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Variational principle for topological pressures on subsets

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ABSTRACT

This paper studies the relations between Pesin–Pitskel topological pressure on an arbitrary subset and measure theoretic pressure of Borel probability measures, which extends Feng and Huang's recent result on entropies [13] for pressures. More precisely, this paper defines the measure theoretic pressure $P_\mu(T,f)$ for any Borel probability measure, and shows that $P_B(T,f,K) = \sup\{P_\mu(T,f) : \mu \in \mathcal{M}(X), \mu(K) = 1\}$, where $\mathcal{M}(X)$ is the space of all Borel probability measures, $K \subseteq X$ is a non-empty compact subset and $P_B(T,f,K)$ is the Pesin–Pitskel topological pressure on K. Furthermore, if $Z \subseteq X$ is an analytic subset, then $P_B(T,f,Z) = \sup\{P_B(T,f,K) : K \subseteq Z \text{ is compact}\}$. This paper also shows that Pesin–Pitskel topological pressure can be determined by the measure theoretic pressure.

1. Introduction

Throughout this paper X is a compact metric space with metric d and $T: X \to X$ is a continuous transformation, such a pair (X,T) is a *topological dynamical system* (TDS for short). Let $\mathcal{M}(X)$ be the space of all Borel probability measures on X, and denote by $\mathcal{M}_T$ and $\mathcal{E}_T$ the set of all T-invariant (respectively, ergodic) Borel probability measures on X. For any $\mu \in \mathcal{M}_T$, let $h_\mu(T)$ denote the measure theoretic entropy of μ with respect to T and let $h_{top}(T)$ denote the topological entropy of the system (X,T), see [34] for the precise definitions. It is well-known that entropies constitute essential invariants in the characterization of the complexity of dynamical systems. The classical measure theoretic entropy for an invariant measure and the topological entropy are introduced in [19] and [1] respectively. The basic relation between topological entropy and measure theoretic entropy is the variational principle, e.g., see [34].

Topological pressure is a non-trivial and natural generalization of topological entropy. One of the most fundamental dynamical invariants that associate to a continuous map is the topological pressure with

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a potential function. It roughly measures the orbit complexity of the iterated map on the potential function. Ruelle [29] introduced topological pressure of a continuous function for $\mathbb{Z}^n$-actions on compact spaces and established the variational principle for topological pressure when the action is expansive and satisfies the specification property. Later, Walters [33] generalized the variational principle for a $\mathbb{Z}_+$-action without these assumptions. Misiurewicz [23] gave an elegant proof of the variational principle for $\mathbb{Z}_+^n$-action. See [16,26, 25,30–32] for the variational principle for amenable group actions and [11,18] for sofic groups actions. And we would like to mention, Barreira [2–4], Cao, Feng and Huang [8], Mummert [24], Zhao and Cheng [39,40] dealing with variational principle for topological pressure with nonadditive potentials, and Huang and Yi [17] and Zhang [36], where variational principle for the local topological pressure are also considered. This paper conducts research for $\mathbb{Z}$ or $\mathbb{Z}_+$-actions.

From a viewpoint of dimension theory, Pesin and Pitskel' [28] defined the topological pressure on noncompact sets which is a generalization of Bowen's definition of topological entropy on noncompact sets [5], and they proved the variational principle under some supplementary conditions. The notions of the topological pressure, variational principle and equilibrium states play a fundamental role in statistical mechanics, ergodic theory and dynamical systems (see the books [6,34]).

Motivated by Feng and Huang's recent work [13], where the authors studied the variational principle between Bowen topological entropy on an arbitrary subset and measure theoretic entropy for Borel probability measures (not necessarily invariant), recently, Wang and Chen generalized Feng–Huang's result to BS-dimension [35]. As a natural generalization of topological entropy, topological pressure is a quantity which belongs to one of the concepts in the thermodynamic formalism. This study defines measure theoretic pressure for a Borel probability measure and investigates its variational relation with the Pesin–Pitskel topological pressure. Moreover, it is proved that Pesin–Pitskel topological pressure is determined by measure theoretic pressure of Borel probability measures. The outline of the paper is as follows. The main results, as well as those definitions of the measure theoretic pressure and topological pressures, are given in Section 2. The proof of the main results and related propositions are given in Section 3.

2. Definitions and the statement of main results

This section first gives the definition of measure theoretic pressure for any Borel probability measure, and then recalls different kinds of definitions of the topological pressure. The main results of this paper is given in the end of this section.

We first give some necessary notations. For any $n \in \mathbb{N}$ and $\epsilon > 0$, let $d_n(x,y) = \max\{d(T^i(x), T^i(y)) : 0 \le i < n\}$ for any $x, y \in X$ and $B_n(x,\epsilon) = \{y \in X : d_n(x,y) < \epsilon\}$. A set $E \subseteq X$ is said to be an (n,ϵ)-separated subset of X with respect to T if $x, y \in E, x \neq y$, implies $d_n(x,y) > \epsilon$. A set $F \subseteq X$ is said to be an (n,ϵ)-spanning subset of X with respect to T if $\forall x \in X$, $\exists y \in F$ with $d_n(x,y) \le \epsilon$. Let $C(X)$ denote the Banach space of all continuous functions on X equipped with the supremum norm $\|\cdot\|$.

2.1. Measure theoretic pressure

Let $\mu \in \mathcal{M}(X)$ and $f \in C(X)$, the *measure theoretic pressure* of μ for T (w.r.t. f) is defined by

$$P_\mu(T,f) := \int P_\mu(T,f,x)\,\mathrm{d}\mu(x)$$

where $P_\mu(T,f,x) := \lim_{\epsilon\to 0}\liminf_{n\to\infty}(\frac{1}{n}\log[e^{f_n(x)}\cdot\mu(B_n(x,\epsilon))^{-1}])$ and $f_n(x) := \sum_{i=0}^{n-1} f(T^i x)$.

