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Compactness for the commutators of singular integral operators with rough variable kernels

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Let T_Ω be the singular integral operator with variable kernel defined by

$$T_\Omega f(x) = \mathrm{p.\,v.} \int\limits_{\mathbb{R}^n} \frac{\Omega(x,\, x-y)}{|x-y|^n} f(y)\mathrm{d}y,$$

where $\Omega(x,\, y)$ is homogeneous of degree zero in the second variable y, and $\int_{S^{n-1}} \Omega(x,\, z')\mathrm{d}\sigma(z') = 0$ for any $x \in \mathbb{R}^n$. In this paper, the authors prove that if $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 2(n-1)/n$, then the commutator generated by a $\mathrm{CMO}(\mathbb{R}^n)$ function and T_Ω, and the associated lacunary maximal operator, are compact on $L^2(\mathbb{R}^n)$. The associated continuous maximal commutator is also considered.

1. Introduction

We will work on $\mathbb{R}^n$, $n \geq 2$. Let $\Omega(x,\, z)$ be a function on $\mathbb{R}^n \times \mathbb{R}^n$, which is homogeneous of degree zero with respect to the variable z. Throughout this paper, for such a function $\Omega(x,\, z)$, we assume that Ω satisfies the vanishing condition that

$$\int\limits_{S^{n-1}} \Omega(x,\, z')\mathrm{d}\sigma(z') = 0 \tag{1.1}$$

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for any $x \in \mathbb{R}^n$, here and in what follows, S^{n-1} denotes the unit sphere in $\mathbb{R}^n$, and for $z \in \mathbb{R}^n$, $z' = z/|z|$. For a fixed $q \in [1, \infty]$, we say that $\Omega(x, z') \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$, if

$$\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})} = \sup_{x\in\mathbb{R}^n} \Big(\int\limits_{S^{n-1}} |\Omega(x, z')|^q \mathrm{d}\sigma(z') \Big)^{1/q} < \infty.$$

The singular integral operator with variable kernel, associated with Ω, is defined by

$$T_\Omega f(x) = \mathrm{p.\,v.} \int\limits_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^n} f(y)\mathrm{d}y$$

initially for $f \in \mathcal{S}(\mathbb{R}^n)$. These operators were introduced by Calderón and Zygmund in their celebrated works [5,6], and are relevant in second order linear elliptic equations with variable coefficients. Calderón and Zygmund [5,6] proved that if $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 2(n-1)/n$, then T_Ω is bounded on $L^2(\mathbb{R}^n)$ and the condition $q > 2(n-1)/n$ is optimal in the sense that the $L^2(\mathbb{R}^n)$ boundedness of T may fail if $q \le 2(n-1)/n$. Moreover, Calderón and Zygmund [7] showed that T_Ω is bounded on $L^p(\mathbb{R}^n)$ for $p \in (2, \infty)$ if $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ with $\frac{1}{q} < \frac{1}{p}\frac{n}{n-1} + \big(1 - \frac{2}{p}\big)$.

The maximal singular integral operator associated with T_Ω is given by

$$T^\star_\Omega f(x) = \sup_{\epsilon>0} \Big| \int\limits_{|x-y|\ge\epsilon} \frac{\Omega(x, x-y)}{|x-y|^n} f(y)\mathrm{d}y \Big|.$$

This operator plays an important role in studying the almost everywhere convergence of the singular integral. Aguilera and Harboure [1] proved that if $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 4(n-1)/(2n-1)$, then $T^\star_\Omega$ is bounded on $L^2(\mathbb{R}^n)$. Cowling and Mauceri [15], Christ, Duoandikoetxea and Rubio de Francia [13], proved that if $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 2(n-1)/n$, then $T^\star_\Omega$ is bounded on $L^2(\mathbb{R}^n)$.

Now let $b \in \mathrm{BMO}(\mathbb{R}^n)$, the space of functions of bounded mean oscillation which was introduced by John and Nirenberg. The commutator generated by b and T_Ω is defined by

$$T_{\Omega,\,b} f(x) = b(x) T_\Omega f(x) - T_\Omega(bf)(x), \tag{1.2}$$

initially for $f \in \mathcal{S}(\mathbb{R}^n)$. We define the maximal operator $T^\star_{\Omega,\,b}$, corresponding to $T_{\Omega,\,b}$, by

$$T^\star_{\Omega,\,b} f(x) = \sup_{\epsilon>0} \Big| \int\limits_{|x-y|>\epsilon} \big(b(x) - b(y)\big) \frac{\Omega(x, x-y)}{|x-y|^n} f(y)\mathrm{d}y \Big|. \tag{1.3}$$

Chiarenza et al. [12] proved that if $\Omega \in L^\infty(\mathbb{R}^n) \times C^\infty(S^{n-1})$, then $T_{\Omega,\,b}$ is bounded on $L^2(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Di Fazio and Ragusa [17] considered the boundedness of $T_{\Omega,\,b}$ on Morrey spaces. By subtle Fourier transform estimates and Littlewood–Paley theory, Chen and Ding [8] proved that $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 2(n-1)/n$ is sufficient for $T_{\Omega,\,b}$ to be bounded on $L^2(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Furthermore, Chen, Ding and Li [9] showed that $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 2(n-1)/n$ also implies that $T^\star_{\Omega,\,b}$ is bounded on $L^2(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$.

Uchiyama [20] considered the compactness of the commutator of singular integral operators. Let $\mathrm{CMO}(\mathbb{R}^n)$ be the closure of $C^\infty_0(\mathbb{R}^n)$ in the $\mathrm{BMO}(\mathbb{R}^n)$ topology, which coincides with $\mathrm{VMO}(\mathbb{R}^n)$, the space of functions of vanishing mean oscillation introduced by Coifman and Weiss in [14], see also [4]. Uchiyama proved that if S is a Calderón–Zygmund operator, and $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $[b, S]$, the commutator of S and b, as in (1.2), is a compact operator on $L^p(\mathbb{R}^n)$ ($p \in (1, \infty)$) if and only if $b \in \mathrm{CMO}(\mathbb{R}^n)$. This shows that for $\mathrm{CMO}(\mathbb{R}^n)$ functions b, the properties of $[b, S]$ maybe better than that of the operator S. Since then, many authors have considered the compactness of commutators of classical operators on various function spaces, among them, we mention the papers [10,2,18,19,3] and the references therein.

The purpose of this paper is to consider the compactness on $L^2(\mathbb{R}^n)$ for $T_{\Omega,\,b}$ and the associated maximal operators. To formulate our main result, we first recall the definition of compact operator.

Definition 1.1. Let $\mathcal{X}$, $\mathcal{Y}$ be two Banach spaces and T be a bounded operator from $\mathcal{X}$ to $\mathcal{Y}$. Suppose that for each bounded set $\mathcal{G} \subset \mathcal{X}$, $T\mathcal{G} = \{Tx : x \in \mathcal{G}\}$ is a strongly pre-compact set in $\mathcal{Y}$, then T is called a compact operator from $\mathcal{X}$ to $\mathcal{Y}$.

Our first result can be stated as follows.

Theorem 1.2. *Let $\Omega(x, z')$ satisfy (1.1) and $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 2(n-1)/n$. Then for $b \in \mathrm{CMO}(\mathbb{R}^n)$, the operator $T_{\Omega,\,b}$ and the lacunary maximal operator $T^{\star\star}_{\Omega,\,b}$ defined by*

$$T^{\star\star}_{\Omega,\,b}f(x) = \sup_{u\in\mathbb{Z}}\Big|\int\limits_{|x-y|>2^u}\big(b(x)-b(y)\big)\frac{\Omega(x,\,x-y)}{|x-y|^n}f(y)\mathrm{d}y\Big|$$

are compact on $L^2(\mathbb{R}^n)$.

We are very interested in the compactness on $L^2(\mathbb{R}^n)$ for the maximal operator $T^{\star}_{\Omega,\,b}$ defined by (1.3). Although we do not know if $T^{\star}_{\Omega,\,b}$ is compact on $L^2(\mathbb{R}^n)$, we can prove that, in some sense, $T^{\star}_{\Omega,\,b}$ enjoys the compactness on $L^2(\mathbb{R}^n)$. As usual, for $\{f_k\} \subset L^p(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$, $f_k \rightharpoonup f$ means that for any $g \in L^{p'}(\mathbb{R}^n)$ $(p' = p/(p-1))$,

$$\lim_{k\to\infty}\int\limits_{\mathbb{R}^n} g(x)\big(f_k(x)-f(x)\big)\mathrm{d}x = 0.$$

Theorem 1.3. *Let $\Omega(x, z')$ satisfy (1.1) and $\Omega \in L^\infty(\mathbb{R}^n) \times L^q(S^{n-1})$ for some $q > 2(n-1)/n$. Let $b \in \mathrm{CMO}(\mathbb{R}^n)$ and $T^{\star}_{\Omega,\,b}$ be the operator defined by (1.3). Then for $\{f_k\} \subset L^2(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$,*

$$|f_k - f| \rightharpoonup 0 \Rightarrow \|T^{\star}_{\Omega,\,b}f_k - T^{\star}_{\Omega,\,b}f\|_{L^2(\mathbb{R}^n)} \to 0.$$

Remark 1.4. In $L^p(\mathbb{R}^n)$, it is obvious that

$$|f_k - f| \rightharpoonup 0 \Rightarrow f_k \rightharpoonup f.$$

However, $f_k \rightharpoonup f$ does not imply $|f_k - f| \rightharpoonup 0$. For example, let $g(x) = \chi_{[0,\,1]}(x)$ and $g_m(x) = \exp(2m\pi ix)g(x)$. It is easy to verify that $\{g_m\}_{m\in\mathbb{Z}}$ is an orthonormal system of $L^2(\mathbb{R})$, and so in $L^2(\mathbb{R})$, $g_m \rightharpoonup 0$ $(|m| \to \infty)$, but $|g_m| = \chi_{[0,\,1]} \not\rightharpoonup 0$. This shows that the statement of Theorem 1.3 is strictly weaker than the compactness of $T^{\star\star}_{\Omega,\,b}$.

Remark 1.5. To prove our theorems, we will invoke the idea of approximating the operator T_Ω and the corresponding lacunary maximal operator, by integral operators whose kernels enjoy appropriate regularity. It should be pointed out that this idea can be dated back to the paper of Watson [21], and was used in [11] to prove the compactness of the commutator of rough homogeneous singular integral operators on $L^p(\mathbb{R}^n)$. However, the argument for the maximal commutator of singular integral operators with variable kernels is more complicated.

We make some conventions. In what follows, C always denotes a positive constant that is independent of the main parameters involved but whose value may differ from line to line. We use the symbol $A \lesssim B$ to denote that there exists a positive constant C such that $A \le CB$. Also, we use the notation $A \approx B$ to denote that there exist two positive constant C_1 and C_2 such that $C_1 A \le B \le C_2 A$. For a set $E \subset \mathbb{R}^n$, χ_E denotes its characteristic function. For $p \in [1, \infty]$, we use p' to denote the dual exponent of p, namely, $p' = p/(p-1)$. For suitable function f, $\widehat{f}$ denotes the Fourier transform of f. Let M be the Hardy–Littlewood maximal operator. For $r \in (0, \infty)$, we use M_r to denote the operator $M_r f(x) = \big(M(|f|^r)(x)\big)^{1/r}$.

2. Approximation

This section is to devoted to some approximations which will be used in the proof of theorems.

