0 which is a subset of $Y(x)$ for all $x < x_0$. It follows that $h([0, \sigma))$ is contained in the horizontal strip $\{w: \operatorname{Im} w \in Y_0\}$, which in its turn is contained in the domain Ω . According to [11, Theorem 2.5] this means that σ is a boundary regular fixed point. The contradiction obtained proves the claim of this step.

Step 4. Assume finally that σ is a boundary regular fixed point. We are going to show that the limit set of $\operatorname{Im} h$ at σ is not a singleton.

Assume first that σ is not the DW-point. Then by [10, Lemma 1] there exists a non-empty interval $Y_0 \subset \mathbb{R}$ such that the strip $V := \{w: \operatorname{Im} w \in Y_0\}$ is contained in Ω and for every $w \in V$, the $h^{-1}(w - t) \rightarrow \sigma$ as $t \rightarrow +\infty$. It immediately follows that the limit set⁹ of $\operatorname{Im} h(z)$ as $\mathbb{D} \ni z \rightarrow \sigma$ contains Y_0 .

The proof for the case of the DW-point $\sigma = \tau$ is very similar. Take any $w \in \Omega$. Then $h^{-1}(w + t) = \phi_t(h^{-1}(w)) \rightarrow \tau$ as $t \rightarrow +\infty$. This means that the limit set of $\operatorname{Im} h$ at τ coincides with the closure of $\{\operatorname{Im} w: w \in \Omega\}$. The proof is now complete. \square

Proposition 4.16. Suppose that h has no continuous extension to the DW-point τ . Then there exist a line L parallel to the real axis and a ray $R \subset L$ with $\sup_{w \in R} \operatorname{Re} w = +\infty$ such that $L \cap \Omega = \emptyset$ and $R \subset \partial\Omega$. In particular, the domain \mathcal{D} defined in Remark 4.12 is either half-plane or a horizontal strip.

Proof. Let P be the prime end of Ω that corresponds under the mapping h to the DW-point $\tau = 1$. Since by the hypothesis h has no continuous extension to τ , the prime end P is not trivial. Then it follows from Lemma 4.11 and Proposition 4.14(A) that the impression $I(P)$ contains a ray R parallel to the real axis with $\sup_{w \in R} \operatorname{Re} w = +\infty$. In particular, $R \subset \partial\Omega$. Then by the translational invariance of Ω , the straight line L containing R cannot intersect Ω . The proof is complete. \square

Proposition 4.17. The following two statements are equivalent:

- (A) Ω does not contain any half-plane H bounded by a line parallel to \mathbb{R} ;
- (B) the family

$$\Phi := (\overline{\mathbb{D}} \ni z \mapsto \phi_t(z) \in \overline{\mathbb{D}})_{t \geq 0}$$

is equicontinuous at the DW-point $\tau = 1$ of (ϕ_t) .

Proof. First of all we notice that if (A) fails to hold, i.e., if there exists a half-plane $H \subset \Omega$ whose boundary is parallel to \mathbb{R} , then the family Φ is not equicontinuous at τ . Indeed, take any line $L \subset H$ parallel to \mathbb{R} . Then the rays $L^+ := \{w \in L: \operatorname{Re} w \geq 0\}$, $L^- := \{w \in L: \operatorname{Re} w \leq 0\}$ are two equivalent slits in Ω and, as

⁹ We have shown a bit more: Y_0 is contained, in fact, in the non-tangent limit set

$\{y \in [-\infty, +\infty]: \exists (z_n) \subset \mathbb{D} \text{ s.t. } z_n \rightarrow \sigma \text{ non-tangentially and } \operatorname{Im} h(z_n) \rightarrow y\}.$

it follows from [Remarks 4.4 and 4.5](#), their preimages $h^{-1}(L^+)$, $h^{-1}(L^-)$ land at the DW-point $\tau = 1$. Take any sequence $(w_n) \subset L^-$ tending to ∞ . Write $t_n := -\operatorname{Re} w_n$. Then, on the one hand, $z_n := h^{-1}(w_n) \rightarrow \tau$ as $n \rightarrow \infty$. However, on the other hand, $\phi_{t_n}(z_n) = h^{-1}(w_n + t_n)$ is the same point in \mathbb{D} for all $n \in \mathbb{N}$. This shows that the family Φ is not equicontinuous at τ .

It now remains to prove that (A) implies (B). The idea of the proof is as follows. Choose a point in $w_0 \in \Omega$. By [Remark 4.5](#) the preimage $h^{-1}(R_0)$ of the ray $R_0 := \{w_0 + t: t \geq 0\}$ is a slit landing at the DW-point $\tau = 1$. In other words, R_0 as a slit in Ω , converges to the prime end P corresponding under the map h to the DW-point τ . We will construct a null-chain (C_n) that represents the prime end P and which has the following property: for each $n \in \mathbb{N}$, the connected component D_n of the set $\Omega \setminus C_n$ that contains C_{n+1} is invariant w.r.t. the translations $w \mapsto w + t$, $t \geq 0$.

