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The Cauchy problem for a two-component Novikov equation in the critical Besov space ☆

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ABSTRACT

In this paper, we consider the Cauchy problem for a two-component Novikov equation in the critical Besov space $B^{5/2}_{2,1}$. We first derive a new *a priori* estimate for the 1-D transport equation in $B^{3/2}_{2,\infty}$, which is the endpoint case. Then we apply this *a priori* estimate and the Osgood lemma to prove the local existence. Moreover, we also show that the solution map $u_0 \mapsto u$ is Hölder continuous in $B^{5/2}_{2,1}$ equipped with weaker topology. It is worth mentioning that our method is different from the previous one that involves extracting a convergent subsequence from an iterative sequence in critical Besov spaces.

1. Introduction

In this paper, we consider the Cauchy problem for the two-component system

$$\begin{cases} m_t + uvm_x + 3vu_x m = 0, & t > 0,\ x \in \mathbb{R}, \\ n_t + uvn_x + 3uv_x n = 0, & t > 0,\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

associated with the initial conditions

$$u(0,x) = u_0(x), \qquad v(0,x) = v_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $m = u - u_{xx}$, $n = v - v_{xx}$.

System (1.1) is proposed by Geng and Xue in [15], in which they show that (1.1) is exactly a negative flow in the hierarchy and admits exact solutions with N-peakons and an infinite sequence of conserved quantities. Moreover, a reduction of this hierarchy and its Hamiltonian structures are discussed. The bi-Hamiltonian structure for (1.1) is also studied in [23].

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When $v = 1$, (1.1) reduces to the Degasperis–Procesi (DP) equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx}, \tag{1.3}$$

which was derived as an integrable model by Degasperis and Procesi in [11]. Eq. (1.3) was also derived by Dullin, Gottwald and Holm [12] as a shallow water approximation to the Euler equation. Its complete integrability, bi-Hamilton structure and exact peakon solutions were studied in [10]. The local well-posedness for the strong solutions, the global well-posedness for the weak solutions, the blow-up and non-uniformly continuous dependence on the initial data can be seen in [14,19,24,33,34] and references therein.

When $v = u$, (1.1) becomes the Novikov equation [27],

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}. \tag{1.4}$$

In [15,21], it is derived that (1.4) possesses a bi-Hamiltonian structure and an infinite sequence of conserved quantities and admits exact peaked solutions $u(t,x) = \pm\sqrt{c}e^{-|x-ct|}$, $c > 0$, as well as the explicit formulas for multipeakon solutions [20,21]. By using the Littlewood–Paley decomposition and Kato's theory, the well-posedness of the Novikov equation has been studied in Besov spaces $B^s_{p,r}(\mathbb{R})$ and in the Sobolev space $H^s(\mathbb{R})$ (see [26,30,31]). Jiang and Ni [22] established some results about blow-up phenomena of the strong solution to the Cauchy problem for (1.4).

Another important two-component system is the Camassa–Holm system:

$$\begin{cases} m_t + um_x + 2u_x m + \rho\rho_x = 0, \\ \rho_t + (u\rho)_x = 0, \\ m = u - u_{xx}, \end{cases} \tag{1.5}$$

which has been studied by many physicists and mathematicians. Local well-posedness and blow up at finite time for (1.5) with the initial data in Hilbert spaces or Sobolev spaces have been established in [6,13]. The blow-up phenomena and global existence to (1.5) in Hilbert spaces have been derived in [16]. For the related two-component generalized Camassa–Holm system, we refer to [17,18,35] and references therein.

Denote $P(D) = (1 - \partial_x^2)^{-1}$, then we can rewrite (1.1), (1.2) as follows:

$$\begin{cases} u_t + uvu_x + F^1(u,v) = 0, \quad t > 0,\ x \in \mathbb{R}, \\ v_t + uvv_x + F^2(u,v) = 0, \quad t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.6}$$

where

$$F^1(u,v) = \sum_{i=1}^4 F_i^1(u,v), \qquad F^2(u,v) = \sum_{i=1}^4 F_i^2(u,v), \tag{1.7}$$

and

$$\begin{cases} F_1^1(u,v) = 3P(D)(uvu_x), \quad F_2^1(u,v) = 2P(D)\big(u_x^2 v_x\big), \\ F_3^1(u,v) = 2P(D)(uv_x u_{xx}), \quad F_4^1(u,v) = P(D)(uu_x v_{xx}), \\ F_1^2(u,v) = 3P(D)(vuv_x), \quad F_2^2(u,v) = 2P(D)\big(u_x v_x^2\big), \\ F_3^2(u,v) = 2P(D)(vu_x v_{xx}), \quad F_4^2(u,v) = P(D)(vv_x u_{xx}). \end{cases} \tag{1.8}$$

Recently, Mi et al. [25] studied the Cauchy problem for (1.1) (or (1.6)) and they established the local well-posedness in $B^s_{p,r} \times B^s_{p,r}$ with $1 \le p,r \le \infty$ and $s > \max\{5/2, 2+1/p\}$. By calculation, we find that

$F^j : B^{5/2}_{2,1} \times B^{5/2}_{2,1} \to B^{5/2}_{2,1} \times B^{5/2}_{2,1}$ $(j = 1, 2)$ is also continuous (see Lemma 2.7 below). Therefore, it is reasonable to expect the existence of local solutions in $B^{5/2}_{2,1} \times B^{5/2}_{2,1}$. Moreover, $s = 5/2$ is the critical point for the embedding $B^{s-2}_{2,1} \hookrightarrow L^\infty$ to be true (notice that $P(D)$ is an isometry from $B^{s-2}_{2,1}$ into $B^{s}_{2,1}$). In this paper, we consider the Cauchy problem (1.1), (1.2) in $B^{5/2}_{2,1} \times B^{5/2}_{2,1}$ and the continuous properties of the corresponding solution map.

Notation. Before stating the main results, we explain the notation and conventions used throughout this paper. We use $\lesssim$ to denote estimates that hold up to some universal constant which may change from line to line but whose meaning is clear from the context. $f \approx g$ stands for $f \lesssim g$ and $f \gtrsim g$. $\mathscr{S}(\mathbb{R})$ is the space of rapidly decreasing functions on $\mathbb{R}$ and $\mathscr{S}'(\mathbb{R})$ is its dual space. All function spaces are over $\mathbb{R}$ and we drop $\mathbb{R}$ in our notations of function spaces if there is no ambiguity. C is also a generic constant that may assume different values in different lines.

The main results of this paper are as follows:

Theorem 1.1. *When $(u_0, v_0) \in B^{5/2}_{2,1} \times B^{5/2}_{2,1}$, we have the following results:*

(1) *There is a $T = T(u_0, v_0) > 0$ such that (1.6) has a solution*

$$(u, v) \in C\big([0, T]; B^{5/2}_{2,1} \times B^{5/2}_{2,1}\big) \cap C^1\big([0, T]; B^{3/2}_{2,1} \times B^{3/2}_{2,1}\big).$$

Furthermore, for $0 \le t \le T = \frac{3}{16C(\|u_0\|_{B^{5/2}_{2,1}} + \|v_0\|_{B^{5/2}_{2,1}})^2}$, (u, v) satisfies the estimate

$$\big\|u(t)\big\|_{B^{5/2}_{2,1}} + \big\|v(t)\big\|_{B^{5/2}_{2,1}} \le 2\big(\|u_0\|_{B^{5/2}_{2,1}} + \|v_0\|_{B^{5/2}_{2,1}}\big), \tag{1.9}$$

where $C > 0$ is a constant.

Theorem 1.2. *The solution map $(u_0, v_0) \mapsto (u, v)$ defined by (1.6) is Hölder continuous as a map from $B(0, R) \subset B^{5/2}_{2,1} \times B^{5/2}_{2,1}$, with the $B^{3/2}_{2,\infty} \times B^{3/2}_{2,\infty}$ topology, to $C([0, T], B^{s'}_{2,1} \times B^{s'}_{2,1})$, where $3/2 < s' < 5/2$ and $T = T(R)$. More precisely, if $(u_0, v_0), (w_0, q_0) \in B(0, R) \subset B^{5/2}_{2,1} \times B^{5/2}_{2,1}$ and (u, v), (w, q) are the solutions corresponding to the initial data (u_0, v_0), (w_0, q_0), respectively, then we have*

$$\big(\|u - w\|_{B^{s'}_{2,1}} + \|v - q\|_{B^{s'}_{2,1}}\big) \lesssim \big(\|u_0 - w_0\|_{B^{3/2}_{2,\infty}} + \|v_0 - q_0\|_{B^{3/2}_{2,\infty}}\big)^\alpha, \tag{1.10}$$

where $\alpha = (5/2 - q)\exp\{-C_R\}$ with the constant $C_R > 0$ depending on the radius R.

