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Accurate approximations for the complete elliptic integral of the second kind

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ABSTRACT

In this paper, we prove that the double inequality

$$\lambda S_{11/4,7/4}(1, r') < \mathcal{E}(r) < \mu S_{11/4,7/4}(1, r')$$

holds for all $r \in (0, 1)$ if and only if $\lambda \leq \pi/2 = 1.570796\cdots$ and $\mu \geq 11/7 = 1.571428\cdots$, where $r' = (1-r^2)^{1/2}$, $\mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1-r^2\sin^2(t)}dt$ is the complete elliptic integral of the second kind, and $S_{p,q}(a, b) = [q(a^p - b^p)/(p(a^q - b^q))]^{1/(p-q)}$ is the Stolarsky mean of a and b.

1. Introduction

For $r \in (0, 1)$, Legendre's complete elliptic integrals [1] of the first kind and the second kind are given by

$$\mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin^2(t)}},$$

$$\mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2} \sqrt{1-r^2\sin^2(t)}dt,$$

respectively. It is well known that

$$\mathcal{K}(0^+) = \mathcal{E}(0^+) = \frac{\pi}{2}, \quad \mathcal{K}(1^-) = \infty, \quad \mathcal{E}(1^-) = 1. \tag{1.1}$$

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They are the particular cases of the Gaussian hypergeometric function

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}\quad(-1<x<1),$$

where $(a)_n=\Gamma(a+n)/\Gamma(a)$ and $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}dt$ $(x>0)$ is the gamma function. Indeed, we have

$$\mathcal{K}(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2}r^{2n}, \tag{1.2}$$

$$\mathcal{E}(r)=\frac{\pi}{2}F\left(-\frac{1}{2},\frac{1}{2};1;r^2\right)=\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n}{(n!)^2}r^{2n}. \tag{1.3}$$

The complete elliptic integrals and Gaussian hypergeometric function have important applications in quasiconformal mappings, number theory, and other fields of the mathematical and mathematical physics. For instance, the Gaussian arithmetic–geometric mean AGM and the modulus of the plane Grötzsch ring can be expressed in terms of the complete elliptic integral of the first kind, and the complete elliptic integral of the second kind gives the formula of the perimeter of an ellipse. Moreover, Ramanujan modular equation and continued fraction in number theory are both related to the Gaussian hypergeometric function $F(a,b;c;x)$. In particular, many remarkable inequalities and properties for the complete elliptic integrals and Gaussian hypergeometric function can be found in the literature [2–6,8,11,13,14,17].

Recently, the bounds for the complete elliptic integral of the second kind $\mathcal{E}(r)$ have attracted the attention of many researchers. In [15], Vuorinen conjectured that the inequality

$$\mathcal{E}(r)\geq\frac{\pi}{2}M_{3/2}(1,r') \tag{1.4}$$

holds for all $r\in(0,1)$, where and in what follows $r'=(1-r^2)^{1/2}$, $M_p(a,b)$ is the pth power mean of a and b which is given by

$$M_p(a,b)=\left(\frac{a^p+b^p}{2}\right)^{1/p}\quad(p\neq 0),\qquad M_0(a,b)=\sqrt{ab}.$$

Inequality (1.4) was proved by Barnard et al. in [6]. In [18, Theorem 2.4] and [16, Corollary 3.1], the authors proved that the double inequalities

$$\begin{aligned}&\frac{23A(1,r')-5H(1,r')-2M_2(1,r')}{16}\\&\quad<\frac{2}{\pi}\mathcal{E}(r)<\frac{(24-5\sqrt{2}\pi)A(1,r')-(8-\pi-\sqrt{2}\pi)H(1,r')-(16-5\pi)M_2(1,r')}{2\pi(3-2\sqrt{2})}\end{aligned} \tag{1.5}$$

and

$$\frac{(9r'^2+14r'+9)^2}{128(1+r')^3}<\frac{2}{\pi}\mathcal{E}(r)<\frac{\sqrt{4r'^2+(\pi^2-8)r'+4}}{\pi} \tag{1.6}$$

hold for all $r\in(0,1)$, where $A(a,b)=M_1(a,b)=(a+b)/2$ and $H(a,b)=M_{-1}(a,b)=2ab/(a+b)$ are respectively the arithmetic and harmonic means of a and b.

Very recently, Hua and Qi [9, Theorem 1.3] proved that the double inequality

$$\frac{1+r'+r'^2}{2(1+r')}+\frac{1+r'}{8}<\frac{2}{\pi}\mathcal{E}(r)<\left(\frac{8}{\pi}-2\right)\frac{1+r'+r'^2}{1+r'}+\left(2-\frac{6}{\pi}\right)(1+r') \tag{1.7}$$

is valid for all $r \in (0,1)$.

Let $p,q \in \mathbb{R}$ with $p \neq q$ and $pq \neq 0$, and $a,b>0$. Then the Stolarsky mean $S_{p,q}(a,b)$ [12] is defined by

$$S_{p,q}(a,b)=\left[\frac{q(a^p-b^p)}{p(a^q-b^q)}\right]^{1/(p-q)} \quad (a \neq b), \quad S_{p,q}(a,a)=a. \tag{1.8}$$

The main purpose of this paper is to prove that the double inequality

$$\lambda S_{11/4,7/4}(1,r')<\mathcal{E}(r)<\mu S_{11/4,7/4}(1,r')$$

holds for all $r \in (0,1)$ with the best possible parameters $\lambda = \pi/2 = 1.570796\cdots$ and $\mu = 11/7 = 1.571428\cdots$, see Theorem 3.1. Moreover, the lower bound here is better than any of the lower bounds (1.4)–(1.7), see Remark 3.4. Some complicated computations are carried out using Mathematica computer algebra system.

