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Global existence of the two-dimensional QGE with sub-critical dissipation

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ABSTRACT

In this paper, we study the sub-critical dissipative quasi-geostrophic equations $(\mathbf{S}_\alpha)$. We prove that there exists a unique local-in-time solution for any large initial data θ_0 in the space $\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$ defined by (1). Moreover, we show that $(\mathbf{S}_\alpha)$ has a global solution in time if the norms of the initial data in $\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$ are bounded by 1/4. Also, we prove a blow-up criterion of the local-in-time solution of $(\mathbf{S}_\alpha)$.

1. Introduction

In this article, we study the initial value problem for the two-dimensional quasi-geostrophic equation with sub-critical dissipation $(\mathbf{S}_\alpha)$, $1/2 < \alpha \leq 1$.

$$\begin{cases} \partial_t\theta + (-\Delta)^\alpha\theta + u.\nabla\theta = 0 \\ \theta(0) = \theta_0 \end{cases} \tag{$\mathbf{S}_\alpha$}$$

where $1/2 < \alpha \leq 1$ is a real number. The variable θ represents the potential temperature and $u = (\partial_2(-\Delta)^{-1/2}\theta, -\partial_1(-\Delta)^{-1/2}\theta)$ is the fluid velocity. In the following, we are interested in the case when $1/2 < \alpha \leq 1$.

The mathematical study of the non-dissipative case has first been proposed by Constantin, Majda and Tabak in [9] where it is shown to be an analogue to the $3D$ Euler equations. The dissipative case has then been studied by Constantin and Wu in [10] when $\alpha > 1/2$ where the authors studied global existence and decay of solutions in Sobolev spaces.

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In this paper, we study ($\mathbf{S}_\alpha$) in scaling invariant spaces. Solvability of evolution equations in scaling invariant spaces is well-developed in the context of the Navier–Stokes equations. For example, if we restrict the function spaces to the energy spaces, the optimal result is due to Fujita and Kato in [14]. Later, Chemin [8] proved similar results in the framework of Besov spaces $\dot{B}^{\frac{q}{p}-1}_{p,q}$. Let us find the scaling invariant critical spaces for ($\mathbf{S}_\alpha$). The equation is invariant under the following scaling:

$$\theta_\lambda(t,x) = \lambda^{2\alpha-1}\theta\big(\lambda^{2\alpha}t, \lambda x\big), \quad \text{with initial data } \theta_\lambda(0,x) = \theta^0_\lambda(x) = \lambda^{2\alpha-1}\theta^0(\lambda x).$$

So $L^{\frac{2}{2\alpha-1}}$, $\dot{H}^{2-2\alpha}$ and $\dot{B}^{1+\frac{2}{p}-2\alpha}_{p,q}$ are critical spaces. The global well-posedness for small initial data in critical Besov spaces with $p<\infty$ was obtained in [3,21]. The global well-posedness for large data in critical spaces was obtained by several authors; in Lebesgue space $L^{\frac{2}{2\alpha-1}}$ by Carrillo and Ferreira [7] for $\alpha > \frac{1}{2}$, in energy space H^1 by Dong and Du [11], and in Besov spaces by Abidi and Hmidi [1], Hmidi and Keraani [16] and Wang and Zhang [20] with $p=\infty$.

In this paper, we will solve the system ($\mathbf{S}_\alpha$) in a critical space, whose definition is based on the Fourier transform, by means of a contraction argument. Thus, we can obtain a unique local solution to the system ($\mathbf{S}_\alpha$) for any initial data in the critical space and prove the corresponding solution will be global if the initial data is sufficient small. Before giving our main result, let us first define our setting.

For $\sigma \in \mathbb{R}$, we define the functional space

$$\mathcal{X}^\sigma\big(\mathbb{R}^2\big) = \big\{f \in \mathcal{S}'\big(\mathbb{R}^2\big);\ \big(\xi \mapsto |\xi|^\sigma \widehat{f}(\xi)\big) \in L^1\big(\mathbb{R}^2\big)\big\}, \tag{1}$$

which is equipped with the norm

$$\|f\|_{\mathcal{X}^\sigma} = \int_{\mathbb{R}^2} |\xi|^\sigma \big|\widehat{f}(\xi)\big| d\xi.$$

In [18], Z. Lei and F. Lin showed well-posedness of solutions in $\mathcal{X}^\sigma(\mathbb{R}^2)$ for Navier–Stokes equations with $\sigma = -1$. The space (1) belongs to a class whose definition of the norm is based on Fourier transform, but it is not contained in L^2, i.e., there is $f \in \mathcal{X}^\sigma(\mathbb{R}^2)$ with infinite L^2-norm. Several examples of spaces with definition based on Fourier transform and containing singular data with infinite L^2-norm have been used to study well-posedness of PDEs of parabolic, elliptic and dispersive types. The reader is referred to [4–6,19,12,15,17,2,13] and their references.

Our local and global existence results read as follows.

Theorem 1. *Let $\theta^0 \in \mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$. There is a time $T>0$ and a unique solution $\theta \in \mathcal{C}([0,T], \mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$ of ($\mathbf{S}_\alpha$) such that $\theta \in L^1([0,T], \mathcal{X}^1(\mathbb{R}^2))$. If $\|\theta^0\|_{\mathcal{X}^{1-2\alpha}} < 1/4$, then the solution θ is global and*

$$\|\theta\|_{\mathcal{X}^{1-2\alpha}} + \big(1 - 4\big\|\theta^0\big\|_{\mathcal{X}^{1-2\alpha}}\big)\int_0^t \|\theta\|_{\mathcal{X}^1} \le \big\|\theta^0\big\|_{\mathcal{X}^{1-2\alpha}}, \quad \forall t \ge 0. \tag{2}$$

The proof of the above theorem is based on fixed point theorem. We will establish a local existence result and be able to get global existence that is essentially based on a pointwise estimate of the Fourier transform of θ.

We state now our second main result.

Theorem 2. *Let $\theta \in \mathcal{C}([0,T^*),\mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$ be the maximal solution of* ($\mathbf{S}_\alpha$) *given by Theorem 1. Then*

$$T^* < \infty \quad \Longrightarrow \quad \int_0^{T^*} \|\theta\|_{\mathcal{X}^1} = \infty. \tag{3}$$

The rest of this paper is divided into three sections. Section 2 is divided into two. Section 2.1 fixes notations and Section 2.2 contains preliminary results. Section 3 is divided into three subsections: 3.1 deals with the existence of solution with any large initial data in $\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$, Section 3.2 deals with the uniqueness of solution, and Section 3.3 is devoted to global existence. Section 4 proves a blow-up criterion of the local-in-time solution given by Theorem 1.