Now we outline the proof and state the main difficulties we are confronted with. To prove the local existence, we construct the approximate solutions $\{(u_k, v_k)\}$ via the standard iteration method. Then we prove the convergence of the approximate solutions $\{(u_k, v_k)\}$ and prove that the limit is the solution. The first difficulty appears in the proof of the convergence of the iterative sequence. Because we cannot obtain the solution by simply taking limit in an iterative equation, we can only expect to prove that $\{(u_k, v_k)\}$ is a Cauchy sequence. We will use the Osgood lemma (see Lemma 2.5) to overcome this difficulty. Actually, we firstly obtain the convergence of $\{(u_k, v_k)\}$ in $B^{3/2}_{2,\infty} \times B^{3/2}_{2,\infty}$, from which the second difficulty arises, and then obtain the convergence of $\{(u_k, v_k)\}$ in $B^{3/2}_{2,1} \times B^{3/2}_{2,1}$. The second difficulty is on the lack of the appropriate *a priori* estimate of the transport equation in $B^{1+1/p}_{p,r}$ with $r \neq 1$ (in the 1-D case, see [2,7,9]). We obtain a new *a priori* estimate in $B^{3/2}_{2,\infty}$ and use it to obtain the convergence of the iterative sequence in $B^{3/2}_{2,\infty} \times B^{3/2}_{2,\infty}$. Our proof is motivated by those presented in [2] and [5]. In [2], the authors used the Osgood lemma to prove the Cauchy–Lipschitz Theorem (Theorem 3.2 in [2]). In [5], Chen et al. studied the local well-posedness for a two-component integrable system.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we prove Theorem 1.1. In Section 4, we prove Theorem 1.2.

2. Preliminaries

In this section, we state some preliminary lemmas that we shall use in this paper. We start with the Littlewood–Paley decomposition and we refer to [1,2,7,9,29,28] for the elementary properties of them.

Let χ, ϕ be two smooth radial functions $0 \le \chi, \phi \le 1$, such that χ is supported in the ball $B = \{\xi \in \mathbb{R}\colon |\xi| \le \frac{4}{3}\}$ and ϕ is supported in the ring $C = \{\xi \in \mathbb{R}\colon \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. Furthermore,

$$\chi(\xi) + \sum_{j\in\mathbb{N}} \phi\big(2^{-j}\xi\big) = 1, \quad \forall \xi \in \mathbb{R},$$
$$\operatorname{supp}\phi\big(2^{-j}\cdot\big) \cap \operatorname{supp}\phi\big(2^{-j'}\cdot\big) = \varnothing \quad \text{if } \big|j - j'\big| \ge 2,$$
$$\operatorname{supp}\chi(\cdot) \cap \operatorname{supp}\phi\big(2^{-j}\cdot\big) = \varnothing \quad \text{if } j \ge 1.$$

For $u \in \mathscr{S}'(\mathbb{R})$, we define the nonhomogeneous dyadic block operators as

$$\Delta_{-1}u = \chi(D)u = \mathcal{F}_x^{-1}\chi\mathcal{F}_x u,$$
$$\Delta_j u = \phi\big(2^{-j}D\big)u = \mathcal{F}_x^{-1}\phi\big(2^{-j}\xi\big)\mathcal{F}_x u, \quad \text{if } j \ge 0,$$

where $\mathcal{F}_x u$ is the Fourier transform in x. Then we have

$$u = \sum_{j=-1}^{\infty} \Delta_j u \quad \text{converges in } \mathscr{S}'(\mathbb{R}) \text{ or in } H^s(\mathbb{R}).$$

We define the low frequency cut-off S_j as $S_j u = \sum_{i=-1}^{j-1} \Delta_i u$. Direct computation implies that for any $1 \le p \le \infty$,

$$\Delta_i\Delta_j u \equiv 0, \quad \text{if } |i - j| \ge 2,\ \forall u, v \in \mathscr{S}'(\mathbb{R}),$$
$$\Delta_j(S_{i-1}u\Delta_i v) \equiv 0, \quad \text{if } |i - j| \ge 5,\ \forall u, v \in \mathscr{S}'(\mathbb{R}),$$
$$\|\Delta_i u\|_{L^p} \le C\|u\|_{L^p}, \qquad \|S_j u\|_{L^p} \le C\|u\|_{L^p}, \quad \forall u \in L^p(\mathbb{R}).$$

Definition 2.1 *(Besov spaces).* Let $s \in \mathbb{R}$, $1 \le p, r \le +\infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbb{R})$ is defined by

$$B^s_{p,r} = \big\{f \in \mathscr{S}'\colon \|f\|_{B^s_{p,r}} < \infty\big\},$$

where $\|f\|_{B^s_{p,r}} = \|2^{js}\Delta_j f\|_{l^r(L^p)} = \|(2^{js}\|\Delta_j f\|_{L^p})_{j\ge -1}\|_{l^r}$. In particular, $B^\infty_{p,r} = \bigcap_{s'\in\mathbb{R}} B^{s'}_{p,r}$.

Then we list some basic properties of Besov spaces that will be frequently used in this paper. Their proofs can be found in [1,2,7,9,29,28].

Lemma 2.1. *Letting $s \in \mathbb{R}$, $1 \le p, r, p_j, r_j \le \infty$, $j = 1, 2$, we have:*

(1) $B^{s_1}_{p_1,r_1} \hookrightarrow B^{s_2}_{p_2,r_2}$ *if $p_1 \le p_2$, $r_1 \le r_2$, and $s_2 = s_1 - (\frac{1}{p_1} - \frac{1}{p_2})$.*

(2) $\forall s > 0$, $B^s_{p,r} \cap L^\infty$ *is a Banach algebra.* $B^s_{p,r}$ *is a Banach algebra* $\Leftrightarrow B^s_{p,r} \hookrightarrow L^\infty \Leftrightarrow s > \frac{1}{p}$ (or $s \geq \frac{1}{p}$ and $r = 1$).

(3) $\forall \theta \in [0, 1]$, $s = \theta s_1 + (1 - \theta)s_2$,

$$\|f\|_{B^s_{p,r}} \leq C\|f\|^\theta_{B^{s_1}_{p,r}} \|f\|^{1-\theta}_{B^{s_2}_{p,r}}, \quad \forall f \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r}.$$

(4) $\forall \theta \in (0, 1)$, $s_1 > s_2$, $s = \theta s_1 + (1 - \theta)s_2$,

$$\|u\|_{B^s_{p,1}} \leq \frac{C(\theta)}{s_1 - s_2} \|u\|^\theta_{B^{s_1}_{p,\infty}} \|u\|^{1-\theta}_{B^{s_2}_{p,\infty}}, \quad \forall u \in B^{s_1}_{p,\infty}.$$

(5) *If* $\{u_n\}_{n\in\mathbb{N}}$ *is bounded in* $B^s_{p,r}$ *and* u_n *converges to* u *in* $\mathscr{S}'$, *then* $u \in B^s_{p,r}$ *and*

$$\|u\|_{B^s_{p,r}} \leq \liminf_{n\to\infty} \|u_n\|_{B^s_{p,r}}.$$

(6) *For any* $s \in \mathbb{R}$, $P(D) = (1 - \partial_x^2)^{-1}$ *is an isometry from* $B^s_{p,r}$ *into* $B^{s+2}_{p,r}$.

Remark 2.1. Based on the above lemma, for any $f \in B^{-1/2}_{2,1}$, $g \in B^{1/2}_{2,1}$, we have

$$\|fg\|_{B^{-1/2}_{2,\infty}} \lesssim \|f\|_{B^{-1/2}_{2,1}} \|g\|_{B^{1/2}_{2,1}}. \tag{2.1}$$

The essential part of the above estimate has been given in [8], that is, the usual product is continuous from $B^{-1/2}_{2,1} \times (B^{1/2}_{2,\infty} \cap L^\infty)$ to $B^{-1/2}_{2,\infty}$, i.e.

$$\|fg\|_{B^{-1/2}_{2,\infty}} \leq C\|f\|_{B^{-1/2}_{2,1}} \|g\|_{B^{1/2}_{2,\infty} \cap L^\infty}.$$

From the embedding $B^{1/2}_{2,1} \hookrightarrow B^{1/2}_{2,\infty} \cap L^\infty$, we obtain (2.1).