2. Lemmas

In order to prove our main result we need several lemmas, which we present in this section.

Lemma 2.1. *(See [4, Theorem 1.25].) Let $-\infty<a<b<\infty$, $f,g:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x)\neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemma 2.2. *(See [10,19].) The double inequality*

$$\frac{1}{(x+a)^{1-a}}<\frac{\Gamma(x+a)}{\Gamma(x+1)}<\frac{1}{x^{1-a}}$$

holds for all $x>0$ and $a \in (0,1)$.

Lemma 2.3. *(See [7, Lemma 1.7].) Let $\psi(x)=\Gamma'(x)/\Gamma(x)$ be the psi function. Then the inequality*

$$\psi(x+1)>\log\left(x+\frac{1}{2}\right)$$

holds for all $x>0$.

Lemma 2.4. *(See [20, Lemma 7].) Let $n\in\mathbb{N}$ and $m\in\mathbb{N}\cup\{0\}$ with $n>m$, $a_i\geq 0$ for all $0\leq i\leq n$, $a_na_m>0$ and*

$$P_n(t)=-\sum_{i=0}^{m}a_it^i+\sum_{i=m+1}^{n}a_it^i.$$

Then there exists $t_0\in(0,\infty)$ such that $P_n(t_0)=0$, $P_n(t)<0$ for $t\in(0,t_0)$ and $P_n(t)>0$ for $t\in(t_0,\infty)$.

Lemma 2.5. *Let $n \geq 0$, a_n, b_n, c_n, d_n, D_n, $g_1(n)$ and g_n be respectively defined by*

$$a_n = \frac{(5n+9)\left(\frac{1}{2}\right)_{n+1}^2}{n!(n+3)!}, \quad b_n = \frac{\left(\frac{7}{8}\right)_n}{n!}, \quad c_n = \frac{(n-1)(n+18)(2n+1)\left(\frac{1}{2}\right)_n^2}{n!(n+3)!}, \tag{2.1}$$

$$d_n = 8a_n + \sum_{k=0}^{n} b_{n-k}c_k, \quad D_n = \frac{8}{7}d_{n+1} - d_n, \tag{2.2}$$

$$g_1(n) = \sum_{k=2}^{n} \frac{(n-k)(k-1)(k+18)}{(n-k+1)(k+1)(k+2)(k+3)} \tag{2.3}$$

and

$$g(n) = \frac{4\Gamma(7/8)n(2n+1)}{\pi(n+1)(n+2)(n+3)} - \frac{3n^{7/8}}{7(n+1)} + \frac{128n^{7/8}}{7\pi(64n-9)}g_1(n). \tag{2.4}$$

Then

$$D_n > \frac{g(n)}{\Gamma\left(\frac{7}{8}\right)}$$

for $n \geq 4$.

Proof. From (2.1) and (2.2) we clearly see that

$$b_0 = 1, \quad c_0 = -3, \quad c_1 = 0, \quad c_k > 0 \quad (k \geq 2),$$

$$a_{k+1} = \frac{(2k+3)^2(5k+14)}{4(5k+9)(k+1)(k+4)}a_k, \quad b_{k+1} = \frac{k+7/8}{k+1}b_k,$$

$$c_{k+1} = \frac{k(2k+1)(2k+3)(k+19)}{4(k-1)(k+1)(k+4)(k+18)}c_k \quad (k \geq 2),$$

$$\begin{aligned}
D_n &= \frac{8}{7}\left[8a_{n+1} + b_{n+1}c_0 + \sum_{k=2}^{n+1} b_{n+1-k}c_k\right] - \left[8a_n + b_nc_0 + \sum_{k=2}^{n} b_{n-k}c_k\right] \\
&= 8\left(\frac{8}{7}a_{n+1} - a_n\right) - 3\left(\frac{8}{7}b_{n+1} - b_n\right) + \frac{8}{7}b_0c_{n+1} + \sum_{k=2}^{n}\left(\frac{8}{7}b_{n+1-k} - b_{n-k}\right)c_k \\
&= 8\left[\frac{2(2n+3)^2(5n+14)}{7(5n+9)(n+1)(n+4)} - 1\right]a_n - 3\left[\frac{8n+7}{7(n+1)} - 1\right]b_n \\
&\quad + \frac{2n(2n+1)(2n+3)(n+19)}{7(n-1)(n+1)(n+4)(n+18)}c_n + \sum_{k=2}^{n}\left(\frac{8(n-k)+7}{7(n-k+1)}\right)b_{n-k}c_k \\
&= \frac{8n(5n^2-6n-29)}{7(5n+9)(n+1)(n+4)}a_n - \frac{3n}{7(n+1)}b_n \\
&\quad + \frac{2n(2n+1)(2n+3)(n+19)}{7(n-1)(n+1)(n+4)(n+18)}c_n + \frac{1}{7}\sum_{k=2}^{n}\frac{n-k}{n-k+1}b_{n-k}c_k.
\end{aligned} \tag{2.5}$$