For any $\mu \in \mathcal{M}_T$, using Birkhoff's ergodic theorem (e.g. see [34]) and Brin–Katok's entropy formula [7], for μ-almost every $x \in X$ we have that

$$P_\mu(T,f,x) = h_\mu(T,x) + f^*(x),$$

moreover, we know that $f^* \circ T = f^*$, $\int f^* \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu$ and $\int h_\mu(T, x) \, \mathrm{d}\mu = h_\mu(T)$. Particularly, if $\mu \in \mathcal{E}_T$ we have that $P_\mu(T, f, x) = h_\mu(T) + \int f \, \mathrm{d}\mu$ for μ-almost every $x \in X$. We refer the reader to [9,10,14,37,38] for more details on the measure theoretic pressure of invariant measures for a large class of potentials.

In the following subsections, we turn to recall definitions of *upper capacity topological pressure*, *Pesin–Pitskel topological pressure* and *weighted topological pressure*.

2.2. Upper capacity topological pressure

Recall that the *upper capacity topological pressure* of T on a subset $Z \subseteq X$ with respect to a continuous function f is given by

$$P(T, f, Z) = \lim_{\epsilon \to 0} P(T, f, Z, \epsilon)$$

where

$$P(T, f, Z, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log P_n(T, f, Z, \epsilon),$$

$$P_n(T, f, Z, \epsilon) = \sup\Big\{ \sum_{x \in E} e^{f_n(x)} : E \text{ is an } (n, \epsilon)\text{-separated subset of } Z \Big\}.$$

2.3. Pesin–Pitskel topological pressure

Let $Z \subseteq X$ be a subset of X, which does not have to be compact nor T-invariant. Fix $\epsilon > 0$, we call $\Gamma = \{B_{n_i}(x_i, \epsilon)\}_i$ a *cover of* Z if $Z \subseteq \bigcup_i B_{n_i}(x_i, \epsilon)$. For $\Gamma = \{B_{n_i}(x_i, \epsilon)\}_i$, set $n(\Gamma) = \min_i \{n_i\}$.

Let $f \in C(X)$ and $s \in \mathbb{R}$, put

$$M(Z, f, s, N, \epsilon) = \inf_{\Gamma} \sum_i \exp\Big(-s n_i + \sup_{y \in B_{n_i}(x_i, \epsilon)} f_{n_i}(y) \Big),$$

where the infimum is taken over all covers Γ of Z with $n(\Gamma) \geq N$. The quantity $M(Z, f, s, N, \epsilon)$ is monotone in N and hence, we let

$$m(Z, f, s, \epsilon) = \lim_{N \to \infty} M(Z, f, s, N, \epsilon).$$

It is easy to show that there is a critical value

$$P_B(T, f, Z, \epsilon) = \inf\big\{ s : m(Z, f, s, \epsilon) = 0 \big\} = \sup\big\{ s : m(Z, f, s, \epsilon) = +\infty \big\}.$$

Definition 2.1. We call the quantity

$$P_B(T, f, Z) = \lim_{\epsilon \to 0} P_B(T, f, K, \epsilon)$$

the *Pesin–Pitskel topological pressure* of T on the set Z (w.r.t. f).

The Pesin–Pitskel topological pressure can be defined in the following alternative way, see [2] or [27] for more details.

Suppose $\mathcal{U}$ is a finite open cover of X. Denote the diameter of the open cover by $|\mathcal{U}| := \max\{\operatorname{diam}(U) : U \in \mathcal{U}\}$. For $n \geq 1$ we denote by $\mathcal{W}_n(\mathcal{U})$ the collection of strings $\mathbf{U} = U_1 \ldots U_n$ with $U_i \in \mathcal{U}$. For $\mathbf{U} \in \mathcal{W}_n(\mathcal{U})$ we call the integer $m(\mathbf{U}) = n$ the length of $\mathbf{U}$ and define

$$X(\mathbf{U}) = U_1 \cap T^{-1} U_2 \cap \ldots \cap T^{-(n-1)} U_n = \big\{ x \in X : T^{j-1} x \in U_j \text{ for } j = 1, \ldots, n \big\}.$$

Given a subset $Z \subseteq X$. We say that $\Lambda \subset \bigcup_{n\geq 1} \mathcal{W}_n(\mathcal{U})$ covers Z if $\bigcup_{\mathbf{U}\in\Lambda} X(\mathbf{U}) \supset Z$. For $s \in \mathbb{R}$, define

$$M_N^s(\mathcal{U}, f, Z) = \inf_{\Lambda} \sum_{\mathbf{U}\in\Lambda} \exp\Big(-sm(\mathbf{U}) + \sup_{y\in X(\mathbf{U})} f_{m(\mathbf{U})}(y)\Big)$$

where the infimum is taken over all $\Lambda \subset \bigcup_{n\geq N} \mathcal{W}_n(\mathcal{U})$ that cover Z and $\sup_{y\in X(\mathbf{U})} f_{m(\mathbf{U})}(y) = -\infty$ if $X(\mathbf{U}) = \emptyset$. Clearly $M_N^s(\mathcal{U}, f, \cdot)$ is a finite outer measure on X, and

$$M_N^s(\mathcal{U}, f, Z) = \inf\big\{M_N^s(\mathcal{U}, f, G) : G \supset Z, G \text{ is open}\big\}.$$

Clearly, $M_N^s(\mathcal{U}, f, Z)$ increases as N increases, define

$$M^s(\mathcal{U}, f, Z) := \lim_{N\to\infty} M_N^s(\mathcal{U}, f, Z)$$

and

$$P_B(T, f, \mathcal{U}, Z) := \inf\big\{s : M^s(\mathcal{U}, f, Z) = 0\big\} = \sup\big\{s : M^s(\mathcal{U}, f, Z) = +\infty\big\},$$

set

$$P_B(T, f, Z) := \sup_{\mathcal{U}} P_B(T, f, \mathcal{U}, Z)$$

and it is not difficult to prove that $\sup_{\mathcal{U}} P_B(T, f, \mathcal{U}, Z) = \lim_{|\mathcal{U}|\to 0} P_B(T, f, \mathcal{U}, Z)$.