2. Notations and preliminary results

2.1. Notations

In this short paragraph, we give some notations:

- For f, we denote

$$u_f = \big(\partial_2(-\Delta)^{-1/2} f, -\partial_1(-\Delta)^{-1/2} f\big)$$

- The Fourier transform $\mathcal{F}(f)$ of a tempered distribution f on $\mathbb{R}^2$ is defined as

$$\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^2} e^{-ix\xi} f(x)\, dx.$$

- The inverse Fourier formula is

$$\mathcal{F}^{-1}(f)(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{ix\xi} f(x)\, dx.$$

- For any Banach space $(B, \|.\|)$, any real number $1 \le p \le \infty$ and any time $T > 0$, we will denote by $L^p_T(B)$ the space of all measurable functions $t \in [0,T] \longmapsto f(t) \in B$ such that $(t \longmapsto \|f(t)\|) \in L^p([0,T])$.
- The fractional Laplacian operator $(-\Delta)^\alpha$ for a real number α is defined through the Fourier transform, namely

$$\widehat{(-\Delta)^\alpha} f(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi).$$

2.2. Preliminary results

The main results of this section are the following lemmas that will play a crucial role in the proof of our main theorem.

Lemma 3. *Let $\alpha \in]\frac{1}{2},1]$. We have the following inequalities*

$$\|f\|_{\mathcal{X}^{2-2\alpha}} \le \|f\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \|f\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}} \tag{4}$$

and

$$\|f\|_{\mathcal{X}^0} \le \|f\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \|f\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \tag{5}$$

Proof of (4). By choosing $(p,q) = (2\alpha, \frac{2\alpha}{2\alpha-1})$ and applying Hölder's inequality, we deduce that

$$\begin{aligned}
\|f\|_{\mathcal{X}^{2-2\alpha}} &= \int_{\mathbb{R}^2} |\xi|^{2-2\alpha} \big|\widehat{f}(\xi)\big| d\xi \\
&\le \int_{\mathbb{R}^2} \left(|\xi|\big|\widehat{f}(\xi)\big|\right)^{\frac{1}{2\alpha}} \left(|\xi|^{1-2\alpha}\big|\widehat{f}(\xi)\big|\right)^{\frac{2\alpha-1}{2\alpha}} d\xi \\
&\le \Bigg(\int_{\mathbb{R}^2} |\xi|\big|\widehat{f}(\xi)\big| d\xi\Bigg)^{\frac{1}{2\alpha}} \Bigg(\int_{\mathbb{R}^2} |\xi|^{1-2\alpha}\big|\widehat{f}(\xi)\big| d\xi\Bigg)^{\frac{2\alpha-1}{2\alpha}}.
\end{aligned}$$

Hence,

$$\|f\|_{\mathcal{X}^{2-2\alpha}} \le \|f\|_{\mathcal{X}^1}^{1/2\alpha} \|f\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}}. \qquad \square$$

Proof of (5). To estimate $\|f\|_{\mathcal{X}^0}$, we choose $p = 2\alpha$ and $q = \frac{2\alpha}{2\alpha-1}$ and apply Hölder's inequality. We obtain

$$\begin{aligned}
\|f\|_{\mathcal{X}^0} &= \int_{\mathbb{R}^2} |\xi|^0 \big|\widehat{f}(\xi)\big| d\xi \\
&\le \int_{\mathbb{R}^2} |\xi|^{\frac{2\alpha-1}{2\alpha}} \big|\widehat{f}(\xi)\big|^{\frac{2\alpha-1}{2\alpha}} |\xi|^{\frac{1-2\alpha}{2\alpha}} \big|\widehat{f}(\xi)\big|^{\frac{1}{2\alpha}} d\xi \\
&\le \Bigg(\int_{\mathbb{R}^2} |\xi|\big|\widehat{f}(\xi)\big| d\xi\Bigg)^{1-\frac{1}{2\alpha}} \Bigg(\int_{\mathbb{R}^2} |\xi|^{1-2\alpha}\big|\widehat{f}(\xi)\big| d\xi\Bigg)^{1/2\alpha}
\end{aligned}$$

Hence,

$$\|f\|_{\mathcal{X}^0} \le \|f\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \|f\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}}. \qquad \square$$

The following lemma, which is a direct consequence of the preceding one, will be useful in the proof of Theorem 1.

Lemma 4. *Let $f, g \in \mathcal{X}^{1-2\alpha}(\mathbb{R}^2) \cap \mathcal{X}^1(\mathbb{R}^2)$. We have the following inequalities*

$$\|fg\|_{\mathcal{X}^{2-2\alpha}} \le 2\|f\|_{\mathcal{X}^{2-2\alpha}} \|g\|_{\mathcal{X}^0} + 2\|f\|_{\mathcal{X}^0} \|g\|_{\mathcal{X}^{2-2\alpha}}. \tag{6}$$

$$\begin{aligned}
\|fg\|_{\mathcal{X}^{2-2\alpha}} \le{}& 2\|f\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \|f\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}} \|g\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \|g\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \\
&+ 2\|f\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \|f\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \|g\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \|g\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}}.
\end{aligned} \tag{7}$$

Particularly

$$\|f.f\|_{\mathcal{X}^{2-2\alpha}} \le 4\|f\|_{\mathcal{X}^{1-2\alpha}} \|f\|_{\mathcal{X}^1}. \tag{8}$$

Proof of (6).

$$\begin{aligned}\|fg\|_{\mathcal{X}^{2-2\alpha}} &= \int_{\mathbb{R}^2} |\xi|^{2-2\alpha}\big|\widehat{fg}(\xi)\big|d\xi \\ &= \int_{\mathbb{R}^2} |\xi|^{2-2\alpha}\big|\widehat{f}*\widehat{g}(\xi)\big|d\xi \\ &= \int_{\mathbb{R}^2} |\xi|^{2-2\alpha}\bigg|\int_{\mathbb{R}^2}\widehat{f}(\eta)\widehat{g}(\xi-\eta)d\eta\bigg|d\xi.\end{aligned}$$

Then

$$\begin{aligned}\|fg\|_{\mathcal{X}^{2-2\alpha}} &\leq \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\xi|^{2-2\alpha}\big|\widehat{f}(\eta)\big|\big|\widehat{g}(\xi-\eta)\big|d\eta d\xi \\ &\leq \int_{\mathbb{R}^2} |\xi|^{2-2\alpha}\int_{\mathbb{R}^2}\big|\widehat{f}(\eta)\big|\big|\widehat{g}(\xi-\eta)\big|d\eta d\xi \\ &\leq I_1+I_2,\end{aligned}$$

where

$$\begin{aligned}I_1 &:= \int_{\mathbb{R}^2} |\xi|^{2-2\alpha}\int_{|\eta|<|\xi-\eta|}\big|\widehat{f}(\eta)\big|\big|\widehat{g}(\xi-\eta)\big|d\eta d\xi \\ I_2 &:= \int_{\mathbb{R}^2} |\xi|^{2-2\alpha}\int_{|\eta|>|\xi-\eta|}\big|\widehat{f}(\eta)\big|\big|\widehat{g}(\xi-\eta)\big|d\eta d\xi.\end{aligned}$$

On the one hand, if $|\eta|<|\xi-\eta|$, we have

$$|\xi| \leq |\xi-\eta|+|\eta| \leq 2\max\big(|\xi-\eta|,|\eta|\big)=2|\xi-\eta|.$$

Then

$$\begin{aligned}|\xi|^{2-2\alpha} &\leq 2^{2-2\alpha}|\xi-\eta|^{2-2\alpha} \\ &\leq 2|\xi-\eta|^{2-2\alpha}, \quad \text{for all } \alpha\in(1/2,1].\end{aligned}$$