It follows from Lemma 2.2 and (2.1) together with $\Gamma(1/2) = \sqrt{\pi}$ that

$$a_n = \frac{5n+9}{n!(n+3)!}\left[\frac{\Gamma(n+3/2)}{\Gamma(1/2)}\right]^2 = \frac{(5n+9)(n+1/2)^2}{\pi(n+1)(n+2)(n+3)}\left[\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\right]^2$$
$$> \frac{(5n+9)(n+1/2)^2}{\pi(n+1)(n+2)(n+3)(n+1/2)} = \frac{(5n+9)(2n+1)}{2\pi(n+1)(n+2)(n+3)}, \tag{2.6}$$

$$\frac{1}{\Gamma\left(\frac{7}{8}\right)\left(n+\frac{7}{8}\right)^{1/8}} < b_n = \frac{\Gamma(n+7/8)}{n!\Gamma(7/8)} = \frac{1}{\Gamma(7/8)}\frac{\Gamma(n+7/8)}{\Gamma(n+1)} < \frac{1}{\Gamma\left(\frac{7}{8}\right)n^{1/8}}, \tag{2.7}$$

$$c_n = \frac{(n-1)(n+18)(2n+1)}{n!(n+3)!}\left[\frac{\Gamma(n+1/2)}{\Gamma(1/2)}\right]^2$$
$$= \frac{(n-1)(n+18)(2n+1)}{\pi(n+1)(n+2)(n+3)}\left[\frac{\Gamma(n+1/2)}{\Gamma(n+1)}\right]^2 > \frac{2(n-1)(n+18)}{\pi(n+1)(n+2)(n+3)} \quad (n \geq 2). \tag{2.8}$$

Making use of the inequality $(1-x)^p < 1 - px$ for all $p, x \in (0,1)$ we get

$$\frac{1}{(n-k+7/8)^{1/8}} \geq \frac{1}{(n-2+7/8)^{1/8}} = \frac{1}{n^{1/8}\left(1-\frac{9}{8n}\right)^{1/8}} > \frac{64n^{7/8}}{64n-9} \tag{2.9}$$

for all $2 \leq k \leq n$.

Note that

$$5n^2 - 6n - 29 > 0 \tag{2.10}$$

for $n \geq 4$.

It follows from (2.5)–(2.10) that

$$D_n > \frac{8n(5n^2-6n-29)}{7(5n+9)(n+1)(n+4)} \times \frac{(2n+1)(5n+9)}{2\pi(n+1)(n+2)(n+3)}$$
$$- \frac{3n}{7\Gamma(7/8)(n+1)n^{1/8}} + \frac{2n(2n+1)(2n+3)(n+19)}{7(n-1)(n+1)(n+4)(n+18)} \times \frac{2(n-1)(n+18)}{\pi(n+1)(n+2)(n+3)}$$
$$+ \frac{1}{7}\sum_{k=2}^{n}\left[\frac{n-k}{n-k+1} \times \frac{1}{\Gamma(7/8)(n-k+7/8)^{1/8}} \times \frac{2(k-1)(k+18)}{\pi(k+1)(k+2)(k+3)}\right]$$
$$> \frac{4n(2n+1)}{\pi(n+1)(n+2)(n+3)} - \frac{3n^{7/8}}{7\Gamma(7/8)(n+1)}$$
$$+ \frac{128n^{7/8}}{7\pi\Gamma(7/8)(64n-9)}\sum_{k=2}^{n}\frac{(n-k)(k-1)(k+18)}{(n-k+1)(k+1)(k+2)(k+3)} = \frac{g(n)}{\Gamma(7/8)}$$

for all $n \geq 4$. □

Lemma 2.6. *Let $g_1(n)$ be defined by (2.3). Then*

$$g_1(n) = \frac{n^3+7n^2-12n+24}{(n+2)(n+3)(n+4)}[\psi(n+1)+\gamma] - \frac{n(11n^2+8n+21)}{(n+1)(n+2)(n+3)(n+4)},$$

where $\gamma = \lim_{n\to\infty}(\sum_{k=1}^{n} 1/k - \log n) = 0.57721566\ldots$ is the Euler–Mascheroni constant.

Proof. It is not difficult to verify that

$$\frac{(n-k)(k-1)(k+18)}{(n-k+1)(k+1)(k+2)(k+3)} = -\frac{17(n+1)}{(n+2)(k+1)} + \frac{48(n+2)}{(n+3)(k+2)} - \frac{30(n+3)}{(n+4)(k+3)} - \frac{n(n+19)}{(n+2)(n+3)(n+4)(n-k+1)}. \tag{2.11}$$

From (2.3) and (2.11) we have

$$\begin{aligned} g_1(n) &= -\frac{17(n+1)}{(n+2)}\left(\sum_{k=1}^{n}\frac{1}{k} + \frac{1}{n+1} - \frac{3}{2}\right) + \frac{48(n+2)}{n+3}\left(\sum_{k=1}^{n}\frac{1}{k} + \frac{1}{n+1} + \frac{1}{n+2} - \frac{11}{6}\right) \\ &\quad - \frac{30(n+3)}{(n+4)}\left(\sum_{k=1}^{n}\frac{1}{k} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} - \frac{25}{12}\right) - \frac{n(n+19)}{(n+2)(n+3)(n+4)}\left(\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{n}\right) \\ &= \frac{n^3+7n^2-12n+24}{(n+2)(n+3)(n+4)}\sum_{k=1}^{n}\frac{1}{k} - \frac{n(11n^2+8n+21)}{(n+1)(n+2)(n+3)(n+4)} \\ &= \frac{n^3+7n^2-12n+24}{(n+2)(n+3)(n+4)}[\psi(n+1)+\gamma] - \frac{n(11n^2+8n+21)}{(n+1)(n+2)(n+3)(n+4)}. \qquad \square \end{aligned}$$