For each $n \in \mathbb{N}$ denote $x_n := \operatorname{Re} w_0 + n$, $w_n := x_n + i \operatorname{Im} w_0 \in R_0$, and let \tilde{C}_n stand for the connected component of the set $\{w \in \Omega: \operatorname{Re} w = x_n\}$ that contains the point w_n . The following four cases exhaust all possibilities:

Case 1: for each $n \in \mathbb{N}$ the set \tilde{C}_n is bounded.

Case 2: there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\inf\{\operatorname{Im} w: w \in \tilde{C}_n\} \in \mathbb{R}, \quad \sup\{\operatorname{Im} w: w \in \tilde{C}_n\} = +\infty.$$

Case 3: there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\inf\{\operatorname{Im} w: w \in \tilde{C}_n\} = -\infty, \quad \sup\{\operatorname{Im} w: w \in \tilde{C}_n\} \in \mathbb{R}.$$

Case 4: there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\inf\{\operatorname{Im} w: w \in \tilde{C}_n\} = -\infty, \quad \sup\{\operatorname{Im} w: w \in \tilde{C}_n\} = +\infty.$$

In *Case 1*, the sets \tilde{C}_n are of the form $\tilde{C}_n = (x_n + iy'_n, x_n + iy''_n)$, where $-\infty < y'_n < y''_n < +\infty$, and we set $C_n := \tilde{C}_n$ for all $n \in \mathbb{N}$. Because of the translational invariance of Ω , $(y'_n, y''_n) \subset (y'_{n+1}, y''_{n+1})$ for every $n \in \mathbb{N}$.

In *Case 2*, taking if necessary w_{n_0} instead of w_0 , we may assume that $n_0 = 0$. Condition (A) implies that there exists a strictly increasing unbounded sequence $(y''_n) \subset (\operatorname{Im} w_0, +\infty)$ such that for every $n \in \mathbb{N}$ the line

$$L''_n := \{w \in \mathbb{C}: \operatorname{Im} w = y''_n\}$$

is not a subset of Ω . Note that $x_n + iy''_n \in L''_n \cap \Omega$. Now set $y'_n := \inf\{\operatorname{Im} w: w \in \tilde{C}_n\}$, $x''_n := \inf\{\operatorname{Re} w: w \in L''_n \cap \Omega\}$ and let

$$C_n := (x_n + iy'_n, x_n + iy''_n] \cup [x_n + iy''_n, x''_n + iy''_n).$$

Note that again we have $(y'_n, y''_n) \subset (y'_{n+1}, y''_{n+1})$ for every $n \in \mathbb{N}$.

Case 3 can be reduced to the previous case by considering $\overline{\phi_t(z)}$ instead of $\phi_t(z)$, which leads to passing from $h(z)$ to $\overline{h(\bar{z})}$. So we may skip Case 3.

In *Case 4* we also will assume that $n_0 = 0$. Fix a strictly increasing unbounded sequence $(y''_n) \subset (\operatorname{Im} w_0, +\infty)$ and a strictly decreasing unbounded sequence $(y'_n) \subset (-\infty, \operatorname{Im} w_0)$ such that for every $n \in \mathbb{N}$ the lines

$$L'_n := \{w \in \mathbb{C}: \operatorname{Im} w = y'_n\} \quad \text{and} \quad L''_n := \{w \in \mathbb{C}: \operatorname{Im} w = y''_n\}$$

are not subsets of Ω . Note that $x_n + iy'_n \in L'_n \cap \Omega$ and $x_n + iy''_n \in L''_n \cap \Omega$. Now set $x'_n := \inf\{\operatorname{Re} w : w \in L'_n \cap \Omega\}$, $x''_n := \inf\{\operatorname{Re} w : w \in L''_n \cap \Omega\}$ and let

$$C_n := (x'_n + iy'_n, x_n + iy'_n] \cup [x_n + iy'_n, x_n + iy''_n] \cup [x_n + iy''_n, x''_n + iy''_n).$$

Clearly, in all the cases for each $n \in \mathbb{N}$, C_n is a cross-cut in Ω , the closures $\overline{C_n}$ are pairwise disjoint, and $\operatorname{diam}_{\mathbb{C}} C_n \rightarrow 0$ as $n \rightarrow +\infty$. To prove that (C_n) is a null-chain, it remains to show that for every $n \geq 2$, C_{n-1} and C_{n+1} are contained in two different connected components of $\Omega \setminus C_n$. Arguing as in the proof of [Lemma 4.9](#), one can easily conclude that the connected components of $\Omega \setminus C_n$ are $\Omega \setminus \overline{G_n}$ and $\Omega \cap G_n$, where

$$G_n := \{w \in \mathbb{C} : \operatorname{Re} w < x_n, y'_n < \operatorname{Im} w < y''_n\}.$$

By construction $C_{n+1} \cap G_n = \emptyset$, while $C_{n-1} \subset G_n$ because $(y'_{n-1}, y''_{n-1}) \subset (y'_n, y''_n)$.