Let $m \in \mathbb{N}$, $\mathcal{H}_m$ be the space of surface spherical harmonic of degree m on S^{n-1}, D_m be the dimension of $\mathcal{H}_m$. Denote by $\{Y_{m,j}\}_{j=1}^{D_m}$ the normalized complete system in $\mathcal{H}_m$. As it was pointed out in [7], $D_m \approx m^{2\lambda}$ with $\lambda = (n-2)/2$, and for fixed $m \in \mathbb{N}$ and $\xi' \in S^{n-1}$,

$$\sum_{j=1}^{D_m} |Y_{m,j}(\xi')|^2 \approx m^{2\lambda}. \tag{2.1}$$

For a fixed $m, j \in \mathbb{N}$ with $1 \le j \le D_m$, set $K_{m,j}(y) = \frac{Y_{m,j}(y')}{|y|^n}$ and

$$\sigma_{m,j;u}(y) = K_{m,j}(y)\chi_{\{2^{u-1}<|x|\le 2^u\}}(y).$$

For each fixed $m, j \in \mathbb{N}$ with $1 \le j \le D_m$, set

$$T_{m,j}f(x) = \text{p. v.} \int_{\mathbb{R}^n} \frac{Y_{m,j}((x-y)')}{|x-y|^n} f(y)\mathrm{d}y.$$

It was proved in [8, Lemma 2.2] that for $\beta \in (0, 1)$ and $u \in \mathbb{Z}$,

$$\big|\widehat{\sigma_{m,j;u}}(\xi)\big| \lesssim m^{-1-\lambda+\beta/2} \min\{|2^u\xi|, |2^u\xi|^{-\beta/2}\}\big|Y_{m,j}(\xi')\big| \tag{2.2}$$

and

$$\big|\widehat{\sigma_{m,j;u}}(\xi)\big| \lesssim m^{-1-\lambda}\big|Y_{m,j}(\xi')\big|. \tag{2.3}$$

Note that for each $m, j \in \mathbb{N}$ with $1 \le j \le D_m$ and each $\xi \in \mathbb{R}^n\backslash\{0\}$,

$$\sum_{u\in\mathbb{Z}} \big|\widehat{\sigma_{m,j;u}}(\xi)\big| \lesssim m^{-1-\lambda+\beta/2}.$$

It follows from the Plancherel theorem that

$$\|T_{m,j}f\|_{L^2(\mathbb{R}^n} \lesssim m^{-1-\lambda+\beta/2}\|f\|_{L^2(\mathbb{R}^n)}.$$

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a nonnegative function such that $\int_{\mathbb{R}^n} \phi(x)\mathrm{d}x = 1$, $\operatorname{supp}\phi \subset \{x : |x| \le 1/4\}$. For $l \in \mathbb{Z}$, let $\phi_l(y) = 2^{-nl}\phi(2^{-l}y)$. It is easy to verify that for $\xi \in \mathbb{R}^n$ and $0 < \rho \le 1$,

$$|\widehat{\phi_l}(\xi)| = |\widehat{\phi}(2^l\xi)| \lesssim 1, \ \ |\widehat{\phi_l}(\xi) - 1| \lesssim \min\{1, |2^l\xi|^\rho\}. \tag{2.4}$$

For a positive integer l, let

$$K_{m,\,j;\,l}(y) = \sum_{u\in\mathbb{Z}} \sigma_{m,j;\,u} * \phi_{u-l}(y), \tag{2.5}$$

and $T_{m,\,j;\,l}$ be the operator given by

$$T_{m,\,j;\,l}f(x) = \mathrm{p.\,v.}\int_{\mathbb{R}^n} K_{m,\,j;\,l}(x-y)f(y)\mathrm{d}y. \tag{2.6}$$

Lemma 2.1. *Let $m,\, l \in \mathbb{N}$, $1 \le j \le D_m$ and $K_{m,\,j;\,l}$ be the function defined by (2.5). Then*

(i) *for any $R > 0$, $y \in \mathbb{R}^n$ with $|y| < R/4$,*

$$\begin{aligned}&\sum_{u\in\mathbb{Z}}\int_{|x|>R} \left|\sigma_{m,\,j;\,u} * \phi_{u-l}(x-y) - \sigma_{m,\,j;\,u} * \phi_{u-l}(x)\right|\mathrm{d}x \\ &\lesssim \min\{l,\, 2^l|y|/R\};\end{aligned} \tag{2.7}$$

(ii) *the operator $T_{m,j;\,l}$ defined by (2.6), and its lacunary maximal operator defined by*

$$T^{\star\star}_{m,\,j;\,l}f(x) = \sup_{v\in\mathbb{Z}}\Big|\sum_{u=v}^{\infty}\int_{\mathbb{R}^n} \sigma_{m,\,j;\,u} * \phi_{u-l}(x-y)f(y)\,\mathrm{d}y\Big|$$

are bounded on $L^2(\mathbb{R}^n)$ with bound $Cm^{-1-\lambda+\beta/2}$;

(iii) *the maximal singular integral operator $T^{\star}_{m,\,j;\,l}$ defined by*

$$T^{\star}_{m,\,j;\,l}f(x) = \sup_{\epsilon>0}\Big|\int_{|x-y|>\epsilon} K_{m,j;\,l}(x-y)f(y)\mathrm{d}y\Big|$$

is bounded on $L^p(\mathbb{R}^n)$ with bound C.

Proof. Observe that $\|Y_{m,\,j}\|_{L^1(S^{n-1})} \lesssim \|Y_{m,\,j}\|_{L^2(S^{n-1})} \lesssim 1$, $\mathrm{supp}\,\sigma_{m,\,j;\,u} * \phi_{u-l} \subset \{x : 2^{u-2} \le |x| \le 2^{u+2}\}$ and

$$\big\|\phi_{u-l}(\cdot - y) - \phi_{u-l}(\cdot)\big\|_{L^1(\mathbb{R}^n)} \lesssim \min\{1,\, 2^{l-u}|y|\}.$$

It follows that

$$\begin{aligned}&\sum_{u\in\mathbb{Z}}\int_{2^{k-1}R<|x|\le 2^kR} \left|\sigma_{m,\,j;\,u} * \phi_{u-l}(x-y) - \sigma_{m,\,j;\,u} * \phi_{u-l}(x)\right|\mathrm{d}x \\ &= \sum_{u:\,2^u\approx 2^kR} \|\sigma_{m,\,j;\,u}\|_{L^1(\mathbb{R}^n)}\|\phi_{u-l}(\cdot - y) - \phi_{u-l}(\cdot)\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \sum_{u:2^u\approx 2^kR} \big\|\phi_{u-l}(\cdot - y) - \phi_{u-l}(\cdot)\big\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \min\{1,\, 2^{l-k}|y|/R\}.\end{aligned}$$

The conclusion (i) now follows directly.

We now prove (ii). For each $\xi \in \mathbb{R}^n \setminus \{0\}$, we have by the Fourier transform estimates (2.2) and (2.4) that

$$\sum_{k\in\mathbb{Z}} |\widehat{\sigma_{m,\,j;\,k}}(\xi)||\widehat{\phi_{k-l}}(\xi)| \lesssim \sum_{k\in\mathbb{Z}} \left|\widehat{\sigma_{m,\,j;\,k}}(\xi)\right| \lesssim m^{-1-\lambda+\beta/2}|Y_{m,\,j}(\xi')|.$$

This, via (2.1) and the Plancherel theorem, tells us that

$$\|T_{m,\,j;\,l}f\|_{L^2(\mathbb{R}^n)} \lesssim m^{-1-\lambda+\beta/2}\|f\|_{L^2(\mathbb{R}^n)}.$$

To estimate $\|T^{\star\star}_{m,\,j;\,l}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}$, we employ the idea from [16]. Let $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $\|\psi\|_{L^\infty(\mathbb{R}^n)} = 1$, and

$$\operatorname{supp}\psi \subset \{x \in \mathbb{R}^n : |x| \le 2\},\ \ \psi(x) \equiv 1 \text{ if } |x| \le 1.$$

For each integer v, let $\Psi_v \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\Psi}_v(\xi) = \psi(2^v\xi)$. Write

$$\begin{aligned}
&\sum_{u=v}^{\infty} \sigma_{m,\,j;\,u} * \phi_{u-l} * f(x)\\
&\quad= \Psi_u * \big(T_{m,\,j;\,l}f\big)(x) - \Psi_v * \Big(\sum_{u=-\infty}^{v-1} \sigma_{m,\,j;\,u} * \phi_{u-l} * f\Big)(x)\\
&\qquad + \sum_{u=v}^{\infty}(\delta - \Psi_v) * \sigma_{m,\,j;\,u} * \phi_{u-l} * f(x)\\
&\quad= \mathrm{I}^v_{m,\,j;\,l}f(x) + \mathrm{II}^v_{m,\,j;\,l}f(x) + \mathrm{III}^v_{m,\,j;\,l}f(x),
\end{aligned}$$

with δ the Dirac distribution. For each fixed $v \in \mathbb{Z}$, it is obvious that

$$\left|\mathrm{I}^v_{m,\,j;\,l}f(x)\right| \lesssim M\big(T_{m,\,j;\,l}f\big)(x).$$

By the boundedness of operators M, $T_{m,\,j}$ and $T_{m,\,j;\,l}$ on $L^2(\mathbb{R}^n)$,

$$\Big\|\sup_{v\in\mathbb{Z}}\left|\mathrm{I}^v_{m,j;l}f\right|\Big\|^2_{L^2(\mathbb{R}^n)} \lesssim \left\|T_{m,j;l}f - T_{m,j}f\right\|^2_{L^2(\mathbb{R}^n)} \lesssim m^{-2-2\lambda+\beta}\|f\|^2_{L^2(\mathbb{R}^n)}.$$

For the term $\sup_{v\in\mathbb{Z}}\left|\mathrm{II}^v_{m,\,j;\,l}f(x)\right|$, write

$$\sup_{v\in\mathbb{Z}}\left|\mathrm{II}^v_{m,\,j;\,l}f(x)\right| \lesssim \Big(\sum_{v=-\infty}^{\infty}\Big|\Psi_v * \sum_{u=-\infty}^{v-1}\sigma_{m,\,j;\,u} * \phi_{u-l} * f(x)\Big|^2\Big)^{1/2}.$$

For all $\xi \in \mathbb{R}^n \setminus \{0\}$, it follows from (2.2) and (2.4) that

$$\begin{aligned}
&\Big|\psi(2^v\xi)\sum_{u=-\infty}^{v-1}\widehat{\sigma_{m,\,j;\,u}}(\xi)\widehat{\phi}(2^{u-l}\xi)\Big|\\
&\quad\lesssim m^{-1-\lambda+\beta/2}\left|\psi(2^v\xi)\right|\sum_{u=-\infty}^{v-1}|2^u\xi||Y_{m,\,j}(\xi')|\\
&\quad\lesssim m^{-1-\lambda+\beta/2}\left|\psi(2^v\xi)|2^v\xi||Y_{m,\,j}(\xi')\right|.
\end{aligned}$$

We thus have by the Plancherel theorem that

$$\sum_{v=-\infty}^{\infty}\Big\|\Psi_v*\sum_{u=-\infty}^{v-1}\sigma_{m,\,j;\,u}*\phi_{u-l}*f\Big\|_{L^2(\mathbb{R}^n)}^2$$
$$=\sum_{v=-\infty}^{\infty}\int_{\mathbb{R}^n}\Big|\sum_{u=-\infty}^{v-1}\widehat{\sigma_{m,\,j;\,v}}(\xi)\widehat{\phi}(2^{u-l}\xi)\Big|^2\big|\psi(2^v\xi)\widehat{f}(\xi)\big|^2\mathrm{d}\xi$$
$$\lesssim m^{-2-2\lambda+\beta}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|Y_{m,\,j}(\xi')|^2\sum_{v=-\infty}^{\infty}|\psi(2^v\xi)|^2|2^v\xi|^2\big|\widehat{f}(\xi)\big|^2\,\mathrm{d}\xi.$$

The fact that $|\psi|\le 1$ and $\operatorname{supp}\psi\subset\{x:\,|x|\le 2\}$ now tell us that for each $\xi\in\mathbb{R}^n$,

$$\sum_{v=-\infty}^{\infty}|\psi(2^v\xi)|^2|2^v\xi|^2\lesssim\sum_{v\in\mathbb{Z}:\,|2^v\xi|\le 2}|2^v\xi|^2\lesssim 1.$$

Therefore, using (2.1),

$$\big\|\sup_{v\in\mathbb{Z}}\big|\mathrm{II}_{m,\,j;\,l}^{v}f\big|\big\|_{L^2(\mathbb{R}^n)}^2\lesssim m^{-2-2\lambda+\beta}\|f\|_{L^2(\mathbb{R}^n)}^2$$

As for the term $\sup_{v\in\mathbb{Z}}\big|\mathrm{III}_{m,\,j;\,l}^{v}f\big|$, write

$$\sup_{v\in\mathbb{Z}}\big|\mathrm{III}_{m,j;\,l}^{v}f(x)\big|\le\sum_{k=0}^{\infty}\sup_{v\in\mathbb{Z}}\big|(\delta-\Psi_v)*\sigma_{m,\,j;\,u+k}*\phi_{u+k-l}*f(x)\big|$$
$$\lesssim\sum_{k=0}^{\infty}\Big(\sum_{u\in\mathbb{Z}}\Big|(\delta-\Psi_{u-k})*\sigma_{m,\,j;\,u}*\phi_{u-l}*f(x)\Big|^2\Big)^{\frac{1}{2}}.$$