Now we state some results of the transport equation in the Besov spaces in the form which is convenient for our purposes.

Lemma 2.2. *Consider the following* 1*-D linear transport equation:*

$$f_t + vf_x = F, \quad t > 0,\ x \in \mathbb{R}, \tag{2.2}$$

$$f(x, 0) = f_0, \quad x \in \mathbb{R}, \tag{2.3}$$

where the functions $v, F : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $f_0 : \mathbb{R} \to \mathbb{R}$ *are given. Suppose that* $v \in L^\rho(0, T; B^{-M}_{\infty,\infty})$ *for some* $\rho > 1, M > 0$. *Then we have the following results:*

(i) *If* $f_0 \in B^{5/2}_{2,1}$, $v_x \in L^1(0, T; B^{3/2}_{2,1})$ *and* $F \in L^1(0, T; B^{5/2}_{2,1})$, *then* (2.2), (2.3) *has a unique solution* $f \in C([0, T]; B^{5/2}_{2,1})$. *Moreover, for* $t \in [0, T]$, *we have*

$$\|f(t)\|_{B^{5/2}_{2,1}} \leq \exp\{CV_1(t)\} \left(\|f_0\|_{B^{5/2}_{2,1}} + \int_0^t \exp\{-CV_1(\tau)\} \|F(\tau)\|_{B^{5/2}_{2,1}} \,\mathrm{d}\tau \right) \tag{2.4}$$

with $V_1(t) = \int_0^t \|v_x(\tau)\|_{B^{3/2}_{2,1}} \,\mathrm{d}\tau$.

(ii) *If* $f_0 \in B^{3/2}_{2,\infty}$, $v_x \in L^1(0,T;B^{1/2}_{2,1})$ *and* $F \in L^1(0,T;B^{3/2}_{2,\infty})$, *then* (2.2), (2.3) *has a unique solution* $f \in C([0,T];B^{3/2}_{2,\infty})$. *Moreover, for* $t \in [0,T]$, *we have*

$$\|f(t)\|_{B^{3/2}_{2,\infty}} \le \exp\{CV_2(t)\}\left(\|f_0\|_{B^{3/2}_{2,\infty}} + \int_0^t \exp\{-CV_2(\tau)\}\|F(\tau)\|_{B^{3/2}_{2,\infty}}\,\mathrm{d}\tau\right) \tag{2.5}$$

with $V_2(t) = \int_0^t \|v_x(\tau)\|_{B^{1/2}_{2,1}}\,\mathrm{d}\tau$.

Proof. We only prove (ii) since (i) is a special case of Theorem 3.14 and Theorem 3.19 in [2]. Now we prove that if $f \in L^\infty(0,T;B^{3/2}_{2,\infty})$ is a solution to (2.2), (2.3), then (2.5) holds true. We first split the equation in dyadic block. More precisely, applying Δ_j to (2.2), (2.3), we arrive at

$$\begin{cases} \partial_t \Delta_j f + v \cdot \partial_x \Delta_j f = \Delta_j F + R_j, \\ \Delta_j f|_{t=0} = \Delta_j f_0, \end{cases} \tag{2.6}$$

where $R_j = v \cdot \partial_x \Delta_j f - \Delta_j(v \cdot \partial_x f)$.

Multiplying both sides of (2.6) by $\Delta_j f$ and integrating over $\mathbb{R}$ give rise to

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j f(t)\|^2_{L^2} &\le \frac{1}{2}\int_{\mathbb{R}} v_x(\Delta_j f)^2\,\mathrm{d}x + \int_{\mathbb{R}} (\Delta_j F \Delta_j f + R_j \Delta_j f)\,\mathrm{d}x \\ &\le \frac{1}{2}\|v_x\|_\infty \|\Delta_j f\|^2_{L^2} + \left(\|\Delta_j F\|_{L^2} + \|R_j\|_{L^2}\right)\|\Delta_j f\|_{L^2}. \end{aligned}$$

That is

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_j f(t)\|_{L^2} \le \frac{1}{2}\|v_x\|_\infty \|\Delta_j f\|_{L^2} + \|\Delta_j F\|_{L^2} + \|R_j\|_{L^2}.$$

Integrating over $(0,t)$, we obtain

$$\|\Delta_j f(t)\|_{L^2} \le \|\Delta_j f_0\|_{L^2} + \int_0^t \|\Delta_j F\|_{L^2}\,\mathrm{d}\tau + \int_0^t \|R_j\|_{L^2}\,\mathrm{d}\tau + \frac{1}{2}\int_0^t \|v_x\|_{L^\infty}\|\Delta_j f\|_{L^2}\,\mathrm{d}\tau.$$

Multiplying both sides of the above inequality by $2^{\frac{3}{2}j}$ and taking l^∞ norm yield

$$\|f(t)\|_{B^{3/2}_{2,\infty}} \le \|f_0\|_{B^{3/2}_{2,\infty}} + \int_0^t \|F\|_{B^{3/2}_{2,\infty}}\,\mathrm{d}\tau + \int_0^t \left\|\left(2^{\frac{3}{2}j}\|R_j\|_{L^2}\right)_{j\ge -1}\right\|_{l^\infty}\mathrm{d}\tau + \frac{1}{2}\int_0^t \|v_x\|_{L^\infty}\|f\|_{B^{3/2}_{2,\infty}}\,\mathrm{d}\tau.$$

By (2.55) in [2, page 112], we have

$$\sup_{j\ge -1}\left(2^{-\frac{1}{2}j}\|R_j\|_{L^2}\right)_{j\ge -1} \le C\|v_x\|_{B^{1/2}_{2,1}}\|f\|_{B^{-1/2}_{2,\infty}}.$$

From the above estimate and Definition 2.1, we get

$$\sup_{j\ge -1}\left(2^{-\frac{1}{2}j}\|R_j\|_{L^2}\right) \le C\|v_x\|_{B^{1/2}_{2,1}}\left(\sup_{j\ge -1}\left(2^{-\frac{1}{2}j}\|\Delta_j f\|_{L^2}\right)_{j\ge -1}\right).$$

By multiplying both sides of the above inequality by 2^{2j}, we obtain

$$\sup_{j\geq -1}\left(2^{\frac{3}{2}j}\|R_j\|_{L^2}\right)_{j\geq -1}\leq C\|v_x\|_{B^{1/2}_{2,1}}\Big(\sup_{j\geq -1}\left(2^{\frac{3}{2}j}\|\Delta_j f\|_{L^2}\right)_{j\geq -1}\Big)\leq C\|v_x\|_{B^{1/2}_{2,1}}\|f\|_{B^{3/2}_{2,\infty}}.$$

Thus, using the embedding $B^{1/2}_{2,1}\hookrightarrow L^\infty$ yields

$$\|f(t)\|_{B^{3/2}_{2,\infty}}\leq \|f_0\|_{B^{3/2}_{2,\infty}}+\int_0^t\|F\|_{B^{3/2}_{2,\infty}}\,\mathrm{d}\tau + C\int_0^t\|v_x\|_{B^{1/2}_{2,1}}\|f\|_{B^{3/2}_{2,\infty}}\,\mathrm{d}\tau.$$

Applying Gronwall inequality gives

$$\|f(t)\|_{B^{3/2}_{2,\infty}}\leq \left(\|f_0\|_{B^{3/2}_{2,\infty}}+\int_0^t\|F\|_{B^{3/2}_{2,\infty}}\,\mathrm{d}\tau\right)\exp\left\{C\int_0^t\|v_x\|_{B^{1/2}_{2,1}}\,\mathrm{d}\tau\right\},$$

which is (2.5).

