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Existence of standing waves for the complex Ginzburg–Landau equation

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ABSTRACT

We study the existence of standing wave solutions of the complex Ginzburg–Landau equation

$$\varphi_t - e^{i\theta}(\rho I - \Delta)\varphi - e^{i\gamma}|\varphi|^\alpha\varphi = 0 \tag{GL}$$

in $\mathbb{R}^N$, where $\alpha > 0$, $(N-2)\alpha < 4$, $\rho > 0$ and $\theta, \gamma \in \mathbb{R}$. We show that for any $\theta \in (-\pi/2, \pi/2)$ there exists $\varepsilon > 0$ such that (GL) has a non-trivial standing wave solution if $|\gamma - \theta| < \varepsilon$. Analogous result is obtained in a ball $\Omega \in \mathbb{R}^N$ for $\rho > -\lambda_1$, where λ_1 is the first eigenvalue of the Laplace operator with Dirichlet boundary conditions.

1. Introduction

The complex Ginzburg–Landau equation

$$\psi_t = z_1\Delta\psi + z_2|\psi|^\alpha\psi + z_3\psi, \tag{1.1}$$

for $\alpha = 2$, $z_1, z_2, z_3 \in \mathbb{C}$, with $\Re z_1 \geq 0$ was proposed independently by Diprima, Eckhaus, Segel [8] and Stewartson, Stuart [22] to model the interaction of plane waves in fluid flows and plays a central role in the study of the development of nonlinear instabilities in fluid dynamics. See [5,24] and the references cited therein for a discussion of various problems where the complex Ginzburg–Landau equation applies. Local

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(global for $\Re z_2 < 0$) well-posedness of (1.1) (for $\alpha > 0$) was derived in both $\mathbb{R}^N$ and a domain $\Omega \subset \mathbb{R}^N$, under various boundary conditions and assumptions on the parameters, see [9,10,16,20] and the references therein.

The existence of special solutions of (1.1) (holes, fronts, pulses, sources, sinks, etc.) is discussed in numerous works, see e.g. [6,14,15,18,19,21,24]. We look for standing wave solutions. Replacing φ by $e^{i\eta t}\varphi$ for some $\eta \in \mathbb{R}$ and rescaling the equation, we rewrite (1.1) as

$$\partial_t \varphi + e^{i\theta}(\rho\varphi - \Delta\varphi) = e^{i\gamma}|\varphi|^\alpha \varphi, \tag{1.2}$$

where $\rho \in \mathbb{R}$. Given $\omega \in \mathbb{R}$, a standing wave of the form $\varphi = e^{i\omega t}u(x)$ is a solution of (1.2) if and only if u satisfies

$$i\omega u + e^{i\theta}(\rho u - \Delta u) = e^{i\gamma}|u|^\alpha u. \tag{1.3}$$

Plane waves $\varphi = e^{i(kx-\omega t)}$, where $k, \omega \in \mathbb{R}$ are particular standing waves. It is easy to see that (1.2) admits plane wave solutions in $\mathbb{R}^N$ for all values of ρ, θ, γ and α. Stationary solutions are also standing waves of special kind. In the case of the nonlinear heat equation $\theta = \gamma = 0$ or $\theta = 0$, $\gamma = \pi$ then $\omega = 0$, so that Eq. (1.3) reduces to the nonlinear elliptic equation $\rho u - \Delta u = \pm|u|^\alpha u$. The case of the nonlinear Schrödinger equation $\theta = \pm\gamma = \pm\frac{\pi}{2}$ leads to the equation $(\rho \pm \omega)u - \Delta u = \pm|u|^\alpha u$.

We obtain here solutions that are different from these particular ones. In fact, using well known results of the theory of nonlinear elliptic equations for the case $\omega = 0$ and $\theta = \gamma$, we show the existence of nontrivial standing wave solutions for $\theta \approx \gamma$ by a perturbation argument, as we describe below.

Eq. (1.3) will be considered both in the whole space $\Omega = \mathbb{R}^N$ or in a ball $\Omega \subset \mathbb{R}^N$ with Dirichlet boundary condition, for $N \geq 1$. We suppose $\theta, \gamma \in (-\pi/2, \pi/2)$ and α subcritical, i.e.

$$0 < \alpha, \qquad (N-2)\alpha < 4, \tag{1.4}$$

which includes the relevant case $\alpha = 2$, for $N \leq 3$. For $\theta = \gamma$ and $\omega = 0$, (1.3) reduces to

$$\rho u - \Delta u - |u|^\alpha u = 0. \tag{1.5}$$

Consider first $\Omega = \mathbb{R}^N$, in which case we assume that $\rho > 0$. It was shown in [13] that (1.5) has a unique positive radially symmetric solution $U \in C^2(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$. ($C_0(\mathbb{R}^N)$ is the space of continuous functions which tend to zero at infinity.) In fact, $U \in H^2_{\mathrm{rad}}(\mathbb{R}^N)$, the subspace of radial functions of $H^2(\mathbb{R}^N)$. Note that (1.5) is phase invariant, i.e., $Ue^{i\beta} \in \mathbf{H}^2_{\mathrm{rad}}(\mathbb{R}^N)$ is also a solution for all $\beta \in \mathbb{R}$. Here and in the rest of this paper we consider real spaces composed of complex-valued functions, and distinguish them from real spaces of real-valued functions by using bold face typing.