Invoking the Fourier transform estimates (2.2) and (2.3), we deduce that

$$\Big\|\Big(\sum_{u\in\mathbb{Z}}\Big|(\delta-\Psi_{u-k})*\sigma_{m,\,j;\,u}*\phi_{u-l}*f(x)\Big|^2\Big)^{1/2}\Big\|_{L^2(\mathbb{R}^n)}^2$$
$$=\sum_{u\in\mathbb{Z}}\int_{\mathbb{R}^n}\big|1-\psi(2^{u-k}\xi)\big|^2\big|\widehat{\sigma_{k,\,j;\,u}}(\xi)\big|^2|\widehat{f}(\xi)|^2\mathrm{d}\xi$$
$$\lesssim m^{-2-2\lambda+\beta}\int_{\mathbb{R}^n}\sum_{u\in\mathbb{Z}}|1-\psi(2^{u-k}\xi)|^2|2^u\xi|^{-\beta}\big|Y_{m,\,j}(\xi')\widehat{f}(\xi)\big|^2\,\mathrm{d}\xi$$
$$\lesssim m^{-2-2\lambda+\beta}\int_{\mathbb{R}^n}|Y_{m,\,j}(\xi')|^2|\widehat{f}(\xi)|^2\mathrm{d}\xi,$$

since for each $\xi\in\mathbb{R}^n\backslash\{0\}$,

$$\sum_{u=-\infty}^{\infty}|1-\psi(2^{u-k}\xi)|^2|2^u\xi|^{-\beta}\lesssim\sum_{u:\,|2^u\xi|\ge 2^k}|2^u\xi|^{-\beta}\lesssim 1.$$

This, together with (2.1), implies that

$$\big\|\sup_{v\in\mathbb{Z}}\big|\mathrm{III}_{m,\,j;\,l}^{v}f\big|\big\|_{L^2(\mathbb{R}^n)}^2\lesssim m^{-2-2\lambda+\beta/2}\|f\|_{L^2(\mathbb{R}^n)}^2.$$

The estimates for $\sup_{v\in\mathbb{Z}}|\mathrm{I}^v_{m,\,j;\,l}f|$, $\sup_{v\in\mathbb{Z}}|\mathrm{II}^v_{m,\,j;\,l}f|$ and $\sup_{v\in\mathbb{Z}}|\mathrm{III}^v_{m,j;l}f|$ give the desired $L^2(\mathbb{R}^n)$ bound for $T^{\star\star}_{m,j;l}$.

It remains to prove conclusion (iii). As in [16], we can verify that

$$T^{\star}_{m,\,j;\,l}f(x)\le T^{\star\star}_{m,\,j;\,l}f(x)+M_{m,\,j}Mf(x).$$

Here and in what follows, $M_{m,\,j}$ is the operator defined by

$$M_{m,\,j}f(x)=\sup_{u\in\mathbb{Z}}\int_{\mathbb{R}^n}|\sigma_{m,\,j,\,u}(x-y)f(y)|\mathrm{d}y.$$

By the fact that $\|Y_{m,\,j}\|_{L^2(S^{n-1})}=1$ and method of rotation of Calderón–Zygmund, we know that $M_{m,\,j}$ is bounded on $L^p(\mathbb{R}^n)$ with bound independent of m and j. This via conclusion (ii) tells us that $T^{\star}_{m,\,j}$ is bounded on $L^2(\mathbb{R}^n)$ with bound independent of m and j. $\square$

It was pointed out in [7] that

$$\Omega(x,z')=\sum_{m\ge 0}\sum_{j=1}^{D_m}a_{mj}(x)Y_{m,j}(z').$$

The vanishing moment (1.1) implies that $a_{0,j}=0$ for any $1\le j\le D_m$. Let

$$a_m(x)=\Big(\sum_{j=1}^{D_m}|a_{mj}(x)|^2\Big)^{1/2},\ d_{m,\,j}(x)=\frac{a_{m,j}(x)}{a_m(x)}.$$

It then follows that $\sum_{j=1}^{D_m}d^2_{m,j}=1$, and

$$\Omega(x,\,z')=\sum_{m\ge 1}a_m(x)\sum_{j=1}^{D_m}d_{m,j}(x)Y_{m,j}(z').$$

Define the operator $T_{\Omega,\,l}$ by

$$T_{\Omega,\,l}f(x)=\sum_{m\ge 1}a_m(x)\sum_{j=1}^{D_m}d_{m,j}(x)T_{m,\,j;\,l}f(x), \tag{2.8}$$

with $T_{m,\,j;\,l}$ the operator defined by (2.6).

The following result plays an important role in the proof of our theorems.

Theorem 2.2. *Let $\beta\in(0,\,1)$, $\Omega(x,\,z')$ satisfy (1.1) and $\Omega\in L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})$ for some $q>2(n-1)/n$, $T_{\Omega,\,l}$ be the operator defined by (2.8). Then there exist constants $\gamma_1,\,\gamma_2,\,\gamma_3\in(0,\,1)$, such that*

$$\|T_\Omega f-T_{\Omega,\,l}f\|_{L^2(\mathbb{R}^n)}\lesssim 2^{-\gamma_1 l}\|f\|_{L^2(\mathbb{R}^n)}, \tag{2.9}$$

$$\sum_{j=1}^{D_m}\Big\|\sup_{u\in\mathbb{Z}}\Big|\sum_{v=u}^{\infty}V_{m,\,j;\,l,\,v}*f\Big|\Big\|^2_{L^2(\mathbb{R}^n)}\lesssim m^{-2+\beta}2^{-\gamma_2 l}\|f\|^2_{L^2(\mathbb{R}^n)}, \tag{2.10}$$

and

$$\sum_{j=1}^{D_m}\Big\|\sup_{v\in\mathbb{Z}}\Big|V_{m,\,j;\,l,\,v}*f\Big|\Big\|_{L^2(\mathbb{R}^n)}^2\lesssim m^{-2+\beta}2^{-\gamma_3 l}\|f\|_{L^2(\mathbb{R}^n)}^2, \tag{2.11}$$

where and in the following, for $v\in\mathbb{Z}$ and $l\in\mathbb{N}$, $V_{m,\,j,\,l;\,v}$ is defined by

$$V_{m,\,j,\,l;\,v}(y)=\sigma_{m,\,j;\,v}*\phi_{v-l}(y)-\sigma_{m,\,j;\,v}(y).$$

Proof. Note that

$$T_\Omega f(x)=\sum_{m\ge1}a_m(x)\sum_{j=1}^{D_m}d_{m,j}(x)T_{m,\,j}f(x).$$

A straightforward computation involving the Hölder inequality, leads to that

$$\begin{aligned}\Big|T_\Omega f(x)-T_{\Omega,l}f(x)\big|&\le\sum_{m\ge1}|a_m(x)|\sum_{j=1}^{D_m}|d_{m,j}(x)|\big|T_{m,j}f(x)-T_{m,j;l}f(x)\big|\\&\le\Big\{\mathrm{I}_\alpha(x)\sum_{m=1}^{\infty}m^\alpha\sum_{j=1}^{D_m}\Big|T_{m,j}f(x)-T_{m,\,j;\,l}f(x)\Big|^2\Big\}^{1/2}\end{aligned}$$

with $\alpha\in(0,\,1)$ and $\mathrm{I}_\alpha(x)=\sum_{m=1}^\infty a_m^2(x)m^{-\alpha}$. We know from [7, p. 230] that for $x\in\mathbb{R}^n$,

$$\{\mathrm{I}_\alpha(x)\}^{1/2}\lesssim\|\Omega\|_{L^\infty(\mathbb{R}^n)\times L^q(S^{n-1})} \tag{2.12}$$

provided that $q>2(n-1)/n$ and $\alpha\in(0,\,1)$ sufficiently close to 1. If we can prove that for any $m,\,l\in\mathbb{N}$,

$$\sum_{j=1}^{D_m}\|T_{m,\,j}f-T_{m,\,j;\,l}f\|_{L^2(\mathbb{R}^n)}^2\lesssim m^{-2+\beta}2^{-\gamma_1 l}\|f\|_{L^2(\mathbb{R}^n)}^2, \tag{2.13}$$

we then deduce the inequality (2.9) by taking $\beta\in(0,\,1)$ in (2.13) small enough such that $\beta+\alpha<1$.

The proof of (2.13) is fairly standard. In fact, for each fixed $\xi\in\mathbb{R}^n\backslash\{0\}$, we have by the estimates (2.2) that

$$\begin{aligned}\sum_{u\in\mathbb{Z}}\big|\widehat{\sigma_{m,\,j;\,u}}(\xi)\big|\big|1-\widehat{\phi}(2^{u-l}\xi)\big|&\lesssim m^{-1-\lambda+\beta/2}\sum_{u:\,|2^u\xi|<W}|2^u\xi||Y_{m,\,j}(\xi')|\\&\quad+m^{-1-\lambda+\beta/2}\sum_{u:\,|2^u\xi|\ge W}|2^u\xi|^{-\beta/2}|2^{u-l}\xi|^{\beta/4}|Y_{m,\,j}(\xi')|\\&\lesssim 2^{-\gamma_1 l}m^{-1-\lambda+\beta/2}|Y_{m,\,j}(\xi')|,\end{aligned}$$

if we choose $W=2^{-l\beta/(4+\beta)}$ and $\gamma_1=\beta/(4+\beta)$. This, via the Plancherel theorem, leads to that

$$\begin{aligned}\|T_{m,j}f-T_{m,j;\,l}f\|_{L^2(\mathbb{R}^n)}^2&=\int_{\mathbb{R}^n}\Big|\sum_{u\in\mathbb{Z}}\widehat{\sigma_{m,j;\,u}}(\xi)\big(1-\widehat{\phi}(2^{u-l}\xi)\big)\Big|^2|\widehat{f}(\xi)|^2d\xi\\&\lesssim 2^{-2\gamma_1 l}m^{-2-2\lambda+\beta}\|f\|_{L^2(\mathbb{R}^n)}^2.\end{aligned}$$

The inequality (2.9) now follows from the last inequality and (2.1).

We now turn our attention to the estimate (2.10). Let $\psi \in C_0^\infty(\mathbb{R}^n)$, $\Psi_v \in \mathcal{S}(\mathbb{R}^n)$ with $v \in \mathbb{Z}$ be the same as in the proof of Lemma 2.1. Write

$$\begin{aligned}\sum_{v=u}^{\infty} V_{m,\,j,\,l;\,v} * f(x) &= \Psi_u * \big(T_{m,\,j;\,l}f - T_{m,\,j}f\big)(x) - \Psi_u * \Big(\sum_{v=-\infty}^{u-1} V_{m,\,j,\,l;\,v} * f\Big)(x) \\ &\quad + \sum_{v=u}^{\infty}(\delta - \Psi_u) * V_{m,\,j,\,l;\,v} * f(x) \\ &= \mathrm{J}^{u,\,1}_{m,\,j;\,l}f(x) + \mathrm{J}^{u,\,2}_{m,\,j;\,l}f(x) + \mathrm{J}^{u,\,3}_{m,\,j;\,l}f(x),\end{aligned}$$

with δ the Dirac distribution. We deduce from (2.9) that

$$\begin{aligned}\sum_{j=1}^{D_m}\big\|\sup_{u\in\mathbb{Z}}|\mathrm{J}^{u,\,1}_{m,\,j;\,l}f|\big\|^2_{L^2(\mathbb{R}^n)} &\lesssim \sum_{j=1}^{D_m}\big\|M(T_{m,\,j;\,l}f - T_{m,\,j}f)\big\|^2_{L^2(\mathbb{R}^n)} \\ &\lesssim m^{-2+\beta}2^{-2\gamma_1 l}\|f\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

For all $\xi \in \mathbb{R}^n\backslash\{0\}$, it follows from (2.2) and (2.4) that

$$\begin{aligned}\Big|\sum_{v=-\infty}^{u-1}\widehat{\sigma_{m,\,j;\,v}}(\xi)\big(\widehat{\phi}(2^{v-l}\xi) - 1\big)\Big| &\lesssim m^{-1-\lambda+\beta/2}\sum_{v=-\infty}^{u-1}|2^{v-l}\xi||Y_{m,\,j}(\xi')| \\ &\lesssim m^{-1-\lambda+\beta/2}2^{-l}|2^u\xi||Y_{m,\,j}(\xi')|.\end{aligned}$$