Thus (C_n) is a null-chain, and $D_n = \Omega \setminus \overline{G_n}$ for all $n \in \mathbb{N}$. Hence it can be seen easily from the construction that the slit R_0 converges to the prime end P represented by (C_n) . Moreover, $\bigcap_{n \in \mathbb{N}} D_n = \emptyset$, see, e.g., [\[9, pp. 170–171\]](#). Thus, discarding a finite number of cross-cuts in (C_n) , we may assume that $h(0) \notin D_n$ for all $n \in \mathbb{N}$.

Now fix any $n \in \mathbb{N}$. Since $t + G_n \supset G_n$ for any $t \geq 0$, we see that D_n is invariant w.r.t. the translations $w \mapsto w + t, t \geq 0$. This means that the set $U_n := h^{-1}(D_n)$ is invariant w.r.t. the semigroup (ϕ_t) . Note that $h^{-1}(C_k)$, $k \in \mathbb{N}$, are cross-cuts in \mathbb{D} , whose closures $\Gamma_k := \overline{h^{-1}(C_k)}$ are pairwise disjoint, see [Remark 4.4](#). Note that $0 \notin U_n$ by construction. Therefore, $U_n = W_n \cap \mathbb{D}$, where W_n is the bounded Jordan domain with ∂W_n formed by Γ_n together with its reflection Γ_n^* w.r.t. \mathbb{T} . Furthermore, $\operatorname{diam}_{\mathbb{C}} C_n \rightarrow 0$ implies, according to [Proposition 3.3](#), that $\operatorname{diam}_{\mathbb{C}} \Gamma_n \rightarrow 0$ as $n \rightarrow +\infty$. Hence $\operatorname{diam}_{\mathbb{C}} W_n \rightarrow 0$ as $n \rightarrow +\infty$. Finally, note that

$$\overline{h^{-1}(R_0)} = \overline{\{\phi_t(h^{-1}(w_0)) : t \geq 0\}}$$

is a Jordan arc that joins $z = 0 \notin W_n$ with the DW-point τ and which, by the construction, intersects ∂W_n exactly once and in a non-tangential way. It follows that $\tau \in W_n$ for all $n \in \mathbb{N}$. Thus $(W_n)_{n \in \mathbb{N}}$ is a neighbourhood basis of the point τ with the property that $\phi_t(W_n \cap \mathbb{D}) \subset W_n$ for all $t \geq 0$ and all $n \in \mathbb{N}$. It follows that (B) holds and thus the proof is completed. \square

Proposition 4.18. *For each $T > 0$ the family*

$$\Phi_T := (\overline{\mathbb{D}} \ni z \mapsto \phi_t(z) \in \overline{\mathbb{D}})_{t \in [0, T]}$$

is equicontinuous at the DW-point $\tau = 1$ of (ϕ_t) .

Proof. Essentially we will use the same idea as for the proof of implication (A) \Rightarrow (B) in [Proposition 4.17](#).

In view of [Propositions 4.6 and 4.17](#), we may assume that the prime end P that corresponds under the mapping h to the DW-point τ is non-trivial and, employing also [Proposition 4.16](#), that there exist two half-planes H_1, H_2 with $\partial_{\mathbb{C}} H_j$ parallel to \mathbb{R} , $j = 1, 2$, such that $H_1 \subset \Omega \subset H_2$. Without loss of generality we may assume that H_j 's are of the form

$$H_j := \{w \in \mathbb{C} : \operatorname{Im} w < y_j\}$$

for some $y_1 < y_2 \in \mathbb{R}$. Fix any $w_0 \in H_1$.

Let $n \in \mathbb{N}$. Denote $C_n := \tilde{C}_{-n} \cup \{w : \operatorname{Im}(w - w_0) \leq 0, |w - w_0| = n\} \cup \tilde{C}_n$, where \tilde{C}_m , $m \in \mathbb{Z}$, stands for the connected component of $\{w \in \Omega : \operatorname{Re}(w - w_0) = m, \operatorname{Im}(w - w_0) \geq 0\}$ that contains the point $w_0 + m$. Clearly, C_n 's are cross-cuts in Ω with pairwise disjoint closures. Moreover, $\operatorname{diam}_{\mathbb{C}} C_n \rightarrow 0$ as $n \rightarrow +\infty$.