The existence and uniqueness of the solution follow from (i) and the estimate (2.5). Like the process as in Theorem 3.19 in [2, page 136], for $f_0\in B^{3/2}_{2,\infty}$ and $F\in L^1(0,T;B^{3/2}_{2,\infty})$, we choose $f_{0k}\in C([0,T];B^{5/2}_{2,1})$ and $F_k\in L^1(0,T;B^{5/2}_{2,1})$ such that $f_{0k}\to f_0$ in $B^{3/2}_{2,\infty}$ and $F_k\to F$ in $L^1(0,T;B^{3/2}_{2,\infty})$. Then, by (i) and the estimate (2.5), there exists a sequence of solutions $\{f_k\}\subset C([0,T];B^{5/2}_{2,1})$ such that $f_k\to f$ in $C([0,T];B^{3/2}_{2,\infty})$ for some $f\in C([0,T];B^{3/2}_{2,\infty})$ and f is the unique solution to (2.2), (2.3). $\square$

We will use the following Moser-type estimates frequently.

Lemma 2.3 *(Moser-type estimates). (See [2,4,7,9].) If $1\leq p,r\leq +\infty$, then we have the following estimates:*

(i) *For $s>0$,*

$$\|fg\|_{B^s_{p,r}}\leq C\big(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty}+\|f\|_{L^\infty}\|g\|_{B^s_{p,r}}\big)\quad \forall f,g\in B^s_{p,r}\cap L^\infty.$$

(ii) *For all $s_1\leq \frac{1}{p}<s_2$ ($s_2\geq \frac{1}{p}$ if $r=1$) and $s_1+s_2>0$,*

$$\|fg\|_{B^{s_1}_{p,r}}\leq C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}}\quad \forall f\in B^{s_1}_{p,r},\ g\in B^{s_2}_{p,r}.$$

Lemma 2.4. *(See [8,9].) There exists a constant $C>0$ such that for all $s\in\mathbb{R}$, $\varepsilon>0$ and $1\leq p\leq\infty$, we have*

$$\|f\|_{B^s_{p,1}}\leq C\frac{1+\varepsilon}{\varepsilon}\|f\|_{B^s_{p,\infty}}\ln\left(\mathrm{e}+\frac{\|f\|_{B^{s+\varepsilon}_{p,\infty}}}{\|f\|_{B^s_{p,\infty}}}\right),\quad \forall f\in B^{s+\varepsilon}_{p,\infty}.$$

The following Osgood lemma appears as a substitute for Gronwall's lemma. It can be seen as Lemma 3.4 in [2] or as Lemma 5.2.1 in [3].

Lemma 2.5 *(Osgood lemma). (See [2,3].) Let $\rho\geq 0$ be a measurable function, $\gamma>0$ be a locally integrable function and μ be a continuous and increasing function. Assume that, for some nonnegative real number c, the function ρ satisfies*

$$\rho(t)\leq c+\int_{t_0}^t\gamma(t')\mu\big(\rho(t')\big)\,\mathrm{d}t'.$$

If $c>0$, then $-\mathcal{M}(\rho(t))+\mathcal{M}(c)\le\int_{t_0}^t\gamma(t')\,\mathrm{d}t'$ with $\mathcal{M}=\int_x^1\frac{\mathrm{d}r}{\mu(r)}$.

If $c=0$ and μ verifies the condition $\int_0^1\frac{\mathrm{d}r}{\mu(r)}=+\infty$, then the function $\rho=0$.

Now we establish some estimates on the terms $F^1(u,v)$ and $F^2(u,v)$.

Lemma 2.6. *The F^j $(j=1,2)$ defined by (1.7), (1.8) map $B^{5/2}_{2,1}\times B^{5/2}_{2,1}$ into $B^{5/2}_{2,1}\times B^{5/2}_{2,1}$ and satisfy that for all $(u,v)\in B^{5/2}_{2,1}\times B^{5/2}_{2,1}$,*

$$\big\|F^j(u,v)\big\|_{B^{5/2}_{2,1}}\lesssim\|u\|_{B^{5/2}_{2,1}}\|v\|_{B^{5/2}_{2,1}}\big(\|u\|_{B^{5/2}_{2,1}}+\|v\|_{B^{5/2}_{2,1}}\big)\lesssim\big(\|u\|_{B^{5/2}_{2,1}}+\|v\|_{B^{5/2}_{2,1}}\big)^3. \tag{2.7}$$

Proof. We only prove the estimate for $\|F^1(u,v)\|_{B^{5/2}_{2,1}}$ since the proof for $\|F^2(u,v)\|_{B^{5/2}_{2,1}}$ is similar. By the triangle inequality, we have $\|F^1(u,v)\|_{B^{5/2}_{2,1}}\le\sum_{1\le i\le4}\|F^1_i(u,v)\|_{B^{5/2}_{2,1}}$ with $F^1_i(u,v)$ given in (1.8). Note that $P(D)$ is an isometry from $B^{s-2}_{2,1}$ into $B^s_{2,1}$ and $B^{5/2-2}_{2,1}=B^{1/2}_{2,1}$ is an algebra, then we have

$$\big\|F^1_1(u,v)\big\|_{B^{5/2}_{2,1}}\lesssim\|u\|^2_{B^{5/2}_{2,1}}\|v\|_{B^{5/2}_{2,1}},\qquad\big\|F^1_2(u,v)\big\|_{B^{5/2}_{2,1}}\lesssim\|u\|^2_{B^{5/2}_{2,1}}\|v\|_{B^{5/2}_{2,1}},$$
$$\big\|F^1_3(u,v)\big\|_{B^{5/2}_{2,1}}\lesssim\|u\|^2_{B^{5/2}_{2,1}}\|v\|_{B^{5/2}_{2,1}},\qquad\big\|F^1_4(u,v)\big\|_{B^{5/2}_{2,1}}\lesssim\|u\|^2_{B^{5/2}_{2,1}}\|v\|_{B^{5/2}_{2,1}}.$$

Similarly, (2.7) holds true for $\|F^2(u,v)\|_{B^{5/2}_{2,1}}$ and hence we obtain (2.7). $\square$

In several cases, for $(u,v),(w,q)\in B^s_{2,1}\times B^s_{2,1}$, we need to estimate the difference $F^j(u,v)-F^j(w,q)$ $(j=1,2)$ in terms of the quantity $(u-w,v-q)$. Using (1.8), (2.1) and Lemma 2.3, we can obtain the following estimate easily and we omit the proof here.

Lemma 2.7. *When $3/2\le s\le5/2$, for all $(u,v),(w,q)\in B^s_{2,1}\times B^s_{2,1}$, we have*

$$\big\|F^j(u,v)-F^j(w,q)\big\|_{B^s_{2,\infty}}\lesssim\big(\|u\|_{B^s_{2,1}}+\|v\|_{B^s_{2,1}}+\|q\|_{B^s_{2,1}}+\|w\|_{B^s_{2,1}}\big)^2\big(\|u-w\|_{B^s_{2,1}}+\|v-q\|_{B^s_{2,1}}\big). \tag{2.8}$$

Lemma 2.8. *If $(u,v),(w,q)\in C([0,T];\mathscr{S}'\times\mathscr{S}')\cap L^\infty(0,T;B^{5/2}_{2,1}\times B^{5/2}_{2,1})$ are two solutions to (1.6) for some $T>0$ with initial data $(u_0,v_0),(w_0,v_0)\in B^{5/2}_{2,1}\times B^{5/2}_{2,1}$ respectively, then for any $s'\in(3/2,5/2)$ and $t\in[0,T]$,*

$$\big(\|u-w\|_{B^{s'}_{2,1}}+\|v-q\|_{B^{s'}_{2,1}}\big)\lesssim\big(\|u_0-w_0\|_{B^{3/2}_{2,\infty}}+\|v_0-q_0\|_{B^{3/2}_{2,\infty}}\big)^{\theta\exp\{-CT\}}, \tag{2.9}$$

where $\theta=\frac{5}{2}-s'\in(0,1)$ and $C>0$ is a constant.