Lemma 2.7. *Let $g(n)$ be defined by (2.4), and $g_2(n)$ and $g_3(n)$ be respectively defined by*

$$g_2(n) = \psi(n+1) + \gamma + \frac{7\Gamma(7/8)n^{1/8}(n+4)(2n+1)(64n-9)}{32(n+1)(n^3+7n^2-12n+24)} - \frac{3\pi(n+2)(n+3)(n+4)(64n-9)}{128(n+1)(n^3+7n^2-12n+24)} - \frac{n(11n^2+8n+21)}{(n+1)(n^3+7n^2-12n+24)} \tag{2.12}$$

and

$$g_3(n) = \log(n+1/2) + \gamma + \frac{7\Gamma(7/8)n^{1/8}(n+4)(2n+1)(64n-9)}{32(n+1)(n^3+7n^2-12n+24)} - \frac{192n^4+2141n^3+5069n^2+4746n-648}{40(n+1)(n^3+7n^2-12n+24)}. \tag{2.13}$$

Then

$$\begin{aligned} g(n) &= \frac{128n^{7/8}(n^3+7n^2-12n+24)}{7\pi(64n-9)(n+2)(n+3)(n+4)}g_2(n) \\ &> \frac{128n^{7/8}(n^3+7n^2-12n+24)}{7\pi(64n-9)(n+2)(n+3)(n+4)}g_3(n) \end{aligned}$$

for all $n \ge 1$.

Proof. It follows from Lemma 2.3 and Lemma 2.6, (2.4), (2.12), (2.13) and $3\pi/128 < 3/40$ that

$$\begin{aligned} g(n) &= \frac{4\Gamma(7/8)n(2n+1)}{\pi(n+1)(n+2)(n+3)} - \frac{3n^{7/8}}{7(n+1)} + \frac{128n^{7/8}}{7\pi(64n-9)} \\ &\quad \times \left[\frac{n^3+7n^2-12n+24}{(n+2)(n+3)(n+4)}(\psi(n+1)+\gamma) - \frac{n(11n^2+8n+21)}{(n+1)(n+2)(n+3)(n+4)}\right] \\ &= \frac{128n^{7/8}(n^3+7n^2-12n+24)}{7\pi(64n-9)(n+2)(n+3)(n+4)}g_2(n) \end{aligned} \tag{2.14}$$

and

$$g_2(n) > \log(n+1/2) + \gamma + \frac{7\Gamma(7/8)n^{1/8}(n+4)(2n+1)(64n-9)}{32(n+1)(n^3+7n^2-12n+24)} - \frac{3(n+2)(n+3)(n+4)(64n-9)}{40(n+1)(n^3+7n^2-12n+24)} - \frac{n(11n^2+8n+21)}{(n+1)(n^3+7n^2-12n+24)} = g_3(n). \tag{2.15}$$

Therefore, Lemma 2.7 follows easily from (2.14) and (2.15). □

Lemma 2.8. *Let $g_3(x)$ be defined by*

$$g_3(x) = \log(x+1/2) + \gamma + \frac{7\Gamma(7/8)x^{1/8}(x+4)(2x+1)(64x-9)}{32(x+1)(x^3+7x^2-12x+24)} - \frac{192x^4+2141x^3+5069x^2+4746x-648}{40(x+1)(x^3+7x^2-12x+24)}. \tag{2.16}$$

Then $g_3(x)$ is strictly increasing on $[10,\infty)$.

Proof. Let $g_4(x)$, $g_5(x)$, $g_6(x)$ and $g_7(x)$ be defined by

$$g_4(x) = 80x^8 + 2490x^7 + 29\,041x^6 + 124\,744x^5 + 86\,839x^4 - 483\,876x^3 - 735\,606x^2 - 434\,112x - 75\,600, \tag{2.17}$$

$$g_5(x) = 896x^7 + 7346x^6 + 41\,033x^5 - 3438x^4 - 149\,813x^3 - 227\,064x^2 - 40\,824x + 864, \tag{2.18}$$

$$g_6(x) = \log\left[\frac{g_4(x)}{40(2x+1)}\right] - \log\left[\frac{7\Gamma(7/8)g_5(x)}{256x^{7/8}}\right] \tag{2.19}$$

and

$$g_7(x) = 1\,003\,520x^{12} + 6\,215\,360x^{11} - 76\,861\,608x^{10} - 194\,717\,340x^9 - 1\,129\,301\,614x^8 - 7\,502\,136\,933x^7 - 1\,325\,597\,720x^6 - 6\,754\,007\,727x^5 - 29\,245\,627\,200x^4 - 18\,436\,239\,066x^3 - 4\,240\,833\,192x^2 - 348\,047\,280x - 19\,051\,200. \tag{2.20}$$

