Doctopic: Miscellaneous ARTICLE IN PRESS YJMAA:18573

J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Complete convergence for moving average processes associated to heavy-tailed distributions and applications

Wei Li [a], Pingyan Chen [b], Tien-Chung Hu [c,*]

[a] *College of Computation Science, Zhongkai University of Agriculture and Engineering, Guangzhou, 510225, PR China*
[b] *Department of Mathematics, Jinan University, Guangzhou, 510632, PR China*
[c] *Dept. of Math., National Tsing Hua University, Hsinchu 30013, Taiwan, ROC*

A R T I C L E   I N F O

Article history:
Received 22 March 2014
Available online xxxx
Submitted by V. Pozdnyakov

Keywords:
Complete convergence
Heavy-tailed distribution
Integral test
Law of the iterated logarithm
Moving average process

A B S T R A C T

The complete convergence is obtained for the moving average processes associated to heavy-tailed distributions via integral test. As the applications, two versions of Chover's law of the iterated logarithm are deduced.

1. Introduction and main results

The concept of complete convergence was first introduced by Hsu and Robbins [13] as follows. A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant C if $\sum_{n=1}^{\infty} P\{|U_n - C| > \varepsilon\} < \infty$ for all $\varepsilon > 0$. By the Borel–Cantelli lemma, this implies $U_n \to C$ almost surely and the converse implication is not necessarily true if $\{U_n, n \geq 1\}$ are not independent. Hsu and Robbins [13] proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite. And Erdös [9,10] proved that the converse is also true.

This result has been generalized and extended in several directions, for example, see Katz [18], Baum and Katz [2], Bai and Su [1], Gut [12], Hu et al. [15], etc. It is worthwhile to point that Katz [18], Baum and Katz [2] obtained the following results: if $0 < p < 2$ and $r \geq 1$ then $E|X|^{rp} < \infty$ if and only if

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\left|\sum_{j=1}^{n} X_j - nb\right| > \varepsilon n^{1/p}\right\} < \infty \quad \text{for all } \varepsilon > 0, \tag{1.1}$$

* Corresponding author.
E-mail address: tchu@math.nthu.edu.tw (T.C. Hu).

http://dx.doi.org/10.1016/j.jmaa.2014.05.071
0022-247X/

Please cite this article in press as: W. Li et al., Complete convergence for moving average processes associated to heavy-tailed distributions and applications, J. Math. Anal. Appl. (2014), http://dx.doi.org/10.1016/j.jmaa.2014.05.071

if and only if

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j - kb\right| > \varepsilon n^{1/p}\right\} < \infty \quad \text{for all } \varepsilon > 0, \tag{1.2}$$

where $b = EX$ if $rp \ge 1$ and $b = 0$ if $0 < rp < 1$.

Recently the rate of complete convergence for sequences of dependent random variables has attracted lots of attention. One of those investigations is to investigate the rate of complete convergence for moving average processes based on the independent, identically distributed random variables. The concept of the moving average processes and relevant limit results is stated in the following:

Assume that $\{X_i, -\infty < i < +\infty\}$ is a doubly infinite sequence of identically distributed random variables. Let $\{a_i, -\infty < i < +\infty\}$ be a sequence of real numbers with

$$\sum_{i=-\infty}^{\infty} |a_i|^{\delta} < \infty \tag{1.3}$$

for some $0 < \delta \le 1$ and define the moving average process as

$$Y_n = \sum_{i=-\infty}^{\infty} a_i X_{i+n}, \quad n \ge 1. \tag{1.4}$$

When $\{X, X_i, -\infty < i < +\infty\}$ is a sequence of independent and identically distributed random variables, many limiting results have been obtained for the moving average process $\{Y_n, n \ge 1\}$. For example, Ibragimov [17] established the central limit theorem, Burton and Dehling [4] obtained a large deviation principle assuming $E\exp\{tX\} < \infty$ for all t, and Li et al. [20] obtained the complete convergence result for $\{Y_n, n \ge 1\}$. All those show that the partial sums of $\{Y_n, n \ge 1\}$ have similar limiting behavior properties in comparison with the limiting properties of independent and identically distributed random variables.

For example, Hsu–Robbins result was extended by Li et al. [20] for moving average processes.