For $m \in \mathbb{Z}$ denote $y'_m := \sup\{\operatorname{Im} w : w \in \tilde{C}_m\}$. Note that $y'_m < y_2 < +\infty$ because $\Omega \subset H_2$. By the translational invariance of Ω , we have $y'_m \leq y'_k$ whenever $m < k$. It follows that for any $n \geq 2$, $C_{n+1} \subset \Omega \setminus G_n$ and $C_{n-1} \subset G_n$, where

$$\begin{aligned} G_n &:= \{w : \operatorname{Im}(w - w_0) < 0, |w - w_0| < n\} \\ &\cup \{w : |\operatorname{Re}(w - w_0)| < n, \operatorname{Im} w_0 \leq \operatorname{Im} w < y'_n\} \\ &\cup \{w : \operatorname{Re}(w - w_0) \leq -n, y'_{-n} < \operatorname{Im} w < y'_n\}. \end{aligned}$$

(The last set in the union may be empty.) Arguing as in the proof of [Lemma 4.9](#), one can conclude that for each $n \geq 2$ the sets $D_n := \Omega \setminus \overline{G_n} \supset C_{n+1}$ and $D'_n := \Omega \cap G_n \supset C_{n-1}$ are the two connected components of $\Omega \setminus C_n$. Thus (C_n) is a null-chain.

We also notice that if $k > n$, then $\overline{G_n} \subset \overline{G_k} + t$ for all $t \in [0, k - n]$. It follows that

$$D_k + t = (\Omega \setminus \overline{G_k}) + t = (\Omega + t) \setminus (\overline{G_k} + t) \subset \Omega \setminus (\overline{G_k} + t) \subset \Omega \setminus \overline{G_n} = D_n \quad (4.3)$$

for all $t \in [0, k - n]$.

Fix $T > 0$. Then according to (4.3), $\phi_t(U_{k(n)}) \subset U_n$ for all $t \in [0, T]$ and all $n \in \mathbb{N}$, where $U_n := h^{-1}(D_n)$, $k(n) := n + [T] + 1$, and $[\cdot]$ stands for the integer part of a real number. Note that $w_0 \notin D_n$ for all $n \in \mathbb{N}$ and that $R_0 := \{w_0 + t : t \geq 0\}$ intersect each of C_n 's exactly once and non-tangentially. Arguing now as in the proof of [Proposition 4.17](#), we conclude that the family Φ_T is equicontinuous at τ . \square

4.3. Proof of the main theorem ([Theorem 4.1](#))

As we mentioned in [Section 2](#), without loss of generality we may assume that the one-parameter semigroup (ϕ_t) has the DW-point at $\tau = 1$.

Let σ be a repulsive or super-repulsive boundary fixed point of (ϕ_t) . Then by [Corollary 4.10](#), h has the unrestricted limit at σ . Therefore, by [Proposition 4.6](#), for every $T > 0$ the family Φ_T is equicontinuous at σ .

Note that by [Proposition 4.18](#), the family Φ_T is also equicontinuous at the DW-point τ .

It remains to notice that if (ϕ_t) is of hyperbolic type, which means that the angular derivative $\phi'_t(\tau) < 1$ for $t > 0$, then by [\[11, Theorem 2.1\]](#) the domain Ω is contained in a horizontal strip. Consequently, in this case by [Proposition 4.17](#), the family Φ is equicontinuous at τ . This completes the proof. \square

5. A few examples

5.1. Local behaviour near the boundary DW-point

Recall that if (ϕ_t) is a one-parameter semigroup in \mathbb{D} , then ϕ_t converges, as $t \rightarrow +\infty$, to the DW-point τ *locally uniformly* in \mathbb{D} . In the case of a hyperbolic semigroup (ϕ_t) , although the DW-point τ belongs to \mathbb{T} , according to [Theorem 4.1](#) the family Φ is equicontinuous at τ . It is interesting to notice that these two facts do **not** imply that there exists a neighbourhood W of τ such that $\phi_t \rightarrow \tau$ *uniformly* in $\mathbb{D} \cap W$ as $t \rightarrow +\infty$. For instance, there can be infinitely many boundary fixed points in any neighbourhood of τ , as the following example shows.

Example 1. Denote $S := \{w : |\operatorname{Im} w| < 1\}$, $I'_n := \{x + i(1 - 1/n) : x \leq n\}$, and $I''_n := \{x - i(1 - 1/n) : x \leq n\}$. Then

$$\Omega := S \setminus \bigcup_{n=2}^{+\infty} (I'_n \cup I''_n)$$

is a domain invariant w.r.t. the translations $w \mapsto w + t$, $t \geq 0$. Denote by h the conformal mapping of \mathbb{D} onto Ω normalized by the conditions $h(0) = 0$ and $\lim_{t \rightarrow +\infty} h^{-1}(t) = 1$. Then the formula $\phi_t(z) := h^{-1}(h(z) + t)$ defines a one-parameter semigroup with the DW-point $\tau = 1$ and h is its Koenigs function.