Then it follows from Lemma 2.4 and (2.16)–(2.20) that

$$g_3'(x) = \frac{1}{(x+1)^2(x^3+7x^2-12x+24)^2}\left[\frac{g_4(x)}{40(2x+1)} - \frac{7\Gamma(7/8)g_5(x)}{256x^{7/8}}\right], \tag{2.21}$$

$$g_6'(x) = \frac{(x+1)(x^3+7x^2-12x+24)g_7(x)}{8x(2x+1)g_4(x)g_5(x)}, \tag{2.22}$$

$$g_4(10) = 74\,721\,936\,680 > 0, \tag{2.23}$$

$$g_5'(10) = 12\,667\,971\,966 > 0, \tag{2.24}$$

$$g_5(10) = 20\,201\,993\,224 > 0, \tag{2.25}$$

$$g_6(10) = \log\frac{597\,775\,493\,440}{371\,211\,625\,491} - \frac{1}{8}\log 10 - \log\Gamma(7/8) = 0.1027\cdots > 0, \tag{2.26}$$

$$g_7(10) = 471\,458\,730\,683\,390\,800 > 0 \tag{2.27}$$

and there exist $x_1, x_2, x_3 \in (0,\infty)$ such that the following statements are true:

Statement *A*. $g_4(x) < 0$ for $x \in (0, x_1)$ and $g_4(x) > 0$ for $x \in (x_1, \infty)$.
Statement *B*. $g_5'(x) < 0$ for $x \in (0, x_2)$ and $g_5'(x) > 0$ for $x \in (x_2, \infty)$.
Statement *C*. $g_7(x) < 0$ for $x \in (0, x_3)$ and $g_7(x) > 0$ for $x \in (x_3, \infty)$.

From (2.23), (2.24) and (2.27) together with Statements A–C we know that

$$g_4(x) > 0, \tag{2.28}$$

$$g_5'(x) > 0, \tag{2.29}$$

$$g_7(x) > 0 \tag{2.30}$$

for $x \geq 10$.

Inequalities (2.25) and (2.29) imply that

$$g_5(x) > 0 \tag{2.31}$$

for $x \geq 10$.

It follows from (2.22) and (2.26) together with (2.30) that

$$g_6(x) > 0 \tag{2.32}$$

for $x \geq 10$.

Therefore, Lemma 2.8 follows easily from (2.19), (2.21), (2.28), (2.31) and (2.32). □

Lemma 2.9. *Let d_n, D_n and $g_3(n)$ be defined by (2.2) and (2.13). Then*

$$d_n > 0$$

for all $n \geq 25$.

Proof. From (2.13) and Lemma 2.8 that

$$\begin{aligned} g_3(n) \geq g_3(25) = \gamma + \log\frac{51}{2} + \frac{16\,471\,623}{16\,410\,368}\sqrt[8]{25}\Gamma(7/8) \\ - \frac{27\,934\,813}{5\,128\,240} = 0.00413\cdots > 0 \end{aligned} \tag{2.33}$$

for $n \geq 25$.

It follows from (2.2), Lemma 2.5, Lemma 2.7, (2.33) and the numerical computation result

$$d_{25} = \frac{19\,971\,006\,130\,817\,673\,428\,973\,665\,371}{158\,456\,325\,028\,528\,675\,187\,087\,900\,672} > 0$$

that

$$D_n = \frac{8}{7}d_{n+1} - d_n > 0 \tag{2.34}$$

for $n \geq 25$.

Therefore, Lemma 2.9 follows easily from (2.34). □

Lemma 2.10. *Let $r \in (0,1)$, $I_1(r)$, $I_2(r)$, $I_3(r)$, $I_4(r)$ and $I_5(r)$ be respectively defined by*

$$I_1(r) = M_{3/2}(1,r), \quad I_2(r) = \frac{23A(1,r) - 5H(1,r) - 2M_2(1,r)}{16}, \tag{2.35}$$

$$I_3(r) = \frac{(9r^2+14r+9)^2}{128(1+r)^3}, \quad I_4(r) = \frac{1+r+r^2}{2(1+r)} + \frac{1+r}{8} \tag{2.36}$$

and

$$I_5(r) = S_{11/4,7/4}(1,r). \tag{2.37}$$

Then the inequalities

$$I_5(r) > I_3(r) > I_2(r) > I_1(r) > I_4(r)$$

hold for all $r \in (0,1)$.

Proof. It follows from (1.8) and (2.35)–(2.37) that

$$\begin{aligned} I_5(r) - I_3(r) &= \frac{7\left(1-r^{11/4}\right)}{11\left(1-r^{7/4}\right)} - \frac{\left(9r^2+14r+9\right)^2}{128\left(1+r\right)^3} \\ &= \frac{\left(1-r^{1/4}\right)^5}{1408\left(1-r^{7/4}\right)(1+r)^3} h(r), \end{aligned} \tag{2.38}$$

where

$$\begin{aligned} h(r) = {} & 5r^4 + 35r^{15/4} + 140r^{7/2} + 420r^{13/4} + 966r^3 + 1722r^{11/4} \\ & + 2268r^{5/2} + 2415r^{9/4} + 2362r^2 + 2415r^{7/4} + 2268r^{3/2} \\ & + 1722r^{5/4} + 966r + 420r^{3/4} + 140r^{1/2} + 35r^{1/4} + 5, \end{aligned} \tag{2.39}$$