Theorem A. *Let $\{X, X_i, -\infty < i < \infty\}$ be a sequence of independent and identically distributed random variables with $EX = 0$ and $EX^2 < \infty$, $\{a_i, -\infty < i < +\infty\}$ be a sequence of real numbers satisfying (1.3) for $\delta = 1$. Suppose $\{Y_n, n \ge 1\}$ is the moving average processes defined as (1.4). Then $\sum_{n=1}^{\infty} P\{|\sum_{j=1}^{n} Y_j| > \varepsilon n\} < \infty$ for all $\varepsilon > 0$.*

Using a method different from that in Li et al. [20], Chen et al. [6] obtained the complete convergence for the maximum sums, which extended the results in Katz [18], Baum and Katz [2] partly to moving average processes.

Theorem B. *Let $\{X, X_i, -\infty < i < \infty\}$ be a sequence of independent and identically distributed random variables with $EX = 0$ and $E|X|^{rp} < 0$ for $r \ge 1$, $1 \le p < 2$, and let $\{a_i, -\infty < i < +\infty\}$ be a sequence of real numbers satisfying (1.3) for $\delta = 1$ if $rp > 1$ and $0 < \delta < 1$ if $rp = 1$. Suppose $\{Y_n, n \ge 1\}$ is the moving average process defined as (1.4). Then*

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} Y_j\right| > \varepsilon n^{1/p}\right\} < \infty \quad \textit{for all } \varepsilon > 0.$$

In fact, Chen et al. [6] obtained their result under the dependent setup. And if we further assume that

$$a=\sum_{i=-\infty}^{\infty} a_i \neq 0, \tag{1.5}$$

using the symmetrical method and the comparison principle, the converse of Theorems A and B also holds. The argument is similar to the proof of the divergence part of the main result.

The goal of the present investigation is to establish a version of complete convergence for moving average processes associated to integral test and heavy tailed distributions.

Before the main results are stated, we should mention that the moment assumption in Theorem B, $E|X|^{rp}<\infty$ implies $\lim_{x\to\infty} x^{rp}P\{|X|>x\}=0$ and the converse is not necessary true. Of course, the condition $\lim_{x\to\infty} x^{rp}P\{|X|>x\}=c\in(0,\infty)$ does not imply the moment condition $E|X|^{rp}<\infty$.

The assumption of a random variable with the heavy tailed distribution of order rp which is more general than $\lim_{x\to\infty} x^{rp}P\{|X|>x\}=c\in(0,\infty)$ is introduced as the following:

First we recall that a measurable function $l(\cdot)$ is said to be slowly varying at infinity if it is positive on $[0,\infty)$ and $\lim_{x\to\infty} l(\lambda x)/l(x)=1$ for each $\lambda>0$. We say that a random variable X with the heavy tailed distribution of order rp if

$$\lim_{x\to\infty} xP\bigl\{|X|>\varphi(x)\bigr\}=c\in(0,\infty), \tag{1.6}$$

where $\varphi(x)=x^{\frac{1}{rp}}l(x)$, $r\ge 1$, $0<p<2$, and the function $l(x)$ is slowly varying at infinity. From Bingham et al. [3], we can obtain that $P\{|X|>x\}$ is a regularly varying function with index $-rp$, i.e. for any $\lambda>0$

$$\lim_{x\to\infty}\frac{P\{|X|>\lambda x\}}{P\{|X|>x\}}=\lambda^{-rp}.$$

We refer to Bingham et al. [3] for other equivalent definitions and for detailed and comprehensive study of properties of regularly varying functions and slowly varying functions.

Obviously, $\lim_{x\to\infty} x^{rp}P\{|X|>x\}=c\in(0,\infty)$ is the special case of a heavy tail distribution and (1.6) is equivalent to

$$\lim_{x\to\infty} x^rP\bigl\{|X|>\varphi\bigl(x^r\bigr)\bigr\}=c\in(0,\infty). \tag{1.6$'$}$$

Now we are ready to state the main results and however all those proofs will be detailed in the next section.

Theorem 1.1. *Assume that $\varphi(x)=x^{\frac{1}{rp}}l(x)$ for $0<p<2$, $r\ge 1$, where $l(x)>0$ $(x>0)$ is a slowly varying function at infinity, and $f>0$ is a non-decreasing function. Let $\{X,X_i,-\infty<i<\infty\}$ be a sequence of independent and identically distributed random variables satisfying (1.6) and $EX=0$ if $rp>1$, $\{a_i,-\infty<i<+\infty\}$ be a sequence of real numbers satisfying (1.5) and (1.3) for $\delta=1$ if $rp>1$ and $0<\delta<rp$ if $rp\le 1$. Suppose $\{Y_n,n\ge 1\}$ is the moving average process defined as in (1.4). Then for all $\varepsilon>0$*

$$\sum_{n=1}^{\infty} n^{r-2}P\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} Y_j\right|>\varepsilon\varphi\bigl(\bigl(nf(n)\bigr)^r\bigr)\right\}<\infty \quad or \quad =\infty \tag{1.7}$$

according to $\int_2^\infty \frac{dx}{xf^r(x)}<\infty$ or $=\infty$.