Therefore

$$\begin{aligned}I_1 &= \int_{\mathbb{R}^2} |\xi|^{2-2\alpha}\int_{|\eta|<|\xi-\eta|}\big|\widehat{f}(\eta)\big|\big|\widehat{g}(\xi-\eta)\big|d\eta d\xi \\ &\leq 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\big|\widehat{f}(\eta)\big||\xi-\eta|^{2-2\alpha}\big|\widehat{g}(\xi-\eta)\big|d\eta d\xi \\ &\leq 2\big\||\widehat{f}|*|\xi|^{2-2\alpha}|\widehat{g}|\big\|_{L^1}.\end{aligned}$$

By Young's inequality, we get

$$I_1 \leq 2\|f\|_{\mathcal{X}^0}\|g\|_{\mathcal{X}^{2-2\alpha}}. \tag{9}$$

On the other hand, if $|\eta| > |\xi - \eta|$, we have

$$|\xi| \leq |\xi - \eta| + |\eta| \leq 2\max\big(|\xi - \eta|, |\eta|\big) = 2|\eta|.$$

Then

$$\begin{aligned} |\xi|^{2-2\alpha} &\leq 2^{2-2\alpha}|\eta|^{2-2\alpha} \\ &\leq 2|\eta|^{2-2\alpha}, \quad \text{for all } \alpha \in (1/2, 1]. \end{aligned}$$

Hence

$$\begin{aligned} I_2 &= \int\limits_{\mathbb{R}^2} |\xi|^{2-2\alpha} \int\limits_{|\eta|>|\xi-\eta|} \big|\widehat{f}(\eta)\big|\big|\widehat{g}(\xi - \eta)\big| d\eta d\xi \\ &\leq 2 \int\limits_{\mathbb{R}^2}\int\limits_{\mathbb{R}^2} |\eta|^{2-2\alpha}\big|\widehat{f}(\eta)\big|\big|\widehat{g}(\xi - \eta)\big| d\eta d\xi \\ &\leq 2\big\||\xi|^{2-2\alpha}|\widehat{f}| * |\widehat{g}|\big\|_{L^1}. \end{aligned}$$

Young's inequality gives

$$I_2 \leq 2\|f\|_{\mathcal{X}^{2-2\alpha}}\|g\|_{\mathcal{X}^0}. \tag{10}$$

Combining (9) and (10), we get (6). □

Proof of (7). By including the inequalities (4) and (5) in (6), we obtain

$$\|fg\|_{\mathcal{X}^{2-2\alpha}} \leq 2\|f\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}}\|f\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}}\|g\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}}\|g\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} + 2\|f\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}}\|f\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}}\|g\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}}\|g\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}}. \quad \square$$

The proof of Theorem 1 requires the following lemma.

Lemma 5. *Let $\theta \in L_T^\infty(\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)) \cap L_T^1(\mathcal{X}^1(\mathbb{R}^2))$. Then*

$$\bigg\|\int\limits_0^t e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta) dz\bigg\|_{\mathcal{X}^{1-2\alpha}} \leq C_\alpha \|\theta\|_{L_T^\infty(\mathcal{X}^{1-2\alpha})}\|\theta\|_{L_T^1(\mathcal{X}^1)}.$$

Proof. We have

$$\begin{aligned} \bigg\|\int\limits_0^t e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta) dz\bigg\|_{\mathcal{X}^{1-2\alpha}} &\leq \int\limits_0^t \big\|e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta)\big\|_{\mathcal{X}^{1-2\alpha}} dz \\ &\leq \int\limits_0^t \big\|e^{-(t-z)|D|^{2\alpha}} (\theta u_\theta)\big\|_{\mathcal{X}^{2-2\alpha}} dz \\ &\leq \int\limits_0^t \|\theta u_\theta\|_{\mathcal{X}^{2-2\alpha}} dz. \end{aligned}$$

Using Lemma 4, we obtain

$$\left\|\int_0^t e^{-(t-z)|D|^{2\alpha}}\operatorname{div}(\theta u_\theta)dz\right\|_{\mathcal{X}^{1-2\alpha}} \le 2\int_0^t \|\theta\|_{\mathcal{X}^{2-2\alpha}}\|u_\theta\|_{\mathcal{X}^0} + \|\theta\|_{\mathcal{X}^0}\|u_\theta\|_{\mathcal{X}^{2-2\alpha}}dz.$$

Using Lemma 3 and the fact $\|u_\theta\|_{\mathcal{X}^1} \le \|\theta\|_{\mathcal{X}^1}$ and $\|u_\theta\|_{\mathcal{X}^{1-2\alpha}} \le \|\theta\|_{\mathcal{X}^{1-2\alpha}}$, we get

$$\begin{aligned}\left\|\int_0^t e^{-(t-z)|D|^{2\alpha}}\operatorname{div}(\theta u_\theta)dz\right\|_{\mathcal{X}^{1-2\alpha}} &\le 4\int_0^t \|\theta\|_{\mathcal{X}^{1-2\alpha}}\|\theta\|_{\mathcal{X}^1}\,dz \\ &\le 4\|\theta\|_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|\theta\|_{L^1_T(\mathcal{X}^1)}. \qquad \square\end{aligned}$$

Also the following lemma will be useful in the sequel.

Lemma 6. *Let $\theta \in L^\infty_T(\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)) \cap L^1_T(\mathcal{X}^1(\mathbb{R}^2))$. Then*

$$\int_0^T\left\|\int_0^t e^{-(t-z)|D|^{2\alpha}}\operatorname{div}(\theta u_\theta)dz\right\|_{\mathcal{X}^1} dt \le 4\|\theta\|_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|\theta\|_{L^1_T(\mathcal{X}^1)}.$$

Proof. We have

$$\begin{aligned}\int_0^T\left\|\int_0^t e^{-(t-z)|D|^{2\alpha}}\operatorname{div}(\theta u_\theta)dz\right\|_{\mathcal{X}^1} dt &\le \int_0^T\int_0^t \left\|e^{-(t-z)|D|^{2\alpha}}\operatorname{div}(\theta u_\theta)(z)\right\|_{\mathcal{X}^1}dzdt \\ &\le \int_0^T\int_0^t\int_{\mathbb{R}^2} e^{-(t-z)|\xi|^{2\alpha}}|\xi|^2\left|\mathcal{F}(\theta u_\theta)(z,\xi)\right|d\xi dzdt \\ &\le \int_{\mathbb{R}^2}|\xi|^2\left(\int_0^T\int_0^t e^{-(t-z)|\xi|^{2\alpha}}\left|\mathcal{F}(\theta u_\theta)(z,\xi)\right|dzdt\right)d\xi.\end{aligned}$$

By integrating the function $e^{-(t-z)|\xi|^{2\alpha}}$ twice with respect to $z \in [0,t]$ and with respect to $t \in [0,T]$, we get

$$\int_0^T\int_0^t e^{-(t-z)|\xi|^{2\alpha}}\left|\mathcal{F}(\theta u_\theta)(z,\xi)\right|dz\,dt = \int_0^T\left|\mathcal{F}(\theta u)(z,\xi)\right|\left(\int_z^T e^{-(t-z)|\xi|^{2\alpha}}dt\right)dz.$$