By [11, Theorem 2.1], (ϕ_t) is of hyperbolic type. Thus Φ is equicontinuous at τ . However, h has no unrestricted limit at τ , because the impression $I(P)$ of the prime end P that corresponds to τ under the map h , is the whole boundary of S . To see that any neighbourhood of τ contains infinitely many boundary fixed points let us return to the proof of implication (A) \Rightarrow (B) in Proposition 4.17. Take $w_0 := 0$. Then for each $n \geq 2$, the domain D_n constructed in the proof of that proposition, contains the strip $S_n := \{w: 1/n < \operatorname{Im} w < 1/(n+1)\}$. According to [11, Theorem 2.5], there exists a repulsive boundary fixed point σ_n such that for every $w \in S_n$, $h^{-1}(w - t) \rightarrow \sigma_n$ as $t \rightarrow +\infty$. Hence $\sigma_n \in \overline{W_n}$, where the domains W_n were defined on page 218 in the proof of Proposition 4.17. It remains to recall that the sequence (W_n) form a neighbourhood basis of the point τ .

The reason why in the above example the uniform convergence of $\phi_t \rightarrow \tau$ fails in $W \cap \mathbb{D}$ for any neighbourhood W of τ is the presence of repulsive boundary fixed points. In fact, the following statement holds.

Remark 5.1. Let (ϕ_t) be any one-parameter semigroup in $\operatorname{Hol}(\mathbb{D}, \mathbb{D})$ with the DW-point $\tau \in \partial\mathbb{D}$. Let $\sigma \in \overline{\mathbb{D}}$. Then either $\phi_t(\sigma) = \sigma$ for all $t \geq 0$, or $\phi_t(\sigma) \rightarrow \tau$ as $t \rightarrow +\infty$. (Recall that for the case $\sigma \in \mathbb{T}$, $\phi_t(\sigma)$ stands for the angular limit of ϕ_t at σ .) Indeed, for $\sigma \in \mathbb{D}$ this follows from the Denjoy–Wolff Theorem. So assume that $\sigma \in \mathbb{T}$. Then by Proposition 3.4, $\Gamma := h([0, \sigma))$, where h is the Koenigs function of (ϕ_t) , is a slit in the domain $\Omega := h(\mathbb{D})$. For $t \geq 0$ denote $\Gamma_t := h(\phi_t([0, \sigma))) = t + \Gamma$. Let us assume first that Γ is bounded (as a subset of \mathbb{C}), i.e., it lands at some point of $\partial\Omega \cap \mathbb{C}$. We claim that $\sup\{|h^{-1}(w) - \tau| : w \in \Gamma_t\} \rightarrow 0$ as $t \rightarrow +\infty$, which is equivalent to $\phi_t \rightarrow \tau$ uniformly on $[0, \sigma)$ and hence implies that $\phi_t(\sigma) \rightarrow \tau$ as $t \rightarrow +\infty$. Indeed, recall again that $L_w := \{w + x : x \geq 0\} \subset \Omega$ for any $w \in \Omega$. By boundedness of Γ ,

$$\sup_{w \in \Gamma_t} \operatorname{diam}_{\overline{\mathbb{C}}}(L_w) = \sup_{\substack{w \in \Gamma \\ x \geq 0}} \frac{1}{\sqrt{1 + |w + t + x|^2}} \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

Since $h^{-1}(L_w)$ is a slit in \mathbb{D} landing at the DW-point τ , our claim follows now from Proposition 3.3.

It remains to consider the case when Γ is not bounded, i.e., the case when Γ lands at ∞ . In this case Γ_t is also a slit in Ω landing at ∞ for any $t \geq 0$. Moreover, Γ and Γ_t are *two equivalent slits* in Ω for any $t \geq 0$, because $[w, w + t] \subset \Omega$ for any $w \in \Gamma$. This means that $h^{-1}(\Gamma_t)$ lands at the same point as $h^{-1}(\Gamma) = [0, \sigma)$, i.e., at the point σ . Thus $\phi_t(\sigma) = \sigma$ for all $t \geq 0$.

Remark 5.2. It might be interesting to compare the statement of the previous remark with analogous results for *discrete iteration* in \mathbb{D} , see, e.g., [31] and [14, Section 5], asserting that under some additional conditions on $\phi \in \operatorname{Hol}(\mathbb{D}, \mathbb{D})$, the orbits $(\phi^{o_n}(\sigma))_{n \in \mathbb{N}}$ converge to the DW-point of ϕ for a.e. $\sigma \in \mathbb{T}$ (where $\phi(\sigma)$ stands again for the angular limit at σ , whenever it exists).

In view of Remark 5.1, it might look to be a plausible conjecture that if the family Φ is equicontinuous at the boundary DW-point τ and if there is a neighbourhood W of τ that does not contain other boundary fixed points, then $\phi_t \rightarrow \tau$ as $t \rightarrow +\infty$ uniformly in $W \cap \mathbb{D}$. However, the following example disproves this conjecture.