2.4. Weighted topological pressure

For any bounded function $g : X \to \mathbb{R}$, $f \in C(X)$, $\epsilon > 0$ and $N \in \mathbb{N}$, define

$$W(g, f, s, N, \epsilon) = \inf \sum_i c_i \exp\Big(-sn_i + \sup_{y\in B_{n_i}(x_i,\epsilon)} f_{n_i}(y)\Big)$$

where the infimum is taken over all finite or countable families $\{B_{n_i}(x_i, \epsilon), c_i\}$ such that $0 < c_i < \infty, x_i \in X, n_i \geq N$ and

$$\sum_i c_i \chi_{B_i} \geq g,$$

where $B_i := B_{n_i}(x_i, \epsilon)$ and χ_A denotes the characteristic function on a subset $A \subseteq X$. For $K \subseteq X$ and $g = \chi_K$ we set

$$W(K, f, s, N, \epsilon) := W(\chi_K, f, s, N, \epsilon).$$

The quantity $W(K, f, s, N, \epsilon)$ does not decrease as N increases, hence the following limit exists:

$$w(K, f, s, \epsilon) = \lim_{N\to\infty} W(K, f, s, N, \epsilon).$$

Clearly, there exists a critical value of the parameter s. Hence, we define

$$P_W(T, f, K, \epsilon) = \inf\big\{s : w(K, f, s, \epsilon) = 0\big\} = \sup\big\{s : w(K, f, s, \epsilon) = \infty\big\}$$

Since the quantity $P_W(T, f, K, \epsilon)$ is monotone with respect to ϵ, the following limit exists:

$$P_W(T, f, K) = \lim_{\epsilon \to 0} P_W(T, f, K, \epsilon).$$

The term $P_W(T, f, K)$ is called a *weighted topological pressure* of T on the set K (with respect to f).

Remark 2.1. The weighted topological pressure is a generalization of Feng–Huang's weighted entropy [13]. Indeed, let $f \equiv 0$, then the quantity $P_W(T, 0, K)$ is the same as Feng–Huang's weighted entropy [13].

Now we collect some properties of the pressures which will be used in the proof of the main results, see [2] or [27] for their proofs.

Proposition 2.1. *Let f be a continuous function on X, then the following statements hold:*

(i) For $Z_1 \subseteq Z_2$, $\mathcal{P}(T, f, Z_1) \leq \mathcal{P}(T, f, Z_2)$, where $\mathcal{P}$ is P, P_B or P_W;
(ii) For $Z = \bigcup_{i=1}^{\infty} Z_i$, we have $P_B(T, f, Z) = \sup_{i \geq 1} P_B(T, f, Z_i)$ and $P(T, f, Z) \leq \sup_{i \geq 1} P(T, f, Z_i)$;
(iii) For any $Z \subseteq X$, $P_B(T, f, Z) \leq P(T, f, Z)$. Moreover, we have $P_B(T, f, Z) = P(T, f, Z)$ if Z is T-invariant and compact.

2.5. Statement of main results

The following variational relation of the Pesin–Pitskel topological pressure and the measure theoretic pressure is the main finding of this paper. We give the statements first and postpone the proof to the next section. To give the statements of our results, we recall the definition of *analytic set*. A set in a metric space is *analytic* if it is a continuous image of the set $\mathcal{N}$ of infinite sequences of natural numbers (with its product topology). In a Polish space, the analytic subsets are closed under countable unions and intersections, and any Borel set is analytic (cf. [12, 2.2.10]).

The first theorem shows that the Pesin–Pitskel topological pressure is determined by measure theoretic pressure of Borel probability measures, which extends the result in [21] for Pesin–Pitskel topological pressure.

Theorem A. *Let f be a continuous function on X, μ a Borel probability measure on X and $E \subset X$ a Borel subset. For $s \in \mathbb{R}$, the following properties hold:*

(1) If $P_\mu(T, f, x) \leq s$ for all $x \in E$, then $P_B(T, f, E) \leq s$;
(2) If $P_\mu(T, f, x) \geq s$ for all $x \in E$ and $\mu(E) > 0$, then $P_B(T, f, E) \geq s$.

The next theorem gives the variational relation between Pesin–Pitskel topological pressure on arbitrary subsets and measure theoretic pressure of Borel probability measures, and studies the Pesin–Pitskel topological pressure on an analytic subset.

Theorem B. *Let f be a continuous function on X, the following statements hold:*

(1) If $K \subseteq X$ is non-empty and compact, then

$$P_B(T, f, K) = \sup\{P_\mu(T, f) : \mu \in \mathcal{M}(X), \mu(K) = 1\};$$

(2) If the topological entropy of the system is finite, i.e., $h_{top}(T) < \infty$, and $Z \subseteq X$ is analytic, then

$$P_B(T, f, Z) = \sup\{P_B(T, f, K) : K \subseteq Z, K \text{ is compact}\}.$$

3. Proof of the main result

This section gives the proof of Theorem A and Theorem B.

We first give the proof of Theorem A.

Proof of Theorem A. Modifying Ma and Wen's arguments for entropy [21], we give the proof.

(1) For a fixed $r > 0$, let

$$E_k = \left\{x \in E : \liminf_{n\to\infty} \frac{1}{n} \log\left[e^{f_n(x)}.\mu\big(B_n(x,\epsilon)\big)^{-1}\right] < s + r, \forall \epsilon \in \left(0, \frac{1}{k}\right)\right\}.$$

Then we have $E = \bigcup_{k=1}^{\infty} E_k$, since $P_\mu(T, f, x) \le s$ for all $x \in E$.