Theorem 1.1. *Assume (1.4) holds and suppose $\rho > 0$. Let $U \in H^2_{\mathrm{rad}}(\mathbb{R}^N)$ be the unique positive radial solution of (1.5). Given $\theta \in (-\pi/2, \pi/2)$ and $\beta \in \mathbb{R}$ there exists $0 < \varepsilon < \min\{\pi/2 - \theta, \pi/2 + \theta\}$ and a C^1 mapping $g : (\theta - \varepsilon, \theta + \varepsilon) \to \mathbb{R} \times \mathbf{H}^2_{\mathrm{rad}}(\mathbb{R}^N)$, $g(\gamma) = (\omega_\gamma, u_\gamma)$, satisfying $\omega_\theta = 0$, $u_\theta = Ue^{i\beta}$ and such that $\varphi_\gamma = e^{i\omega_\gamma t}u_\gamma$ is a solution of (1.2).*

In the bounded domain case of the unitary ball Ω of $\mathbb{R}^N$, we suppose that

$$\rho > -\lambda_1, \tag{1.6}$$

where λ_1 is the first eigenvalue associate to the Laplace–Dirichlet operator in Ω. As in the case of the whole space, (1.5) admits a unique positive solution $U \in H^2(\Omega) \cap H^1_0(\Omega)$, which is radial and radially decreasing. The following result is analogous to Theorem 1.1.

Theorem 1.2. *Assume (1.4), (1.6) hold and let $U \in H^2(\Omega) \cap H_0^1(\Omega)$ be the positive solution of (1.5). Given $\theta \in (-\pi/2, \pi/2)$ and $\beta \in \mathbb{R}$ there exists $0 < \varepsilon < \min\{\pi/2 - \theta, \pi/2 + \theta\}$ and a C^1 mapping $g : (\theta - \varepsilon, \theta + \varepsilon) \to \mathbb{R} \times (\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega))$, $g(\gamma) = (\omega_\gamma, u_\gamma)$, satisfying $\omega_\theta = 0$, $u_\theta = Ue^{i\beta}$ and such that $\varphi_\gamma = e^{i\omega_\gamma t}u_\gamma$ is a solution of (1.2).*

In the proofs of Theorem 1.1 and Theorem 1.2 we apply the Implicit Function Theorem to $F(\omega, u, \gamma) = i\omega u + e^{i\theta}(\rho u - \Delta u) - e^{i\gamma}|u|^\alpha u = 0$ in a neighborhood of $\omega = 0$, $u = Ue^{i\beta}$ and $\gamma = \theta$. Analogous approach was considered in [3] to obtain standing wave solutions to (1.2) in a bounded domain for α small, where an eigenvector of the Laplace–Dirichlet operator is used as a starting point. Our point of view allows us to obtain solutions for α satisfying (1.4) and for the case of the whole space. We are lead to study the linearized operator $L_\beta = \partial_u F(0, Ue^{i\beta}, \theta)$ in an appropriate setting. In fact, it will be sufficient to consider $L = \partial_u F(0, U, \theta)$, see Section 5.

We address some comments about the hypothesis in Theorem 1.1 and Theorem 1.2.

Remark 1.3.

(1) The assumption $\theta \in (-\pi/2, \pi/2)$ yields an accretive linear operator associated to the problem and corresponds to $\Re z_1 > 0$ in (1.1). We also obtain $\gamma \in (-\pi/2, \pi/2)$, i.e., standing waves appear in the focusing case. In the defocusing case $\gamma \in (\pi/2, 3\pi/2)$, multiplying the equation by $\overline{\varphi}$ and integrating, we see that $\|\varphi(t)\|_{L^2(\mathbb{R}^N)}$ decreases in time. Thus there cannot be any non-trivial standing wave in that case.

(2) In Theorems 1.1 and 1.2, we obtain solutions for γ close to θ. This important restriction seems to be technical and it is reasonable to conjecture that the branches of solutions we construct could be extended to the whole interval $(-\pi/2, \pi/2)$.

(3) The restriction to radial solutions in Theorem 1.1 seems to be necessary in our proof. It ensures the compactness of the linear operator K introduced in the proof. It also ensures that $\ker L$ is one-dimensional, which allows for the application of the Implicit Function Theorem. As discussed in Section 3, $\ker L$ is $(N+1)$-dimensional in $L^2(\mathbb{R}^N)$.

(4) The assumption that Ω is a ball in Theorem 1.2 ensures that $\ker L$ is one-dimensional. We don't know if this is true in general. The standing waves in Theorem 1.2 can be constructed such that they are radially symmetric, see Remark 5.1.

This paper is organized as follows. In Section 2 we recall some well stablished properties of the positive solution U, both in the bounded and in the unbounded domain cases. A spectral analysis of the operator L is developed in Section 4 for the case where Ω is a ball, and in Section 3 when Ω is the whole space. Finally, in Section 5 we prove Theorem 1.1 and Theorem 1.2.

2. The starting point $\theta = \gamma$

In this section we recall some well known properties of solutions $u \in H_0^1(\Omega)$ of (1.5) which will be useful later, in the cases where Ω is a ball or the whole space

2.1. The case of a ball

Let Ω be a ball of $\mathbb{R}^N$ and we assume (1.6). Then (1.5) admits infinitely many real solutions and, in particular, one positive radially symmetric solution U [1]. (Non-radial complex solutions were obtained in [17].) Eq. (1.5) is phase invariant: if u solves (1.5) so does $e^{i\beta}u$ for all $\beta \in \mathbb{R}$.

The positive solution U, which was shown to be unique in [13], can be obtained by ode methods [2]. It can also be derived by solving the minimization problem

$$\min_{u\in S}\int_{\Omega}\rho|u|^2+|\nabla u|^2,\tag{2.1}$$

where

$$S=\left\{u\in H_0^1(\Omega),\ \int|u|^{\alpha+2}=1\right\}.\tag{2.2}$$

Using that $H^1(\Omega)$ is compactly injected in $L^{\alpha+2}(\Omega)$ one easily sees that (2.1) has a (unique) positive solution $\tilde{U}$. It is also clear that $U=k\tilde{U}$ solves (1.5) for a judicious choice of k. It then follows from standard symmetrization arguments that U is radial and radially decreasing.