Please cite this article in press as: W. Li et al., Complete convergence for moving average processes associated to heavy-tailed distributions and applications, J. Math. Anal. Appl. (2014), http://dx.doi.org/10.1016/j.jmaa.2014.05.071

We conclude two applications from Theorem 1.1 for $r = 1$ and $r = 2$ as two corollaries which are related to Chover's type law of iterated logarithm for sequences and arrays respectively.

Corollary 1.1. *Under the conditions of Theorem 1.1, let $r = 1$, then with probability one we have*

$$\limsup_{n\to\infty} \frac{1}{\varphi(nf(n))}\left|\sum_{j=1}^{n} Y_j\right| = \begin{cases} 0, \\ \infty \end{cases} \Leftrightarrow \int_2^\infty \frac{dx}{xf(x)} \begin{cases} < \infty, \\ = \infty. \end{cases} \tag{1.8}$$

Furthermore

$$\limsup_{n\to\infty} \left|\frac{\sum_{j=1}^{n} Y_j}{\varphi(n)}\right|^{1/\log\log n} = e^{1/p} \quad a.s. \tag{1.9}$$

Corollary 1.2. *Let $\{X, X_{ni}, n \geq 1, -\infty < i < \infty\}$ be an array of independent and identically distributed random variables. Suppose X and $\{a_i, -\infty < i < \infty\}$ as Theorem 1.1 and $r = 2$. Set*

$$Y_{nj} = \sum_{i=-\infty}^{\infty} a_i X_{n,i+j}, \quad n \geq 1,\ 1 \leq j \leq n.$$

Then with probability one we have

$$\limsup_{n\to\infty} \frac{1}{\varphi((nf(n))^2)}\left|\sum_{j=1}^{n} Y_{nj}\right| = \begin{cases} 0, \\ \infty \end{cases} \Leftrightarrow \int_2^\infty \frac{dx}{xf^2(x)} \begin{cases} < \infty, \\ = \infty. \end{cases} \tag{1.10}$$

Furthermore

$$\limsup_{n\to\infty} \left|\frac{\sum_{j=1}^{n} Y_{nj}}{\varphi(n^2)}\right|^{1/\log\log n} = e^{1/(2p)} \quad a.s. \tag{1.11}$$

Remark 1.1. Formula (1.9) is called Chover's type law of the iterated logarithm. The pioneer work is due to Chover [8] for symmetrical stable random variables. For more general results, we refer to Chen and Hu [7] and the references therein.

Remark 1.2. In fact, Corollary 1.1 is a special case of Corollary 3.4 in Chen and Hu [7]. But we provide a different and interesting proof of the convergence part.

Remark 1.3. Corollary 1.2 is new even for partial sums of arrays of independent and identically distributed random variables. The strong law for arrays is studied by Hu et al. [16] firstly. There are more results can be found in Hu et al. [14] and the references therein.

2. Lemmas and proofs

We need the following lemmas in order to prove our main results. Throughout this section C represents a positive constant, which may be vary in different places.

Lemma 2.1. *Let X be the same as in Theorem 1.1, then*

(1) if $t < rp$, $E|X|^t I(|X| > x) \sim c_1 x^t P\{|X| > x\}$ for some $c_1 = c_1(t, rp) > 0$,
(2) if $t > rp$, $E|X|^t I(|X| \leq x) \sim c_2 x^t P\{|X| > x\}$ for some $c_2 = c_2(t, rp) > 0$,

where $a(x) \sim b(x)$ means $\lim_{x\to\infty} a(x)/b(x) = 1$.

Proof. Theses results follow from Theorem 1 on p. 273 of Feller [11] directly. □

Lemma 2.2 *(Rosenthal's type moment inequality). (See Petrov [21].) Let $t \geq 2$, $\{W_n, n \geq 1\}$ be a sequence of centered and independent random variables with $E|W_n|^t < \infty$ for all $n \geq 1$. Then*

$$E \max_{1\leq k\leq n}\left|\sum_{j=1}^{k} W_j\right|^t \leq C_t\left(\sum_{j=1}^{n} E|W_j|^t + \left(\sum_{j=1}^{n} EW_j^2\right)^{t/2}\right),$$

where the positive constant C_t depends only on t. In particular

$$E \max_{1\leq k\leq n}\left|\sum_{j=1}^{k} W_j\right|^2 \leq C_2\sum_{j=1}^{n} EW_j^2.$$

Lemma 2.3. *Let X as Theorem 1.1, $f > 0$ be a non-decreasing function with $\int_2^\infty \frac{dx}{xf^r(x)} < \infty$. Set $b_n = \varphi((nf(n))^r)$, then*

(1) if $t < rp$, $\sum_{n=1}^{\infty} n^{r-1}b_n^{-t}E|X|^t I(|X| > b_n) < \infty$,
(2) if $t > rp$, $\sum_{n=1}^{\infty} n^{r-1}b_n^{-t}E|X|^t I(|X| \leq b_n) < \infty$.