Proof. Since (u,v), (w,q) are two solutions to (1.6) with initial data $(u_0,v_0),(w_0,q_0)\in B^{5/2}_{2,1}$, respectively, we have

$$u_t+uvu_x+F^1(u,v)=0,\qquad w_t+wqw_x+F^1(w,q)=0, \tag{2.10}$$
$$v_t+uvv_x+F^2(u,v)=0,\qquad q_t+wqq_x+F^2(w,q)=0, \tag{2.11}$$

where F^j $(j=1,2)$ are given in (1.7) and (1.8). Subtracting the second equations of (2.10), (2.11) from the first ones, respectively, then we obtain that $h=u-w$ and $g=v-q$ satisfy

$$h_t + wq\partial_x h = -(ug + hq)u_x - \big[F^1(u,v) - F^1(w,q)\big], \tag{2.12}$$

$$g_t + uv\partial_x g = -(ug + hq)q_x - \big[F^2(u,v) - F^2(w,q)\big], \tag{2.13}$$

$$h(0) = u_0 - w_0, \qquad g(0) = v_0 - q_0. \tag{2.14}$$

By Lemma 2.2, we arrive at

$$\big\|h(t)\big\|_{B^{3/2}_{2,\infty}} \le \|h_0\|_{B^{3/2}_{2,\infty}} \exp\left\{C\int_0^t \big\|(wq)_x(\tau)\big\|_{B^{1/2}_{2,1}}\,\mathrm{d}\tau\right\} + \int_0^t \exp\left\{C\int_\tau^t \big\|(wq)_x(\tau')\big\|_{B^{1/2}_{2,1}}\,\mathrm{d}\tau'\right\}\|\widehat{F}\|_{B^{3/2}_{2,\infty}}\,\mathrm{d}\tau,$$

where $\widehat{F} = -(ug + hq)u_x - [F^1(u,v) - F^1(w,q)]$. Since $(u,v),(w,q) \in L^\infty(0,T;B^{5/2}_{2,1} \times B^{5/2}_{2,1})$, there is an $M > 0$ such that

$$\big\|u(t)\big\|_{B^{5/2}_{2,1}} + \big\|v(t)\big\|_{B^{5/2}_{2,1}} + \big\|w(t)\big\|_{B^{5/2}_{2,1}} + \big\|q(t)\big\|_{B^{5/2}_{2,1}} \le M, \quad 0 \le t \le T. \tag{2.15}$$

By (2.15), we have $\exp\{C\int_0^t \|(wq)_x(\tau)\|_{B^{1/2}_{2,1}}\,\mathrm{d}\tau\} \lesssim \exp\{CTM^2\} \lesssim 1$. Using Lemma 2.7, the algebra property of $B^{3/2}_{2,\infty}$, the embedding $B^s_{2,1} \hookrightarrow B^s_{2,\infty}$ and (2.15), we arrive at

$$\big\|F^1(u,v) - F^1(w,q)\big\|_{B^{3/2}_{2,\infty}} \lesssim \|h\|_{B^{3/2}_{2,1}} + \|g\|_{B^{3/2}_{2,1}},$$

$$\big\|(ug + hq)u_x\big\|_{B^{3/2}_{2,\infty}} \lesssim \|h\|_{B^{3/2}_{2,1}} + \|g\|_{B^{3/2}_{2,1}}.$$

Hence it follows that $\|\widehat{F}\|_{B^{3/2}_{2,1}} \lesssim \|h\|_{B^{3/2}_{2,1}} + \|g\|_{B^{3/2}_{2,1}}$. Applying Lemma 2.4 with $\varepsilon = 1$, $s = 3/2$ and (2.15) gives rise to

$$\begin{aligned}\|\widehat{F}\|_{B^{3/2}_{2,\infty}} &\lesssim \|h\|_{B^{3/2}_{2,1}} + \|g\|_{B^{3/2}_{2,1}} \\ &\lesssim \|h\|_{B^{3/2}_{2,\infty}} \ln\left(\mathrm{e} + \frac{M}{\|h\|_{B^{3/2}_{2,\infty}}}\right) + \|g\|_{B^{3/2}_{2,\infty}} \ln\left(\mathrm{e} + \frac{M}{\|g\|_{B^{3/2}_{2,\infty}}}\right), \quad 0 \le t \le T.\end{aligned}$$

As the function $x\ln(\mathrm{e} + \frac{1}{x})$ is nondecreasing, we have

$$\|\widehat{F}\|_{B^{3/2}_{2,\infty}} \lesssim \big(\|h\|_{B^{3/2}_{2,\infty}} + \|g\|_{B^{3/2}_{2,\infty}}\big) \ln\left(\mathrm{e} + \frac{M}{\|h\|_{B^{3/2}_{2,\infty}} + \|g\|_{B^{3/2}_{2,\infty}}}\right), \quad 0 \le t \le T.$$

Consequently, for all $t \in [0,T]$, $W(t) = \|h(t)\|_{B^{3/2}_{2,\infty}} + \|g(t)\|_{B^{3/2}_{2,\infty}}$ satisfies

$$\big\|h(t)\big\|_{B^{3/2}_{2,\infty}} \lesssim \big\|h(0)\big\|_{B^{3/2}_{2,\infty}} + C\int_0^t W(\tau)\ln\left(\mathrm{e} + \frac{M}{W(\tau)}\right)\mathrm{d}\tau.$$

Similarly,

$$\big\|g(t)\big\|_{B^{3/2}_{2,\infty}} \lesssim \big\|g(0)\big\|_{B^{3/2}_{2,\infty}} + C\int_0^t W(\tau)\ln\left(\mathrm{e} + \frac{M}{W(\tau)}\right)\mathrm{d}\tau,$$

and therefore

$$W(t) \lesssim W(0) + C \int_0^t W(\tau) \ln\Big(\mathrm{e} + \frac{M}{W(\tau)}\Big)\, \mathrm{d}\tau, \quad 0 \le t \le T. \tag{2.16}$$

Since

$$\ln\Big(\mathrm{e} + \frac{M}{x}\Big) \le \ln(\mathrm{e}+1)\Big(1 - \ln\frac{x}{M}\Big) \quad \text{for } x \in (0, M],$$

from (2.16), we have

$$\frac{W(t)}{M} \lesssim \frac{W(0)}{M} + C \int_0^t \frac{W(\tau)}{M}\Big(1 - \ln\frac{W(\tau)}{M}\Big)\, \mathrm{d}\tau, \quad 0 \le t \le T. \tag{2.17}$$

Thanks to Lemma 2.5, we obtain

$$\frac{W(t)}{\mathrm{e}M} \lesssim \Big(\frac{W(0)}{\mathrm{e}M}\Big)^{\exp\{-CT\}}.$$

Thus, we have

$$\big(\|u-w\|_{B^{3/2}_{2,\infty}} + \|v-q\|_{B^{3/2}_{2,\infty}}\big) \lesssim \big(\|u_0-w_0\|_{B^{3/2}_{2,\infty}} + \|v_0-q_0\|_{B^{3/2}_{2,\infty}}\big)^{\exp\{-CT\}}. \tag{2.18}$$

When $3/2 < s' < 5/2$, by interpolation, the embedding $B^{5/2}_{2,1} \hookrightarrow B^{5/2}_{2,\infty}$ and (2.15), we obtain

$$\|h\|_{B^{s'}_{2,1}} \lesssim \|h\|^{\theta}_{B^{3/2}_{2,\infty}} \|h\|^{1-\theta}_{B^{5/2}_{2,\infty}} \le \|h\|^{\theta}_{B^{3/2}_{2,\infty}} \big(\|u\|_{B^{5/2}_{2,1}} + \|w\|_{B^{5/2}_{2,1}}\big)^{1-\theta} \lesssim \|h\|^{\theta}_{B^{3/2}_{2,\infty}}$$

where $\theta = 5/2 - s'$. Using (2.18), we obtain that

$$\|u-w\|_{B^{s'}_{2,1}} \lesssim \|u-w\|^{\theta}_{B^{3/2}_{2,\infty}} \lesssim \big(\|u_0-w_0\|_{B^{3/2}_{2,\infty}} + \|v_0-q_0\|_{B^{3/2}_{2,\infty}}\big)^{\theta\exp\{-CT\}}. \tag{2.19}$$

Similarly,

$$\|v-q\|_{B^{s'}_{2,1}} \lesssim \|v-q\|^{\theta}_{B^{3/2}_{2,\infty}} \lesssim \big(\|u_0-w_0\|_{B^{3/2}_{2,\infty}} + \|v_0-q_0\|_{B^{3/2}_{2,\infty}}\big)^{\theta\exp\{-CT\}}. \tag{2.20}$$

We finish the proof by combining (2.19) and (2.20). $\square$

3. Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1, which includes the following several steps.