We thus have by the Plancherel theorem that

$$\begin{aligned}&\sum_{j=1}^{D_m}\big\|\sup_{u\in\mathbb{Z}}|\mathrm{J}^{u,\,2}_{m,;\,l;\,l}f|\big\|^2_{L^2(\mathbb{R}^n)} \\ &\quad\le \sum_{j=1}^{D_m}\sum_{u\in\mathbb{Z}}\Big\|\Psi_u * \sum_{l=-\infty}^{u-1}V_{m,\,j,\,l;\,v} * f\Big\|^2_{L^2(\mathbb{R}^n)} \\ &\quad= \sum_{j=1}^{D_m}\sum_{u\in\mathbb{Z}}\int_{\mathbb{R}^n}\Big|\sum_{v=-\infty}^{u-1}\widehat{\sigma_{m,\,j;\,v}}(\xi)\big(\widehat{\phi}(2^{v-l}\xi) - 1\big)\Big|^2\big|\psi(2^u\xi)\widehat{f}(\xi)\big|^2\mathrm{d}\xi \\ &\quad\lesssim 2^{-2l}m^{-2-2\lambda+\beta}\int_{\mathbb{R}^n}\sum_{j=1}^{D_m}|Y_{m,\,j}(\xi')|^2\sum_{u=-\infty}^{\infty}|\psi(2^u\xi)|^2|2^u\xi|^2\big|\widehat{f}(\xi)\big|^2\,\mathrm{d}\xi \\ &\quad\lesssim 2^{-2l}m^{-2-2\lambda+\beta}\|f\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Therefore, using (2.1),

$$\sum_{j=1}^{D_m}\big\|\sup_{u\in\mathbb{Z}}|\mathrm{J}^{u,\,2}_{m,\,j;\,l}f|\big\|^2_{L^2(\mathbb{R}^n)} \lesssim m^{-2+2\beta}2^{-l}\|f\|_{L^2(\mathbb{R}^n)}.$$

For the term $\sup_{u\in\mathbb{Z}}\big|\mathrm{J}^{u,\,3}_{m,\,j;\,l}f\big|$, write

$$\sup_{u\in\mathbb{Z}}\left|\mathrm{J}^{u,\,3}_{m,\,j;\,l}f(x)\right| \le \sum_{k=0}^{\infty}\sup_{u\in\mathbb{Z}}\left|(\delta-\Psi_u)*V_{m,\,j,\,l;\,u+k}*f(x)\right|$$
$$\lesssim \sum_{k=0}^{\infty}\Big(\sum_{u\in\mathbb{Z}}\Big|(\delta-\Psi_{u-k})*V_{m,\,j,\,l;\,u}*f(x)\Big|^2\Big)^{1/2}.$$

Invoking the Fourier transform estimates (2.2) and (2.3), we deduce that

$$\Big\|\Big(\sum_{u\in\mathbb{Z}}\Big|(\delta-\Psi_{u-k})*V_{m,\,j,\,l;\,u}*f(x)\Big|^2\Big)^{1/2}\Big\|^2_{L^2(\mathbb{R}^n)}$$
$$=\sum_{u\in\mathbb{Z}}\int_{\mathbb{R}^n}\left|1-\psi(2^{u-k}\xi)\right|^2\Big|\widehat{\sigma_{k,\,j;\,u}}(\xi)\big(\widehat{\phi}(2^{u-l}\xi)-1\big)\Big|^2|\widehat{f}(\xi)|^2\mathrm{d}\xi$$
$$\lesssim m^{-2-2\lambda+\beta}2^{-k\beta/2}2^{-l\beta/2}\int_{\mathbb{R}^n}|Y_{m,\,j}(\xi')|^2|\widehat{f}(\xi)|^2\mathrm{d}\xi,$$

since by (2.4), for each $\xi\in\mathbb{R}^n\backslash\{0\}$,

$$|\widehat{\phi}(2^{u-l}\xi)-1|\lesssim|2^{u-l}\xi|^{\beta/4}$$

and so

$$\sum_{u\in\mathbb{Z}}\left|1-\psi(2^{u-k}\xi)\right|^2\Big|\widehat{\sigma_{k,\,j;\,u}}(\xi)\big(\widehat{\phi}(2^{u-l}\xi)-1\big)\Big|^2$$
$$\lesssim\sum_{u:\,|2^u\xi|\ge 2^k}|2^u\xi|^{-\beta}|2^{u-l}\xi|^{\beta/2}\lesssim 2^{-k\beta/2}2^{-l\beta/2}.$$

This, together with (2.1), implies that

$$\sum_{j=1}^{D_m}\big\|\sup_{u\in\mathbb{Z}}\big|\mathrm{J}^{u,\,3}_{m,\,j;\,l}f\big|\big\|^2_{L^2(\mathbb{R}^n)}\lesssim m^{-2+\beta}2^{-l\beta/2}\|f\|^2_{L^2(\mathbb{R}^n)}.$$

Combining the estimates for the terms $\sup_{u\in\mathbb{Z}}|\mathrm{J}^{u,\,1}_{m,\,j;\,l}f|$, $\sup_{u\in\mathbb{Z}}|\mathrm{J}^{u,\,2}_{m,\,j;\,l}f|$ and $\sup_{u\in\mathbb{Z}}|\mathrm{J}^{u,\,3}_{m,\,j;\,l}f|$ yields (2.10).

It remains to prove (2.11). By (2.4), for each $\xi\in\mathbb{R}^n\backslash\{0\}$,

$$\sum_{v\in\mathbb{Z}}\left|\widehat{V_{m,j;l,v}}(\xi)\right|^2\lesssim m^{-2-2\lambda+\beta}\sum_{u:|2^u\xi|<W}|2^u\xi|^2|Y_{m,j}(\xi')|^2$$
$$+m^{-2-2\lambda+\beta}\sum_{u:|2^u\xi|\ge W}|2^u\xi|^{-\beta}|2^{u-l}\xi|^{\beta/2}|Y_{m,j}(\xi')|^2$$
$$\lesssim 2^{-\gamma_3 l}m^{-2-2\lambda+\beta}|Y_{m,\,j}(\xi')|^2.$$

This via the Plancherel theorem yields

$$\sum_{j=1}^{D_m}\Big\|\sup_{v\in\mathbb{Z}}\left|V_{m,j;l,v}*f(x)\right|\Big\|^2_{L^2(\mathbb{R}^n)}\le\sum_{j=1}^{D_m}\Big\|\Big(\sum_{v\in\mathbb{Z}}\left|V_{m,j;l,v}*f(x)\right|^2\Big)^{\frac12}\Big\|_{L^2(\mathbb{R}^n)}$$
$$\lesssim m^{-2+\beta}2^{-\gamma_3 l}\|f\|^2_{L^2(\mathbb{R}^n)}.$$

This completes the proof of Theorem 2.2. □

3. Proof of theorems

Let $1 \le p < \infty$, $L^p(l^\infty, \mathbb{R}^n)$ be the space of sequences of functions defined by

$$L^p(l^\infty, \mathbb{R}^n) = \big\{\{f_k\}_{k\in\mathbb{Z}} : \|\{f_k\}\|_{L^p(l^\infty, \mathbb{R}^n)} < \infty\big\},$$

with

$$\|\{f_k\}\|_{L^p(l^\infty, \mathbb{R}^n)} = \Big\| \sup_{k\in\mathbb{Z}} |f_k| \Big\|_{L^p(\mathbb{R}^n)}.$$

This vector linear space (with usual addition and scalar multiplication) is a Banach space.

Let E be a compact set in $\mathbb{R}^n$, and

$$C(l^\infty, E) = \{\vec{f} = \{f_k\}_{k\in\mathbb{Z}} : f_k \in C(E) \text{ for all } k \in \mathbb{Z}\}.$$

It is easy to see that $C(l^\infty, E)$ with the distance

$$d(\vec{f}, \vec{g}) = \sup_{x\in E} \|\vec{f}(x) - \vec{g}(x)\|_{l^\infty}$$

is a metric space. Here

$$\|\vec{f}(x) - \vec{g}(x)\|_{l^\infty} = \sup_{k\in\mathbb{Z}} |f_k(x) - g_k(x)|.$$

Lemma 3.1. *Let $1 \le p < \infty$, $\mathcal{G} \subset L^p(l^\infty, \mathbb{R}^n)$. Suppose that $\mathcal{G}$ satisfies the following four conditions:*

(a) *$\mathcal{G}$ is bounded, that is, there exists a constant C such that for all $\vec{f} = \{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$, $\|\vec{f}\|_{L^p(l^\infty, \mathbb{R}^n)} \le C$;*

(b) *for each fixed $\epsilon > 0$, there exists a constant $A > 0$, such that for all $\{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,*

$$\big\| \sup_{k\in\mathbb{Z}} |f_k| \chi_{\{|\cdot|>A\}}(\cdot) \big\|_{L^p(\mathbb{R}^n)} < \epsilon;$$

(c) *for each fixed $\epsilon > 0$, there exists a constant $\varrho > 0$, such that for all $t \in \mathbb{R}^n$ with $|t| < \varrho$ and $\vec{f} = \{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,*

$$\|\vec{f}(\cdot + t) - \vec{f}(\cdot)\|_{L^p(l^\infty, \mathbb{R}^n)} < \epsilon;$$

(d) *for each fixed $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $\{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,*

$$\sup_{x\in\mathbb{R}^n} \sup_{k>N} |f_k(x)| < \epsilon,$$

and

$$\sup_{k<-N} \|f_k - f_{-N}\|_{L^p(\mathbb{R}^n)} < \epsilon.$$

Then $\mathcal{G}$ is strongly pre-compact in $L^p(l^\infty, \mathbb{R}^n)$.

Proof. Since $L^p(l^\infty, \mathbb{R}^n)$ is a Banach space, it suffices to prove that $\mathcal{G}$ is a totally bounded set in $L^p(l^\infty, \mathbb{R}^n)$, namely, for each $\epsilon > 0$, $\mathcal{G}$ has a finite ϵ-net. To this aim, we employ the argument used in the proof of the Fréchet–Kolmogorov theorem (see [22, p. 275]) with some suitable modifications.

Let T_t be the translation operator defined by

$$T_t(\vec{f})(x) = \vec{f}(x+t)$$

for $\vec{f} = \{f_k\}_{k\in\mathbb{Z}}$. The assumption (c) tells us that

$$\overline{\lim}_{|t|\to 0}\big\|T_t(\vec{f}) - \vec{f}\big\|_{L^p(l^\infty)} = 0 \tag{3.1}$$

uniformly in $\vec{f} \in \mathcal{G}$. For $a > 0$ and $\vec{f} = \{f_k\}_{k\in\mathbb{Z}} \in L^p(l^\infty, \mathbb{R}^n)$, let

$$M_a(\vec{f})(x) = \{M_a f_k\}_{k\in\mathbb{Z}},$$

with

$$M_a f_k(x) = \frac{1}{|B(0, a)|}\int\limits_{|y|<a} f_k(x+y)\mathrm{d}y.$$

An application of the Minkowski inequality gives us that

$$\|M_a(\vec{f}) - \vec{f}\|_{L^p(l^\infty, \mathbb{R}^n)} \le \sup_{|t|\le a}\|T_t(\vec{f}) - \vec{f}\|_{L^p(l^\infty, \mathbb{R}^n)}.$$

For $\epsilon > 0$, we choose $a_0 > 0$ such that

$$\sup_{\vec{f}\in\mathcal{G}}\|M_{a_0}(\vec{f}) - \vec{f}\|_{L^p(l^\infty, \mathbb{R}^n)} < \frac{\epsilon}{12}. \tag{3.2}$$

If we can prove that $M_{a_0}\mathcal{G} = \{M_{a_0}(\vec{f}) : \vec{f} \in \mathcal{G}\}$ has a finite $\epsilon/3$-net $\{M_{a_0}(\vec{h}^{(1)}), \dots, M_{a_0}(\vec{h}^{(j)})\}$ with $\vec{h}^{(1)}, \dots, \vec{h}^{(j)} \in L^p(l^\infty, \mathbb{R}^n)$, we then know $\{\vec{f}^{(1)}, \dots, \vec{f}^{(j)}\}$ is an ϵ-net of $\mathcal{G}$.