Proof. (1) By Lemma 2.1 and (1.6)′

$$\begin{aligned}\sum_{n=1}^{\infty} n^{r-1}b_n^{-t}E|X|^t I\big(|X| > b_n\big) &\leq C\sum_{n=1}^{\infty} n^{r-1}P\big\{|X| > b_n\big\} \leq C\sum_{n=1}^{\infty} n^{r-1}\big(nf(n)\big)^{-r}\\ &= C\sum_{n=1}^{\infty}\big(nf^r(n)\big)^{-1} \leq C\int_2^\infty \frac{dx}{xf^r(x)} < \infty.\end{aligned}$$

(2) By Lemma 2.1 and (1.6)′

$$\sum_{n=1}^{\infty} n^{r-1}b_n^{-t}E|X|^t I\big(|X| \leq b_n\big) \leq C\sum_{n=1}^{\infty} n^{r-1}P\big\{|X| > b_n\big\} < \infty. \qquad \square$$

Proof of Theorem 1.1. For the convergence part, we assume that $\int_2^\infty \frac{dx}{xf^r(x)} < \infty$ and set $b_n = \varphi((nf(n))^r)$. We will show that for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{r-2}P\left\{\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} Y_j\right| > \varepsilon b_n\right\} < \infty. \tag{2.1}$$

Set $Z_{nj} = X_jI(|X_j| > b_n)$ and $W_{nj} = X_jI(|X_j| \leq b_n)$ for $n \geq 1$ and $-\infty < j < \infty$. Obviously, $X_j = Z_{nj} + W_{nj}$. Note that for any $\varepsilon > 0$

$$\begin{aligned}P\left\{\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} Y_j\right| > \varepsilon b_n\right\} &\leq P\left\{\max_{1\leq k\leq n}\left|\sum_{i=-\infty}^{\infty} a_i\sum_{j=i+1}^{i+k} Z_{nj}\right| > \varepsilon b_n/2\right\}\\ &\quad + P\left\{\max_{1\leq k\leq n}\left|\sum_{i=-\infty}^{\infty} a_i\sum_{j=i+1}^{i+k} W_{nj}\right| > \varepsilon b_n/2\right\}.\end{aligned}$$

Hence to prove (2.1), it is enough to show that for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Z_{nj}\right| > \varepsilon b_n\right\} < \infty \tag{2.2}$$

and

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} W_{nj}\right| > \varepsilon b_n\right\} < \infty. \tag{2.3}$$

Since $0 < \delta \le 1$ and $\delta < rp$, by Markov's inequality, c_r-inequality, (1.3) and Lemma 2.3

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Z_{nj}\right| > \varepsilon b_n\right\} \\
&\quad\le C\sum_{n=1}^{\infty} n^{r-2} b_n^{-\delta} E\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} Z_{nj}\right|^{\delta} \\
&\quad\le C\sum_{n=1}^{\infty} n^{r-1} b_n^{-\delta} E|X|^{\delta} I\big(|X| > b_n\big) < \infty,
\end{aligned}$$

i.e. (2.2) holds.

In order to prove (2.3), we have to divide rp into three cases as following:

If $rp < 1$, by Markov's inequality, c_r-inequality, (1.3) and Lemma 2.3

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} W_{nj}\right| > \varepsilon b_n/2\right\} \\
&\quad\le C\sum_{n=1}^{\infty} n^{r-1} b_n^{-1} E|X| I\big(|X| \le b_n\big) < \infty,
\end{aligned}$$

i.e. in the case $0 < rp < 1$, (2.3) holds.