Now fix $k \ge 1$ and $0 < \epsilon < \frac{1}{5k}$. For each $x \in E_k$, there exists a strictly increasing sequence $\{n_j\}_{j=1}^{\infty}$ (for simplicity of notations, we omit the dependence on x if there is no confusion caused) such that

$$\mu\big(B_{n_j}(x,\epsilon)\big) \ge \exp\big(-n_j(s+r) + f_{n_j}(x)\big), \quad \forall j \ge 1.$$

For any $N \ge 1$, the set E_k is contained in the union of the sets in the family

$$\mathcal{F} = \big\{B_{n_j}(x,\epsilon) : x \in E_k, n_j \ge N\big\}.$$

Using Lemma 1 in [21], there exists a finite or countable subfamily $\mathcal{G} = \{B_{n_i}(x_i,\epsilon)\}_{i\in I} \subset \mathcal{F}$ of pairwise disjoint balls such that

$$E_k \subset \bigcup_{i\in I} B_{n_i}(x_i, 5\epsilon).$$

Note that

$$\mu\big(B_{n_i}(x_i,\epsilon)\big) \ge \exp\big(-n_i(s+r) + f_{n_i}(x_i)\big), \quad \forall i \in I.$$

The disjointness of $\{B_{n_i}(x_i,\epsilon)\}_{i\in I}$ yields that

$$\mathcal{M}(E_k, f, s+r, N, 5\epsilon) \le \sum_{i\in I} \exp\big(-n_i(s+r) + f_{n_i}(x_i)\big) \le \sum_{i\in I} \mu\big(B_{n_i}(x_i,\epsilon)\big) \le 1$$

where

$$\mathcal{M}(K, f, s, N, \epsilon) := \inf_{\Gamma} \sum_i \exp\big(-sn_i + f_{n_i}(x_i)\big) \tag{3.1}$$

and the infimum is taken over all covers $\Gamma = \{B_{n_i}(x_i,\epsilon)\}$ of K with $n(\Gamma) \ge N$. It follows that

$$\mathcal{M}(E_k, f, s+r, 5\epsilon) = \lim_{N\to\infty} \mathcal{M}(E_k, f, s+r, N, 5\epsilon) \le 1.$$

Since Pesin–Pitskel pressure does not change if we replace $M(\cdot)$ by $\mathcal{M}(\cdot)$ in the definition of pressure [27], we have that $P_B(T, f, E_k, 5\epsilon) \le s + r$ for any $0 < \epsilon < \frac{1}{5k}$. The arbitrariness of ϵ implies that

$$P_B(T, f, E_k) \le s + r, \quad \forall k \ge 1.$$

Hence,

$$P_B(T,f,E) = P_B\left(T,f,\bigcup_{k=1}^{\infty} E_k\right) = \sup_{k\geq 1} P_B(T,f,E_k) \leq s+r.$$

The arbitrariness of r implies that $P_B(T,f,E) \leq s$.

(2) Fix $r>0$. For each $k \geq 1$, put

$$E_k = \left\{x \in E : \liminf_{n\to\infty} \frac{1}{n} \log\left[e^{f_n(x)}.\mu\big(B_n(x,\epsilon)\big)^{-1}\right] > s-r, \forall \epsilon \in \left(0, \frac{1}{k}\right)\right\}.$$

Since $P_\mu(T,f,x) \geq s$ for all $x \in E$, we have that $E_k \subset E_{k+1}$ and $\bigcup_{k=1}^{\infty} E_k = E$. Fix a sufficiently large $k \geq 1$ with $\mu(E_k) > \frac{1}{2}\mu(E) > 0$. For each $N \geq 1$, put

$$E_{k,N} = \left\{x \in E_k : \liminf_{n\to\infty} \frac{1}{n} \log\left[e^{f_n(x)}.\mu\big(B_n(x,\epsilon)\big)^{-1}\right] > s-r, \forall n \geq N, \epsilon \in \left(0, \frac{1}{k}\right)\right\}.$$

It is easy to see that $E_{k,N} \subset E_{k,N+1}$ and $\bigcup_{N=1}^{\infty} E_{k,N} = E_k$. Thus we can pick $N^* \geq 1$ such that $\mu(E_{k,N^*}) > \frac{1}{2}\mu(E_k) > 0$. For simplicity of notation, let $E^* = E_{k,N^*}$ and $\epsilon^* = \frac{1}{k}$. By the choice of E^*, we have that

$$\mu\big(B_n(x,\epsilon)\big) \leq \exp\big(-n(s-r)+f_n(x)\big), \quad \forall x \in E^*,\ 0<\epsilon<\epsilon^*,\ n \geq N.$$

Fix a sufficiently large $N > N^*$. For each cover $\mathcal{F} = \{B_{n_i}(y_i, \frac{\epsilon}{2})\}_{i\geq 1}$ of E^* with $0<\epsilon<\epsilon^*$ and $n_i \geq N \geq N^*$ for each $i \geq 1$. Without loss of generality, assume that $E^* \cap B_{n_i}(y_i,\frac{\epsilon}{2}) \neq \varnothing,\ \forall i$. Thus, for each $i \geq 1$ there exists $x_i \in E^* \cap B_{n_i}(y_i, \frac{\epsilon}{2})$. Hence,

$$B_{n_i}\left(y_i, \frac{\epsilon}{2}\right) \subset B_{n_i}(x_i,\epsilon).$$

It follows that

$$\sum_{i\geq 1} \exp\Big(-n(s-r) + \sup_{y\in B_{n_i}(y_i,\frac{\epsilon}{2})} f_{n_i}(y)\Big) \geq \sum_{i\geq 1} \exp\big(-n(s-r)+f_{n_i}(x_i)\big) \geq \sum_{i\geq 1} \mu\big(B_{n_i}(x_i,\epsilon)\big) \geq \mu(E^*).$$

Therefore,

$$M\left(E^*, f, s-r, N, \frac{\epsilon}{2}\right) \geq \mu\big(E^*\big) > 0.$$

Consequently

$$M\left(E^*, f, s-r, \frac{\epsilon}{2}\right) = \lim_{N\to\infty} M\left(E^*, f, s-r, N, \frac{\epsilon}{2}\right) \geq \mu\big(E^*\big) > 0,$$

which implies that $P_B(T,f,E^*,\frac{\epsilon}{2}) \geq s-r$. Letting $\epsilon \to 0$, we have that $P_B(T,f,E^*) \geq s-r$. It follows that $P_B(T,f,E) \geq P_B(T,f,E^*) \geq s-r$. The arbitrariness of r implies that $P_B(T,f,E) \geq s$. This completes the proof of the theorem. $\square$

We now recall the Vitali covering lemma which is useful in the proof of the main results.