3.1. Approximate solutions

Let $u_1(t,x) = v_1(t,x) = 0$. We define a sequence of smooth functions (u_k, v_k) $(k \ge 1)$ by solving the following linear transport equations iteratively:

$$\begin{cases} \partial_t u_{k+1} + u_k v_k \partial_x u_{k+1} + F^1(u_k, v_k) = 0, & t > 0,\ x \in \mathbb{R}, \\ \partial_t v_{k+1} + u_k v_k \partial_x v_{k+1} + F^2(u_k, v_k) = 0, & t > 0,\ x \in \mathbb{R}, \\ u_{k+1}(0,x) = S_{k+1}u_0(x), \quad v_{k+1}(0,x) = S_{k+1}v_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.1}$$

where F^1, F^2 are given in (1.7) and (1.8). Since $u_0, v_0 \in B^{5/2}_{2,1}$, then all the initial data $u_{k+1}(0,x), v_{k+1}(0,x) \in B^{\infty}_{2,1}$ and

$$\|u_{k+1}(0)\|_{B^{5/2}_{2,1}} + \|u_{k+1}(0)\|_{B^{5/2}_{2,1}} \le C\big(\|u_0\|_{B^{5/2}_{2,1}} + \|v_0\|_{B^{5/2}_{2,1}}\big). \tag{3.2}$$

By Lemma 2.2(i) and induction with the index k, we see that (3.1) has a sequence of global smooth solution $(u_k, v_k)_{k\in\mathbb{N}}$ belonging to $C(\mathbb{R}^+; B^{\infty}_{2,1} \times B^{\infty}_{2,1})$.

3.2. Uniform bounds of the approximate solutions

We now prove that $(u_k, v_k)_{k\in\mathbb{N}} \in C([0,T]; B^{5/2}_{2,1} \times B^{5/2}_{2,1}) \cap C^1([0,T]; B^{3/2}_{2,1} \times B^{3/2}_{2,1})$ is uniformly bounded for some $T > 0$. For $k \in \mathbb{N}$, set $U_k(t) = \int_0^t \|u_k v_k\|_{B^{5/2}_{2,1}} \,\mathrm{d}\tau$, $W_k(t) = \|u_k(t)\|_{B^{5/2}_{2,1}} + \|v_k(t)\|_{B^{5/2}_{2,1}}$ and $V = \|u_0\|_{B^{5/2}_{2,1}} + \|v_0\|_{B^{5/2}_{2,1}}$, then we have

$$W_{k+1} \le e^{CU_k(t)}\left(V + C\int_0^t e^{-CU_k(\tau)} W_k(\tau)^3 \,\mathrm{d}\tau\right). \tag{3.3}$$

In fact, since $\|u_{k+1}(0,x)\|_{B^{5/2}_{2,1}} = \|S_{k+1}u_0\|_{B^{5/2}_{2,1}} \lesssim \|u_0\|_{B^{5/2}_{2,1}}$, by (i) in Lemma 2.2 and Lemma 2.6, it follows that

$$\begin{aligned}
\|u_{k+1}(t)\|_{B^{5/2}_{2,1}} &\le \|S_{k+1}u_0\|_{B^{5/2}_{2,1}} \exp\left\{C\int_0^t \|\partial_x(u_k v_k)\|_{B^{3/2}_{2,1}} \,\mathrm{d}t'\right\} \\
&\quad + \int_0^t \exp\left\{C\int_\tau^t \|\partial_x(u_k v_k)\|_{B^{3/2}_{2,1}} \,\mathrm{d}t'\right\} \|F^1(u_k, v_k)\|_{B^{5/2}_{2,1}} \,\mathrm{d}\tau \\
&\le \|u_0\|_{B^{5/2}_{2,1}} \exp\left\{C\int_0^t \|u_k v_k\|_{B^{5/2}_{2,1}} \,\mathrm{d}t'\right\} \\
&\quad + \int_0^t \exp\left\{C\int_\tau^t \|u_k v_k\|_{B^{5/2}_{2,1}} \,\mathrm{d}t'\right\} \big(\|u_k\|_{B^{5/2}_{2,1}} + \|v_k\|_{B^{5/2}_{2,1}}\big)^3 \,\mathrm{d}\tau,
\end{aligned}$$

which gives

$$\|u_{k+1}(t)\|_{B^{5/2}_{2,1}} \le e^{CU_k(t)}\left(\|u_0\|_{B^{5/2}_{2,1}} + C\int_0^t e^{-CU_k(\tau)} \big(\|u_k\|_{B^{5/2}_{2,1}} + \|v_k\|_{B^{5/2}_{2,1}}\big)^3 \,\mathrm{d}\tau\right). \tag{3.4}$$

Similarly, for $\|v_{k+1}(t)\|_{B^{5/2}_{2,1}}$, we have

$$\|v_{k+1}(t)\|_{B^{5/2}_{2,1}} \le e^{CU_k(t)}\left(\|v_0\|_{B^{5/2}_{2,1}} + C\int_0^t e^{-CU_k(\tau)} \big(\|u_k\|_{B^{5/2}_{2,1}} + \|v_k\|_{B^{5/2}_{2,1}}\big)^3 \,\mathrm{d}\tau\right). \tag{3.5}$$

Combining (3.4) and (3.5), we obtain (3.3). Let $T > 0$ such that

$$4C\big(\|u_0\|_{B^{5/2}_{2,1}} + \|v_0\|_{B^{5/2}_{2,1}}\big)^2 T = 4CV^2 T < 1.$$

For any $k \in \mathbb{N}$, suppose

$$W_k(t) \leq \frac{V}{(1-4CV^2t)^{1/2}}, \quad \forall t \in [0,T]. \tag{3.6}$$

We now prove that it also holds true for $k+1$. Since $U_n(t) = \int_0^t \|u_k v_k\|_{B^{5/2}_{2,1}} d\tau$, by (3.6), we have

$$e^{CU_k(t)-CU_k(\tau)} \leq \exp\left\{ C\int_\tau^t \frac{V^2}{1-4CV^2t'}\,\mathrm{d}t' \right\} = \left(\frac{1-4CV^2\tau}{1-4CV^2t}\right)^{\frac{1}{4}}. \tag{3.7}$$

Note that $U_k(0)=0$, let $\tau=0$, then we have

$$e^{CU_k(t)} \leq \frac{1}{(1-4CV^2t)^{\frac{1}{4}}}. \tag{3.8}$$

Using (3.3), (3.6), (3.7) and (3.8) gives rise to

$$\begin{aligned}
W_{k+1} &\leq e^{CU_k(t)}V + C\int_0^t e^{CU_k(t)-CU_k(\tau)} W_k(\tau)^3\,\mathrm{d}\tau \\
&\leq \frac{1}{(1-4CV^2t)^{\frac{1}{4}}}V + C\int_0^t \left(\frac{1-4CV^2\tau}{1-4CV^2t}\right)^{\frac{1}{4}} \frac{V^3}{(1-4CV^2\tau)^{3/2}}\,\mathrm{d}\tau \\
&\leq \frac{1}{(1-4CV^2t)^{\frac{1}{4}}}\left[V + V\int_0^t \frac{CV^2}{(1-4CV^2\tau)^{5/4}}\,\mathrm{d}\tau\right] = \frac{V}{(1-4CV^2t)^{1/2}}.
\end{aligned}$$

Therefore, (3.6) holds for all $k \geq 1$. Particularly, if

$$T = \frac{3}{16CV^2} = \frac{3}{16C(\|u_0\|_{B^{5/2}_{2,1}} + \|v_0\|_{B^{5/2}_{2,1}})^2},$$

then for all $k \geq 1$, the solution $\{(u_k, v_k)\}$ exists for $0 \leq t \leq T$ and satisfies

$$\|u_k(t)\|_{B^{5/2}_{2,1}} + \|v_k(t)\|_{B^{5/2}_{2,1}} \leq 2\big(\|u_0\|_{B^{3/2}_{2,1}} + \|v_0\|_{B^{3/2}_{2,1}}\big), \quad 0 \leq t \leq T. \tag{3.9}$$

Furthermore, for $t \in [0,T]$, using Eq. (3.1) yields

$$\|\partial_t u_k\|_{B^{3/2}_{2,1}} \leq \|u_k\|_{B^{3/2}_{2,1}} \|v_k\|_{B^{3/2}_{2,1}} \|\partial_x u_{k+1}\|_{B^{3/2}_{2,1}} + \big\|F^1(u_k, v_k)\big\|_{B^{5/2}_{2,1}} \lesssim \big(\|u_0\|_{B^{5/2}_{2,1}} + \|v_0\|_{B^{5/2}_{2,1}}\big)^3.$$

Similarly,

$$\|\partial_t v_k\|_{B^{3/2}_{2,1}} \lesssim \big(\|u_0\|_{B^{5/2}_{2,1}} + \|v_0\|_{B^{5/2}_{2,1}}\big)^3. \tag{3.10}$$

Hence $\{(u_k, v_k)\} \subset C([0,T]; B^{5/2}_{2,1}) \cap C^1([0,T]; B^{3/2}_{2,1})$ is uniformly bounded.