$$I_3(r) - I_2(r) = \frac{\sqrt{2(1+r^2)}}{16} - \frac{11r^4+36r^3+34r^2+36r+11}{128(1+r)^3}, \tag{2.40}$$

$$\left[\frac{\sqrt{2(1+r^2)}}{16}\right]^2 - \left[\frac{11r^4+36r^3+34r^2+36r+11}{128(1+r)^3}\right]^2 = \frac{(1-r)^6\left(7r^2+18r+7\right)}{16\,384(1+r)^6}, \tag{2.41}$$

$$I_1^3(r) - I_4^3(r) = \frac{\left(1-r^{1/2}\right)^6\left(3r^3+18r^{5/2}-3r^2+28r^{3/2}-3r+18r^{1/2}+3\right)}{512(1+r)^3}. \tag{2.42}$$

Therefore, $I_5(r) > I_3(r)$ for all $r \in (0,1)$ follows from (2.38) and (2.39), $I_3(r) > I_2(r)$ for all $r \in (0,1)$ follows from (2.40) and (2.41) and $I_1(r) > I_4(r)$ for all $r \in (0,1)$ follows from (2.42). While the inequality $I_2(r) > I_1(r)$ for all $r \in (0,1)$ follows from [18, Lemma 3.2]. □

3. Main results

Theorem 3.1. *The double inequality*

$$\lambda S_{11/4,7/4}(1,r') < \mathcal{E}(r) < \mu S_{11/4,7/4}(1,r')$$

holds for all $r \in (0,1)$ if and only if $\lambda \le \pi/2 = 1.570796\cdots$ and $\mu \ge 11/7 = 1.571428\cdots$.

Proof. Let $r \in (0,1)$, a_n, b_n, c_n and d_n be respectively defined by (2.1) and (2.2), and $G(r)$, $f_1(r)$, $f_2(r)$, $f_3(r)$, $f_4(r)$, $f_5(r)$, $f_6(r)$ and $f_7(r)$ be respectively defined by

$$G(r) = \frac{1 - \frac{2}{\pi}\mathcal{E}(r)}{1 - S_{11/4,7/4}(1, r')}, \tag{3.1}$$

$$f_1(r) = 11\left(1 - r'^{7/4}\right)\left(1 - \frac{2}{\pi}\mathcal{E}(r)\right),$$

$$f_2(r) = 11\left(1 - r'^{7/4}\right) - 7\left(1 - r'^{11/4}\right),$$

$$f_3(r) = 1 - \frac{2}{\pi}\mathcal{E}(r) + \frac{8\left(r'^{1/4} - r'^2\right)(\mathcal{K}(r) - \mathcal{E}(r))}{7\pi r^2}, \quad f_4(r) = 1 - r',$$

$$f_5(r) = 32\mathcal{K}(r) - 32\mathcal{E}(r) - 14r^2\mathcal{K}(r) - 3r^4\mathcal{K}(r) - 2r^2\mathcal{E}(r), \tag{3.2}$$

$$f_6(r) = \frac{1}{4}\left[128\mathcal{K}(r) - 128\mathcal{E}(r) - 224r^2\mathcal{K}(r) + 93r^4\mathcal{K}(r) + 160r^2\mathcal{E}(r) - 21r^4\mathcal{E}(r)\right] \tag{3.3}$$

and

$$f_7(r) = f_5(r) - r'^{-7/4} f_6(r). \tag{3.4}$$

Then simple computations lead to

$$G(r) = \frac{f_1(r)}{f_2(r)}, \quad f_1(0^+) = f_2(0^+) = 0, \tag{3.5}$$

$$\frac{f_1'(r)}{f_2'(r)} = \frac{f_3(r)}{f_4(r)}, \quad f_3(0^+) = f_4(0^+) = 0, \tag{3.6}$$

$$\left[\frac{f_3'(r)}{f_4'(r)}\right]' = -\frac{2f_7(r)}{7\pi r^5 r'}. \tag{3.7}$$

It follows from (1.2), (1.3), (3.2) and (3.3) that

$$\begin{aligned}
f_5(r) &= 16\pi \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} r^{2n} - 16\pi \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^2} r^{2n} \\
&\quad - 7\pi \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_{n-1}^2}{[(n-1)!]^2} r^{2n} - \frac{3\pi}{2} \sum_{n=2}^{\infty} \frac{\left(\frac{1}{2}\right)_{n-2}^2}{[(n-2)!]^2} r^{2n} - \pi \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)_{n-1} \left(\frac{1}{2}\right)_{n-1}}{[(n-1)!]^2} r^{2n} \\
&= \frac{3\pi}{2} \sum_{n=3}^{\infty} \frac{n(5n-6)(n-1)(n-2)\left(\frac{1}{2}\right)_{n-2}^2}{(n!)^2} r^{2n} = \frac{3\pi}{2} r^6 \sum_{n=0}^{\infty} a_n r^{2n},
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
f_6(r) &= 16\pi \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^2}{(n!)^2} r^{2n} - 16\pi \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)_n \left(\frac{1}{2}\right)_n}{(n!)^2} r^{2n} - 28\pi \sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_{n-1}^2}{[(n-1)!]^2} r^{2n} \\
&\quad + \frac{93\pi}{8} \sum_{n=2}^{\infty} \frac{\left(\frac{1}{2}\right)_{n-2}^2}{[(n-2)!]^2} r^{2n} + 20\pi \sum_{n=1}^{\infty} \frac{\left(-\frac{1}{2}\right)_{n-1} \left(\frac{1}{2}\right)_{n-1}}{[(n-1)!]^2} r^{2n} - \frac{21\pi}{8} \sum_{n=2}^{\infty} \frac{\left(-\frac{1}{2}\right)_{n-2} \left(\frac{1}{2}\right)_{n-2}}{[(n-2)!]^2} r^{2n} \\
&= -\frac{3\pi}{16} \sum_{n=3}^{\infty} \frac{n(n-1)(n-2)(n-4)(n+15)(2n-5)\left(\frac{1}{2}\right)_{n-3}^2}{(n!)^2} r^{2n} = -\frac{3\pi}{16} r^6 \sum_{n=0}^{\infty} c_n r^{2n}.
\end{aligned} \tag{3.9}$$