Then

$$\begin{aligned}\int_0^T\left\|\int_0^t e^{-(t-z)|D|^{2\alpha}}\operatorname{div}(\theta u_\theta)dz\right\|_{\mathcal{X}^1} dt &\le \int_{\mathbb{R}^2}|\xi|^2\left(\int_0^T\left[\int_z^T e^{-(t-z)|\xi|^{2\alpha}}dt\right]\left|\mathcal{F}(\theta u_\theta)(z,\xi)\right|dz\right)d\xi \\ &\le \int_{\mathbb{R}^2}|\xi|^2\left(\int_0^T\left[\frac{1-e^{-(T-z)|\xi|^{2\alpha}}}{|\xi|^{2\alpha}}\right]\left|\mathcal{F}(\theta u_\theta)(z,\xi)\right|dz\right)d\xi\end{aligned}$$

$$\leq \int_{\mathbb{R}^2} \left(|\xi|^{2-2\alpha} \int_0^T \left|\mathcal{F}(\theta u_\theta)(z,\xi)\right| dz \right) d\xi$$
$$\leq \int_0^T \left\|(\theta u_\theta)(z)\right\|_{\mathcal{X}^{2-2\alpha}} dz.$$

Using Lemma 4 and the fact $\|u_\theta\|_{\mathcal{X}^1} \leq \|\theta\|_{\mathcal{X}^1}$ and $\|u_\theta\|_{\mathcal{X}^{1-2\alpha}} \leq \|\theta\|_{\mathcal{X}^{1-2\alpha}}$, we get

$$\int_0^T \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} \operatorname{div}(\theta u_\theta) dz \right\|_{\mathcal{X}^1} dt \leq 4 \int_0^T \left\|\theta(z)\right\|_{\mathcal{X}^{1-2\alpha}} \left\|\theta(z)\right\|_{\mathcal{X}^1} dz$$
$$\leq 4\|\theta\|_{L^\infty_T(\mathcal{X}^{1-2\alpha})} \|\theta\|_{L^1_T(\mathcal{X}^1)}. \qquad \square$$

3. Proof of Theorem 1

3.1. Existence

The idea of the proof is to write the initial condition as a sum of higher and lower frequencies. For small frequencies, we give a regular solution of the associated linear system to $(\mathbf{S}_\alpha)$ and for the higher frequencies we consider a partial differential equation very similar to $(\mathbf{S}_\alpha)$ with small initial data in $\mathcal{X}^{1-2\alpha}$ for which we can solve it by the fixed point theorem.

- Let r be a real number such that $0 < r < 1/20$.
- Let $N \in \mathbb{N}$, such that

$$\int_{|\xi|>N} |\xi|^{1-2\alpha} \left|\widehat{\theta^0}(\xi)\right| d\xi < \frac{r}{5}.$$

Put a^0 and b^0 defined by

$$a^0 = \mathcal{F}^{-1}\left(\mathbf{1}_{|\xi|<N} \widehat{\theta^0}(\xi)\right)$$
$$b^0 = \mathcal{F}^{-1}\left(\mathbf{1}_{|\xi|>N} \widehat{\theta^0}(\xi)\right).$$

Clearly

$$\left\|b^0\right\|_{\mathcal{X}^{1-2\alpha}} < \frac{r}{5}. \tag{11}$$

And put

$$a = e^{-t|D|^{2\alpha}} a^0,$$

a is the unique solution of the heat equation

$$\begin{cases} \partial_t a + (-\Delta)^\alpha a = 0 \\ a(0) = a^0. \end{cases}$$

We have

$$\|a\|_{\mathcal{X}^{1-2\alpha}} \leq \left\|\theta^0\right\|_{\mathcal{X}^{1-2\alpha}}, \quad \forall t \geq 0,$$

and

$$\begin{aligned}\|a\|_{L^1([0,T],\mathcal{X}^1)} &= \int_0^T \int_{\mathbb{R}^2} e^{-t|\xi|^{2\alpha}} |\xi| \big|\widehat{\theta^0}(\xi)\big| d\xi dt \\ &= \int_{\mathbb{R}^2} \left(\int_0^T e^{-t|\xi|^{2\alpha}} dt \right) |\xi| \big|\widehat{\theta^0}(\xi)\big| d\xi \\ &= \int_{\mathbb{R}^2} \big(1 - e^{-T|\xi|^{2\alpha}}\big) |\xi|^{1-2\alpha} \big|\widehat{\theta^0}(\xi)\big| d\xi.\end{aligned}$$

Using the dominated convergence theorem, we get

$$\lim_{T\to 0^+} \|a\|_{L^1([0,T],\mathcal{X}^1)} = 0. \tag{12}$$

Let $\varepsilon > 0$ such that

$$4\varepsilon \big\|\theta^0\big\|_{\mathcal{X}^{1-2\alpha}} < \frac{r}{5} \tag{13}$$

and

$$2\Big(\big\|\theta^0\big\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}} \varepsilon^{\frac{1}{2\alpha}} + \big\|\theta^0\big\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}} \varepsilon^{1-\frac{1}{2\alpha}}\Big) < \frac{1}{5}. \tag{14}$$

By Eq. (12), there is a time $T = T(\varepsilon) > 0$ such that

$$\|a\|_{L^1([0,T],\mathcal{X}^1)} < \varepsilon. \tag{15}$$

Put $b = \theta - a$, clearly b is a solution of the following system

$$\begin{cases} \partial_t b + (-\Delta)^\alpha b + (u_a + u_b)\nabla(a + b) = 0 \\ b(0) = b^0. \end{cases}$$

The integral form of b is as follows

$$b = e^{-t|D|^{2\alpha}} b^0 - \int_0^t e^{-(t-z)|D|^{2\alpha}} (u_a + u_b)\nabla(a + b) dz.$$

To prove the existence of b, put the following operator

$$\Psi(b) = e^{-t|D|^{2\alpha}} b^0 - \int_0^t e^{-(t-z)|D|^{2\alpha}} (u_a + u_b)\nabla(a + b) dz.$$

- Now, we introduce the space $\mathcal{X}_T$ as follows

$$\mathcal{X}_T = \mathcal{C}\big([0,T], \mathcal{X}^{1-2\alpha}(\mathbb{R}^2)\big) \cap L^1\big([0,T], \mathcal{X}^1(\mathbb{R}^2)\big)$$

with the norm

$$\|f\|_{\mathcal{X}_T} = \|f\|_{L^\infty_T(\mathcal{X}^{1-2\alpha})} + \|f\|_{L^1_T(\mathcal{X}^1)}.$$

Using Lemmas 5 and 6, we can prove $\Psi(\mathcal{X}_T) \subset \mathcal{X}_T$.