3.3. Convergence of the approximate solutions

In this subsection, we will prove that $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T]; B^s_{2,1} \times B^s_{2,1})$ $(0 < s < 5/2)$ by using Lemma 2.5. Actually, the estimate of $(u_{k+l+1} - u_{l+1}, v_{k+l+1} - v_{l+1})$ is essential in the derivation of Lemma 2.8. For simplicity, let

$$H_{k,l}(t) = \big\|(u_{k+l} - u_l)(t)\big\|_{B^{3/2}_{2,\infty}} + \big\|(v_{k+l} - v_l)(t)\big\|_{B^{3/2}_{2,\infty}}, \quad k, l \in \mathbb{N}.$$

Since $\{(u_k, v_k)\} \subset C([0,T]; B^{5/2}_{2,1})$ is uniformly bounded, there is an $M > 0$ such that

$$\|u_{k+l} - u_l\|_{B^{5/2}_{2,\infty}} + \|v_{k+l} - v_l\|_{B^{5/2}_{2,\infty}} \lesssim \|u_{k+l} - u_l\|_{B^{5/2}_{2,1}} + \|v_{k+l} - v_l\|_{B^{5/2}_{2,1}} \le M, \quad k, l \in \mathbb{N}.$$

Using the estimates analogous to those in Lemma 2.8, we obtain that for $k, l \in \mathbb{N}$,

$$H_{k+1,l}(t) \lesssim H_{k+1,l}(0) + C\int_0^t H_{k,l}(t) \ln\left(\mathrm{e} + \frac{M}{H_{k,l}(\tau)}\right) \mathrm{d}\tau, \quad t \in [0,T].$$

According to the definition of S_j (see, e.g. [32, page 2142] for the details), we obtain

$$H_{k+1,l}(0) = \big\|(u_{k+l+1} - u_{l+1})(0)\big\|_{B^{3/2}_{2,\infty}} \lesssim \big\|(u_{k+l+1} - u_{l+1})(0)\big\|_{B^{3/2}_{2,1}} \le C2^{-k}\|u_0\|_{B^{5/2}_{2,1}},$$

and

$$H_{k+1,l}(t) \lesssim \left(2^{-k} + \int_0^t H_k(\tau) \ln\left(\mathrm{e} + \frac{M}{H_k(\tau)}\right) \mathrm{d}\tau\right), \quad t \in [0,T].$$

As the function $x\ln(\mathrm{e} + \frac{M}{x})$ is nondecreasing, we see that $H_k(t) \triangleq \sup_{l\in\mathbb{N}} H_{k,l}(t)$ satisfies

$$H_{k+1}(t) \lesssim \left(2^{-k} + \int_0^t H_k(\tau) \ln\left(\mathrm{e} + \frac{M}{H_k(\tau)}\right) \mathrm{d}\tau\right), \quad t \in [0,T].$$

Let $\widetilde{H}(t) = \limsup_{k\to\infty} H_k(t)$. For any given $\varepsilon > 0$, there exists a $K \in \mathbb{N}$ such that $H_k(t) \le \widetilde{H}(t) + \varepsilon$ for $k > K$. Then we have

$$H_{k+1}(t) \lesssim \left(2^{-k} + \int_0^t \big(\widetilde{H}(\tau) + \varepsilon\big) \ln\left(\mathrm{e} + \frac{M}{\widetilde{H}(\tau) + \varepsilon}\right) \mathrm{d}\tau\right), \quad t \in [0,T], \ k > K.$$

Therefore, for $t \in [0,T]$, we have

$$\begin{aligned}\limsup_{n\to\infty} H_k(t) &\lesssim \limsup_{n\to\infty}\left(2^{-k} + \int_0^t \big(\widetilde{H}(\tau) + \varepsilon\big) \ln\left(\mathrm{e} + \frac{M}{\widetilde{H}(\tau) + \varepsilon}\right) \mathrm{d}\tau\right)\\ &= \int_0^t \big(\widetilde{H}(\tau) + \varepsilon\big) \ln\left(\mathrm{e} + \frac{M}{\widetilde{H}(\tau) + \varepsilon}\right) \mathrm{d}\tau.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\widetilde{H}(t) \lesssim \int_0^t \widetilde{H}(\tau)\ln\left(\mathrm{e} + \frac{M}{\widetilde{H}(\tau)}\right) \mathrm{d}\tau.$$

As the function $x\ln(\mathrm{e}+\frac{1}{x})$ is nondecreasing and $\int_0^1 \frac{1}{x\ln(\mathrm{e}+\frac{M}{x})}\,\mathrm{d}x = +\infty$, from Lemma 2.5, we obtain $\widetilde{H}(t) = 0$. In other words, $\{(u_k, v_k)\}$ is a Cauchy sequence in $C([0,T]; B^{3/2}_{2,\infty} \times B^{3/2}_{2,\infty})$. By (4) in Lemma 2.1, we see

$$\|u_{k+l} - u_l\|_{B^{3/2}_{2,1}} \lesssim \|u_{k+l} - u_l\|_{B^2_{2,1}} \lesssim \|u_{k+l} - u_l\|^{\frac{1}{2}}_{B^{3/2}_{2,\infty}} \|u_{k+l} - u_l\|^{\frac{1}{2}}_{B^{5/2}_{2,\infty}} \lesssim \|u_{k+l} - u_l\|^{\frac{1}{2}}_{B^{3/2}_{2,\infty}}.$$

Similarly, $\|v_{k+l} - v_l\|_{B^{3/2}_{2,1}} \lesssim \|v_{k+l} - v_l\|^{\frac{1}{2}}_{B^{3/2}_{2,\infty}}$. Hence $\{(u_k, v_k)\}_{k\in\mathbb{N}}$ is actually a Cauchy sequence in $C([0,T]; B^{3/2}_{2,1} \times B^{3/2}_{2,1})$ and converges to some $(u,v) \in C([0,T]; B^{3/2}_{2,1} \times B^{3/2}_{2,1})$.

3.4. Existence of the solution

Since $\|u_k\|_{B^{5/2}_{2,1}} + \|v_k\|_{B^{5/2}_{2,1}}$ is uniformly bounded by $2(\|u_0\|_{B^{5/2}_{2,1}} + \|v_0\|_{B^{5/2}_{2,1}})$, the property (5) in Lemma 2.1 guarantees that

$$\|u\|_{B^{5/2}_{2,1}} + \|v\|_{B^{5/2}_{2,1}} \leq 2\big(\|u_0\|_{B^{5/2}_{2,1}} + \|v_0\|_{B^{5/2}_{2,1}}\big), \tag{3.11}$$

which implies that $(u,v) \in L^\infty(0,T; B^{5/2}_{2,1} \times B^{5/2}_{2,1})$. When $3/2 < s' < 5/2$, by interpolation again, we have

$$\|u_k - u\|_{B^{s'}_{2,1}} \leq \|u_k - u\|^{\theta}_{B^{3/2}_{2,1}} \|u_k - u\|^{1-\theta}_{B^{5/2}_{2,1}} \leq \|u_k - u\|^{\theta}_{B^{3/2}_{2,1}} \big(4\|u_0\|_{B^{5/2}_{2,1}}\big)^{1-\theta}, \tag{3.12}$$

where $\theta = 5/2 - s'$. Similarly, we also obtain

$$\|v_k - u\|_{B^{s'}_{2,1}} \leq \|v_k - v\|^{\theta}_{B^{3/2}_{2,1}} \big(4\|v_0\|_{B^{5/2}_{2,1}}\big)^{1-\theta}, \quad \theta = 5/2 - s'. \tag{3.13}$$

Consequently, $(u_k, v_k)_{k\in\mathbb{N}}$ converges to some (u,v) in $C([0,T]; B^s_{2,1} \times B^s_{2,1})$ with $3/2 < s < 5/2$. For the nonlinear terms in (3.1), using (2.8) yields

$$\big\|F^j(u_k, v_k) - F^j(u,v)\big\|_{B^s_{2,1}} \lesssim \big(\|u_k - u\|_{B^s_{2,1}} + \|v_k - v\|_{B^s_{2,1}}\big), \quad 3/2 < s < 5/2,\ j = 1, 2.$$

Taking limits to (3.1), we deduce that u indeed solves Eq. (1.6).