Lemma 3.1. *Let (X, d) be a compact metric space and $\mathcal{B} = \{B(x_i, r_i)\}_{i \in \mathcal{I}}$ a family of closed (or open) balls in X. Then there exists a finite or countable subfamily $\mathcal{B}' = \{B(x_i, r_i)\}_{i \in \mathcal{I}'}$ of pairwise disjoint balls in $\mathcal{B}$ such that*

$$\bigcup_{B \in \mathcal{B}} B \subseteq \bigcup_{i \in \mathcal{I}'} B(x_i, 5r_i)$$

Proof. See [22, Theorem 2.1]. □

The following proposition gives the relations of the Pesin–Pitskel topological pressure with the weighted topological pressure. The argument of the proof is similar as Feng and Huang's [13, Proposition 3.2].

Proposition 3.2. *Let $K \subseteq X$. Then for any $s \in \mathbb{R}$ and $\epsilon, \delta > 0$, we have*

$$\mathcal{M}(K, f, s + \delta, N, 6\epsilon) \le W(K, f, s, N, \epsilon) \le M(K, f, s, N, \epsilon)$$

for all sufficiently large N, where $\mathcal{M}(\cdot)$ is defined as (3.1). Consequently, we have $P_B(T, f, K, 6\epsilon) \le P_W(T, f, K, \epsilon) \le P_B(T, f, K, \epsilon)$ and $P_B(T, f, K) = P_W(T, f, K)$.

Proof. Let $K \subseteq X$, $s \in \mathbb{R}$, $\epsilon, \delta > 0$, taking $c_i = 1$ in the definition of weighted topological pressure, we see that $W(K, f, s, N, \epsilon) \le M(K, f, s, N, \epsilon)$ for each $N \in \mathbb{N}$. In the following, we show that

$$\mathcal{M}(K, f, s + \delta, N, 6\epsilon) \le W(K, f, s, N, \epsilon) \tag{3.2}$$

for all sufficiently large N.

Assume that $N \ge 2$ is such that $n^2 e^{-n\delta} \le 1$ for $n \ge N$. Let $\{B_{n_i}(x_i, \epsilon), c_i\}_{i \in \mathcal{I}}$ be a family so that $\mathcal{I} \subseteq \mathbb{N}$, $x_i \in X$, $0 < c_i < \infty$, $n_i \ge N$ and

$$\sum_i c_i \chi_{B_i} \ge \chi_K$$

where $B_i := B_{n_i}(x_i, \epsilon)$. To prove (3.2), it suffices to prove that

$$\mathcal{M}(K, f, s + \delta, N, 6\epsilon) \le \sum_{i \in \mathcal{I}} c_i \exp\Big(-sn_i + \sup_{y \in B_{n_i}(x_i, \epsilon)} f_{n_i}(y)\Big). \tag{3.3}$$

Denote $\mathcal{I}_n := \{i \in \mathcal{I} : n_i = n\}$ and $\mathcal{I}_{n,k} = \{i \in \mathcal{I}_n : i \le k\}$ for $n \ge N$ and $k \in \mathbb{N}$. For the simplicity of notations, set $B_i := B_{n_i}(x_i, \epsilon)$ and $5B_i := B_{n_i}(x_i, 5\epsilon)$ for $i \in \mathcal{I}$. Without loss of generality, assume that $B_i \ne B_j$ for $i \ne j$. For $t > 0$, set

$$K_{n,t} = \Big\{x \in K : \sum_{i \in \mathcal{I}_n} c_i \chi_{B_i}(x) > t\Big\} \quad \text{and} \quad K_{n,k,t} = \Big\{x \in K : \sum_{i \in \mathcal{I}_{n,k}} c_i \chi_{B_i}(x) > t\Big\}.$$

We divide the proof of (3.3) into the following three steps.

Step 1. For each $n \ge N, k \in \mathbb{N}$ and $t > 0$, there exists a finite set $\mathcal{J}_{n,k,t} \subseteq \mathcal{I}_{n,k}$ such that the balls $B_i (i \in \mathcal{J}_{n,k,t})$ are pairwise disjoint, $K_{n,k,t} \subseteq \bigcup_{i \in \mathcal{J}_{n,k,t}} 5B_i$ and

$$\sum_{i \in \mathcal{J}_{n,k,t}} \exp\Big(-sn + \sup_{y \in B_i} f_n(y)\Big) \le \frac{1}{t} \sum_{i \in \mathcal{I}_{n,k}} c_i \exp\Big(-sn + \sup_{y \in B_n(x_i, \epsilon)} f_n(y)\Big).$$

To prove the previous inequality, using Federer's method [12, 2.10.24] (see also Mattila [22, Lemma 8.16]), we may assume that each c_i is a positive integer (see [13] for details). Let m be the smallest integer satisfying $m \geq t$. Let $\mathcal{B} = \{B_i : i \in \mathcal{I}_{n,k}\}$, and define $u : \mathcal{B} \to \mathbb{Z}$ by $u(B_i) = c_i$. We can inductively define integer-valued functions $v_0, v_1, ..., v_m$ on $\mathcal{B}$ and subfamilies $\mathcal{B}_1, ..., \mathcal{B}_m$ of $\mathcal{B}$ with $v_0 = u$. Using Lemma 3.1 (use the metric d_n instead of d), there exists a pairwise disjoint subfamily $\mathcal{B}_1$ of $\mathcal{B}$ such that $\bigcup_{B\in\mathcal{B}} B \subseteq \bigcup_{B\in\mathcal{B}_1} 5B$, and hence $K_{n,k,t} \subseteq \bigcup_{B\in\mathcal{B}_1} 5B$. Using Lemma 3.1 repeatedly, for $j = 1, ..., m$, we can obtain disjoint subfamilies $\mathcal{B}_j$ of $\mathcal{B}$ such that

$$\mathcal{B}_j \subseteq \big\{B \in \mathcal{B} : v_{j-1}(B) \geq 1\big\}, \qquad K_{n,k,t} \subseteq \bigcup_{B\in\mathcal{B}_j} 5B$$

and the function v_j such that

$$v_j(B) = \begin{cases} v_{j-1}(B) - 1 & \text{for } B \in \mathcal{B}_j, \\ v_{j-1}(B) & \text{for } B \in \mathcal{B} \setminus \mathcal{B}_j. \end{cases}$$