We first prove that for each $a, A, \varepsilon > 0$, there exist $\vec{h}^{(1)}, \dots, \vec{h}^{(j)} \in L^p(l^\infty, \mathbb{R}^n)$, such that $\big\{M_a(\vec{h}^{(1)}), \dots, M_a(\vec{h}^{(j)})\big\}$ is an ε-net of $\{M_a(\vec{f}) : \vec{f} \in \mathcal{G}\}$ in $C(l^\infty, \overline{B(0, A)})$, and for each $1 \le i \le j$,

$$\|M_a(\vec{h}^{(i)})(x) - \vec{h}^{(i)}(x)\|_{l^\infty} \le \|M_a(\vec{f}^{(i)}(x) - \vec{f}^{(i)}(x)\|_{l^\infty}$$

for some $\vec{f}^{(i)} \in \mathcal{G}$. As in [22, p. 276], for $\vec{f} = \{f_k\} \in L^p(l^\infty, \mathbb{R}^n)$, we have by the Hölder inequality that

$$\sup_{k\in\mathbb{Z}}\sup_{x\in\mathbb{R}^n}|M_a f_k(x)| \lesssim a^{-n/p}\|\vec{f}\|_{L^p(l^\infty, \mathbb{R}^n)}, \tag{3.3}$$

and

$$\sup_{k\in\mathbb{Z}}\big|M_a f_k(x_1) - M_a f_k(x_2)\big| \lesssim a^{-n/p}\big\|\vec{f}(\cdot + x_1 - x_2) - \vec{f}(\cdot)\big\|_{L^p(l^\infty, \mathbb{R}^n)}. \tag{3.4}$$

Also, we can verify that for each $N \in \mathbb{N}$ and $\vec{f} \in L^p(l^\infty, \mathbb{R}^n)$,

$$\sup_{x\in\mathbb{R}^n}\sup_{k>N}|M_a f_k(x)| \le \sup_{y\in\mathbb{R}^n}\sup_{k>N}|f_k(y)|, \tag{3.5}$$

and

$$\sup_{x\in\mathbb{R}^n}\sup_{k<-N}\big|M_a f_k(x) - M_a f_{-N}(x)\big| \le a^{-n/p}\|f_k - f_{-N}\|_{L^p(\mathbb{R}^n)}. \tag{3.6}$$

For fixed $\varepsilon > 0$, the assumption (c) states that there exists a constant $\delta > 0$ such that for all $t \in \mathbb{R}^n$ with $|t| < \delta$ and $\vec{f} = \{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$,

$$\|\vec{f}(\cdot + t) - \vec{f}(\cdot)\|_{L^p(l^\infty,\, \mathbb{R}^n)} < a^{n/p}\varepsilon/6;$$

Thus by (3.4), for all $\vec{f} \in \mathcal{G}$ and $x_1,\, x_2 \in \overline{B(0,\, A)}$ with $|x_1 - x_2| < \delta$,

$$\sup_{k\in\mathbb{Z}} |M_a f_k(x_1) - M_a f_k(x_2)| < \varepsilon/6. \tag{3.7}$$

On the other hand, by (3.5), (3.6) and the assumption (d), we can choose $N \in \mathbb{N}$ such that for all $\vec{f} \in \mathcal{G}$

$$\sup_{x\in\mathbb{R}^n} \sup_{k>N} |M_a f_k(x)| < \varepsilon/3, \tag{3.8}$$

and

$$\sup_{x\in\mathbb{R}^n} \sup_{k<-N} |M_a f_k(x) - M_a f_{-N}(x)| < \varepsilon/3. \tag{3.9}$$

Now let $\{x_1,\, \ldots,\, x_m\}$ be a δ-net of $\overline{B(0,\, A)}$ and

$$\mathcal{G}_N = \{\{f_k\}_{-N\le k\le N} : \{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}\}.$$

It follows from assumption (a) and (3.3) that

$$\mathcal{H} = \big\{\big(M_a(\vec{f})(x_1),\, \ldots,\, M_a(\vec{f})(x_m)\big) : \vec{f} \in \mathcal{G}_N\big\}$$

is a bounded set in $(\mathbb{R}^N)^m$. We choose $\{f_k^{(1)}\}_{k\in\mathbb{Z}},\, \ldots,\, \{f_k^{(j)}\}_{k\in\mathbb{Z}} \in \mathcal{G}$, such that

$$\Big\{\{M_a f_k^{(1)}(x_1)\}_{-N\le k\le N},\, \ldots,\, \{M_a f_k^{(j)}(x_m)\}_{-N\le k\le N}\Big\}$$

is the $\varepsilon/6$-net of $\mathcal{H}$. For $1 \le i \le j$, set

$$\vec{h}^{(i)} = \Big\{\ldots,\, f_{-N}^{(i)},\, \ldots,\, f_{-N}^{(i)},\, f_{-N+1}^{(i)},\, \ldots,\, f_{N-1}^{(i)},\, f_N^{(i)},\, 0,\, \ldots,\, 0,\, \ldots\Big\}.$$

We claim that $\big\{M_a(\vec{h}^{(1)}), \ldots, M_a(\vec{h}^{(j)})\big\}$ is the ε-net of $\{M_a(\vec{f}) : \vec{f} \in \mathcal{G}\}$ in $C(l^\infty, \overline{B(0,\, A)})$. In fact, for fixed $x \in B(0,\, A)$ and $\vec{f} = \{f_k\}_{k\in\mathbb{Z}} \in \mathcal{G}$, we take u with $1 \le u \le m$ such that $|x - x_u| < \delta$, and take i with $1 \le i \le j$, such that

$$\sup_{-N\le k\le N} \sup_{1\le v\le m} \big|M_a f_k(x_v) - M_a f_k^{(i)}(x_v)\big| < \frac{\varepsilon}{6}. \tag{3.10}$$

It then follows from (3.7) and (3.10) that,

$$\begin{aligned}
\sup_{-N\le k\le N} \big|M_a f_k(x) - M_a f_k^{(i)}(x)\big| \le{}& \sup_{-N\le k\le N} \big|M_a f_k(x_u) - M_a f_k^{(i)}(x_u)\big| \\
&+ \sup_{-N\le k\le N} \big|M_a f_k(x_u) - M_a f_k(x)\big| \\
&+ \sup_{-N\le k\le N} \big|M_a f_k^{(i)}(x_u) - M_a f_k^{(i)}(x)\big| \\
&< \varepsilon/2.
\end{aligned} \tag{3.11}$$

We also deduce from (3.9) and (3.10) that

$$\begin{aligned}\sup_{k<-N}\left|M_af_k(x)-M_af_{-N}^{(i)}(x)\right| &\le \sup_{k<-N}\left|M_af_k(x)-M_af_{-N}(x)\right| \\ &\quad +\left|M_af_{-N}(x)-M_af_{-N}^{(i)}(x)\right| \\ &< \varepsilon. \end{aligned}\tag{3.12}$$

The inequalities (3.11), (3.12), together with (3.8), lead to that

$$\left\|M_a(\vec{f})(x)-M_a(\vec{h}^{(i)})(x)\right\|_{l^\infty}<\varepsilon,$$

and $\vec{h}^{(1)},\dots,\vec{h}^{(j)}$ are our desired functions.

We are now ready to prove that $\{M_{a_0}\vec{f}:\ \vec{f}\in\mathcal{G}\}$ has a finite $\epsilon/3$-net in $L^p(l^\infty,\mathbb{R}^n)$. By assumption (b), we can choose A large enough such that for all $\vec{f}=\{f_k\}_{k\in\mathbb{Z}}\in\mathcal{G}$,

$$\left\|\sup_{k\in\mathbb{Z}}|f_k|\chi_{\{|\cdot|>A\}}(\cdot)\right\|_{L^p(\mathbb{R}^n)}<\frac{\epsilon}{24}.$$

Let $\{M_{a_0}(\vec{h}^{(1)}),\dots,M_{a_0}(\vec{h}^{(j)})\}$ be an $\epsilon/(12|B(0,A)|^{1/p})$-net of $\{M_{a_0}(\vec{f}):\vec{f}\in\mathcal{G}\}$ in $C(l^\infty,\overline{B(0,A)})$, with $\vec{h}^{(i)}=\{h_k^{(i)}\}_{k\in\mathbb{Z}}$ and for each $1\le i\le j$, there exists $\vec{g}^{(i)}\in\mathcal{G}$ such that

$$\|M_{a_0}(\vec{h}^{(i)})-\vec{h}^{(i)}\|_{L^p(l^\infty,\mathbb{R}^n)}\le\|M_{a_0}\vec{g}^{(i)}-\vec{g}^{(i)}\|_{L^p(l^\infty,\mathbb{R}^n)}.$$

By (3.2), it is obvious that for each i with $1\le i\le j$,

$$\|M_{a_0}(\vec{h}^{(i)})-\vec{h}^{(i)}\|_{L^p(l^\infty,\mathbb{R}^n)}<\frac{\epsilon}{12}.$$

For each $\vec{f}=\{f_k\}_{k\in\mathbb{Z}}\in\mathcal{G}$, we can choose i with $1\le i\le j$ such that

$$\begin{aligned}&\left\|M_{a_0}(\vec{f})-M_{a_0}(\vec{h}^{(i)})\right\|_{L^p(l^\infty,\mathbb{R}^n)}\\ &\le\Big(\int\limits_{|x|\le A}\sup_{k\in\mathbb{Z}}|M_{a_0}f_k(x)-M_{a_0}h_k^{(i)}(x)|^p\mathrm{d}x\Big)^{1/p}\\ &\quad+\Big(\int\limits_{|x|>A}\sup_{k\in\mathbb{Z}}\left|f_k(x)-h_k^{(i)}(x)\right|^p\mathrm{d}x\Big)^{1/p}\\ &\quad+\Big(\int\limits_{|x|>A}\sup_{k\in\mathbb{Z}}\left|h_k^{(i)}(x)-M_{a_0}h_k^{(i)}(x)\right|^p\mathrm{d}x\Big)^{1/p}\\ &\quad+\|M_{a_0}(\vec{f})-\vec{f}\|_{L^p(l^\infty,\mathbb{R}^n)}<\epsilon/3.\end{aligned}$$

Therefore, $\{M_{a_0}(\vec{h}^{(1)}),\ \dots,\ M_{a_0}(\vec{hf}^{(j)})\}$ is the $\epsilon/3$-net of $\{M_{a_0}(\vec{f}):\ \vec{f}\in\mathcal{G}\}$ in $L^p(l^\infty,\mathbb{R}^n)$. This completes the proof of Lemma 3.1. □

Proof of Theorem 1.2. We only consider the compactness of $T^{\star\star}_{\Omega,\,b}$. The argument for the commutator $T_{\Omega,\,b}$ is similar and fairly simpler than that of the operator $T^{\star\star}_{\Omega,\,b}$, and will be omitted. For $m,\,j,\,l\in\mathbb{N}$ and $1\le j\le D_m$, set

$$T^v_{m,\,j;\,l,\,b}f(x)=\sum_{u=v}^{\infty}\int_{\mathbb{R}^n}\big(b(x)-b(y)\big)\sigma_{m,j;\,u}*\phi_{u-l}(x-y)f(y)\mathrm{d}y.$$

We first claim that if $b\in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}b\subset B(0,\,R)$ for some $R>0$, then

(i) for $A>4R$,

$$\big\|\sup_{v\in\mathbb{Z}}\big|T^v_{m,\,j;\,l,b}f\big|\chi_{\{|\cdot|>A\}}\big\|_{L^2(\mathbb{R}^n)}\lesssim\Big(\frac{R}{A}\Big)^{n/4}\|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)};\tag{3.13}$$

(ii) for each $l,\,m,\,j\in\mathbb{N}$ with $1\le j\le D_m$, $t\in\mathbb{R}^n$ with $|t|<1$,

$$\begin{aligned}&\big\|\sup_{v\in\mathbb{Z}}\big|T^v_{m,\,j;\,l,\,b}f(\cdot)-T^v_{m,\,j;\,l,\,b}f(\cdot+t)\big|\big\|_{L^2(\mathbb{R}^n)}\\&\quad\lesssim(\|b\|_{L^\infty(\mathbb{R}^n)}+\|\nabla b\|_{L^\infty(\mathbb{R}^n)})2^l|t|^{1/2}\|f\|_{L^2(\mathbb{R}^n)};\end{aligned}\tag{3.14}$$