- Also, we denote by $\mathbf{B}_r$ the subset of $\mathcal{X}_T$ defined by

$$\mathbf{B}_r = \big\{\theta \in \mathcal{X}_T;\ \|\theta\|_{L^\infty([0,T],\mathcal{X}^{1-2\alpha})} \le r,\ \|\theta\|_{L^1([0,T],\mathcal{X}^1)} \le r\big\}.$$

- For $b \in \mathbf{B}_r$, we prove that $\Psi(b) \in \mathbf{B}_r$. In fact, we have

$$\big\|\Psi(b)(t)\big\|_{\mathcal{X}^{1-2\alpha}} \le \sum_{k=0}^{4} I_k,$$

where

$$\begin{aligned}
I_0 &= \big\|e^{-t|D|^{2\alpha}} b^0\big\|_{\mathcal{X}^{1-2\alpha}}\\
I_1 &= \int_0^t \big\|e^{-(t-z)|D|^{2\alpha}} u_a \nabla a\big\|_{\mathcal{X}^{1-2\alpha}} dz\\
I_2 &= \int_0^t \big\|e^{-(t-z)|D|^{2\alpha}} u_a \nabla b\big\|_{\mathcal{X}^{1-2\alpha}} dz\\
I_3 &= \int_0^t \big\|e^{-(t-z)|D|^{2\alpha}} u_b \nabla a\big\|_{\mathcal{X}^{1-2\alpha}} dz\\
I_4 &= \int_0^t \big\|e^{-(t-z)|D|^{2\alpha}} u_b \nabla b\big\|_{\mathcal{X}^{1-2\alpha}} dz.
\end{aligned}$$

Using Eq. (11), Lemma 5 and the fact $b \in \mathbf{B}_r$, we get

$$\begin{aligned}
I_0 &\le \frac{r}{5}\\
I_1 &\le 4\|a\|_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|a\|_{L^1_T(\mathcal{X}^1)} \le 4\varepsilon\big\|\theta^0\big\|_{\mathcal{X}^{1-2\alpha}} \le \frac{r}{5}\\
I_2, I_3 &\le 2\|a\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|a\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)}\|b\|^{\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|b\|^{1-\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)}\\
&\quad + 2\|a\|^{\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|a\|^{1-\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)}\|b\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|b\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)}\\
&\le 2\big(\big\|\theta^0\big\|^{1-\frac{1}{2\alpha}}_{\mathcal{X}^{1-2\alpha}}\varepsilon^{\frac{1}{2\alpha}} + \big\|\theta^0\big\|^{\frac{1}{2\alpha}}_{\mathcal{X}^{1-2\alpha}}\varepsilon^{1-\frac{1}{2\alpha}}\big)r\\
&\le \frac{r}{5}\\
I_4 &\le 4\|b\|_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|b\|_{L^1_T(\mathcal{X}^1)} \le 4r^2 \le \frac{r}{5}.
\end{aligned}$$

Then

$$\big\|\Psi(b)(t)\big\|_{\mathcal{X}^{1-2\alpha}} \le r. \tag{16}$$

Similarly,

$$\big\|\Psi(b)(t)\big\|_{L^1_T(\mathcal{X}^1)} \leq \sum_{k=0}^{4} J_k,$$

where

$$J_0 = \int_0^T \big\|e^{-t|D|^{2\alpha}} b^0\big\|_{\mathcal{X}^1}\,dt,$$

$$J_1 = \int_0^T \bigg\| \int_0^t e^{-(t-z)|D|^{2\alpha}} u_a \nabla a\,dz \bigg\|_{\mathcal{X}^1} dt,$$

$$J_2 = \int_0^T \bigg\| \int_0^t e^{-(t-z)|D|^{2\alpha}} u_a \nabla b\,dz \bigg\|_{\mathcal{X}^1} dt,$$

$$J_3 = \int_0^T \bigg\| \int_0^t e^{-(t-z)|D|^{2\alpha}} u_b \nabla a\,dz \bigg\|_{\mathcal{X}^1} dt,$$

$$J_4 = \int_0^T \bigg\| \int_0^t e^{-(t-z)|D|^{2\alpha}} u_b \nabla b\,dz \bigg\|_{\mathcal{X}^1} dt.$$

By using Lemma 6, we have

$$\begin{aligned}
J_0 &\leq \frac{r}{5} \\
J_1 &\leq 4\|a\|_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|a\|_{L^1_T(\mathcal{X}^1)} \leq 4\varepsilon\big\|\theta^0\big\|_{\mathcal{X}^{1-2\alpha}} \leq \frac{r}{5} \\
J_2, J_3 &\leq 2\|a\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|a\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)}\|b\|^{\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|b\|^{1-\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)} \\
&\quad + 2\|a\|^{\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|a\|^{1-\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)}\|b\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|b\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)} \\
&\leq 2\big(\big\|\theta^0\big\|^{1-\frac{1}{2\alpha}}_{\mathcal{X}^{1-2\alpha}}\varepsilon^{\frac{1}{2\alpha}} + \big\|\theta^0\big\|^{\frac{1}{2\alpha}}_{\mathcal{X}^{1-2\alpha}}\varepsilon^{1-\frac{1}{2\alpha}}\big)r \\
&\leq \frac{r}{5} \\
J_4 &\leq 4\|b\|_{L^\infty_T(\mathcal{X}^{1-2\alpha})}\|b\|_{L^1_T(\mathcal{X}^1)} \leq 4r^2 \leq \frac{r}{5}.
\end{aligned}$$

Then

$$\big\|\Psi(b)(t)\big\|_{L^1_T(\mathcal{X}^1)} \leq r. \tag{17}$$

Combining Eqs. (16) and (17), we get $\Psi(b) \in \mathbf{B}_r$ and we can deduce

$$\Psi(\mathbf{B}_r) \subset \mathbf{B}_r. \tag{18}$$

- Proof of the estimate

$$\big\|\Psi(b_1) - \Psi(b_2)\big\|_{\mathcal{X}_T} \leq \frac{1}{2}\|b_1 - b_2\|_{\mathcal{X}_T}, \quad b_1, b_2 \in \mathbf{B}_r.$$

Put B_1 and B_2 defined by

$$B_i = \big(\partial_{x_2} \Lambda^{-1} b_i, -\partial_{x_1} \Lambda^{-1} b_i\big).$$

We have

$$\begin{aligned}\Psi(b_1) - \Psi(b_2) &= -\int_0^t e^{-(t-z)|D|^{2\alpha}} (B_2 \nabla b_2 - B_1 \nabla b_1)\\ &= -\int_0^t e^{-(t-z)|D|^{2\alpha}} \big(B_2 \nabla (b_2 - b_1) + (B_2 - B_1)\nabla b_1\big)\end{aligned}$$

and

$$\big\|\Psi(b_1) - \Psi(b_2)\big\|_{\mathcal{X}^{1-2\alpha}} \leq K_1 + K_2,$$

with

$$\begin{aligned}K_1 &= \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} B_2 \nabla (b_2 - b_1) dz \right\|_{\mathcal{X}^{1-2\alpha}},\\ K_2 &= \left\| \int_0^t e^{-(t-z)|D|^{2\alpha}} (B_2 - B_1) \nabla b_1 dz \right\|_{\mathcal{X}^{1-2\alpha}}.\end{aligned}$$

Using Lemmas 4 and 5, we can deduce

$$\begin{aligned}K_1 &\leq 2\|B_2\|^{\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})} \|B_2\|^{1-\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)} \|b_2 - b_1\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})} \|b_2 - b_1\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)}\\ &\quad + 2\|B_2\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})} \|B_2\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)} \|b_2 - b_1\|^{\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})} \|b_2 - b_1\|^{1-\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)}\\ &\leq 2\|b_2\|^{\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})} \|b_2\|^{1-\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)} \|b_2 - b_1\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})} \|b_2 - b_1\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)}\\ &\quad + 2\|b_2\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})} \|b_2\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)} \|b_2 - b_1\|^{\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})} \|b_2 - b_1\|^{1-\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)}\\ &\leq 2r\|b_2 - b_1\|^{1-\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})} \|b_2 - b_1\|^{\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)}\\ &\quad + 2r\|b_2 - b_1\|^{\frac{1}{2\alpha}}_{L^\infty_T(\mathcal{X}^{1-2\alpha})} \|b_2 - b_1\|^{1-\frac{1}{2\alpha}}_{L^1_T(\mathcal{X}^1)}\\ &\leq 4r\|b_2 - b_1\|_{\mathcal{X}_T}.\end{aligned}$$