Example 2. Consider the domain

$$\tilde{S} := S_0 \setminus \bigcup_{n=2}^{+\infty} (J'_n \cup J''_n), \quad \text{where } S_0 := \left\{ u + iv : |v| < 1; \ u > -\frac{1}{1 - |v|} \right\},$$

$$J'_n := \left[-n + i \left(1 - \frac{1}{n} \right), i \left(1 - \frac{1}{n} \right) \right], \quad J''_n := \left[-n - i \left(1 - \frac{1}{n} \right), -i \left(1 - \frac{1}{n} \right) \right].$$

The segments J'_n and J''_n are slits in S_0 landing on the curve $u = -1/(1 - |v|)$, $|v| < 1$. Clearly, there exists no continuous map $F : [0, 1) \rightarrow \tilde{S}$ such that $\lim_{[0, 1) \ni x \rightarrow 1} \operatorname{Re} F(x) = -\infty$.

Denote by $\tilde{S}(y_1, y_2)$, where $y_1 < y_2$, the image of \tilde{S} under the affine map $u + iv \mapsto u + i(av + b)$, $a := (y_2 - y_1)/2$, $b := (y_1 + y_2)/2$, chosen in such a way that the minimal strip containing $\tilde{S}(y_1, y_2)$ is $\{w \in \mathbb{C} : y_1 < \operatorname{Im} w < y_2\}$. Now we consider the domain Ω constructed in [Example 1](#) and “fill in” with $\tilde{S}(y_1, y_2)$ ’s, for appropriately chosen parameters y_1, y_2 , each of the strips one obtains by removing from Ω all the straight lines containing the slits I'_n and I''_n , $n \geq 2$. In a more strict language, we consider the domain

$$\tilde{\Omega} := \Omega \setminus K(-1/2, 1/2) \setminus \bigcup_{n=2}^{+\infty} (K(1 - 1/n, 1 - 1/(1+n)) \cup K(-1 + 1/(n+1), -1 + 1/n)),$$

$$\text{where } K(y_1, y_2) := \{w \in \mathbb{C} : y_1 < \operatorname{Im} w < y_2\} \setminus \tilde{S}(y_1, y_2).$$

The set $\tilde{\Omega}$ is a simply connected domain in \mathbb{C} containing the origin and invariant w.r.t. the translations $w \mapsto w + t$. Therefore, there exists a unique conformal mapping h of \mathbb{D} onto $\tilde{\Omega}$ with $h(0) = 0$, $\lim_{t \rightarrow +\infty} h^{-1}(t) = 1$, which is the Kœnigs function of the one-parameter semigroup $(\phi_t) := (h^{-1} \circ (h + t))$ with the DW-point $\tau := 1$. First of all we notice that this semigroup has no (super-)repulsive boundary fixed points. Indeed, if $\sigma \in \mathbb{T} \setminus \{\tau\}$ is a boundary fixed point, then $[0, 1) \ni x \mapsto F(x) := h(\sigma x)$ is a continuous map into $\tilde{\Omega}$, and by [Proposition 3.7](#), $\operatorname{Re} F(x) \rightarrow -\infty$ as $x \rightarrow 1 - 0$. However, it is easy to see from the definition of $\tilde{\Omega}$ that there exists no mapping with these properties.

Note also that, as in [Example 1](#), (ϕ_t) is of hyperbolic type and thus the family Φ is equicontinuous at $\tau = 1$. It remains to see that there exists no neighbourhood W of τ such that ϕ_t converges uniformly to τ on $W \cap \mathbb{D}$. To this end take $w_0 = 0$ and define (D_n) and (W_n) as in the proof of implication (A) \Rightarrow (B) in [Proposition 4.17](#). Since for each $n \in \mathbb{N}$, we have $\inf\{\operatorname{Re} w : w \in D_n \setminus D_{n+1}\} = -\infty$, there exists no $t \geq 0$ such that $D_n + t \subset D_{n+1}$. Therefore, there exists no $t \geq 0$ such that $\phi_t(W_n \cap \mathbb{D}) \subset W_{n+1} \cap \mathbb{D}$. Recall that (W_n) is a neighbourhood basis of τ . Thus, although $\phi_t(z) \rightarrow \tau$ as $t \rightarrow +\infty$ for all $z \in \mathbb{D}$, this convergence is not uniform in any neighbourhood of τ .

5.2. Contact points

In [Remark 4.2](#) we mentioned that the [Theorem 4.1](#) cannot be extended to *contact points*. To demonstrate this fact, we now present an example of a one-parameter semigroup (ϕ_t) with a contact point at which there exists no unrestricted limit of ϕ_t ’s.