One may also obtain U as a mountain pass solution, see [1]. Consider

$$E(u)=\frac{\rho}{2}\int|u|^2+\frac{1}{2}\int|\nabla u|^2-\frac{1}{\alpha+2}\int|u|^{\alpha+2}\tag{2.3}$$

and

$$\Gamma=\big\{\gamma\in C\big([0,1];H_0^1(\Omega)\big),\ \gamma(0)=0,\ \gamma(1)=u_1\big\},\tag{2.4}$$

where $E(u_1)<0$. Then $E(u)$ is well-defined for $u\in H_0^1(\Omega)$ and Γ is nonempty. In addition,

$$c=\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}E\big(\gamma(t)\big)\tag{2.5}$$

is a critical value of E such that $c=E(U)>0$ and $E'(U)=0$. Moreover, it can be easily shown that U is a ground state solution, i.e., $E(U)\le E(V)$ for all solution $V\neq 0$ of (1.5).

2.2. The case of the whole space

The general picture essentially remains unchanged for real solutions u of (1.5) in all $\mathbb{R}^N$ provided $\rho>0$. It is easy to see that the minimum in (2.1) is reached when $\Omega=\mathbb{R}^N$. Indeed, note that, due to Schwarz symmetrization, one may assume that there exists a minimizing sequence $\{u_j\}_{j\in\mathbb{N}}$ such that u_j is nonnegative, radial and radially decreasing. Let B_r be the ball of radius r of $\mathbb{R}^N$ centered at zero and let V_Nr^N be its volume. We have

$$V_Nr^Nu_j^2(r)\le\int_{B_R}|u_j|^2\le\|u_j\|_{L^2(\mathbb{R}^N)}$$

for all $r>0$. This shows that $u_j(r)$ decays as $r^{N/2}$ as $r\to\infty$ uniformly in j. Using that $H^1(B_R)$ is compactly injected in $L^{\alpha+2}$, a standard argument allow us to obtain a (nonnegative) solution u of (2.1). Then u satisfies $\rho u-\Delta u=\lambda u^{\alpha+2}$ for some $\lambda>0$ so that $U=\lambda^{1/\alpha}u$ solves (1.5) in $\mathbb{R}^N$. It is straightforward to see that U is a ground state, that is, U has the smallest energy E among the nontrivial solutions of (1.5).

For $N\ge 2$, we have the following.

Proposition 2.1. *Let $N\ge 2$ and denote $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ the space of radially symmetric functions of $H^1(\mathbb{R}^N)$. Let Γ be defined by (2.4) where $H_0^1(\Omega)$ is replaced by $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Then c given by (2.5) is a critical value of E and there exists a solution $U>0$ of (1.5) such that $E(U)=c$.*

Proof. The fact that $c > 0$ and that the Palais–Smale condition for E at the c level is a consequence of the fact that $H^1_{\rm rad}(\mathbb{R}^N)$ is compactly injected in $L^{\alpha+2}(\mathbb{R}^N)$ [23]. Moreover, we may take u_1 as $s\varphi$, where $\varphi \neq 0$ and s is large enough. Applying Theorem 2.1 of [1] we obtain that c is a critical value of E. To see that the corresponding critical point U is positive, let us observe that $c = \max\{E(tU),\ t \geq 0\} = \max\{E(t|U|),\ t \geq 0\}$. Furthermore, $E(tu_1) < 0$ for $t \geq 1$ and we can choose s large enough so that $E(s(tu_1 + (1-t)|U|)) < 0$ for all $t \in [0,1]$. Therefore, the polygonal path γ joining u_1, su_1, $s|U|$ and 0 belongs to Γ and $E(|U|) = \max\{E(u),\ u \in \Gamma\}$ if s is sufficiently large. This shows that $|U|$ is a critical point of E and thus $|U|$ solves (1.5). From the maximum principle, we obtain that $U > 0$. $\square$

Remark 2.2.

(1) For $N = 1$ the above proof does not work as $H^1_{\rm rad}(\mathbb{R})$ is no longer compactly injected in $L^{\alpha+2}(\mathbb{R})$.
(2) For $N \geq 2$, it is easy to see that the solution U obtained in Proposition 2.1 is a ground state. For a connection between solutions obtained from constrained minimization and mountain-pass solutions in a more general framework, see [12].
(3) It is likely that Proposition 2.1 and the above remark are also valid for $N = 1$ but this would require, in the proof of the mountain pass theorem, a version of the deformation lemma for radial decreasing functions.
(4) In [13] it is also shown the uniqueness of positive radially symmetric solutions and in [23] it is proven that it decays exponentially. (For an alternative variational characterization of U involving the so-called Gagliardo–Nirenberg quotient, see [25, Proposition 2.6].)

For Ω either the unitary ball or the whole space, we consider the linearized operator

$$Lv = \rho v - \Delta v - U^\alpha v - \alpha U^\alpha \Re v, \tag{2.6}$$

where U is the positive solution of (1.5). More precisely, we set $D(L) = \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega)$ and define $L : D(L) \subset \mathbf{L}^2(\Omega) \to \mathbf{L}^2(\Omega)$ by (2.6). Then $Lv = L_+\Re v + iL_-\Im v$ where

$$L_+v = \rho v - \Delta v - (\alpha+1)|U|^\alpha v, \tag{2.7}$$

$$L_-v = \rho v - \Delta v - |U|^\alpha v. \tag{2.8}$$

We study below the operators L_+ and L_-.

3. The linearized operator: the case $\Omega = \mathbb{R}^N$

In the case $\Omega = \mathbb{R}^N$, under a suitable rescaling we may assume that $\rho = 1$ in (2.6). We want to show that L_+ is an injective operator when restricted to the space $V \overset{\rm def}{=} L^2_{\rm rad}(\mathbb{R}^N)$ of radially symmetric and square integrable functions. We define $D(L_+) = H^2(\mathbb{R}^N) \cap V$ and consider $L_+ : D(L_+) \subset V \to V$ given by (2.7).