Similarly, we get

$$K_2 \leq 4r\|b_2 - b_1\|_{\mathcal{X}_T}.$$

Using the fact $r < 1/20$, we obtain

$$\big\|\Psi(b_2) - \Psi(b_1)\big\|_{\mathcal{X}_T} \leq \frac{1}{2}\|b_2 - b_1\|_{\mathcal{X}_T}, \quad \forall b_1, b_2 \in \mathbf{B}_r. \tag{19}$$

Then, combining Eqs. (18)–(19) and the fixed point theorem, there is a unique $b \in \mathbf{B}_r$ such that $\theta = a+b$ is a solution of $(\mathbf{S}_\alpha)$ with $\theta \in \mathcal{X}_T(\mathbb{R}^2)$.

3.2. Uniqueness

Let $\theta_1, \theta_2 \in \mathcal{C}([0,T], \mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$ be two solution of $(\mathbf{S}_\alpha)$, with $\theta_1 \in L^1([0,T], \mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$ and $\theta_1(0) = \theta_2(0)$. Put u_1, u_2, δ and w defined as follows

$$u_i = \big(\partial_{x_2}\Lambda^{-1}\theta_i, -\partial_{x_1}\Lambda^{-1}\theta_i\big)$$
$$\delta = \theta_1 - \theta_2$$
$$w = u_1 - u_2$$

We have

$$\partial_t\delta + (-\Delta)^\alpha\delta + w\nabla\delta + w\nabla\theta_1 + u_1\nabla\delta = 0 \tag{20}$$

then

$$\partial_t\widehat{\delta} + |\xi|^{2\alpha}\widehat{\delta} + \mathcal{F}(w\nabla\delta) + \mathcal{F}(w\nabla\theta_1) + \mathcal{F}(u_1\nabla\delta) = 0;$$

multiplying this equation by $\bar{\widehat{\delta}}$, we get

$$\partial_t\widehat{\delta}.\bar{\widehat{\delta}} + |\xi|^{2\alpha}|\widehat{\delta}|^2 + \mathcal{F}(w\nabla\delta).\bar{\widehat{\delta}} + \mathcal{F}(w\nabla\theta_1).\bar{\widehat{\delta}} + \mathcal{F}(u_1\nabla\delta).\bar{\widehat{\delta}} = 0. \tag{21}$$

From Eq. (20), we have

$$\partial_t\bar{\widehat{\delta}} + |\xi|^{2\alpha}\bar{\widehat{\delta}} + \overline{\mathcal{F}(w\nabla\delta)} + \overline{\mathcal{F}(w\nabla\theta_1)} + \overline{\mathcal{F}(u_1\nabla\delta)} = 0;$$

multiplying this equation by $\widehat{\delta}$, we get

$$\partial_t\bar{\widehat{\delta}}.\widehat{\delta} + |\xi|^{2\alpha}|\widehat{\delta}|^2 + \overline{\mathcal{F}(w\nabla\delta)}.\widehat{\delta} + \overline{\mathcal{F}(w\nabla\theta_1)}.\widehat{\delta} + \overline{\mathcal{F}(u_1\nabla\delta)}.\widehat{\delta} = 0. \tag{22}$$

By summing (21) and (22), we get

$$\partial_t|\widehat{\delta}|^2 + 2|\xi|^{2\alpha}|\widehat{\delta}|^2 + 2\,Re\big(\mathcal{F}(w\nabla\delta).\bar{\widehat{\delta}}\big) + 2\,Re\big(\mathcal{F}(w\nabla\theta_1).\bar{\widehat{\delta}}\big) + 2\,Re\big(\mathcal{F}(u_1\nabla\delta).\bar{\widehat{\delta}}\big) = 0$$

and

$$\partial_t|\widehat{\delta}|^2 + 2|\xi|^{2\alpha}|\widehat{\delta}|^2 \le 2\big|\mathcal{F}(w\nabla\delta)\big|.|\bar{\widehat{\delta}}| + 2\big|\mathcal{F}(w\nabla\theta_1)\big|.|\bar{\widehat{\delta}}| + 2\big|\mathcal{F}(u_1\nabla\delta)\big|.|\bar{\widehat{\delta}}|.$$

For $\varepsilon > 0$, we have

$$\partial_t|\widehat{\delta}|^2 = \partial_t\big(|\widehat{\delta}|^2 + \varepsilon^2\big) = 2\sqrt{|\widehat{\delta}|^2 + \varepsilon^2}\,\partial_t\sqrt{|\widehat{\delta}|^2 + \varepsilon^2}$$

then

$$\begin{aligned}2\partial_t\sqrt{|\widehat{\delta}|^2 + \varepsilon^2} + 2|\xi|^{2\alpha}\frac{|\widehat{\delta}|^2}{\sqrt{|\widehat{\delta}|^2 + \varepsilon^2}} &\le 2\big|\mathcal{F}(w\nabla\delta)\big|\frac{|\widehat{\delta}|}{\sqrt{|\widehat{\delta}|^2 + \varepsilon^2}} + 2\big|\mathcal{F}(w\nabla\theta_1)\big|\frac{|\widehat{\delta}|}{\sqrt{|\widehat{\delta}|^2 + \varepsilon^2}} \\ &\quad + 2\big|\mathcal{F}(u_1\nabla\delta)\big|\frac{|\widehat{\delta}|}{\sqrt{|\widehat{\delta}|^2 + \varepsilon^2}} \\ &\le 2\big|\mathcal{F}(w\nabla\delta)\big| + 2\big|\mathcal{F}(w\nabla\theta_1)\big| + 2\big|\mathcal{F}(u_1\nabla\delta)\big|.\end{aligned}$$

By integrating with respect to time

$$\sqrt{|\widehat{\delta}|^2+\varepsilon^2}+\int_0^t |\xi|^{2\alpha}\frac{|\widehat{\delta}|^2}{\sqrt{|\widehat{\delta}|^2+\varepsilon^2}} \le \int_0^t \left|\mathcal{F}(w\nabla\delta)\right| + \int_0^t \left|\mathcal{F}(w\nabla\theta_1)\right| + \int_0^t \left|\mathcal{F}(u_1\nabla\delta)\right|.$$

Letting $\varepsilon \to 0$, we get

$$|\widehat{\delta}|+\int_0^t |\xi|^{2\alpha}|\widehat{\delta}| \le \int_0^t \left|\mathcal{F}(w\nabla\delta)\right| + \int_0^t \left|\mathcal{F}(w\nabla\theta_1)\right| + \int_0^t \left|\mathcal{F}(u_1\nabla\delta)\right|.$$