Example 3. Consider the domain

$$\Omega := \mathbb{H}_i \setminus \bigcup_{n \in \mathbb{N}} \{x + i/n : x \leq 0\}.$$

Clearly, this domain is invariant w.r.t. the right translations. Therefore with an appropriate choice of a conformal mapping h of \mathbb{D} onto Ω , we obtain a one-parameter semigroup $\phi_t := h^{-1} \circ (h + t)$, $t \geq 0$, with the DW-point $\tau = 1$. Moreover, Ω has a unique prime end P whose impression is $(-\infty, 0]$ and this prime end contains an accessible boundary point, *i.e.*, there is a slit Γ in Ω that converges to P (*e.g.*, we can take $\Gamma := (0, 1 + i]$). This slit lands at the point $w_0 = 0$. For $t > 0$ the translation $\Gamma_t := \Gamma + t$ of Γ is also a slit in Ω , with landing point at $w = t \in \partial\Omega$. Note that $\{w : \operatorname{Re} w > 0, \operatorname{Im} w > 0\} \subset \Omega \subset \mathbb{H}_i$ and hence h^{-1} has a continuous injective extension to $\Omega \cup (0, +\infty)$. It follows that for each $t > 0$, ϕ_t has a contact point at the preimage σ_0 of the prime end P under h , but does not have the unrestricted limit at σ_0 . In this

example, σ_0 is not a regular contact point, *i.e.*, $\phi'_t(\sigma_0) = \infty$ for all $t > 0$. However, a simple modification of this example (take, *e.g.*, the domain $\Omega' := \Omega \cup \{w: |w - i| < 1\}$ instead of Ω) shows that even if we consider a regular contact point, there still do not need to exist the unrestricted limit at that point.

6. A remark on evolution families admitting continuous extension to the boundary

The notion of an evolution family in $\text{Hol}(\mathbb{D}, \mathbb{D})$ is a natural extension for that of a one-parameter semigroup to the non-autonomous setting. It goes back to the seminal paper [30] that gave rise to a theory which is now known as Loewner Theory and which has proved to be a powerful tool in the Geometric Function Theory and its applications, see, *e.g.*, the survey [2]. We use the general definition of an evolution family introduced in [7], see also [22].

Definition 6.1. Let with $d \in [1, +\infty]$. A family $(\varphi_{s,t})_{0 \leq s \leq t < +\infty} \subset \text{Hol}(\mathbb{D}, \mathbb{D})$ is said to be an *evolution family of order d in the unit disk \mathbb{D}* if it satisfies the following conditions:

EF1. $\varphi_{s,s} = \text{id}_{\mathbb{D}}$, for all $s \geq 0$,

EF2. $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $0 \leq s \leq u \leq t < +\infty$,

EF3. for all $z \in \mathbb{D}$ and for all $T > 0$ there exists a non-negative function $k_{z,T} \in L^d([0, T], \mathbb{R})$ such that

$$|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_{z,T}(\xi) d\xi$$

whenever $0 \leq s \leq u \leq t \leq T$.

Similar to one-parameter semigroups, any evolution family in \mathbb{D} is formed by solutions to initial value problems for a specific first order ODE, driven by the so-called *Herglotz vector fields*. These non-autonomous vector fields can be regarded as locally integrable families of infinitesimal generators; in particular, one-parameter semigroups and their infinitesimal generators are special cases of evolution families and their corresponding Herglotz vector fields, see [7] for the details. Therefore, the class of holomorphic mappings $\varphi \in \text{Hol}(\mathbb{D}, \mathbb{D})$ that can be embedded into an evolution family in \mathbb{D} , *i.e.*, the class of all φ such that $\varphi_{s,t} = \varphi$ for some $s \geq 0$, $t \geq s$ and some evolution family $(\varphi_{s,t})$, is much larger than that for one-parameter semigroups in \mathbb{D} . In fact, using [12, Lemma 2.8] one can deduce from the Parametric Representation of bounded normalized univalent functions, see, *e.g.*, [4, Theorem 5, p. 70], that this class, regardless the order of the evolution families to be considered, coincides with the set of all univalent holomorphic self-maps of \mathbb{D} .

Therefore, the results of this paper on one-parameter semigroups cannot be extended, in general, to evolution families. However, an analogue of Proposition 3.2 holds under additional condition that

CNT1. $(\varphi_{s,t})_{t \geq s \geq 0} \subset \mathcal{A}(\mathbb{D})$, where $\mathcal{A}(\mathbb{D})$ stands for the class of all holomorphic functions in \mathbb{D} admitting continuous extension to the closed unit disk $\overline{\mathbb{D}}$.

We endow $\mathcal{A}(\mathbb{D})$ with the Chebyshev (supremum) norm $\|\varphi\|_{\mathcal{A}(\mathbb{D})} := \sup_{z \in \mathbb{D}} |\varphi(z)|$. Each element φ of $\mathcal{A}(\mathbb{D})$ is identified with its extension to $\overline{\mathbb{D}}$.