(iii) for each $N,\,l,\,m,\,j\in\mathbb{N}$ with $1\le j\le D_m$,

$$\sup_{x\in\mathbb{R}^n}\sup_{v>N}\big|T^v_{m,\,j;\,l,\,b}f(x)\big|\lesssim 2^{-nN/2}\|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)},\tag{3.15}$$

and

$$\sup_{v<-N}\Big\|T^v_{m,\,j;\,l,b}f-T^{-N}_{m,\,j;\,l,b}f\Big\|_{L^2(\mathbb{R}^n)}\lesssim 2^{-N}\|\nabla b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}.\tag{3.16}$$

To prove (3.13), set $K^v_{m,\,j;\,l}(y)=\sum_{u=v}^\infty\sigma_{m,\,j,\,u}*\phi_{u-l}(y)$. For each $v\in\mathbb{Z}$, by the Hölder inequality, we have that for $x\in\mathbb{R}^n$ with $|x|>4R$,

$$\begin{aligned}\int_{|y|<R}\big|K^v_{m,\,j;\,l}(x-y)f(y)\big|\mathrm{d}y&\le\Big\{\int_{|y|<R}\big|K^v_{m,\,j;\,l}(x-y)\big||f(y)|^2\mathrm{d}y\Big\}^{1/2}\\&\quad\times\Big\{\int_{|y|<R}\big|K^v_{m,\,j;\,l}(x-y)\big|^2\mathrm{d}y\Big\}^{1/4}R^{n/4}.\end{aligned}$$

Note that for $x\in\mathbb{R}^n$ with $|x|>4R$,

$$\begin{aligned}\Big\{\int_{|y|<R}\big|K^v_{m,\,j;\,l}(x-y)\big|^2\mathrm{d}y\Big\}^{1/2}&\lesssim\sum_{k\in\mathbb{Z}:\,2^k\approx|x|}\|\sigma_{m,\,j;\,k}\|_{L^2(\mathbb{R}^n)}\|\phi_{k-l}\|_{L^1(\mathbb{R}^n)}\\&\lesssim|x|^{-n/2}.\end{aligned}$$

Therefore, for $|x|>4R$,

$$\big\{\sup_{v\in\mathbb{Z}}\big|T^v_{m,\,j;\,b,\,l}f(x)\big|\big\}^2\lesssim R^{n/2}|x|^{-n/2}\int_{|y|<R}\sup_{v\in\mathbb{Z}}\big|K^v_{m,\,j;\,l}(x-y)\big||f(y)|^2\mathrm{d}y.$$

On the other hand, since

$$\begin{aligned}
&\int\limits_{|x|>A} \sum_{k\in\mathbb{Z}} |\sigma_{m,\,j;\,l} * \phi_{k-l}(x)| \frac{\mathrm{d}x}{|x|^{n/2}} \\
&\lesssim \sum_{v=1}^{\infty} (2^v A)^{-\frac{n}{2}} \sum_{k\in\mathbb{Z}} \int\limits_{2^{v-1}A<|x|\le 2^v A} \big|\sigma_{m,j;\,k} * \phi_{k-l}(x)\big| \mathrm{d}x \\
&\lesssim \sum_{v=1}^{\infty} (2^v A)^{-\frac{n}{2}} \sum_{k\in\mathbb{Z}:2^k\approx 2^v A} \|\sigma_{m,j;\,k}\|_{L^1(\mathbb{R}^n)} \|\phi_{k-l}\|_{L^1(\mathbb{R}^n)} \\
&\lesssim A^{-n/2},
\end{aligned}$$

we thus get when $A > 4R$,

$$\int\limits_{|x|>A} \big(\sup_{v\in\mathbb{Z}} |T^v_{m,\,j;\,b,\,l} f(x)|\big)^2 \mathrm{d}x \lesssim \Big(\frac{R}{A}\Big)^{n/2} \|b\|^2_{L^\infty(\mathbb{R}^n)} \|f\|^2_{L^2(\mathbb{R}^n)}.$$

We turn our attention to the estimate (3.14). For each fixed $t \in \mathbb{R}^n$ with $|t| < 1$, let $A_t = 4|t|^{1/2}$. For each fixed $v \in \mathbb{Z}$,

$$\begin{aligned}
&\big|T^v_{m,\,j;\,l\,,b} f(x) - T^v_{m,\,j;\,l,\,b} f(x+t)\big| \\
&\lesssim \big|b(x+t) - b(x)\big| \Big| \int\limits_{|x-y|>A_t} K^v_{m,\,j;\,l}(x-y) f(y)\mathrm{d}y \Big| \\
&\quad + \Big| \int\limits_{|x-y|>A_t} U^v_{m,\,j;\,l}(x,\,y;\,t)\big(b(y) - b(x+t)\big) f(y)\mathrm{d}y \Big| \\
&\quad + \Big| \int\limits_{|x-y|\le A_t} K^v_{m,\,j;\,l}(x-y)\big(b(y) - b(x)\big) f(y)\mathrm{d}y \Big| \\
&\quad + \Big| \int\limits_{|x-y|\le A_t} K^v_{m,\,j;\,l}(x+t-y)\big(b(y) - b(x+t)\big) f(y)\mathrm{d}y \Big| \\
&= \mathrm{I}^{v,\,1}_{m,\,j;\,l} f(x,\,t) + \mathrm{I}^{v,\,2}_{m,\,j;\,l} f(x,\,t) + \mathrm{I}^{v,\,3}_{m,\,j;\,l} f(x,\,t) + \mathrm{I}^{v,\,4}_{m,\,j;\,l} f(x,\,t),
\end{aligned}$$

where

$$U^v_{m,\,j;\,l}(x,\,y;\,t) = K^v_{m,\,j;\,l}(x-y) - K^v_{m,\,j;\,l}(x+t-y).$$

By the fact that $\operatorname{supp} \sigma_{m,\,j;\,u} * \phi_{l-u} \subset \{2^{u-2} \le |x| \le 2^{u+2}\}$, a trivial computation gives us that

$$\begin{aligned}
&\int\limits_{|x-y|>A_t} \Big|K^v_{m,j;l}(x-y) - K_{m,j;l}(x-y)\chi_{\{|x-y|>2^v\}}(x-y)\Big| |f(y)|\mathrm{d}y \\
&\lesssim \sum_{u=v-2}^{v+2} \int\limits_{|x-y|>A_t} \big|\sigma_{m,\,j;\,u} * \phi_{u-l}(x-y)\big| |f(y)|\mathrm{d}y \\
&\lesssim \sup_{v\in\mathbb{Z}} \big(|\sigma_{m,\,j;\,v}| * \phi_{v-l} * |f|\big)(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Big|\int\limits_{|x-y|>A_t} K^{v}_{m,\,j;\,l}(x-y)f(y)\mathrm{d}y\Big| \\
&\quad\le \Big|\int\limits_{|x-y|>A_t} K_{m,\,j;\,l}(x-y)\chi_{\{|x-y|>2^v\}}(x-y)f(y)\mathrm{d}y\Big| \\
&\qquad+\int\limits_{\mathbb{R}^n}\Big|K^{v}_{m,\,j;\,l}(x-y)-K_{m,\,j;\,l}(x-y)\chi_{\{|x-y|>2^v\}}(x-y)\Big||f(y)|\mathrm{d}y \\
&\quad\lesssim T^{\star}_{m,\,j;\,l}f(x)+M_{m,\,j}Mf(x).
\end{aligned}$$

This, together with (iii) of Lemma 2.1 and the $L^2(\mathbb{R}^n)$ boundedness of $M_{m,j}$, implies that

$$\begin{aligned}
\big\|\sup_{v\in\mathbb{Z}}\big|\mathrm{I}^{v,\,1}_{m,j;\,l}f(\cdot,t)\big|\big\|_{L^2(\mathbb{R}^n)} &\lesssim |t|\|\nabla b\|_{L^\infty(\mathbb{R}^n)}\big(\|T^{\star}_{m,j;\,l}f\|_{L^2(\mathbb{R}^n)}+\|M_{m,j}Mf\|_{L^2(\mathbb{R}^n)}\big) \\
&\lesssim l|t|\|\nabla b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

The inequality (2.7) shows that

$$\begin{aligned}
&\Big\|\sup_{v\in\mathbb{Z}}\Big|K^{v}_{m,j;l}(\cdot)-K^{v}_{m,j;l}(\cdot+t)\Big|\chi_{\{|\cdot|>A_t\}}(\cdot)\Big\|_{L^1(\mathbb{R}^n)} \\
&\quad\lesssim\sum_{u\in\mathbb{Z}}\Big\|\big(\sigma_{m,j;\,u}*\phi_{u-l}(\cdot+t)-\sigma_{m,j;\,u}*\phi_{u-l}(\cdot)\big)\chi_{\{|\cdot|>A_t\}}(\cdot)\Big\|_{L^1(\mathbb{R}^n)} \\
&\quad\lesssim 2^l\frac{|t|}{A_t}.
\end{aligned}$$

Therefore, by the Young inequality,

$$\big\|\sup_{v\in\mathbb{Z}}\big|\mathrm{I}^{v,\,2}_{m,j;l}f(\cdot,t)\big|\big\|_{L^2(\mathbb{R}^n)}\lesssim 2^l|t|^{1/2}\|b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}.$$

Let

$$\widetilde{K_{m,\,j;\,l}}(x)=|x|\sum_{u\in\mathbb{Z}}|\sigma_{m,\,j;\,u}|*\phi_{u-l}(x).$$

A trivial computation shows that

$$\|\widetilde{K_{m,j;l}}\chi_{\{|\cdot|\le A_t\}}\|_{L^1(\mathbb{R}^n)}\lesssim\sum_{u\in\mathbb{Z}:2^{u-2}\le A_t}2^u\|\sigma_{m,j;u}\|_{L^1(\mathbb{R}^n)}\|\phi_{u-l}\|_{L^1(\mathbb{R}^n)}\lesssim A_t.$$

Note that

$$\sup_{v\in\mathbb{Z}}\big|\mathrm{I}^{v,\,3}_{m,\,j;\,l}f(x,\,t)\big|\lesssim\|\nabla b\|_{L^\infty(\mathbb{R}^n)}\big(\widetilde{K_{m,\,j;\,l}}\chi_{\{|\cdot|<A_t\}}\big)*|f|(x).$$

It then follows that

$$\begin{aligned}
\big\|\sup_{v\in\mathbb{Z}}\big|\mathrm{I}^{v,\,3}_{m,j;l}f(\cdot,t)\big|\big\|_{L^2(\mathbb{R}^n)} &\lesssim\|\nabla b\|_{L^\infty(\mathbb{R}^n)}\|f\|_{L^2(\mathbb{R}^n)}\|\widetilde{K_{m,j;l}}\chi_{\{|\cdot|<A_t\}}\|_{L^1(\mathbb{R}^n)} \\
&\lesssim\|\nabla b\|_{L^\infty(\mathbb{R}^n)}A_t\|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

Similarly, we have that

$$\Big\| \sup_{v\in\mathbb{Z}} \big| \mathrm{I}^{v,\,4}_{m,\,j;\,l} f(\cdot,\,t) \big| \Big\|_{L^2(\mathbb{R}^n)} \lesssim \|\nabla b\|_{L^\infty(\mathbb{R}^n)} (A_t + |t|) \|f\|_{L^2(\mathbb{R}^n)}.$$

Combining the estimates for $\sup_{v\in\mathbb{Z}} \big| \mathrm{I}^{v,\,k}_{m,\,j;\,l} f(x,\,t) \big|$ $(k = 1,\,2,\,3,\,4)$ yields (3.14).