Multiplying by $|\xi|^{1-2\alpha}$ and integrating with respect to ξ, we get

$$\begin{aligned}\|\delta\|_{\mathcal{X}^{1-2\alpha}}+\int_0^t \|\delta\|_{\mathcal{X}^1} &\le \int_0^t \|w\nabla\delta\|_{\mathcal{X}^{1-2\alpha}} + \int_0^t \|w\nabla\theta_1\|_{\mathcal{X}^{1-2\alpha}} + \int_0^t \|u_1\nabla\delta\|_{\mathcal{X}^{1-2\alpha}} \\ &\le \int_0^t \|\delta w\|_{\mathcal{X}^{2-2\alpha}} + \int_0^t \|\theta_1 w\|_{\mathcal{X}^{2-2\alpha}} + \int_0^t \|\delta u_1\|_{\mathcal{X}^{2-2\alpha}}.\end{aligned}$$

By Lemma 4, we get

$$\begin{aligned}\|\delta\|_{\mathcal{X}^{1-2\alpha}}+\int_0^t \|\delta\|_{\mathcal{X}^1} \le{}& 4\int_0^t \|\delta\|_{\mathcal{X}^{1-2\alpha}}\|\delta\|_{\mathcal{X}^1} + 2\int_0^t \|\delta\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}}\|\delta\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}}\|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}}\|\theta_1\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \\ &+ 2\int_0^t \|\delta\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}}\|\delta\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}}\|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}}\|\theta_1\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}}.\end{aligned}$$

Using the elementary inequality

$$xy \le \frac{x^p}{p}+\frac{y^q}{q}$$

with

$$p=\frac{1}{2\alpha}, \qquad q=\frac{2\alpha}{2\alpha-1},$$

we get

$$\|\delta\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}}\|\delta\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}}\|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}}\|\theta_1\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}} \le \frac{1}{4}\|\delta\|_{\mathcal{X}^1}+c_\alpha\|\delta\|_{\mathcal{X}^{1-2\alpha}}\|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha-1}}\|\theta_1\|_{\mathcal{X}^1}$$

and

$$\|\delta\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha}}\|\delta\|_{\mathcal{X}^1}^{1-\frac{1}{2\alpha}}\|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{1-\frac{1}{2\alpha}}\|\theta_1\|_{\mathcal{X}^1}^{\frac{1}{2\alpha}} \le \frac{1}{4}\|\delta\|_{\mathcal{X}^1}+c'_\alpha\|\delta\|_{\mathcal{X}^{1-2\alpha}}\|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha-1}}\|\theta_1\|_{\mathcal{X}^1}.$$

Then

$$\|\delta\|_{\mathcal{X}^{1-2\alpha}}+\int_0^t \|\delta\|_{\mathcal{X}^1} \le \left(c_\alpha+c'_\alpha\right)\int_0^t \|\delta\|_{\mathcal{X}^{1-2\alpha}}\|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha-1}}\|\theta_1\|_{\mathcal{X}^1}.$$

Using Gronwall's Lemma and the fact $(t \mapsto \|\theta_1\|_{\mathcal{X}^{1-2\alpha}}^{\frac{1}{2\alpha-1}} \|\theta_1\|_{\mathcal{X}^1}) \in L^1([0,T])$, we can deduce that $\delta = 0$ in $[0,T]$ which gives the uniqueness.

3.3. Small initial data

In this section, we suppose that $\|\theta^0\|_{\mathcal{X}^{1-2\alpha}} < 1/4$.

Passing to Fourier transform of the first equation of $(\mathbf{S}_\alpha)$ and multiplying it by $\bar{\widehat{\theta}}$, we get

$$\partial_t \widehat{\theta}.\bar{\widehat{\theta}} + |\xi|^{2\alpha}|\widehat{\theta}|^2 + \mathcal{F}(u\nabla\theta).\bar{\widehat{\theta}} = 0. \tag{23}$$

Similarly, we have

$$\partial_t \bar{\widehat{\theta}} + |\xi|^{2\alpha}\bar{\widehat{\theta}} + \overline{\mathcal{F}(u\nabla\theta)} = 0$$

and

$$\partial_t \bar{\widehat{\theta}}.\widehat{\theta} + |\xi|^{2\alpha}|\widehat{\theta}|^2 + \overline{\mathcal{F}(u\nabla\theta)}.\widehat{\theta} = 0. \tag{24}$$

By summing (23) and (24), we get

$$\partial_t |\widehat{\theta}|^2 + 2|\xi|^{2\alpha}|\widehat{\theta}|^2 + 2\,Re\big(\mathcal{F}(u\nabla\theta).\bar{\widehat{\theta}}\big) = 0$$

and

$$\partial_t |\widehat{\theta}|^2 + 2|\xi|^{2\alpha}|\widehat{\theta}|^2 \le 2\big|\mathcal{F}(u\nabla\theta)\big|.|\bar{\widehat{\theta}}|.$$

For $\varepsilon > 0$, we have

$$\partial_t |\widehat{\theta}|^2 = \partial_t\big(|\widehat{\theta}|^2 + \varepsilon^2\big) = 2\sqrt{|\widehat{\theta}|^2 + \varepsilon^2}\,\partial_t\sqrt{|\widehat{\theta}|^2 + \varepsilon^2}.$$

Then

$$\begin{aligned} 2\partial_t\sqrt{|\widehat{\theta}|^2 + \varepsilon^2} + 2|\xi|^{2\alpha}\frac{|\widehat{\theta}|^2}{\sqrt{|\widehat{\theta}|^2 + \varepsilon^2}} &\le 2\big|\mathcal{F}(u\nabla\theta)\big|.\frac{|\widehat{\theta}|}{\sqrt{|\widehat{\theta}|^2 + \varepsilon^2}} \\ &\le 2\big|\mathcal{F}(u\nabla\theta)\big|. \end{aligned}$$

By integrating with respect to time

$$\sqrt{|\widehat{\theta}|^2 + \varepsilon^2} + \int_0^t |\xi|^{2\alpha}\frac{|\widehat{\theta}|^2}{\sqrt{|\widehat{\delta}|^2 + \varepsilon^2}} \le \sqrt{\big|\widehat{\theta^0}\big|^2 + \varepsilon^2} + \int_0^t \big|\mathcal{F}(u\nabla\theta)\big|.$$

Letting $\varepsilon \to 0$, we get

$$|\widehat{\theta}| + \int_0^t |\xi|^{2\alpha}|\widehat{\theta}| \le \big|\widehat{\theta^0}\big| + \int_0^t \big|\mathcal{F}(u\nabla\theta)\big|.$$

Multiplying by $|\xi|^{1-2\alpha}$ and integrating with respect to ξ, we get

$$\|\theta\|_{\mathcal{X}^{1-2\alpha}} + \int_0^t \|\theta\|_{\mathcal{X}^1} \le \|\theta^0\|_{\mathcal{X}^{1-2\alpha}} + \int_0^t \|u\nabla\theta\|_{\mathcal{X}^{1-2\alpha}}.$$

By Lemma 4, we get

$$\begin{aligned}\|\theta\|_{\mathcal{X}^{1-2\alpha}} + \int_0^t \|\theta\|_{\mathcal{X}^1} &\le \|\theta^0\|_{\mathcal{X}^{1-2\alpha}} + 4\int_0^t \|\theta\|_{\mathcal{X}^{1-2\alpha}}\|\theta\|_{\mathcal{X}^1} \\ &\le \|\theta^0\|_{\mathcal{X}^{1-2\alpha}} + 4\sup_{z\in[0,t]}\|\theta(z)\|_{\mathcal{X}^{1-2\alpha}}\int_0^t \|\theta\|_{\mathcal{X}^1}.\end{aligned}$$

Let $T = \sup\{t > 0; \sup_{z\in[0,t]} \|\theta(z)\|_{\mathcal{X}^{1-2\alpha}} < \frac{1}{4}\}$. By the above equation, we have

$$\|\theta(t)\|_{\mathcal{X}^{1-2\alpha}} \le \|\theta^0\|_{\mathcal{X}^{1-2\alpha}} < \frac{1}{4}, \quad \forall t \in [0, T),$$

then $T = \infty$. Therefore, the global existence and inequality (2) are proved.