Remark 6.2. From [7, Proposition 3.5] it follows that if $(\varphi_{s,t})$ is an evolution family in \mathbb{D} of some order $d \in [1, +\infty]$, then

CNT2. for each $z \in \mathbb{D}$ the mapping $(s, t) \mapsto \varphi_{s,t}(z)$ from $\{(s, t) \in \mathbb{R}^2: 0 \leq s \leq t\}$ to \mathbb{D} is separately continuous in s and t .

Moreover, all elements of an evolution family are univalent functions.

Proposition 6.3. *If a family $(\varphi_{s,t})_{0 \leq s \leq t < +\infty}$ satisfies conditions EF1, EF2, CNT1, and CNT2 and none of the functions $\varphi_{s,t}$ is constant, then for each $s \geq 0$ the mapping $[s, +\infty) \ni t \mapsto \varphi_{s,t} \in \mathcal{A}(\mathbb{D})$ is continuous.*

Proof. Let us fix $s \geq 0$. Consider any convergent sequence $(t_n) \subset [s, +\infty)$ and denote by t_0 the limit of (t_n) . We have to prove that $\|\varphi_{s,t_n} - \varphi_{s,t_0}\|_{\mathcal{A}(\mathbb{D})} \rightarrow 0$ as $n \rightarrow +\infty$. Owing to the Arzelà–Ascoli Theorem, it follows from condition CNT2 that we only have to show that the sequence $(\varphi_{s,t_n})_{n \in \mathbb{N}}$ is equicontinuous on $\overline{\mathbb{D}}$. Suppose it is not the case. Then one can find a sequence of Jordan arcs (γ_n) lying in \mathbb{D} such that $\text{diam}_{\mathbb{C}}(\gamma_n) \rightarrow 0$ as $n \rightarrow +\infty$, but $\text{diam}_{\mathbb{C}}(\varphi_{s,t_n}(\gamma_n)) > \varepsilon$ for all $n \in \mathbb{N}$ and some $\varepsilon > 0$ not depending on n .

Choose $T > 0$ such that $t_n < T$ for all $n \in \mathbb{N}$. Then

$$\varphi_{t_n,T} \circ \varphi_{s,t_n} = \varphi_{s,T}, \quad n \in \mathbb{N}. \quad (6.1)$$

Denote $C_n := \varphi_{s,t_n}(\gamma_n)$. By (6.1), we have $\text{diam}_{\mathbb{C}}(\varphi_{t_n,T}(C_n)) = \text{diam}_{\mathbb{C}}(\varphi_{s,T}(\gamma_n))$. Since $\varphi_{s,T} \in \mathcal{A}(\mathbb{D})$, the latter quantity tends to 0 as $n \rightarrow +\infty$. At the same time $\text{diam}_{\mathbb{C}}(C_n) > \varepsilon$ for each $n \in \mathbb{N}$. By the Schwarz–Pick theorem,

$$|\varphi'_{t_n,T}(z)| \leq \frac{1}{1 - |z|^2}, \quad z \in \mathbb{D}, \quad n \in \mathbb{N}. \quad (6.2)$$

It follows now from [32, Theorem 9.2, p. 265] that the sequence $\varphi_{t_n,T}$ has a subsequence converging to a constant. This fact contradicts the hypothesis and thus completes the proof. \square

Remark 6.4. Note that the hypothesis of the above proposition does not imply the continuity of $[0, t] \ni s \mapsto \varphi_{s,t} \in \mathcal{A}(\mathbb{D})$. Indeed, the well-known in Loewner Theory example constructed by Kufarev [29] (see also [4, p. 43]) reveals an evolution family $(\varphi_{s,t}) \subset \mathcal{A}(\mathbb{D})$ that fails to be continuous in s at $s := 0$ w.r.t. the norm $\|\cdot\|_{\mathcal{A}(\mathbb{D})}$. In this example, for fixed $t > 0$ and for each $s \in (0, t)$ the function $\varphi_{s,t}$ maps \mathbb{D} onto \mathbb{D} minus the slit along a part $\Gamma_{s,t}$ of a hyperbolic geodesic Γ_t , while the mapping $\varphi_{0,t}$ maps \mathbb{D} onto the connected component of $\mathbb{D} \setminus \Gamma_t$ that contains the origin. Since $\varphi_{s,t}(\overline{\mathbb{D}}) = \overline{\mathbb{D}} \neq \varphi_{0,t}(\overline{\mathbb{D}})$ for all $s \in (0, t)$, the norm $\|\varphi_{s,t} - \varphi_{0,t}\|_{\mathcal{A}(\mathbb{D})}$ does not tend to zero as $s \rightarrow +0$.

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