Set $\sigma = U^\alpha$ and denote V_σ the space $L^2(\mathbb{R}^N, \sigma dx)$, where dx is the usual Lebesgue measure. We also introduce $K : V_\sigma \to V_\sigma$ such that for $v \in V_\sigma$

$$Kv = (\alpha+1)(I-\Delta)^{-1}U^\alpha v. \tag{3.1}$$

We have that K is a positive, symmetric operator. Using that U decays to zero at infinity, a standard argument shows that K is compact. Denote $\{\varphi_j\}_{j\in\mathbb{N}}$ the orthonormal basis of eigenvectors of K and $\{\mu_j\}_{j\in\mathbb{N}}$ the corresponding set of eigenvalues. Then $\mu_j > 0$ and $K\varphi_j = \mu_j\varphi_j$ is equivalent to

$$\varphi_j - \Delta\varphi_j = \frac{\alpha+1}{\mu_j} U^\alpha \varphi_j. \tag{3.2}$$

Note that, up to a normalization, $\varphi_1 = U$ and $\mu_1 = \alpha + 1$. Setting

$$c = \int |U|^2 + |\nabla U|^2 = \int U^{\alpha+2}, \tag{3.3}$$

we will now prove that $\mu_2 \leq 1$. This is a consequence of the fact that U satisfies

$$\int |U|^2 + |\nabla U|^2 \leq \int |u|^2 + |\nabla u|^2,$$

for all $u \in S_c$, where

$$S_c = \left\{ u \in H_0^1(\Omega), \int |u|^{\alpha+2} = c \right\}. \tag{3.4}$$

Lemma 3.1. *$\mu_2 \leq 1$.*

Proof. Let φ_2 be an eigenvector of K associated to μ_2, so that

$$\int U^{\alpha+1}\varphi_2 = \int \sigma\varphi_1\varphi_2 = 0. \tag{3.5}$$

Consider $G : \mathbb{R}^2 \to \mathbb{R}$ such that

$$G(s,t) = \int \left|(1+s)U + t\varphi_2\right|^{\alpha+2}.$$

Then G is C^2 and

$$\partial_s G(s,t) = (\alpha+2) \int \left|(1+s)U + t\varphi_2\right|^\alpha \big((1+s)U + t\varphi_2\big)U,$$

$$\partial_t G(s,t) = (\alpha+2) \int \left|(1+s)U + t\varphi_2\right|^\alpha \big((1+s)U + t\varphi_2\big)\varphi_2,$$

so that $G(0,0) = c$, $\partial_s G(0,0) = (\alpha+2)\int U^{\alpha+2}$, $\partial_t G(0,0) = (\alpha+2)\int U^{\alpha+1}\varphi_2 = 0$, see (3.3), (3.5). Using the Implicit Function Theorem, we see that there exists $\varepsilon > 0$ and a C^1 function $t \to s(t)$ defined for $|t| < \varepsilon$ such that $s(0) = 0$ and $G((1+s(t))U + t\varphi_2) = c$. Moreover, $s'(0) = -(\partial_s G(0,0))^{-1}\partial_t G(0,0) = 0$.

Set $w(t) = s(t)U + t\varphi_2$. From (3.3) and the Taylor–Lagrange formula we get

$$\begin{aligned} c = \int |U+w|^{\alpha+2} &= \int U^{\alpha+2} + (\alpha+2)\int U^{\alpha+1} w \\ &\quad + (\alpha+2)(\alpha+1) w^2 \int_0^1 \int |U + \sigma w|^\alpha (1-\sigma)\, d\sigma \\ &= c + (\alpha+2)\int U^{\alpha+1}w + \frac{(\alpha+2)(\alpha+1)}{2} w^2 U^\alpha \\ &\quad + (\alpha+2)(\alpha+1) w^2 \int_0^1 \int \big(|U+\sigma w|^\alpha - U^\alpha\big)(1-\sigma)\, d\sigma. \end{aligned}$$

Using that $s = O(t^2)$ and (3.5) it follows that

$$2sc + (\alpha+1)t^2 \int U^\alpha \varphi_2^2 = o(t^2). \tag{3.6}$$

Note that $s < 0$ for t small. Furthermore, using again that $s = O(t^2)$, from (3.3), (3.2), (3.5) we get

$$\begin{aligned}\int |U+w|^2 + \left|\nabla(U+w)\right|^2 &= \int |U|^2 + |\nabla U|^2 + 2\int U(I-\Delta)w + \int w(I-\Delta)w \\ &= c + 2sc + t^2 \int \varphi_2(I-\Delta)\varphi_2 + o(t^2) \\ &= c + 2sc + t^2 \frac{\alpha+1}{\mu_2} \int U^\alpha \varphi_2^2 + o(t^2). \end{aligned}\tag{3.7}$$

Using the minimization characterization of U and the fact that Schwarz symmetrization decreases the H^1 norm. It follows then from (3.6) that

$$c \le \int |U+w|^2 + \left|\nabla(U+w)\right|^2 = c + 2sc\left(1-\mu_2^{-1}\right) + o(s).$$

This shows that $\mu_2 \le 1$. □

We observe that the bound $\mu_2 \le 1$ is also a consequence of the fact that U is a mountain-pass solution (for $N \ge 2$). We present a simple proof below, which uses the specific form of the function $E(u)$. For the proof that general critical points of mountain-pass type have Morse index equal to one, see [11].

Alternative proof of Lemma 3.1 for $N \ge 2$. We first remark that for $k > 1$ large enough $\gamma_0(t) = ktU \in \Gamma$, see (2.4). In addition, $\max_{u\in\gamma_0} E(u) = E(U)$.

We argue by contradiction and suppose that $\mu_2 > 1$. We get from (3.2) that

$$\langle L_+\varphi_2, \varphi_2\rangle = (\alpha+1)\left(\mu_2^{-1} - 1\right) \int U^\alpha \varphi_2^2 < 0. \tag{3.8}$$

Consider now the plane P containing U and $\operatorname{span}\{\varphi_2\}$. Given $\delta > 0$, let

$$\gamma_1 = \left\{U - \delta(\cos tU + \sin t\varphi_2),\ t \in [0,\pi]\right\} \tag{3.9}$$

be an arc of circle in P joining $(1-\delta)U$ and $(1+\delta)U$. For $u \in \gamma_1$, we have that

$$E(u) = E(U) + E'(U)(u-U) + \frac{1}{2}\left\langle L_+(u-U), u-U\right\rangle + o\left(\delta^2\right). \tag{3.10}$$

Moreover, using that $\langle L_+U, \varphi_2\rangle = -\alpha\langle U, \varphi_2\rangle_\sigma = 0$ we get

$$\left\langle L_+(u-U), u-U\right\rangle = \delta^2\left(\cos^2 t\langle L_+U, U\rangle + \sin^2 t\langle L_+\varphi_2, \varphi_2\rangle\right) < 0. \tag{3.11}$$

Using this, (3.10) and that $E'(U) = 0$, we see that we can choose δ small enough so that $E(u) < E(U)$ for all $u \in \gamma_1$.