We now verify claim (iii). Let $N \in \mathbb{N}$, it follows from the Young inequality that

$$\begin{aligned} \sum_{u>N} \int_{\mathbb{R}^n} & |\sigma_{m,\,j;\,u} * \phi_{u-l}(x-y)||f(y)|\mathrm{d}y \\ &\lesssim \|f\|_{L^2(\mathbb{R}^n)} \sum_{v>N} \|\sigma_{m,\,j;\,v}\|_{L^2(\mathbb{R}^n)} \|\phi_{v-l}\|_{L^1(\mathbb{R}^n)} \\ &\lesssim 2^{-nN/2} \|f\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

This gives us (3.15) immediately. On the other hand, observe that for $N \in \mathbb{N}$ and $v < -N$,

$$\begin{aligned} & \Big| T^v_{m,\,j;\,l,b} f(x) - T^{-N}_{m,\,j;\,l,b} f(x) \Big| \\ &\le \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \sum_{u=-\infty}^{-N} \int_{\mathbb{R}^n} |x-y||\sigma_{m,\,j;\,u} * \phi_{u-l}(x-y)||f(y)|\mathrm{d}y, \end{aligned}$$

another application of the Young inequality gives us that

$$\begin{aligned} & \Big\| T^v_{m,\,j;\,l,b} f - T^{-N}_{m,\,j;\,l,b} f \Big\|_{L^2(\mathbb{R}^n)} \\ &\le \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)} \sum_{u=-\infty}^{-N} 2^u \|\sigma_{m,\,j;\,u}\|_{L^1(\mathbb{R}^n)} \|\phi_{u-l}\|_{L^1(\mathbb{R}^n)} \\ &\lesssim 2^{-N} \|\nabla b\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

which gives (3.16).

We can now conclude the proof of Theorem 1.2. For $v \in \mathbb{Z}$, let

$$T^v_{\Omega,\,b} f(x) = \int_{|x-y|>2^v} \big(b(x)-b(y)\big) \frac{\Omega(x,\,x-y)}{|x-y|^n} f(y)\mathrm{d}y,$$

and

$$\Lambda_{\Omega,\,b} f(x) = \{T^v_{\Omega,\,b} f(x)\}_{v\in\mathbb{Z}}.$$

Also, let

$$T^v_{\Omega,\,l,\,b} f(x) = \sum_{m\ge 1} a_m(x) \sum_{j=1}^{D_m} d_{m,\,j}(x) T^v_{m,\,j;\,l,\,b} f(x), \tag{3.17}$$

and $\Theta_{\Omega,\,l,\,b}$ be the operator given by

$$\Theta_{\Omega,\,l,\,b} f(x) = \{T^v_{\Omega,\,l,\,b} f(x)\}_{v\in\mathbb{Z}}.$$

Recall that $T^{\star}_{m,\,j;\,l}$ is bounded on $L^2(\mathbb{R}^n)$. Our claims (3.13)–(3.16) and Lemma 3.1 show that for $b \in C_0^{\infty}(\mathbb{R}^n)$ and $l,\, m,\, j \in \mathbb{N}$ with $1 \le j \le D_m$, the operator $\Gamma_{m,\,j;\,l}$ defined by

$$\Gamma_{m,\,j;\,l} f(x) = \{T^{v}_{m,\,j;\,l,\,b} f(x)\}_{v\in\mathbb{Z}}$$

is compact from $L^2(\mathbb{R}^n)$ to $L^2(l^{\infty},\, \mathbb{R}^n)$. Since $|d_{m,\,j}(x)| \le 1$ and $|a_m(x)| \lesssim m^{\varepsilon}$ for ε as in (2.12), the operator $\widetilde{\Gamma}_{m,\,j;\,l}$ defined by

$$\widetilde{\Gamma}_{m,\,j;\,l} f(x) = \{a_m(x) d_{m,\,j}(x) T^{v}_{m,\,j;\,l,\,b} f(x)\}_{v\in\mathbb{Z}}$$

is also compact from $L^2(\mathbb{R}^n)$ to $L^2(l^{\infty},\, \mathbb{R}^n)$. As it was shown in the proof of Theorem 2.2, for $N_1,\, N_2 \in \mathbb{N}$ with $N_2 > N_1$, we can obtain from the conclusion (ii) in Lemma 2.1 and the Hölder inequality that

$$\begin{aligned}
&\Big\|\sup_{v\in\mathbb{Z}}\Big|\sum_{m=N_1}^{N_2} a_m(x)\sum_{j=1}^{D_m} d_{m,\,j}(x) T^{v}_{m,\,j;\,l,\,b} f(x)\Big|\Big\|^2_{L^2(\mathbb{R}^n)}\\
&\lesssim \Big(\sup_{y\in\mathbb{R}^n}\sum_{m=1}^{\infty}\{a_m(y)\}^2 m^{-\alpha}\Big)\sum_{m=N_1}^{N_2} m^{\alpha}\sum_{j=1}^{D_m}\Big\|\sup_{v\in\mathbb{Z}}|T^{v}_{m,\,j;\,l,\,b} f|\Big\|^2_{L^2(\mathbb{R}^n)}\\
&\lesssim \|b\|^2_{L^{\infty}(\mathbb{R}^n)}\|f\|^2_{L^2(\mathbb{R}^n)}\sum_{m=N_1}^{N_2} m^{-2+\beta+\alpha}.
\end{aligned}$$

Choosing $\alpha \in (0,\, 1)$ sufficiently close to 1 and $\beta \in (0,\, 1)$ sufficient small such that $\alpha + \beta < 1$, we see that $\big\{\sum_{m=1}^{N}\tilde{\Gamma}_{m,\,j;\,l}\big\}_{N\in\mathbb{N}}$ is a Cauchy sequence in the space of linear bounded operators from $L^2(\mathbb{R}^n)$ to $L^2(L^{\infty},\, \mathbb{R}^n)$. Therefore, $\Theta_{\Omega,l,\,b}$ is also compact from $L^2(\mathbb{R}^n)$ to $L^2(l^{\infty},\, \mathbb{R}^n)$. On the other hand, applying the Hölder inequality twice, we can write

$$\begin{aligned}
&\big\|\Theta_{\Omega,\,l,\,b} f(x) - \Lambda_{\Omega,\,b} f(x)\big\|_{l^{\infty}}\\
&\lesssim \|b\|_{L^{\infty}(\mathbb{R}^n)}\sup_{u\in\mathbb{Z}}\sum_{m\ge 1}|a_m(x)|\sum_{j=1}^{D_m}|d_{m,\,j}(x)|\Big|\sum_{v=u}^{\infty} V_{m,\,j;\,l,\,v} f(x)\Big|\\
&\quad + \sup_{u\in\mathbb{Z}}\sum_{m\ge 1}|a_m(x)|\sum_{j=1}^{D_m}|d_{m,\,j}(x)|\Big|\sum_{v=u}^{\infty} V_{m,\,j;\,l,\,v}(bf)(x)\Big|\\
&\lesssim \|b\|_{L^{\infty}(\mathbb{R}^n)}\Big(\sum_{m\ge 1} m^{\alpha}\sum_{j=1}^{D_m}\Big(\sup_{u\in\mathbb{Z}}\Big|\sum_{v=u}^{\infty} V_{m,\,j;\,l,\,v} f(x)\Big|\Big)^2\Big)^{1/2}\\
&\quad + \Big(\sum_{m\ge 1} m^{\alpha}\sum_{j=1}^{D_m}\Big(\sup_{u\in\mathbb{Z}}\Big|\sum_{v=u}^{\infty} V_{m,\,j;\,l,\,v}(bf)(x)\Big|\Big)^2\Big)^{1/2}.
\end{aligned}$$

The conclusion (ii) in Theorem 2.2 now tells us that

$$\big\|\Theta_{\Omega,\,l,\,b} f - \Lambda_{\Omega,\,b} f\big\|_{L^2(l^{\infty},\,\mathbb{R}^n)} \lesssim 2^{-\gamma_2 l}\|f\|_{L^2(\mathbb{R}^n)}.$$

Thus, for $b \in C_0^{\infty}(\mathbb{R}^n)$, $\Lambda_{\Omega,\,b}$ is compact from $L^2(\mathbb{R}^n)$ to $L^2(l^{\infty},\, \mathbb{R}^n)$. Observe that for $f_1,\, f_2 \in L^2(\mathbb{R}^n)$,

$$\|T^{\star\star}_{\Omega,b} f_1 - T^{\star\star}_{\Omega,b} f_2\|_{L^2(\mathbb{R}^n)} \le \|T^{\star\star}_{\Omega,b}(f_1 - f_2)\|_{L^2(\mathbb{R}^n)} = \|\Lambda_{\Omega,b} f_1 - \Lambda_{\Omega,b} f_2\|_{L^2(l^{\infty},\mathbb{R}^n)}.$$

This shows that if $\{\Lambda_{\Omega,\,b} f_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2(l^\infty,\,\mathbb{R}^n)$, then $\{T^{\star\star}_{\Omega,\,b} f_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$. We thus know that $T^{\star\star}_{\Omega,\,b}$ is compact on $L^2(\mathbb{R}^n)$. Recalling that $T^{\star\star}_{\Omega,\,b}$ is bounded on $L^2(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$ (see [9]), we then know $T^{\star\star}_{\Omega,\,b}$ is compact on $L^2(\mathbb{R}^n)$ for $b \in \mathrm{CMO}(\mathbb{R}^n)$. $\square$

For $v \in \mathbb{Z}$, let

$$M^v_{\Omega,\,b} f(x) = \int\limits_{2^v\le|x-y|<2^{v+1}} |b(x)-b(y)|^2 \frac{\Omega(x,\,x-y)}{|x-y|^n} f(y)\mathrm{d}y.$$

Lemma 3.2. *Let $b \in C_0^\infty(\mathbb{R}^n)$. Under the hypothesis of Theorem 1.2, the operator $\Delta_{\Omega,\,b}$ defined by*

$$\Delta_{\Omega,\,b} f(x) = \{M^v_{\Omega,\,b} f(x)\}_{v\in\mathbb{Z}} \tag{3.18}$$

is compact from $L^2(l^\infty,\,\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Proof. For $v \in \mathbb{Z}$, let

$$M^v_{\Omega,l;b} f(x) = \sum_{m\ge1} a_m(x) \sum_{j=1}^{D_m} d_{m,j}(x) \int\limits_{\mathbb{R}^n} |b(x)-b(y)|^2 \sigma_{m,j;v} * \phi_{v-l}(x-y) f(y)\mathrm{d}y,$$

and define the operator $\Delta_{\Omega,\,l;\,b}$ defined by

$$\Delta_{\Omega,\,l;\,b} f(x) = \{M^v_{\Omega,l;b} f(x)\}_{v\in\mathbb{Z}}.$$

As the argument for $\Theta_{\Omega,\,l,\,b}$ in the proof of Theorem 1.2, we can prove that $\Delta_{\Omega,\,l;\,b}$ is bounded from $L^2(\mathbb{R}^n)$ to $L^2(l^\infty,\,\mathbb{R}^n)$. On the other hand, an argument involving the Hölder inequality which is familiar to us, yields

$$\begin{aligned}
&\big\|\Delta_{\Omega,\,b} f(x) - \Delta_{\Omega,\,l;\,b} f(x)\big\|_{L^\infty}\\
&\le \|b\|^2_{L^\infty(\mathbb{R}^n)} \Big(\sum_{m\ge1} m^\alpha \sum_{j=1}^{D_m} \Big(\sup_{v\in\mathbb{Z}} \Big|V_{m,\,j;\,l,\,v} f(x)\Big|\Big)^2\Big)^{1/2}\\
&\quad + \Big(\sum_{m\ge1} m^\alpha \sum_{j=1}^{D_m} \Big(\sup_{v\in\mathbb{Z}} \Big|V_{m,\,j;\,l,\,v} (|b|^2 f)(x)\Big|\Big)^2\Big)^{1/2}\\
&\quad + \|b\|_{L^\infty(\mathbb{R}^n)} \Big(\sum_{m\ge1} m^\alpha \sum_{j=1}^{D_m} \Big(\sup_{v\in\mathbb{Z}} \Big|V_{m,\,j;\,l,\,v} \big(f\mathrm{Re}b\big)(x)\Big|\Big)^2\Big)^{1/2}\\
&\quad + \|b\|_{L^\infty(\mathbb{R}^n)} \Big(\sum_{m\ge1} m^\alpha \sum_{j=1}^{D_m} \Big(\sup_{v\in\mathbb{Z}} \Big|V_{m,\,j;\,l,\,v} \big(f\mathrm{Im}b\big)(x)\Big|\Big)^2\Big)^{1/2},
\end{aligned}$$

which, via (2.11) in Theorem 2.2, show that

$$\|\Delta_{\Omega,\,b} f - \Delta_{\Omega,\,l;\,b} f\|_{L^2(l^\infty,\,\mathbb{R}^n)} \le 2^{-\gamma_3 l} \|f\|_{L^2(\mathbb{R}^n)}.$$

Therefore, $\Delta_{\Omega,\,b}$ is compact from $L^2(l^\infty,\,\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. $\square$