If $rp = 1$, note that $f(x) \to \infty$ and $P\{|X| > x\}$ is a regularly varying function with index -1, by the dominated convergence theorem

$$b_n^{-1} n\big|EXI\big(|X| \le b_n\big)\big| \le b_n^{-1} nE|X| I\big(|X| \le b_n\big) \le \int_0^1 nP\big\{|X| > xb_n\big\}dx \to 0.$$

If $rp > 1$, in this case $EX = 0$, by Lemma 2.1 and $(1.6)'$

$$\begin{aligned}
b_n^{-1} n\big|EXI\big(|X| \le b_n\big)\big| &= b_n^{-1} n\big|EXI\big(|X| > b_n\big)\big| \le b_n^{-1} nE|X| I\big(|X| > b_n\big) \\
&\le CnP\big\{|X| > b_n\big\} \le C/f^r(n) \to 0.
\end{aligned}$$

Hence in the case $rp \ge 1$, to prove (2.3), it is enough to prove that for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k} (W_{nj} - EW_{nj})\right| > \varepsilon b_n\right\} < \infty. \tag{2.4}$$

If $1 \le rp < 2$, by Markov's inequality, Hölder's inequality, Lemmas 2.2 and 2.3

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k}(W_{nj}-EW_{nj})\right| > \varepsilon b_n/2\right\} \\
&\quad\le C\sum_{n=1}^{\infty} n^{r-2} b_n^{-2} E\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k}(W_{nj}-EW_{nj})\right|^2 \\
&\quad\le C\sum_{n=1}^{\infty} n^{r-2} b_n^{-2}\left(\sum_{i=-\infty}^{\infty}|a_i|\right)\sum_{i=-\infty}^{\infty}|a_i| E\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}(W_{nj}-EW_{nj})\right|^2 \\
&\quad\le C\sum_{n=1}^{\infty} n^{r-1} b_n^{-2} E|X|^2 I\big(|X|\le b_n\big) < \infty.
\end{aligned}$$

If $rp \ge 2$, taking $t > rp$ large enough such that $r-2+t/2-t/p < -1$, by Markov's inequality, Hölder's inequality, Lemma 2.2

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k}(W_{nj}-EW_{nj})\right| > \varepsilon b_n/2\right\} \\
&\quad\le C\sum_{n=1}^{\infty} n^{r-2} b_n^{-t} E\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+k}(W_{nj}-EW_{nj})\right|^t \\
&\quad\le C\sum_{n=1}^{\infty} n^{r-2} b_n^{-t}\left(\sum_{i=-\infty}^{\infty}|a_i|\right)^{t-1}\sum_{i=-\infty}^{\infty}|a_i| E\max_{1\le k\le n}\left|\sum_{j=i+1}^{i+k}(W_{nj}-EW_{nj})\right|^t \\
&\quad\le C\sum_{n=1}^{\infty} n^{r-2} b_n^{-t}\left\{\big(nE|X|^2 I\big(|X|\le b_n\big)\big)^{t/2} + nE|X|^t I\big(|X|\le b_n\big)\right\}.
\end{aligned}$$

For any $s \in (0,1)$, $E|X|^{2-2s} < \infty$. Take s small enough such that $r-2+t/2-(1-s)t/p < -1$, and note that $b_n = (nf(n))^{1/p} l((nf(n))^r)$, hence

$$\sum_{n=1}^{\infty} n^{r-2} b_n^{-t}\left\{\big(nE|X|^2 I\big(|X|\le b_n\big)\big)\right\}^{t/2} \le C\sum_{n=1}^{\infty} n^{r-2+t/2} b_n^{-t} b_n^{st} < \infty,$$

and by Lemma 2.3

$$\sum_{n=1}^{\infty} n^{r-2} b_n^{-t} nE|X|^t I\big(|X|\le b_n\big) < \infty.$$

Then (2.4) follows from above two formulas.

For divergence part, we assume that $\int_2^{\infty}\frac{dx}{xf^r(x)} = \infty$. We will show that for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\max_{1\le k\le n}\left|\sum_{j=1}^{k} Y_j\right| > \varepsilon\varphi\big(\big(nf(n)\big)^r\big)\right\} = \infty. \tag{2.5}$$

By Lemma 2.2 in Chen [5], without loss of generality, we can assume that $f(x) \to \infty$. In fact, it is enough to show that for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\left|\sum_{j=1}^{n} Y_j\right| > \varepsilon\varphi\big(\big(nf(n)\big)^r\big)\right\} = \infty. \tag{2.6}$$

We will prove (2.6) by contradiction. Suppose that for some $\varepsilon_0 > 0$

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\left|\sum_{j=1}^{n} Y_j\right| > \varepsilon_0 \varphi\big(\big(nf(n)\big)^r\big)\right\} < \infty.$$

Let $\{X_i', -\infty < i < \infty\}$ be an independent copy of $\{X_i, -\infty < i < \infty\}$. Then the above also holds for $\{X_i', -\infty < i < \infty\}$ and hence