Let now γ be the curve obtained by replacing the path of γ_0 going from $(1-\delta)U$ to $(1+\delta)U$ by γ_1. Then $\gamma \in \Gamma$ and $\max_{u\in\gamma} E(u) < E(U)$, leading to a contradiction. This shows that $\mu_2 \le 1$. □

Lemma 3.2. *Suppose $L_+\varphi = 0$, $\varphi \ne 0$. Then there exists a unique $r^* > 0$ such that $\varphi(r^*) = 0$.*

Proof. Let B_R be the ball of radius R of $\mathbb{R}^N$. For $v \in L^2_{\text{rad}}(B_R)$ let $u \in H^2_{\text{rad}}(B_R) \cap H^1_0(B_R)$ satisfy $(I-\Delta)u = (\alpha+1)U^\alpha v$. We define $K^R : L^2_{\text{rad}}(B_R) \to L^2_{\text{rad}}(B_R)$ such that $K^R v = u$. Then K^R is a compact operator, which is symmetric and positive for the scalar product

$$\langle u, v\rangle_{\sigma,R} = \int\limits_{B_R} U^\alpha uv. \tag{3.12}$$

Denote $\{\varphi_j^R\}_{j\in\mathbb{N}}$ an orthonormal basis of eigenvectors of K^R, associated to the set $\{\mu_j^R\}_{j\in\mathbb{N}}$ of eigenvalues, so that

$$(I-\Delta)\varphi_j^R = \frac{\alpha+1}{\mu_j^R} U^\alpha \varphi_j^R \tag{3.13}$$

and

$$\int\limits_{B_R} U^\alpha \varphi_i^R \varphi_j^R = \delta_{ij}. \tag{3.14}$$

We recall the Courant minimax characterization

$$\mu_j^R = \inf_{\substack{u\in V_j^R \\ \|u\|_{H^1_0(B_R)}=1}} (\alpha+1)\int\limits_{B_R} U^\alpha u^2 = \sup_{V\in S_j^R} \inf_{\substack{u\in V \\ \|u\|_{H^1_0(B_R)}=1}} (\alpha+1)\int\limits_{B_R} U^\alpha u^2,$$

where $V_j^R = \text{span}\{\varphi_1, \varphi_2, \dots, \varphi_j\}$ and S_j^R is the set of all subspaces of $H^1_0(B_R)$ of dimension j. It follows that for $j \geq 2$

$$\mu_j^R < \mu_1^R < \mu_1, \tag{3.15}$$

where μ_1 is the first eigenvalues of K given by (3.1). Moreover, given $R' < R$ denote $\tilde{V}_j^R$ the subspace of $H^1_0(B_R)$ obtained by extending to B_R the functions of $V_j^{R'}$ as zero outside $B_{R'}$. Then $\tilde{V}_j^R \in S_j^R$ and

$$\mu_j^{R'} = \inf_{\substack{u\in V_j^{R'} \\ \|u\|_{H^1_0(B_{R'})}=1}} (\alpha+1)\int\limits_{B_{R'}} U^\alpha u^2 = \inf_{\substack{u\in \tilde{V}_j^R \\ \|u\|_{H^1_0(B_R)}=1}} (\alpha+1)\int\limits_{B_R} U^\alpha u^2 \leq \mu_j^R.$$

In this way, there exists μ_j^∞ such that

$$\mu_j^R \nearrow \mu_j^\infty \leq \mu_1 \tag{3.16}$$

as $R \to \infty$. We extend $\varphi_j^R(r) = 0$ for $r > R$. Using (3.13) and (3.14) we see that

$$\int |\varphi_j^R|^2 + |\nabla \varphi_j^R|^2 = \frac{\alpha+1}{\mu_j^R} \int\limits_{B_R} U^\alpha |\varphi_j^R|^2 = \frac{\alpha+1}{\mu_j^R}. \tag{3.17}$$

It follows then from (3.16) that $\{\varphi_j^R\}_{R\geq\underline{R}}$ is uniformly bounded in $H^1(\mathbb{R}^N)$ for all $\underline{R} > 0$. Upon considering a subsequence, we may write that there exists φ^∞ in $H^1(\mathbb{R}^N)$ such that $\varphi_j^R \rightharpoonup \varphi_j^\infty$ weakly in $H^1(\mathbb{R}^N)$ as $R \to \infty$. Using that $U(r) \underset{r\to\infty}{\longrightarrow} 0$, we readily obtain from (3.13) and (3.14) that

$$(I-\Delta)\varphi_j^\infty = \frac{\alpha+1}{\mu_j^\infty} U^\alpha \varphi_j^\infty$$

with

$$\int\limits_{\mathbb{R}^N} U^\alpha \varphi_i^\infty \varphi_j^\infty = \delta_{ij}.$$