This is possible since $K_{n,k,t} \subset \{x : \sum_{B\in\mathcal{B}:B\ni x} v_j(B) \geq m - j\}$ for $j < m$, whence every $x \in K_{n,k,t}$ belongs to some ball $B \in \mathcal{B}$ with $v_j(B) \geq 1$. Hence,

$$\begin{aligned}
\sum_{j=1}^{m}\sum_{B\in\mathcal{B}_j} \exp\Big(-sn + \sup_{y\in B} f_n(y)\Big) &= \sum_{j=1}^{m}\sum_{B\in\mathcal{B}_j} \big(v_{j-1}(B) - v_j(B)\big) \exp\Big(-sn + \sup_{y\in B} f_n(y)\Big) \\
&\leq \sum_{B\in\mathcal{B}}\sum_{j=1}^{m} \big(v_{j-1}(B) - v_j(B)\big) \exp\Big(-sn + \sup_{y\in B} f_n(y)\Big) \\
&\leq \sum_{B\in\mathcal{B}} u(B) \exp\Big(-sn + \sup_{y\in B} f_n(y)\Big) \\
&= \sum_{i\in\mathcal{I}_{n,k}} c_i \exp\Big(-sn + \sup_{y\in B_i} f_n(y)\Big).
\end{aligned}$$

Choose $j_0 \in \{1, ..., m\}$ such that $\sum_{B\in\mathcal{B}_{j_0}} \exp(-sn + \sup_{y\in B} f_n(y))$ is the smallest. Then

$$\begin{aligned}
\sum_{B\in\mathcal{B}_{j_0}} \exp\Big(-sn + \sup_{y\in B} f_n(y)\Big) &\leq \frac{1}{m} \sum_{i\in\mathcal{I}_{n,k}} c_i \exp\Big(-sn + \sup_{y\in B_n(x_i,\epsilon)} f_n(y)\Big) \\
&\leq \frac{1}{t} \sum_{i\in\mathcal{I}_{n,k}} c_i \exp\Big(-sn + \sup_{y\in B_n(x_i,\epsilon)} f_n(y)\Big).
\end{aligned}$$

Hence,

$$\mathcal{J}_{n,k,t} = \{i \in \mathcal{I}_{n,k} : B_i \in \mathcal{B}_{j_0}\}.$$

Step 2. For each $n \geq N$ and $t > 0$, we have

$$\mathcal{M}(K_{n,t}, f, s + \delta, N, 6\epsilon) \leq \frac{1}{n^2 t} \sum_{i\in\mathcal{I}_n} c_i \exp\Big(-sn + \sup_{y\in B_n(x_i,\epsilon)} f_n(y)\Big). \tag{3.4}$$

Without loss of generality, assume $K_{n,t} \neq \emptyset$; otherwise there is nothing to prove. Since $K_{n,k,t} \to K_{n,t}$ as $k \to \infty$, $K_{n,k,t} \neq \emptyset$ for all sufficiently large k. Let $\mathcal{J}_{n,k,t}$ be the sets constructed in step 1, then $\mathcal{J}_{n,k,t} \neq \emptyset$ for sufficiently large k. Define $E_{n,k,t} = \{x_i : i \in \mathcal{J}_{n,k,t}\}$. Since the space of all non-empty compact subsets of X

is compact with respect to the Hausdorff distance (cf. [12, 2.10.21]), there exists a subsequence $\{k_j\}_{j\geq 1}$ of positive integers and a non-empty compact set $E_{n,t} \subseteq X$ such that $E_{n,k_j,t}$ converges to $E_{n,t}$ in the Hausdorff distance as $j \to \infty$. Since the distance of any two points in $E_{n,k,t}$ is not less than ϵ (with respect to d_n), so do the points in $E_{n,t}$. Thus $E_{n,t}$ is a finite set and $\sharp(E_{n,k_j,t}) = \sharp(E_{n,t})$ for sufficiently large j. Hence, the following holds

$$\bigcup_{x\in E_{n,t}} B_n(x, 5.5\epsilon) \supseteq \bigcup_{x\in E_{n,k_j,t}} B_n(x, 5\epsilon) = \bigcup_{i\in \mathcal{J}_{n,k_j,t}} 5B_i \supseteq K_{n,k_j,t}$$

for all sufficiently large j, and this yields that $\bigcup_{x\in E_{n,t}} B_n(x, 6\epsilon) \supseteq K_{n,t}$. Since $\sharp(E_{n,k_j,t}) = \sharp(E_{n,t})$ for sufficiently large j, using the result in step 1 we have

$$\begin{aligned}\sum_{x\in E_{n,t}} \exp\bigl(-ns + f_n(x)\bigr) &\leq \sum_{x\in E_{n,k_j,t}} \exp\Bigl(-ns + \sup_{y\in B_n(x,\epsilon)} f_n(y)\Bigr)\\ &\leq \frac{1}{t}\sum_{i\in\mathcal{I}_n} c_i \exp\Bigl(-sn + \sup_{y\in B_n(x_i,\epsilon)} f_n(y)\Bigr).\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{M}(K_{n,t}, f, s+\delta, N, 6\epsilon) &\leq \sum_{x\in E_{n,t}} \exp\bigl(-n(s+\delta) + f_n(x)\bigr)\\ &\leq \frac{1}{e^{n\delta}t}\sum_{i\in\mathcal{I}_n} c_i \exp\Bigl(-sn + \sup_{y\in B_n(x_i,\epsilon)} f_n(y)\Bigr)\\ &\leq \frac{1}{n^2 t}\sum_{i\in\mathcal{I}_n} c_i \exp\Bigl(-sn + \sup_{y\in B_n(x_i,\epsilon)} f_n(y)\Bigr).\end{aligned}$$

Step 3. For any $t \in (0, 1)$, we have

$$\mathcal{M}(K, f, s+\delta, N, 6\epsilon) \leq \frac{1}{t}\sum_{i\in\mathcal{I}} c_i \exp\Bigl(-sn + \sup_{y\in B_n(x_i,\epsilon)} f_n(y)\Bigr).$$

Consequently, (3.3) holds.