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} \big(X_j - X_j'\big)\right| > 2\varepsilon_0 \varphi\big(\big(nf(n)\big)^r\big)\right\} < \infty. \tag{2.7}$$

Set $a_{ni} = \sum_{j=1}^{n} a_{j-i}$. Note that $\sum_{i=-\infty}^{\infty} a_i \sum_{j=i+1}^{i+n} (X_j - X_j') = \sum_{i=-\infty}^{\infty} a_{ni}(X_i - X_i')$. By the comparison principle (see Lemma 6.5 of Ledoux and Talagrand [19]), (2.7) implies that

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\left|\sum_{i=[n/3]}^{[n/2]} a_{ni} \big(X_i - X_i'\big)\right| > 2\varepsilon_0 \varphi\big(\big(nf(n)\big)^r\big)\right\} < \infty. \tag{2.8}$$

Note that $a = \sum_{i=-\infty}^{\infty} a_i \neq 0$. Then for n large enough, $|a_{ni}| \geq |a|/2$ holds uniformly for $[n/3] \leq i \leq [n/2]$. By (2.8) and the comparison principle (see Lemma 6.5 of Ledoux and Talagrand [19]) again,

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\left|\sum_{i=[n/3]}^{[n/2]} \big(X_i - X_i'\big)\right| > 4\varepsilon_0 \varphi\big(\big(nf(n)\big)^r\big)/|a|\right\} < \infty. \tag{2.9}$$

By Lévy's inequality (see Proposition 2.3 of Ledoux and Talagrand [19])

$$\begin{aligned} &P\Big\{\max_{[n/3] \leq i \leq [2/n]} \big|\big(X_i - X_i'\big)\big| > 4\varepsilon_0 \varphi\big(\big(nf(n)\big)^r\big)/|a|\Big\} \\ &\quad \leq 2P\left\{\left|\sum_{i=[n/3]}^{[n/2]} \big(X_i - X_i'\big)\right| > 4\varepsilon_0 \varphi\big(\big(nf(n)\big)^r\big)/|a|\right\}. \end{aligned} \tag{2.10}$$

Note that $f(x) \to \infty$, so it is easy to show that

$$nP\big\{\big|X - X'\big| > \varphi\big(\big(nf(n)\big)^r\big)\big\} \to 0$$

and

$$n\varphi^{-2}\big(\big(nf(n)\big)^r\big) E\big(X - X'\big)^2 I\big(\big|X - X'\big| \leq \varphi\big(\big(nf(n)\big)^r\big)\big) \to 0$$

from (1.6)′ and Lemma 2.1. By Theorem 3.1 in Taylor et al. [22]

$$P\left\{\left|\sum_{i=[n/3]}^{[n/2]} \big(X_i - X_i'\big)\right| > 4\varepsilon_0 \varphi\big(\big(nf(n)\big)^r\big)/|a|\right\} \to 0. \tag{2.11}$$

By (2.10) and (2.11), for n large enough, we have

$$P\Big\{\max_{[n/3] \leq i \leq [2/n]} \big|\big(X_i - X_i'\big)\big| > 4\varepsilon_0 \varphi\big(\big(nf(n)\big)^r\big)/|a|\Big\} \leq 1/2.$$

Hence by Lemma 2.6 of Ledoux and Talagrand [19]

$$\sum_{i=[n/3]}^{[n/2]} P\{|(X_i - X_i')| > 4\varepsilon_0\varphi((nf(n))^r)/|a|\}$$
$$\leq 2P\Big\{\max_{[n/3]\leq i\leq [2/n]} |(X_i - X_i')| > 4\varepsilon_0\varphi((nf(n))^r)/|a|\Big\} \tag{2.12}$$

when n large enough. So by (2.9) and (2.12)

$$\sum_{n=1}^{\infty} n^{r-1} P\{|X - X'| > 4\varepsilon_0\varphi((nf(n))^r)/|a|\} < \infty,$$

which ensures that

$$\sum_{n=1}^{\infty} n^{r-1} P\{|X| > 8\varepsilon_0\varphi((nf(n))^r)/|a|\} < \infty.$$

But in the other hand, by (1.6)′ and $\int_2^\infty \frac{dx}{xf^r(x)} = \infty$, for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{r-1} P\{|X| > \varepsilon\varphi((nf(n))^r)\} = \infty$$

which leads a contradiction. Hence (2.5) holds completing the proof of Theorem 1.1. □