Thus, μ_j^∞ is an eigenvalue of K, associated to φ_j^∞. Note that $\varphi_1^\infty > 0$ so that $\mu_1^\infty = \mu_1$. It follows that

$$\mu_j^\infty = \mu_j \tag{3.18}$$

for some $j \geq 2$. Suppose now that $L_+\varphi = 0$ so that $K\varphi = \varphi$. Thus φ is an eigenvector of K having 1 as eigenvalue. As $\mu_1 = \alpha + 1$ and because of Lemma 3.1, we must have $\mu_2 = 1$. In particular, φ is not a first eigenvector and it must change sign. We will show that it changes sign only once. We argue by contradiction and assume that ρ and R are the two first zeroes of φ such that $0 < \rho < R$. Then

$$K^R\varphi = \varphi \quad \text{and} \quad 1 = \mu_j^R \tag{3.19}$$

for some j. As φ changes sign on $(0, R)$, we have $j \geq 2$, so that $\mu_2^R \geq 1$. Let us show that μ_2^R strictly grows with R. Assume that $\mu_2^{R'} = \mu_2^R$ for some $R' < R$. Then $\mu_2^\rho = \mu_2^R$ for all $R' < \rho < R$. Now using (3.13) for φ_2^ρ and φ_2^R in B_ρ and as we deal with radial functions it is easy to see that

$$\varphi_2^R(\rho)\partial_r\varphi_2^\rho(\rho) = 0,$$

so that $\varphi_2^R(\rho) = 0$ for all $\rho \in [R', R]$. This implies $\varphi_2^R = 0$ which gives a contradiction. Therefore $\mu_2^\infty > \mu_2^R \geq 1$. From (3.18) we get that $1 < \mu_2^\infty \leq \mu_2$. But this contradicts Lemma 3.1, showing that $\varphi(r)$ has a single zero $r^* > 0$. □

We next present the ingenious argument of [4] to show that L_+ is injective.

Lemma 3.3. L_+ *is injective.*

Proof. We argue by contradiction and assume that there exists $\varphi \neq 0$ such that $L_+\varphi = 0$. Using Lemma 3.2, we may assume that there exists $r^* > 0$ such that $\varphi(r) > 0$ for $r < r^*$ and $\varphi(r) < 0$ for $r > r^*$. Set now

$$\eta(x) = U(x) + \frac{\alpha}{2} x \cdot \nabla U(x).$$

Since U decays exponentially, $\eta \in H^1_{\text{rad}}(\mathbb{R}^N)$. Moreover, a straightforward calculation yields

$$L_+\eta = -\alpha U.$$

Define $w = U^\alpha(r^*)\eta - U$ and $z = L_+w$. Then $z(r) = \alpha U(r)(U^\alpha(r) - U^\alpha(r^*))$ so that $z(r) > 0$ for $r < r^*$ and $z(r) < 0$ for $r > r^*$. Hence, $z(r)\varphi(r) > 0$ for $r \neq r^*$. However, this is in contradiction with the fact that

$$\langle \varphi, z\rangle = \langle \varphi, L_+w\rangle = \langle L_+\varphi, w\rangle = 0.$$

This shows that L_+ is injective. □

Using decomposition in spherical harmonics, in [25] and in [4] it is proved that the complete kernel of L_+ in $L^2(\mathbb{R}^N)$ is $\ker L_+ = [\partial_1 U, \partial_2 U, \ldots, \partial_N U]$. Note that $\partial_j U$ is not a radial function.

We may now characterize the kernel of L given by (2.6).

Proposition 3.4. *We have* $\ker L = [iU]$.

Proof. If $v \in \ker L$ then $\Re v \in \ker L_+$ and $\Im v \in \ker L_-$. It follows from Lemma 3.3 that $\Re v = 0$. Moreover, if $\varphi \in \ker L_-$ then φ is an eigenvalue of K given by (3.1), associated to $\mu = \alpha + 1$. But $\alpha + 1$ is the first eigenvalue of K and $KU = (\alpha + 1)U$. Hence $\ker L_- = [U]$ so that $\ker L = [iU]$. □

4. The linearized operator: the case of a ball

Let $\Omega \subset \mathbb{R}^N$ be the unitary ball and suppose (1.6) holds. Let U be the unique positive solution of (1.5) and let L be given by (2.6). Then $v \in \ker L$ if and only if $\Re v \in \ker L_+$ and $\Im v \in \ker L_-$. Since $L_- U = 0$ and $U > 0$, it follows that $\ker L_- = [U]$ is a one-dimensional subspace. We will now show that L_+ is injective. This is proved in [7] for $\rho = 0$. For the reader's convenience, we reproduce the arguments here. The two preliminary results, Lemma 4.1 and Lemma 4.2 hold in fact for $\rho > -\lambda_1$ and will be useful in the proof of the general case.

For $x \in \mathbb{R}^N$ write $x = (t, y)$, where $t \in \mathbb{R}$, $y \in \mathbb{R}^{N-1}$, if $N > 1$ or $x = t$ if $N = 1$. Set $\Omega^* = \{x \in \Omega, t < 0\}$, $D(L_+^*) = H^2(\Omega^*) \cap H_0^1(\Omega^*)$ and $L_+^* : D(L_+^*) \subset L^2(\Omega^*) \to L^2(\Omega^*)$ be given by (2.7).

Lemma 4.1. *We have* $\lambda_1^* = \lambda_1(L_+^*) > 0$.

Proof. Let $v = \partial_t U$. It is well known that $v > 0$ over Ω^* with $v > 0$ over $\Gamma^* = \{x \in \overline{\Omega^*},\ |x| = 1\}$. Moreover, taking the derivative with respect to t in (1.5) we see that $L_+^* v = 0$. Consider u_1 a positive eigenvector of L_+^*, so that $L_+^* u_1 = \lambda_1^* u_1$. Then

$$\lambda_1^* \int_{\Omega^*} u_1 v = \int_{\Omega^*} v L_+^* u_1 = -\int_{\partial\Omega^*} v \partial_\eta u_1 > 0. \tag{4.1}$$

This shows that $\lambda_1^* > 0$. □

As a consequence, we have the following.