4. Proof of Theorem 2

The same approach of Theorem 1 is used to obtain a blow-up result of $\theta \in \mathcal{C}([0, T^*), \mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$ if $T^* < \infty$.

Assume that $\int_0^{T^*} \|\theta(t)\|_{\mathcal{X}^1} dt < \infty$. Let $0 < T_0 < T^*$ such that

$$\int_{T_0}^{T^*} \|\theta(t)\|_{\mathcal{X}^1} dt < \frac{1}{2}.$$

For $t \in [T_0, T^*)$ and $s \in [T_0, t]$

$$\begin{aligned}\|\theta(s)\|_{\mathcal{X}^{1-2\alpha}} + \int_{T_0}^s \|\theta(t)\|_{\mathcal{X}^1} &\le \|\theta(T_0)\|_{\mathcal{X}^{1-2\alpha}} + \int_{T_0}^s \|\theta(t)\|_{\mathcal{X}^{1-2\alpha}}\|\theta(t)\|_{\mathcal{X}^1} \\ &\le \|\theta(T_0)\|_{\mathcal{X}^{1-2\alpha}} + \sup_{T_0\le z\le t}\|\theta(z)\|_{\mathcal{X}^{1-2\alpha}}\int_{T_0}^t \|\theta(t)\|_{\mathcal{X}^1} \\ &\le \|\theta(T_0)\|_{\mathcal{X}^{1-2\alpha}} + \frac{1}{2}\sup_{T_0\le z\le t}\|\theta(z)\|_{\mathcal{X}^{1-2\alpha}}.\end{aligned}$$

It follows that

$$\sup_{0\le s\le t}\|\theta(s)\|_{\mathcal{X}^{1-2\alpha}} \le \|\theta(T_0)\|_{\mathcal{X}^{1-2\alpha}} + \frac{1}{2}\sup_{T_0\le z\le t}\|\theta(z)\|_{\mathcal{X}^{1-2\alpha}}.$$

We can deduce that

$$\sup_{0\le s\le t}\|\theta(s)\|_{\mathcal{X}^{1-2\alpha}} \le 2\|\theta(T_0)\|_{\mathcal{X}^{1-2\alpha}}, \quad \forall t \in [T_0, T^*).$$

Put

$$M=\max\Big(2\big\|\theta(T_0)\big\|_{\mathcal{X}^{1-2\alpha}};\ \max_{t\in[0,T_0]}\big\|\theta(t)\big\|_{\mathcal{X}^{1-2\alpha}}\Big).$$

We have

$$\big\|\theta(t)\big\|_{\mathcal{X}^{1-2\alpha}}\leq M,\quad \forall t\in\big[0,T^*\big).$$

Using the integral form of θ, we can write, for $t<t'\in[0,T^*)$,

$$\theta\big(t'\big)-\theta(t)=L_1\big(t,t'\big)+L_2\big(t,t'\big)$$

with

$$L_1=\int_0^t\big(1-e^{-(t'-t)|D|^{2\alpha}}\big)e^{-(t-z)|D|^{2\alpha}}(u.\nabla\theta)(z)dz,$$

$$L_2=\int_t^{t'}e^{-(t'-z)|D|^{2\alpha}}(u.\nabla\theta)(z)dz.$$

We have

$$\begin{aligned}\big\|L_1\big(t,t'\big)\big\|_{\mathcal{X}^{1-2\alpha}}&\leq\int_0^t\int_{\mathbb{R}^2}\big(1-e^{-(t'-t)|\xi|^{2\alpha}}\big)e^{-(t-z)|\xi|^{2\alpha}}|\xi|^{1-2\alpha}\big|\mathcal{F}(u.\nabla\theta)(z,\xi)\big|d\xi dz\\&\leq\int_0^t\int_{\mathbb{R}^2}\big(1-e^{-(t'-t)|\xi|^{2\alpha}}\big)|\xi|^{1-2\alpha}\big|\mathcal{F}(u.\nabla\theta)(z,\xi)\big|d\xi dz.\\\big\|L_2\big(t,t'\big)\big\|_{\mathcal{X}^{1-2\alpha}}&\leq\int_t^{t'}\int_{\mathbb{R}^2}e^{-(t'-z)|\xi|^{2\alpha}}|\xi|^{1-2\alpha}\big|\mathcal{F}(u.\nabla\theta)(z,\xi)\big|d\xi dz\\&\leq\int_t^{t'}\int_{\mathbb{R}^2}|\xi|^{1-2\alpha}\big|\mathcal{F}(u.\nabla\theta)(z,\xi)\big|d\xi dz.\end{aligned}$$

Using Lemmas 4–5 and the dominated convergence theorem, we can deduce that

$$\limsup_{\substack{t,t'\nearrow T^*\\t<t'}}\big\|L_1\big(t,t'\big)\big\|_{\mathcal{X}^{1-2\alpha}}=0,\qquad\limsup_{\substack{t,t'\nearrow T^*\\t<t'}}\big\|L_2\big(t,t'\big)\big\|_{\mathcal{X}^{1-2\alpha}}=0.$$

Therefore

$$\limsup_{\substack{t,t'\nearrow T^*\\t<t'}}\big\|\theta\big(t'\big)-\theta(t)\big\|_{\mathcal{X}^{1-2\alpha}}=0.$$

Then $\theta(t)$ is of Cauchy type at the left of T^* in the Banach space $\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$. Then, there is θ^* an element of $\mathcal{X}^{1-2\alpha}(\mathbb{R}^2)$ such that

$$\lim_{t\nearrow T^*}\theta(t)=\theta^*.$$

Now, consider the following system

$$\begin{cases} \partial_t a + (-\Delta)^\alpha a + u_a.\nabla a = 0 \\ a(0) = \theta^* \end{cases}$$

By Theorem 1, there is a time $t_0 > 0$ and unique solution a such that $a \in \mathcal{C}([0.t_0], \mathcal{X}^{1-2\alpha}(\mathbb{R}^2))$. Then

$$\Theta(t) = \begin{cases} \theta(t), & \text{if } t \in [0, T^*) \\ a(t - T^*), & \text{if } t \in [T^*, T^* + t_0] \end{cases}$$

is a solution of $(\mathbf{S}_\alpha)$ with initial data θ^0 on the interval $[0, T^* + t_0]$ which contradicts the maximality of T^*.

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