Proof of Corollary 1.1. The proof of the divergence part of (1.8) can be seen in Chen and Hu [7], we only prove the convergence part. Assume $\int_2^\infty \frac{dx}{xf(x)} < \infty$. By Lemma 2.2 in Chen [5], we can assume that $\limsup_{x\to\infty} f(2x)/f(x) < \infty$, it follows that

$$\limsup_{n\to\infty} \varphi(2nf(2n))/\varphi(nf(n)) < \infty.$$

By Theorem 1.1, for all $\varepsilon > 0$

$$\sum_{n=1}^{\infty} n^{-1} P\Bigg\{\max_{1\leq k\leq n}\Bigg|\sum_{j=1}^{k} Y_j\Bigg| > \varepsilon\varphi(nf(n))\Bigg\} < \infty.$$

Hence by Theorem 2.1 in Yang et al. [23]

$$\frac{1}{\varphi(nf(n))}\sum_{j=1}^{n} Y_j \to 0 \quad \text{a.s.}$$

For every $\delta > 0$, by taking $f(x) = \log^{1+\delta} x$ in (1.8),

$$\limsup_{n\to\infty} \big(\varphi(n\log^{1+\delta} n)\big)^{-1}\Bigg|\sum_{j=1}^{n} Y_j\Bigg| = 0 \quad \text{a.s.}$$

By the property of slowing varying function (see Bingham et al. [3]), we have for any $\delta' > 0$

$$\limsup_{n\to\infty} \big(\varphi(n)\log^{1/p+\delta/p+\delta'} n\big)^{-1}\Bigg|\sum_{j=1}^{n} Y_j\Bigg| = 0 \quad \text{a.s.},$$

which implies

$$\limsup_{n\to\infty}\left|\bigl(\varphi(n)\bigr)^{-1}\sum_{j=1}^{n}Y_j\right|^{1/\log\log n}\le e^{1/p+\delta/p+\delta'}\quad \text{a.s.}$$

Thus we have

$$\limsup_{n\to\infty}\left|\bigl(\varphi(n)\bigr)^{-1}\sum_{j=1}^{n}Y_j\right|^{1/\log\log n}\le e^{1/p}\quad \text{a.s.} \tag{2.13}$$

For every $\delta>0$, by taking $f(x)=\log^{1-\delta}x$ in (1.8),

$$\limsup_{n\to\infty}\bigl(\varphi\bigl(n\log^{1-\delta}n\bigr)\bigr)^{-1}\left|\sum_{j=1}^{n}Y_j\right|=\infty\quad \text{a.s.}$$

By the property of slowing varying function (see Bingham et al. [3]), we have for any $\delta'>0$

$$\limsup_{n\to\infty}\bigl(\varphi(n)\log^{1/p-\delta/p-\delta'}n\bigr)^{-1}\left|\sum_{j=1}^{n}Y_j\right|=\infty\quad \text{a.s.},$$

which implies

$$\limsup_{n\to\infty}\left|\bigl(\varphi(n)\bigr)^{-1}\sum_{j=1}^{n}Y_j\right|^{1/\log\log n}\ge e^{1/p-\delta/p-\delta'}\quad \text{a.s.}$$

Thus we have

$$\limsup_{n\to\infty}\left|\bigl(\varphi(n)\bigr)^{-1}\sum_{j=1}^{n}Y_j\right|^{1/\log\log n}\ge e^{1/p}\quad \text{a.s.} \tag{2.14}$$

Hence (1.9) holds from (2.13) and (2.14). The proof is completed. □

Proof of Corollary 1.2. Let $\{X_i,-\infty<i<\infty\}$ be a sequence of independent and identically distributed random variables, the common distribution is as X. Then by Theorem 1.1 and its proof, for all $\varepsilon>0$

$$\sum_{n=1}^{\infty}P\left\{\left|\sum_{j=1}^{n}Y_j\right|>\varepsilon\varphi\bigl(\bigl(nf(n)\bigr)^2\bigr)\right\}<\infty\quad\text{or}\quad=\infty$$

according to $\int_2^\infty\frac{dx}{xf^2(x)}<\infty$ or $=\infty$. Hence for all $\varepsilon>0$

$$\sum_{n=1}^{\infty}P\left\{\left|\sum_{j=1}^{n}Y_{nj}\right|>\varepsilon\varphi\bigl(\bigl(nf(n)\bigr)^2\bigr)\right\}<\infty\quad\text{or}\quad=\infty$$

according to $\int_2^\infty\frac{dx}{xf^2(x)}<\infty$ or $=\infty$. Note that $\{\sum_{j=1}^{n}Y_{nj},n\ge1\}$ is a sequence of independent random variables. By the Borel–Cantelli Lemma

$$\limsup_{n\to\infty} \frac{1}{\varphi((nf(n))^2)} \left|\sum_{j=1}^{n} Y_{nj}\right| = 0 \quad \text{or} \quad \infty \quad \text{a.s.}$$

according to $\int_2^\infty \frac{dx}{xf^2(x)} < \infty$ or $= \infty$.