Lemma 4.2. *Let v satisfy $L_+ v = 0$. Then v is radially symmetric.*

Proof. If $v \in \ker L_+$ then $v \circ R \in \ker L_+$ for all unitary transformation R. It thus suffices to show that $v(t, y)$ is symmetric with respect to t. Define $\psi(x) = v(t, y) - v(-t, y)$. Then $L_+^* \psi = 0$, with $\psi = 0$ over $\partial\Omega^*$. It follows from Lemma 4.1 that $\psi = 0$. This ends the proof. □

Lemma 4.3. *Suppose $\rho = 0$. Then the operator L_+ given by (2.8) is injective.*

Proof. Let $v \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfy $L_+ v = 0$. Then

$$0 = \int_\Omega L_+ v U = \int_\Omega L_+ U v = -\alpha \int_\Omega U^{\alpha+1} v. \tag{4.2}$$

Consider now the Pohozaev function $\psi = x \cdot \nabla U$. We have that $\partial_j \psi = \partial_j U + x \cdot \nabla \partial_j U$, so that $\partial_{jj}^2 \psi = 2\partial_{jj}^2 U + x \cdot \nabla \partial_{jj}^2 U$. Thus, $\Delta\psi = 2\Delta U + x \cdot \Delta\nabla U$. In addition, we get from (1.5) that $\Delta\nabla U + (\alpha+1)U^\alpha \nabla U = 0$, and so

$$\Delta\psi = 2\Delta U - (\alpha + 1)U^\alpha x \cdot \nabla U = -2U^{\alpha+1} - (\alpha + 1)U^\alpha \psi. \tag{4.3}$$

It follows from (4.3) and (4.2) that

$$0 = 2\int_{\Omega} U^{\alpha+1}v = \int_{\Omega} L_+\psi v = \int_{\partial\Omega} \psi\partial_\eta v = \int_{\partial\Omega} \partial_\eta U\partial_\eta v. \tag{4.4}$$

We conclude from Lemma 4.2 and Hopf's strong maximum principle that $v'(1) = 0$. Therefore, $v = 0$. □

Remark 4.4. Following [7], Lemma 4.3 holds in the case $N = 2$ and Ω any regular bounded star shaped domain.

The proof that L_+ injective in the case $\rho \neq 0$ follows the arguments of [4], see Lemma 3.3.

Lemma 4.5. *Assume $\rho > -\lambda_1$, $\rho \neq 0$. Then L_+ is injective.*

Proof. Let

$$\eta(r) = 2U(r) + \alpha rU'(r). \tag{4.5}$$

A straightforward calculation gives that

$$L_+\eta = -2\rho\alpha U. \tag{4.6}$$

We want to show that $\ker L_+ = \{0\}$. We argue by contradiction and assume that there exists $\varphi_2 \neq 0$ such that $L_+\varphi_2 = 0$. Since U is a mountain-pass solution of (1.5), we know that $\lambda_2(L_+) \geq 0$, see [11]. Since $\lambda_1(L_+) < 0$, we see that φ_2 is an eigenvector associated to the second eigenvalue $\lambda_2 = 0$. By Lemma 4.2, φ_2 is radial. Using standard comparison arguments, it is easy to see that φ_2 has a single zero r_0 in $(0, 1)$. For $b \in (0, 1)$, define

$$g(r) = \begin{cases} 1 & 0 \leq r < b, \\ 2\frac{1-r}{1-b} - \frac{(1-r)^2}{(1-b)^2} & b \leq r \leq 1. \end{cases} \tag{4.7}$$

Then $g''(r) = g'(r) = 0$ if $r < b$. For $r > b$,

$$g'(r) = \frac{2(b-r)}{(1-b)^2}, \tag{4.8}$$

$$g''(r) = -\frac{2}{(1-b)^2}. \tag{4.9}$$

We remark that we can choose b close enough to 1 so that

$$b > r_0 \tag{4.10}$$

and

$$U^\alpha(r) < U^\alpha(r_0)g(r) \tag{4.11}$$

for $r > b$. Set now $w(r) = g(r)\eta(r)$, see (4.5). It follows from (4.6) that

$$L_+w = gL_+\eta - 2\nabla g \cdot \nabla\eta - \eta\Delta g = -2g\rho\alpha U - 2\nabla g \cdot \nabla\eta - \eta\Delta g. \tag{4.12}$$

Thus $L_+w = -2\rho\alpha U$ if $|x| < b$.

For $|x| > b$, we get from (4.5), (4.8) and (4.9) that

$$\nabla g(r) \cdot \nabla \eta(r) = g'(r)\eta'(r) = \frac{2(b-r)}{(1-b)^2}\big((2+\alpha)U'(r) + \alpha r U''(r)\big) \tag{4.13}$$

and that

$$\Delta g = g'' + \frac{N-1}{r} g' = -\frac{2}{(1-b)^2} - \frac{2(N-1)(r-b)}{r(1-b)^2}. \tag{4.14}$$

Defining $h = -2\nabla g \cdot \nabla \eta - \eta \Delta g$ we get from (4.5), (4.13) and (4.14) that there exists $K > 0$ such that

$$(1-b)^2 h(r) \le 4U(r) + 2\alpha r U'(r) + K(r-b).$$

Since $U(1) = 0$ and $U'(1) < 0$, $1-b$ can be taken eventually smaller so that

$$h(r) < 0 \quad \text{for } r > b. \tag{4.15}$$

Set now $t = U^\alpha(r_0)/(2\rho)$ and $z = L_+(-U + tw)$. Hence, by (4.12) we get

$$z = \alpha U^{\alpha+1} + t(-2g\rho\alpha U + h). \tag{4.16}$$

Let us show that z and φ_2 have the same sign. For $r < b$ we use that $g = 1$ and $h = 0$ to get

$$z(r) = \alpha U(r)\big(U^\alpha(r) - U^\alpha(r_0)\big).$$

It follows that $z(r) > 0$ if $r < r_0$ and $z(r) < 0$ for $r \in (r_0, b)$. In addition, using (4.15) and (4.11), we get for $r > b$ that

$$z(r) < \alpha U(r)\big(U^\alpha(r) - gU^\alpha(r_0)\big) < 0.$$

We see then that $z(r)\varphi_2(r) > 0$ for $r \ne r_0$. But

$$\langle \varphi_2, z\rangle = \big\langle \varphi_2, L_+(-U + \beta w)\big\rangle = \langle L_+\varphi_2, -U + \beta w\rangle = 0,$$

giving a contradiction. This shows that L_+ is injective. □

We present now the main result of this section.

Proposition 4.6. *We have* $\ker L = [iU]$.