Note that

$$r'^{-7/4} = (1-r^2)^{-7/4} = \sum_{n=0}^{\infty} \frac{\left(\frac{7}{8}\right)_n}{n!} r^{2n} = \sum_{n=0}^{\infty} b_n r^{2n}. \tag{3.10}$$

Making use of (3.4) and (3.8)–(3.10) together with Cauchy product we get

$$\begin{aligned} f_7(r) &= \frac{3\pi}{2} r^6 \sum_{n=0}^{\infty} a_n r^{2n} + \frac{3\pi}{16} r^6 \left(\sum_{n=0}^{\infty} c_n r^{2n}\right)\left(\sum_{n=0}^{\infty} b_n r^{2n}\right) \\ &= \frac{3\pi}{16} r^6 \sum_{n=0}^{\infty} \left(8a_n + \sum_{k=0}^{n} b_{n-k} c_k\right) r^{2n} = \frac{3\pi}{16} r^6 \sum_{n=0}^{\infty} d_n r^{2n}. \end{aligned} \tag{3.11}$$

Numerical computations show that

$$d_0 = d_1 = d_2 = d_3 = 0, \quad d_4 = \frac{35}{32\,768}, \quad d_5 = \frac{903}{262\,114}, \quad d_6 = \frac{7343}{1\,048\,576}, \tag{3.12}$$

$$d_7 = \frac{193\,225}{16\,777\,216}, \quad d_8 = \frac{36\,001\,035}{2\,147\,483\,648}, \quad d_9 = \frac{387\,471\,275}{17\,179\,869\,184}, \tag{3.13}$$

$$d_{10} = \frac{7\,897\,834\,945}{274\,877\,906\,944}, \quad d_{11} = \frac{4\,834\,967\,865}{137\,438\,953\,472}, \quad d_{12} = \frac{2\,941\,501\,079\,335}{70\,368\,744\,177\,664}, \tag{3.14}$$

$$d_{13} = \frac{13\,659\,196\,049\,415}{281\,474\,976\,710\,656}, \quad d_{14} = \frac{15\,566\,153\,225\,985}{281\,474\,976\,710\,656}, \tag{3.15}$$

$$d_{15} = \frac{2\,236\,811\,742\,882\,381}{36\,028\,797\,018\,963\,968}, \quad d_{16} = \frac{634\,947\,066\,920\,509\,899}{9\,223\,372\,036\,854\,775\,808}, \tag{3.16}$$

$$d_{17} = \frac{5\,574\,552\,606\,398\,358\,765}{73\,786\,976\,294\,838\,206\,464}, \quad d_{18} = \frac{97\,033\,294\,160\,407\,536\,645}{1\,180\,591\,620\,717\,411\,303\,424}, \tag{3.17}$$

$$d_{19} = \frac{52\,389\,327\,348\,799\,215\,285}{590\,295\,810\,358\,705\,651\,712}, \quad d_{20} = \frac{28\,778\,816\,487\,383\,824\,672\,713}{302\,231\,454\,903\,657\,293\,676\,544}, \tag{3.18}$$

$$d_{21} = \frac{122\,819\,305\,110\,073\,147\,526\,863}{1\,208\,925\,819\,614\,629\,174\,706\,176}, \quad d_{22} = \frac{4\,172\,785\,484\,991\,954\,272\,870\,785}{38\,685\,626\,227\,668\,133\,590\,597\,632}, \tag{3.19}$$

$$d_{23} = \frac{17\,645\,073\,967\,067\,536\,659\,569\,835}{154\,742\,504\,910\,672\,534\,362\,390\,528}, \quad d_{24} = \frac{2\,378\,544\,708\,096\,739\,288\,834\,293\,475}{19\,807\,040\,628\,566\,084\,398\,385\,987\,584}. \tag{3.20}$$

From (3.7) and (3.11)–(3.20) together with Lemma 2.9 we clearly see that the function $f_3'(r)/f_4'(r)$ is strictly decreasing on $(0,1)$. Then (3.5), (3.6) and Lemma 2.1 lead to the conclusion that $G(r)$ is strictly decreasing on $(0,1)$.