The proof of (1.11) is similar to that of (1.9) and so we omit the details. □

Acknowledgments

The authors would like to thank the referees and the editors for the helpful comments and suggestions. Li's work is supported by National Natural Science Foundation of China (No. 61374067). Chen's work is supported by National Natural Science Foundation of China (No. 11271161). Hu's work is partially supported by National Science Council Taiwan under the grant number NSC 101-2118-M-007-001-MY2.

References

[1] Z. Bai, C. Su, On complete convergence for independent sums, Sci. China Ser. A 5 (1985) 399–412 (in Chinese).
[2] L.E. Baum, M. Katz, Convergence rates in the law of large numbers, Trans. Amer. Math. Soc. 120 (1965) 108–123.
[3] N.H. Bingham, C.M. Goldie, J.L. Teugels, Regular Variation, Cambridge University, New York, 1987.
[4] R.M. Burton, H. Dehling, Large deviations for some weakly dependent random processes, Statist. Probab. Lett. 9 (1990) 397–401.
[5] P. Chen, Limiting behavior of weighted sums with stable distribution, Statist. Probab. Lett. 60 (2002) 367–375.
[6] P. Chen, T.-C. Hu, A. Volodin, Limiting behavior of moving average processes under φ-mixing assumption, Statist. Probab. Lett. 79 (2009) 105–111.
[7] P. Chen, T.-C. Hu, Limiting behavior for random elements with heavy tail, Taiwanese J. Math. 16 (2012) 217–236.
[8] J. Chover, A law of the iterated logarithm for stable summands, Proc. Amer. Soc. 17 (1966) 441–443.
[9] P. Erdös, On a theory of Hsu and Robbins, Ann. Math. Stat. 20 (1949) 286–291.
[10] P. Erdös, Remark on my paper "On a theory of Hsu and Robbins", Ann. Math. Stat. 21 (1950) 138.
[11] W. Feller, An Introduction to Probability Theory and Its Applications, vol. II, third ed., Wiley, New York, 1966.
[12] A. Gut, Convergence rates for probabilities of moderate deviations for sums of random variables with multidimensional indices, Ann. Probab. 8 (1980) 298–313.
[13] P. Hsu, H. Robbins, Complete convergence and the law of large numbers, Proc. Natl. Acad. Sci. USA 33 (1947) 25–31.
[14] T.-C. Hu, P. Chen, N.C. Weber, A remark on the strong law for B-valued arrays of random elements, Bull. Aust. Math. Soc. 82 (2010) 31–43.
[15] T.-C. Hu, A. Rosalsky, D. Szynal, A. Volodin, On complete convergence for arrays of rowwise independent random elements in Banach spaces, Stoch. Anal. Appl. 17 (1999) 963–992.
[16] T.-C. Hu, N.C. Weber, On the rate of convergence in the strong law of large numbers for arrays, Bull. Aust. Math. Soc. 45 (1992) 279–282.
[17] I.A. Ibragimov, Some limit theorem for stationary processes, Theory Probab. Appl. 7 (1992) 349–382.
[18] M. Katz, The probability in the tail of a distribution, Ann. Math. Stat. 34 (1969) 312–318.
[19] M. Ledoux, M. Talagrand, Probability in Banach Spaces, Spring-Verlag, Berlin, 1991.
[20] D. Li, M.B. Rao, X. Wang, Complete convergence of moving average processes, Statist. Probab. Lett. 14 (1992) 111–114.
[21] V.V. Petrov, Limit Theorems of Probability Theory: Sequence of Independent Random Variables, Clarendon Press, Oxford, 1995.
[22] R.L. Taylor, R.F. Patterson, A. Bonorgnia, Weak law of large numbers for arrays of rowwise negatively dependent random variables, J. Appl. Math. Stoch. Anal. 14 (2001) 227–236.
[23] S. Yang, C. Su, K. Yu, A general method to the strong law of large numbers and its applications, Statist. Probab. Lett. 78 (2008) 794–803.

Please cite this article in press as: W. Li et al., Complete convergence for moving average processes associated to heavy-tailed distributions and applications, J. Math. Anal. Appl. (2014), http://dx.doi.org/10.1016/j.jmaa.2014.05.071