Proof. Let $v \in \ker L$. Then $\Re v \in \ker L_+$ and $\Im v \in \ker L_-$. It follows from Lemma 4.3 and Lemma 4.5 that $\Re v = 0$. Moreover, as discussed in the beginning of this section $\ker L_- = [U]$. This closes the proof. □

5. Proofs of Theorem 1.1 and Theorem 1.2

In this section we denote $\mathbf{L}^p(\Omega)$ the real Banach space whose elements are complex-valued functions. In particular, $\mathbf{L}^2(\Omega)$ is a Hilbert space for the scalar product

$$(u, v) = \Re \int_\Omega u\bar{v}. \tag{5.1}$$

Accordingly, $\mathbf{H}^m(\Omega)$ denotes a real Hilbert space having complex elements.

Proof of Theorem 1.1. For a fixed $\theta \in (-\pi/2, \pi/2)$ set $X = \mathbb{R} \times \mathbf{H}^2_{\mathrm{rad}}(\mathbb{R}^N)$ and $F : (-\pi/2, \pi/2) \times X \to \mathbf{L}^2_{\mathrm{rad}}(\mathbb{R}^N)$ such that

$$F(\gamma, \omega, u) = \rho u - \Delta u - e^{i(\gamma-\theta)}|u|^\alpha u - i\omega e^{-i\theta} u. \tag{5.2}$$

Note that F is well defined due to Sobolev embedding $\mathbf{H}^2(\mathbb{R}^N) \hookrightarrow \mathbf{L}^{2\alpha+2}(\mathbb{R}^N)$.

Then $\varphi_\gamma = e^{i\omega_\gamma t}u_\gamma$ is a solution of (1.2) if and only if $F(\gamma, \omega_\gamma, u_\gamma) = 0$. Note that $F(\theta, 0, Ue^{i\beta}) = F(\theta, g(\theta)) = 0$. In addition, it is immediate to see that F is a C^1 function such that

$$\begin{aligned}
&\frac{\partial F}{\partial \omega}(\gamma, \omega, u)\mu = -ie^{-i\theta} u\mu, \\
&\frac{\partial F}{\partial u}(\gamma, \omega, u)v = \rho v - \Delta v - e^{i(\gamma-\theta)}\big[|u|^\alpha v + \alpha|u|^{\alpha-2}u\Re(\bar{u}v)\big] - i\omega e^{-i\theta} v.
\end{aligned}$$

By the surjective form of the Implicit Function Theorem [26, Theorem 4.H, p. 177], the proof will be completed once we show that $\partial_{\omega,u}F(\theta, 0, Ue^{i\beta}) : X \to \mathbf{L}^2_{\mathrm{rad}}(\mathbb{R}^N)$ is surjective. Note that

$$\frac{\partial F}{\partial \omega}(\theta, 0, Ue^{i\beta}) = -ie^{-i\theta} Ue^{i\beta}, \tag{5.3}$$

$$\frac{\partial F}{\partial u}(\theta, 0, Ue^{i\beta})v = \rho v - \Delta v - U^\alpha v - \alpha U^\alpha e^{i\beta}\Re\big(e^{-i\beta}v\big), \tag{5.4}$$

so that

$$\partial_{\omega,u}F\big(\theta, 0, Ue^{i\beta}\big)(\mu, v) = e^{i\beta}\partial_{\omega,u}F(\theta, 0, U)\big(\mu, e^{-i\beta}v\big).$$

It thus suffices to consider the case $\beta = 0$.

Given $f \in \mathbf{L}^2_{\mathrm{rad}}(\mathbb{R}^N)$, $\partial_{\omega,u}F(\theta, 0, U)(\mu, v) = f$ is equivalent to

$$-ie^{-i\theta}U\mu + Lv = f, \tag{5.5}$$

where L is given by (2.6). Note that L is a self-adjoint operator in $\mathbf{L}^2_{\mathrm{rad}}(\mathbb{R}^N)$ for the scalar product (5.1). Using that $\ker L = [iU]$, see Proposition 3.4, we choose μ such that

$$\tilde{f} = f + ie^{-i\theta}U\mu \in (iU)^\perp,$$

i.e.,

$$\mu = -\frac{1}{\cos\theta \|U\|^2_{L^2(\mathbb{R}^N)}} \int_{\mathbb{R}^N} \Im f U. \tag{5.6}$$

The fact that $Lv = \tilde{f}$ has a solution for $\tilde{f} \in (iU)^\perp$ follows from the Fredholm Alternative applied to the compact operator $K = (\rho - \Delta)^{-1}U^\alpha$, see Section 3. This shows that L is surjective and closes the proof. $\square$

Proof of Theorem 1.2. Set $X = \mathbb{R} \times (\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega))$ and define $F : (-\pi/2, \pi/2) \times X \to \mathbf{L}^2(\Omega)$ by (5.2). The arguments of the proof of Theorem 1.1 are still valid in this case where Ω is a ball. The fact that $\ker L = [iU]$ was stablished in Proposition 4.6. □

Remark 5.1.

(1) Let Ω be the unitary ball of $\mathbb{R}^N$ and let $\tilde{X} = \mathbb{R} \times \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega) \cap (iU)^\perp$. Given $\tilde{f} \in (iU)^\perp$, there exists a unique $\tilde{z} \in (iU)^\perp$ such that $L\tilde{z} = \tilde{f}$. We may thus modify the proof of Theorem 1.2 and apply the standard Implicit Function Theorem to find a unique curve of $g_\gamma = (\omega_\gamma, u_\gamma)$ in $\tilde{X}$ such that $\varphi_\gamma = e^{i\omega_\gamma} u_\gamma$ is a standing wave solution of (1.2). Since the equation is invariant under unitary transformations and U is radially symmetric, it follows by the uniqueness of g_γ that φ_γ is radially symmetric.

(2) Theorem 1.2 is still valid in the case $N = 2$, Ω any bounded regular star shaped domain and $\rho = 0$. Using Remark 4.4 the arguments of the proof apply without any change.

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