3.5. Regularity and uniqueness of the solution

For the regularity of (u,v), we note that $(u,v) \in L^\infty(0,T; B^{5/2}_{2,1} \times B^{5/2}_{2,1})$. Therefore, $(uv)_x \in L^\infty(0,T; B^{3/2}_{2,1})$ and by Lemma 2.6 we see that $F^j(u,v) \in L^1(0,T; B^{5/2}_{2,1} \times B^{5/2}_{2,1})$ $(j = 1, 2)$. Via Lemma 2.2, we see that $(u,v) \in C([0,T]; B^{5/2}_{2,1} \times B^{5/2}_{2,1})$. Moreover, from (1.6) and Lemma 2.6, we finally obtain that

$$(u,v) \in C\big([0,T]; B^{5/2}_{2,1} \times B^{5/2}_{2,1}\big) \cap C^1\big([0,T]; B^{3/2}_{2,1} \times B^{3/2}_{2,1}\big).$$

The uniqueness is a corollary of Lemma 2.8. Hereto we complete the proof of Theorem 1.1.

4. Proof of Theorem 1.2

For the initial data $(u_0, v_0) \in B(0,R) \subset B^{5/2}_{2,1} \times B^{5/2}_{2,1}$, by (1.9), we see that the lifespan $T_{u,v}$ of the corresponding solution (u,v) to (1.6) satisfies

$$T_{u,v} \geq \frac{3}{16C(\|u_0\|_{B^{5/2}_{2,1}} + \|v_0\|_{B^{5/2}_{2,1}})^2} > \frac{C}{R^2} \triangleq \widetilde{T}.$$

Clearly, $\widetilde{T}$ does not depend on (u,v). Therefore, we can find a $\widetilde{T} > 0$ such that for all $(u_0, v_0) \in B(0,R) \subset B^{5/2}_{2,1} \times B^{5/2}_{2,1}$, the corresponding solution $(u,v) \in C([0,\widetilde{T}]; B^{5/2}_{2,1} \times B^{5/2}_{2,1})$. Directly from (1.9) and Lemma 2.8, Theorem 1.2 is proved.

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References

[1] W. Arendt, S. Bu, Operator-valued Fourier multipliers on periodic Besov spaces and applications, Proc. Edinb. Math. Soc. 47 (2004) 15–33.
[2] H. Bahouri, J.Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren Math. Wiss., vol. 343, Springer, Berlin–Heidelberg–New York, 2011.
[3] J.Y. Chemin, Perfect Incompressible Fluids, Oxford Lecture Ser. Math. Appl., vol. 14, The Clarendon Press, Oxford University Press, New York, 1998.
[4] J.Y. Chemin, Localization in Fourier space and Navier–Stokes, in: Phase Space Analysis of Partial Differential Equations, in: CRM Series, Scuola Norm. Sup. Pisa, 2004, pp. 53–136.
[5] D. Chen, Y. Li, W. Yan, The Cauchy problem for a new two-component integrable system, to appear in Contin. Dyn. Syst. Ser. A.
[6] A. Constantin, R. Ivanov, On an integrable two-component Camassa–Holm shallow water system, Phys. Lett. A 372 (2008) 7129–7132.
[7] R. Danchin, A few remarks on the Camassa–Holm equation, Differential Integral Equations 14 (2001) 953–988.
[8] R. Danchin, A note on well-posedness for Camassa–Holm equation, J. Differential Equations 192 (2003) 429–444.
[9] R. Danchin, Fourier analysis methods for PDE's, Lecture Notes, November 14, 2005.
[10] A. Degasperis, D.D. Holm, A.N.W. Hone, A new integral equation with peakon solutions, Theoret. Math. Phys. 133 (2002) 1463–1474.
[11] A. Degasperis, M. Procesi, Asymptotic Integrability Symmetry and Perturbation Theory, World Sci. Publ., 1999, pp. 23–37.
[12] H. Dullin, G. Gottwald, D. Holm, Camassa–Holm, Korteweg–de Vries-5 and other asymptotically equivalent equations for shallow water waves, Fluid Dyn. Res. 33 (2003) 73–95.
[13] J. Escher, O. Lechtenfeld, Z. Yin, Well-posedness and blow-up phenomena for the 2-component Camassa–Holm equation, Discrete Contin. Dyn. Syst. 19 (2007) 493–513.
[14] J. Escher, Y. Liu, Z. Yin, Global weak solutions and blow-up structure for the Degasperis–Procesi equation, J. Funct. Anal. 241 (2006) 457–485.
[15] X. Geng, B. Xue, An extension of integrable peakon equations with cubic nonlinearity, Nonlinearity 22 (2009) 1847–1856.
[16] C. Guan, Z. Yin, Global existence and blow-up phenomena for an integrable two-component Camassa–Holm shallow water system, J. Differential Equations 248 (2010) 2003–2014.
[17] Z. Guo, Y. Zhou, On solutions to a two-component generalized Camassa–Holm equation, Stud. Appl. Math. 124 (3) (2010) 307–322.
[18] Z. Guo, M. Zhu, Wave breaking for a modified two component Camassa–Holm system, J. Differential Equations 252 (2012) 2759–2770.
[19] A. Himonas, C. Holliman, On well-posedness of the Degasperis–Procesi equation, Discrete Contin. Dyn. Syst. Ser. A 31 (2011) 469–488.
[20] A. Hone, H. Lundmark, J. Szmigielski, Explicit multipeakon solutions of Novikov's cubically nonlinear integrable Camassa–Holm type equation, Dyn. Partial Differ. Equ. 6 (2009) 253–289.
[21] A. Hone, J. Wang, Integrable peakon equations with cubic nonlinearity, J. Phys. A 41 (2008) 372002, 10 pp.
[22] Z. Jiang, L. Ni, Blow-up phenomenon for the integrable Novikov equation, J. Math. Anal. Appl. 385 (2012) 551–558.
[23] N. Li, Q.P. Liu, On bi-Hamiltonian structure of two-component Novikov equation, Phys. Lett. A 377 (2013) 257–261.
[24] Y. Liu, Z. Yin, Global existence and blow-up phenomena for the Degasperis–Procesi equation, Comm. Math. Phys. 267 (2006) 801–820.

[25] Y. Mi, C. Mu, W. Tao, On the Cauchy problem for the two-component Novikov equation, Adv. Math. Phys. (2014), http://dx.doi.org/10.1155/2013/810725.
[26] L. Ni, Y. Zhou, Well-posedness and persistence properties for the Novikov equation, J. Differential Equations 250 (2011) 3002–3201.
[27] V. Novikov, Generalizations of the Camassa–Holm equation, J. Phys. A 42 (2009) 342002, 14 pp.
[28] H.J. Schmeisser, H. Triebel, Topics in Fourier Analysis and Function Spaces, Wiley, Chichester, New York, 1987.
[29] H. Triebel, Theory of Function Space, Birkhäuser, Basel, 1983.
[30] X. Wu, Z. Yin, A note on the Cauchy problem of the Novikov equation, Appl. Anal. 92 (6) (2013), http://dx.doi.org/10.1080/00036811.2011.649735.
[31] W. Yan, Y. Li, Y. Zhang, The Cauchy problem for the integrable Novikov equation, J. Differential Equations 253 (2012) 298–318.
[32] K. Yan, Z. Yin, On the Cauchy problem for a two-component Degasperis–Procesi system, J. Differential Equations 252 (2012) 2131–2159.
[33] Z. Yin, On the Cauchy problem for an integrable equation with peakon solutions, Illinois J. Math. 47 (2003) 649–666.
[34] Y. Zhou, Blow-up phenomenon for the integrable Degasperis–Procesi equation, Phys. Lett. A 328 (2004) 157–162.
[35] M. Zhu, Blow-up, global existence and persistence properties for the coupled Camassa–Holm equations, Math. Phys. Anal. Geom. 14 (3) (2011) 197–209.