To prove this, fix $t \in (0, 1)$. Note that $\sum_{n=N}^{\infty} n^{-2} < 1$ and $K \subseteq \bigcup_{n=N}^{\infty} K_{n,n^{-2}t}$. By (3.4) we have

$$\begin{aligned}\mathcal{M}(K, f, s+\delta, N, 6\epsilon) &\leq \sum_{n=N}^{\infty} \mathcal{M}(K_{n,t}, f, s+\delta, N, 6\epsilon)\\ &\leq \sum_{n=N}^{\infty} \frac{1}{t}\sum_{i\in\mathcal{I}_n} c_i \exp\Bigl(-sn + \sup_{y\in B_n(x_i,\epsilon)} f_n(y)\Bigr)\\ &\leq \frac{1}{t}\sum_{i\in\mathcal{I}} c_i \exp\Bigl(-sn_i + \sup_{y\in B_{n_i}(x_i,\epsilon)} f_{n_i}(y)\Bigr).\end{aligned}$$

To finish the proof of this proposition, note that the Pesin–Pitskel topological pressure does not change if we replace $\sup_{y\in B_n(x,\epsilon)} f_n(y)$ by any number in the interval $[\inf_{y\in B_n(x,\epsilon)} f_n(y), \sup_{y\in B_n(x,\epsilon)} f_n(y)]$ in the definition of the Pesin–Pitskel topological pressure, see [2, Corollary 1.2] or [27] for a proof of this fact. □

The following lemma is an analogue of Feng and Huang's approximation and classical Frostman's lemma, see [13, Lemma 3.4]. The argument is ultimately adapted from Howroyd's elegant argument (cf. [15, Theorem 2], [22, Theorem 8.17]). It can be proven by the same arguments as in [13] except adding a potential function, so we omit the detailed proof.

Lemma 3.3. *Let K be a nonempty compact subset of X and $f \in C(X)$. Let $s \in \mathbb{R}$, $N \in \mathbb{N}$ and $\epsilon > 0$. Suppose that $c := W(K, f, s, N, \epsilon) > 0$. Then there is a Borel probability measure μ on X such that $\mu(K) = 1$ and*

$$\mu\big(B_n(x,\epsilon)\big) \le \frac{1}{c} \exp\Big[-ns + \sup_{y \in B_n(x,\epsilon)} f_n(y)\Big], \quad \forall x \in X,\ n \ge N.$$

Now we are ready to prove the first statement in Theorem B.

Proof of Theorem B(1). Assume $\mu(K) = 1$, since $P_\mu(T, f) = \int P_\mu(T, f, x)\,\mathrm{d}\mu$, it follows that the set

$$K_\delta = \big\{x \in K : P_\mu(T, f, x) \ge P_\mu(T, f) - \delta\big\}$$

has positive μ-measure for all $\delta > 0$. Thus

$$P_B(T, f, K_\delta) \ge P_\mu(T, f) - \delta$$

by (2) of Theorem A. Since $K_\delta \subset K$ for all $\delta > 0$, we have $P_B(T, f, K) \ge P_\mu(T, f)$. Hence,

$$P_B(T, f, K) \ge \sup\big\{P_\mu(T, f) : \mu \in \mathcal{M}(X), \mu(K) = 1\big\}.$$

We next show that

$$P_B(T, f, K) \le \sup\big\{P_\mu(T, f) : \mu \in \mathcal{M}(X), \mu(K) = 1\big\}. \tag{3.5}$$

We can assume that $P_B(T, f, K) \ne -\infty$, otherwise there is nothing to prove. By Proposition 3.2 we have $P_B(T, f, K) = P_W(T, f, K)$. Fix a small number $\beta > 0$. Let $s = P_B(T, f, K) - \beta$. Since

$$\lim_{\epsilon\to 0} \liminf_{n\to\infty} \frac{1}{n}\Big[f_n(x) - \sup_{y \in B_n(x,\epsilon)} f_n(y)\Big] = 0$$

for all $x \in X$, we have that

$$\liminf_{n\to\infty} \frac{1}{n}\Big[f_n(x) - \sup_{y \in B_n(x,\epsilon)} f_n(y)\Big] > -\beta, \quad \forall x \in X$$

for all sufficiently small $\epsilon > 0$. Take such an $\epsilon > 0$ and an $N \in \mathbb{N}$ such that $c := W(K, f, s, N, \epsilon) > 0$. By Lemma 3.3, there exists $\mu \in \mathcal{M}(X)$ with $\mu(K) = 1$ such that

$$\mu\big(B_n(x,\epsilon)\big) \le \frac{1}{c} \exp\Big[-ns + \sup_{y \in B_n(x,\epsilon)} f_n(y)\Big]$$

for any $x \in X$ and $n \ge N$. Therefore

$$P_\mu(T, f, x) \ge P_\mu(T, f, x, \epsilon) \ge s + \liminf_{n\to\infty} \frac{1}{n}\Big[f_n(x) - \sup_{y \in B_n(x,\epsilon)} f_n(y)\Big] \ge P_B(T, f, K) - 2\beta$$

for all $x \in X$. And hence

$$P_\mu(T, f) = \int P_\mu(T, f, x)\,\mathrm{d}\mu \ge P_B(T, f, K) - 2\beta.$$

Consequently, (3.5) immediately follows. □

Next we turn to prove the second statement in Theorem B. We will first prove this result in the case of that X is *zero-dimensional*, and then prove it in general. The proof of the following useful lemma use the same construction as Feng–Huang's method (cf. [13, Proposition 3.7]), so we omit the detailed proof.