Lemma 3.3. *Let $p \in (1, \infty)$, $b \in C_0^\infty(\mathbb{R}^n)$. The operator Υ_b defined by*

$$\Upsilon_b f(x) = \{\Upsilon_b^v f(x)\}_{v\in\mathbb{Z}}, \tag{3.19}$$

with

$$\Upsilon_b^v f(x) = \int\limits_{2^v\le|x-y|<2^{v+1}} \frac{|b(x)-b(y)|^2}{|x-y|^n} f(y)\mathrm{d}y,$$

is compact from $L^p(l^\infty, \mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Proof. For $k \in \mathbb{Z}$, let

$$\Upsilon_{b,\,k} f(x) = \{\Upsilon_b^v f(x)\}_{v\in\mathbb{Z},\, v>k}.$$

It is easy to verify that for $b \in C_0^\infty(\mathbb{R}^n)$,

$$\|\Upsilon_{b,\,k} f(x) - \Upsilon_b f(x)\|_{l^\infty} \le 2^{2k} Mf(x).$$

Thus, it suffices to prove that for $b \in C_0^\infty(\mathbb{R}^n)$ and $k \in \mathbb{Z}_-$, $\Upsilon_{b,\,k}$ is compact from $L^p(l^\infty, \mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Let $b \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp} b \subset B(0, R)$ for some $R > 0$. We claim that

(i) for $A > 4R$,

$$\Big\|\sup_{v\in\mathbb{Z}} \big|\Upsilon_b^v f\big|\chi_{\{|\cdot|>A\}}\Big\|_{L^p(\mathbb{R}^n)} \lesssim \Big(\frac{R}{A}\Big)^{n/p'} \|b\|_{L^\infty(\mathbb{R}^n)}^2 \|f\|_{L^p(\mathbb{R}^n)}; \tag{3.20}$$

(ii) for each $k \in \mathbb{Z}_-$, $t \in \mathbb{R}^n$ with $|t| < \min\{1, 2^{k-2}\}$, and each $r \in (1, p)$,

$$\begin{aligned}&\Big\|\sup_{v\in\mathbb{Z},\, v>k} \big|\Upsilon_b^v f(\cdot) - \Upsilon_b^v f(\cdot+t)\big|\Big\|_{L^p(\mathbb{R}^n)} \\ &\quad\lesssim \big(\|b\|_{L^\infty(\mathbb{R}^n)}^2 + \|\nabla b\|_{L^\infty(\mathbb{R}^n)}^2\big)\Big(\frac{|t|}{2^k}\Big)^{1/r'} \|f\|_{L^p(\mathbb{R}^n)};\end{aligned} \tag{3.21}$$

(iii) for each $N \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}^n} \sup_{v>N} \big|\Upsilon_b^v f(x)\big| \lesssim 2^{-nN/p'} \|b\|_{L^\infty(\mathbb{R}^n)}^2 \|f\|_{L^p(\mathbb{R}^n)}. \tag{3.22}$$

If these claims are true, then by Lemma 3.1, we know that $\Upsilon_{b,k}$ is compact from $L^p(l^\infty, \mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

For each $v \in \mathbb{Z}$, by the Hölder inequality, we have that for $x \in \mathbb{R}^n$ with $|x| > 4R$,

$$\int\limits_{2^v\le|x-y|<2^{v+1},\, |y|<R} \frac{|f(y)|}{|x-y|^n}\mathrm{d}y \lesssim \|f\|_{L^p(\mathbb{R}^n)} \frac{R^{n/p'}}{|x|^n},$$

since $|x - y| \approx |x|$ when $|y| < R$ and $|x| > 4R$. Thus, for $A > 4R$,

$$\Big\|\sup_{v\in\mathbb{Z}} \Upsilon_b^v f\chi_{\{|x|>A\}}\Big\|_{L^p(\mathbb{R}^n)} \lesssim \Big(\frac{R}{A}\Big)^{n/p'} \|b\|_{L^\infty(\mathbb{R}^n)}^2 \|f\|_{L^p(\mathbb{R}^n)}.$$

This gives us (3.20).

We turn our attention to the estimate (3.21). For each fixed $v \in \mathbb{Z}$, let

$$\mathrm{E}^v f(x,t) = \int\limits_{2^v \le |x+t-y| < 2^{v+1}} \Big(\frac{|b(x+t)-b(y)|^2}{|x+t-y|^n} - \frac{|b(x)-b(y)|^2}{|x+t-y|^n} \Big) |f(y)| \mathrm{d}y,$$

$$\mathrm{F}^v f(x,t) = \int\limits_{2^v \le |x+t-y| < 2^{v+1}} \Big(\frac{|b(x)-b(y)|^2}{|x+t-y|^n} - \frac{|b(x)-b(y)|^2}{|x-y|^n} \Big) |f(y)| \mathrm{d}y,$$

and

$$\mathrm{H}^v f(x,t) = \int\limits_{\{2^v \le |x+t-y| < 2^{v+1}\} \triangle \{2^v \le |x-y| < 2^{v+1}\}} \frac{|b(x)-b(y)|^2}{|x-y|^n} |f(y)| \mathrm{d}y.$$

It then follows that

$$\left| \Upsilon_b^v f(x) - \Upsilon_b^v f(x+t) \right| \le \mathrm{E}^v f(x,t) + \mathrm{F}^v f(x,t) + \mathrm{H}^v f(x,t).$$

It is obvious that

$$|\mathrm{E}^v f(x,\, t)| \le |t| \|b\|_{L^\infty(\mathbb{R}^n)} \|\nabla b\|_{L^\infty(\mathbb{R}^n)} Mf(x+t).$$

To estimate F^v, we observe that $2^v \ge 2|t|$ and $2^v \le |x+t-y| < 2^{v+1}$,

$$\Big| \frac{|b(x)-b(y)|^2}{|x+t-y|^n} - \frac{|b(x)-b(y)|^2}{|x-y|^n} \Big| \lesssim \|b\|^2_{L^\infty(\mathbb{R}^n)} \frac{|t|}{|x+t-y|^{n+1}},$$

and then obtain that when $v > k$,

$$|\mathrm{F}^v f(x,\, t)| \lesssim \|b\|^2_{L^\infty(\mathbb{R}^n)} \frac{|t|}{2^k} Mf(x+t).$$

Let $r \in (1,\, p)$. A trivial computation shows that when $v > k$ and $|t| < 2^{v-2}$,

$$\begin{aligned} \mathrm{H}^v f(x,\, t) &\lesssim \|b\|^2_{L^\infty(\mathbb{R}^n)} \Big(\int\limits_{2^v \le |x-y| < 2^{v+1}} \frac{1}{|x-y|^{rn}} |f(y)|^r \mathrm{d}y \Big)^{1/r} \\ &\quad \times \Big(\int\limits_{\{2^v \le |x+t-y| < 2^{v+1}\} \triangle \{2^v \le |x-y| < 2^{v+1}\}} \mathrm{d}y \Big)^{1/r'} \\ &\lesssim \Big(\frac{|t|}{2^k} \Big)^{1/r'} \|b\|^2_{L^\infty(\mathbb{R}^n)} M_r f(x). \end{aligned}$$

Combining the estimates for E^v, F^v and H^v leads to (3.21).

Finally, the inequality (3.22) follows directly from the fact that for each $v \in \mathbb{Z}$ and $x \in \mathbb{R}^n$,

$$|\Upsilon_b^v f(x)| \lesssim \|f\|_{L^p(\mathbb{R}^n)} 2^{-vn/p'} \|b\|^2_{L^\infty(\mathbb{R}^n)}.$$

This completes the proof of Lemma 3.3. □

Proof of Theorem 1.3. Recall that $T^{\star}_{\Omega,\,b}$ is bounded on $L^2(\mathbb{R}^n)$ with bound $C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$ (see [9]). Thus, it suffices to prove that when $b \in C_0^{\infty}(\mathbb{R}^n)$, $|f_k - f| \rightharpoonup 0$ implies that

$$\lim_{k\to\infty} \|T^{\star}_{\Omega,\,b} f_k - T^{\star}_{\Omega,\,b} f\|_{L^2(\mathbb{R}^n)} = 0.$$

Now let

$$M_{\Omega} f(x) = \sup_{v\in\mathbb{Z}} \int\limits_{2^v \le |x-y| < 2^{v+1}} \frac{|\Omega(x,\,x-y)|}{|x-y|^n} |f(y)|\,\mathrm{d}y,$$

$$M_{\Omega,\,b} f(x) = \sup_{v\in\mathbb{Z}} \Big| \int\limits_{2^v \le |x-y| < 2^{v+1}} |b(x)-b(y)|^2 \frac{|\Omega(x,\,x-y)|}{|x-y|^n} |f(y)|\,\mathrm{d}y.$$

A trivial computation then leads to that for $\{f_k\} \subset L^2(\mathbb{R}^n)$ and $f \in L^2(\mathbb{R}^n)$,

$$\left|T^{\star}_{\Omega,\,b} f_k(x) - T^{\star}_{\Omega,\,b} f(x)\right| \le T^{\star\star}_{\Omega,\,b}(f_k - f)(x) + \Big(M_{\Omega,\,b}(|f_k - f|)(x)\Big)^{\frac{1}{2}} \Big(M_{\Omega}(f_k - f)(x)\Big)^{\frac{1}{2}}.$$

Let

$$\widetilde{\Omega}(x,\,z) = |\Omega(x,\,z)| - \frac{1}{|S^{n-1}|} \int\limits_{S^{n-1}} |\Omega(x,\,y')| d\sigma(y').$$

Obviously, $\widetilde{\Omega}(x,\,z)$ is homogeneous of degree zero in the variable z,

$$\widetilde{\Omega}(x,\,z') \in L^{\infty}(\mathbb{R}^n) \times L^q(S^{n-1}), \quad \int\limits_{S^{n-1}} \widetilde{\Omega}(x,\,z')\mathrm{d}\sigma(z') = 0,$$

and

$$\|M_{\Omega,\,b} f\|_{L^2(\mathbb{R}^n)} \lesssim \|\Delta_{\widetilde{\Omega},\,b}(|f|)|_{L^2(l^{\infty},\,\mathbb{R}^n)} + \|\Upsilon_b(|f|)\|_{L^2(l^{\infty},\,\mathbb{R}^n)},$$

with $\Delta_{\widetilde{\Omega},\,b}$, Υ_b defined by (3.18) and (3.19) respectively. Since M_{Ω} is bounded on $L^2(\mathbb{R}^n)$ (see [9]), it follows that

$$\begin{aligned} \left\|T^{\star}_{\Omega,\,b} f_k - T^{\star}_{\Omega,\,b} f\right\|_{L^2(\mathbb{R}^n)} \le{}& \left\|T^{\star\star}_{\Omega,\,b}(f_k - f)\right\|_{L^2(\mathbb{R}^n)} \\ &+ \left\|\Delta_{\widetilde{\Omega},\,b}(|f_k - f|)\right\|^{1/2}_{L^2(l^{\infty},\mathbb{R}^n)} \|f_k - f\|^{1/2}_{L^2(\mathbb{R}^n)} \\ &+ \left\|\Upsilon_b(|f_k - f|)\right\|^{1/2}_{L^2(l^{\infty},\,\mathbb{R}^n)} \|f_k - f\|^{1/2}_{L^2(\mathbb{R}^n)}. \end{aligned}$$

In the proof of Theorem 1.2, we have shown that the operator Λ is compact from $L^2(\mathbb{R}^n)$ to $L^2(l^{\infty},\,\mathbb{R}^n)$, thus $f_k \rightharpoonup f$ in $L^2(\mathbb{R}^n)$ implies that $\|\Lambda(f_k - f)\|_{L^2(l^{\infty},\,\mathbb{R}^n)} \to 0$, and

$$\lim_{k\to\infty} \|T^{\star\star}_{\Omega,\,b}(f_k - f)\|_{L^2(\mathbb{R}^n)} = 0.$$

On the other hand, Lemma 3.2 and Lemma 3.3, show that in $L^2(\mathbb{R}^n)$,

$$|f_k - f| \rightharpoonup 0 \Rightarrow \|\Delta_{\widetilde{\Omega},\,b}(|f_k - f|)\|_{L^2(l^{\infty},\,\mathbb{R}^n)} + \|\Upsilon_b(|f_k - f|)\|_{L^2(l^{\infty},\,\mathbb{R}^n)} \to 0.$$

This completes the proof of Theorem 1.3. $\square$

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