From (1.1), (1.3), (1.8) and (3.1) we have

$$\lim_{r\to 1^-} G(r) = \frac{11(\pi-2)}{4\pi}, \quad \lim_{r\to 0^+} G(r) = 1. \tag{3.21}$$

It follows from (3.1) and (3.21) together the monotonicity of $G(r)$ on the interval $(0,1)$ that

$$S_{11/4,7/4}(1,r') < \frac{2}{\pi}\mathcal{E}(r) < \frac{22-7\pi}{4\pi} + \frac{11(\pi-2)}{4\pi} S_{11/4,7/4}(1,r') \tag{3.22}$$

for all $r \in (0,1)$.

Note that

$$\begin{aligned}&\frac{22-7\pi}{4\pi}+\frac{11(\pi-2)}{4\pi}S_{11/4,7/4}(1,r')-\frac{22}{7\pi}S_{11/4,7/4}(1,r')\\&=-\frac{11(22-7\pi)}{28\pi}\left[S_{11/4,7/4}(1,r')-\frac{7}{11}\right]<0\end{aligned}\tag{3.23}$$

for $r\in(0,1)$ due to $7\pi<22$ and the Stolarsky mean $S_{p,q}(a,b)$ is strictly increasing with respect to its variables a and b.

Inequalities (3.22) and (3.23) lead to

$$\frac{\pi}{2}S_{11/4,7/4}(1,r')<\mathcal{E}(r)<\frac{11}{7}S_{11/4,7/4}(1,r')\tag{3.24}$$

for all $r\in(0,1)$.

Therefore, Theorem 3.1 follows from (3.24) and the fact that

$$\lim_{r\to0^+}\frac{\mathcal{E}(r)}{S_{11/4,7/4}(1,r')}=\frac{\pi}{2},\quad \lim_{r\to1^-}\frac{\mathcal{E}(r)}{S_{11/4,7/4}(1,r')}=\frac{11}{7}.\qquad\square$$

From Theorem 3.1 we get Corollary 3.2 immediately.

Corollary 3.2. *The double inequality*

$$\pi<\frac{2\mathcal{E}(r)}{S_{11/4,7/4}(1,r')}<\frac{22}{7}$$

holds for all $r\in(0,1)$.

Corollary 3.3. *The inequality*

$$\mathcal{E}(r)<\frac{\pi}{2}S_{11/4,7/4}(1,r')+\frac{22-7\pi}{22}$$

holds for all $r\in(0,1)$.

Proof. Let $r\in(0,1)$, $G(r)$ be defined by (3.1) and $H(r)$ be defined by

$$H(r)=\frac{2}{\pi}\mathcal{E}(r)-S_{11/4,7/4}(1,r').\tag{3.25}$$

Then we clearly see that

$$H(r)=[1-G(r)]\left[1-S_{11/4,7/4}(1,r')\right].\tag{3.26}$$

From the proof of Theorem 3.1 we know that $G(r)$ is strictly decreasing on $(0,1)$. Then (3.26) leads to the conclusion that $H(r)$ is strictly increasing on $(0,1)$. Note that

$$H(1^-)=\frac{2}{\pi}-\frac{7}{11}.\tag{3.27}$$

Therefore, Corollary 3.3 follows from (3.25) and (3.27) together with the monotonicity of $H(r)$. $\square$

Remark 3.4. For all $r \in (0,1)$, from Lemma 2.10 we clearly see that the lower bound $\pi S_{11/4,7/4}(1,r')/2$ for $\mathcal{E}(r)$ given in Theorem 3.1 is better than all the lower bounds given in (1.4)–(1.7).

Remark 3.5. Let $I_i(r)$ $(i = 1,2,3,4,5)$ be defined by (2.35)–(2.37), $\triangle_i(r) = 2\mathcal{E}(r)/\pi - I_i(r')$ and $r \to 0^+$. Then making use of power series formulas we get

$$\triangle_1(r) = \frac{1}{2^{14}} r^8 + o\left(r^8\right), \quad \triangle_2(r) = \frac{3}{2^{20}} r^{12} + o\left(r^{12}\right),$$
$$\triangle_3(r) = \frac{1}{2^{20}} r^{12} + o\left(r^{12}\right), \quad \triangle_4(r) = \frac{263}{2^{16}} r^8 + o\left(r^8\right), \quad \triangle_5(r) = \frac{1}{7 \times 2^{21}} r^{12} + o\left(r^{12}\right).$$

Remark 3.6. Let $\triangle_i(r)$ $(i = 1,2,3,4,5)$ be defined as in Remark 3.5. Then (1.4)–(1.7) and Corollary 3.3 lead to

$$\sup_{r\in(0,1)} \triangle_1(r) \geq \triangle_1(1^-) = \frac{2}{\pi} - 2^{-2/3} = 0.00665924\cdots,$$
$$\sup_{r\in(0,1)} \triangle_2(r) \geq \triangle_2(1^-) = \frac{2}{\pi} - \frac{23 - 2\sqrt{2}}{32} = 0.00625812\cdots,$$
$$\sup_{r\in(0,1)} \triangle_3(r) \geq \triangle_3(1^-) = \frac{2}{\pi} - \frac{81}{128} = 0.00380727\cdots,$$
$$\sup_{r\in(0,1)} \triangle_4(r) \geq \triangle_4(1^-) = \frac{2}{\pi} - \frac{5}{8} = 0.01161977\cdots,$$
$$\sup_{r\in(0,1)} \triangle_5(r) \leq \frac{2}{\pi} - \frac{7}{11} = 0.00025613\cdots.$$

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