Lemma 3.4. *Assume that $\mathcal{U}$ is a closed-open partition of X. Let $N \in \mathbb{N}$ and $f \in C(X)$. Then*

(i) If $E_{i+1} \supseteq E_i$ and $\bigcup_i E_i = E$, then $M_N^s(\mathcal{U}, f, E) = \lim_{i\to\infty} M_N^s(\mathcal{U}, f, E_i)$;
(ii) Assume $Z \subset X$ is analytic. Then $M_N^s(\mathcal{U}, f, Z) = \sup\{M_N^s(\mathcal{U}, f, K) : K \subset Z, K \text{ is compact}\}$.

Theorem 3.5. *Assume that X is zero-dimensional, i.e., for any $\delta > 0$, X has a closed-open partition with diameter less than δ. Then for any analytic set $Z \subseteq X$,*

$$P_B(T, f, Z) = \sup\{P_B(T, f, K) : K \subseteq Z, K \text{ is compact}\}.$$

Proof. Let Z be an analytic subset of X with $P_B(T, f, Z) \neq -\infty$, otherwise there is nothing to prove. Let $s < P_B(T, f, Z)$. Since $P_B(T, f, Z) = \sup_{\mathcal{U}} P_B(T, f, \mathcal{U}, Z) = \lim_{|\mathcal{U}|\to 0} P_B(T, f, \mathcal{U}, Z)$, there exists a closed-open partition $\mathcal{U}$ so that $P_B(T, \mathcal{U}, f, Z) > s$ and thus $M^s(\mathcal{U}, f, Z) = \infty$. Hence $M_N^s(\mathcal{U}, f, Z) > 0$ for some $N \in \mathbb{N}$. By Lemma 3.4, we can find a compact set $K \subseteq Z$ such that $M_N^s(\mathcal{U}, f, K) > 0$. This implies $P_B(T, f, K) \geq P_B(T, \mathcal{U}, f, K) \geq s$. This is the result that we need. □

Proposition 3.6. *Let (X, T) be a TDS with $h_{top}(T) < \infty$ and $f \in C(X)$, then there exists a factor $\pi : (Y, S) \to (X, T)$ such that (Y, S) is zero-dimensional and*

$$\sup_{x\in X} P(S, 0, \pi^{-1}(x)) = 0.$$

Proof. Assume that (X, T) is a *TDS* with $h_{top}(T) < \infty$. We obtained the result immediately by Lemma 3.13 in [13]. □

Proposition 3.7. *If $\pi : (Y, S) \to (X, T)$ is a factor map and f is a continuous function on X, then for each subset $E \subset Y$ we have*

$$P_B(T, f, \pi(E)) \leq P_B(S, f \circ \pi, E) \leq P_B(T, f, \pi(E)) + \sup_{x\in X} P(S, 0, \pi^{-1}(x)).$$

Proof. See [20, Theorem 2.1(ii)] for the proof of the second inequality. It is left to prove the first inequality. Fix $\epsilon > 0$. By the uniform continuity of the map π, there exists $\delta > 0$ such that

$$d(x, y) < \delta \quad \Longrightarrow \quad d(\pi(x), \pi(y)) < \epsilon.$$

Fix a positive integer N, consider a cover of E with Bowen balls $\{B_{n_i}(x_i, \delta)\}$, where $n_i \geq N$ for each i. Then it is easy to see that $\{B_{n_i}(\pi(x_i), \epsilon)\}$ is a cover of $\pi(E)$, so we have $M(\pi(E), f, s, N, \epsilon) \leq M(E, f \circ \pi, s, N, \delta)$. And this implies that

$$M(\pi(E), f, s, \epsilon) \leq M(E, f \circ \pi, s, \delta).$$

Hence, $P_B(T, f, \pi(E), \epsilon) \leq P_B(S, f \circ \pi, E, \delta)$. Since $\epsilon \to 0$ implies $\delta \to 0$, letting $\epsilon \to 0$ we have

$$P_B(T, f, \pi(E)) \leq P_B(S, f \circ \pi, E).$$

This completes the proof of the proposition. □

Proposition 3.8. *Let (X,T) be a TDS with $h_{top}(T) < \infty$, then there exists a factor $\pi : (Y,S) \to (X,T)$ such that (Y,S) is zero-dimensional and*

$$P_B\big(T, f, \pi(E)\big) = P_B(S, f \circ \pi, E), \quad \forall E \subseteq Y.$$

Proof. It is the direct combination of Proposition 3.6 and Proposition 3.7. □

Now we turn to prove the second result in Theorem B.

Proof of Theorem B(2). By Proposition 3.8, there exists a factor map $\pi : (Y,S) \to (X,T)$ such that (Y,S) is zero-dimensional and $P_B(T, f, \pi(E)) = P_B(S, f \circ \pi, E)$ for any $f \in C(X)$ and $E \subseteq Y$.

Let Z be an analytic subset of X. Then $\pi^{-1}(Z)$ is also an analytic subset of Y. Using Proposition 3.5, we have

$$\begin{aligned} P_B(T, f, Z) &= P_B\big(S, f \circ \pi, \pi^{-1}(Z)\big) \\ &= \sup\big\{P_B(S, f \circ \pi, E) : E \subseteq \pi^{-1}(Z), E \text{ is compact}\big\} \\ &= \sup\big\{P_B\big(T, f, \pi(E)\big) : E \subseteq \pi^{-1}(Z), E \text{ is compact}\big\} \\ &\leq \sup\big\{P_B(T, f, K) : K \subseteq Z, K \text{ is compact}\big\} \end{aligned}$$

By Proposition 2.1, the reverse inequality is trivial. Hence,

$$P_B(T, f, Z) = \sup\big\{P_B(T, f, K) : K \subseteq Z, K \text{ is compact}\big\}. \qquad \square$$

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