Weighted weak formulation for a nonlinear degenerate parabolic equation arising in chemotaxis or porous media

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A R T I C L E   I N F O

A B S T R A C T

This paper is devoted to the mathematical analysis of a degenerate nonlinear parabolic equation. This kind of equations stems either from the modeling of a compressible two phase flow in porous media or from the modeling of a chemotaxis-fluid process. In the degenerate equation, the strong nonlinearities are technically difficult to be controlled by the degenerate dissipative term because the equation itself presents degenerate terms of order 0 and of order 1. In the case of the sub-quadratic degeneracy of the dissipative term at one point, a weak and classical formulation is possible for the expected solutions. However, in the case of the degeneracy of the dissipative term at two points, we obtain solutions in a weaker sense compared to the one of the classical formulation. Therefore, a degenerate weighted formulation is introduced taking into account the degeneracy of the dissipative term.

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1. Introduction and the nonlinear degenerate model

Let $T > 0$ be a fixed time and Ω be an open bounded subset of $\mathbb{R}^d$, $d = 2, 3$. We set $Q_T := \Omega \times (0, T)$ and $\Sigma_T = \partial\Omega \times (0, T)$. We consider the following nonlinear degenerate parabolic equation

$$\partial_t u - \operatorname{div}\left(a(u)\nabla u - f(u)\mathbf{V}\right) - g(u)\operatorname{div}(\mathbf{V}) + a(u)\nabla u \cdot \tilde{\mathbf{V}} = 0, \quad \text{in } Q_T. \tag{1}$$

The boundary condition is defined by

$$u(\mathbf{x}, t) = 0, \quad \text{on } \Sigma_T. \tag{2}$$

The initial condition is given by

$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \quad \text{in } \Omega. \tag{3}$$

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Models for chemotaxis lead to such kind of degenerate nonlinear parabolic equation (1), where u represents the cell density and $\mathbf{V}$ represents the gradient of the chemical concentration (see e.g. [13,1,8,16,7,17]), and in the case of swimming bacteria, $\tilde{\mathbf{V}}$ represents the velocity of the fluid which transports the cell density and the chemical concentration; in [16,7,17] the authors consider $\tilde{\mathbf{V}}$ as the Navier–Stokes velocity. In the chemotaxis modeling, the functions a and f represent respectively the diffusivity of the cells and the chemosensitivity of the cells to the chemicals. In the specific model in [13], the authors consider the case where the function a degenerates at one side and consider also a relationship between the degeneracy of the functions a and f to establish the existence and uniqueness of weak solutions. Here, we treat the case of two-sidedly degenerate diffusion terms and consider a general model.

Many physical models lead also to degenerate nonlinear parabolic problem. For instance, in [10] the authors analyzed a model of a degenerate nonlinear system arising from compressible two-phase flows in porous media. The described system coupled the saturation (denoted by u) and the global pressure (denoted by p). The global velocity (denoted by $\mathbf{V}$) is taken to be proportional to the gradient of the global pressure. In addition, the functions a and f represent respectively the capillary term and the fractional flow and the velocity $\tilde{\mathbf{V}}$ is considered to be $\tilde{\mathbf{V}} = \gamma \mathbf{V}$ where γ is a nonnegative parameter representing the compressibility factor. Several papers are devoted to the mathematical analysis of nonlinear degenerate parabolic diffusion–convection equations arising in compressible, immiscible displacement models in porous media (see e.g. [12,18]). Here, we consider a generalization of the saturation equation where we assume that the velocity field is given and fixed.

In the paper of Bresch et al. [2], the authors studied the existence of strong and weak solutions for multiphase incompressible fluids models; indeed, they consider the Kazhikhov–Smagulov system where the density equation contains a degenerate diffusion term and first order term.

In [11], the main interest is a nonlinear degenerate parabolic equation where the flux function depends explicitly on the spatial location for which they study the uniqueness and stability of entropy solutions; the studied equation do not contain first and Oth order term. The type of equation (1) arising also in sedimentation–consolidation processes [5,6,4] where the sought u is considered to be the local volume fraction of solids, many constitutive equations imply that there exists a critical number u_c such that $a(u) = 0$ for $u \leq u_c$ which corresponds to the sedimentation step and $a(u) > 0$ in the consolidation step (see eq. (42) in [5]). Consequently, partial differential equations of type (1) model a wide variety of phenomena, ranging from porous media flow, via chemotaxis model, to traffic flow [20].

Our basic requirements on system (1)–(3) are:

(H1) $a \in \mathscr{C}^1([0,1], \mathbb{R})$, $a(u) > 0$ for $0 < u < 1$, $a(0) = 0$, $a(1) = 0$.

Furthermore, there exist $r_1 > 0$, $r_2 > 0$, $m_1, M_1 > 0$, and $0 < u_* < 1$ such that

$$m_1 r_1 u^{r_1-1} \leq a'(u) \leq M_1 r_1 u^{r_1-1}, \text{ for all } 0 \leq u \leq u_*,$$

$$-r_2 M_1 (1-u)^{r_2-1} \leq a'(u) \leq -r_2 m_1 (1-u)^{r_2-1}, \text{ for all } u_* \leq u \leq 1.$$

(H2) f is a differentiable function in $[0,1]$ and $g \in \mathscr{C}^1([0,1])$ verifying

$$g(0) = f(0) = 0, \quad f(1) = g(1) = 1, \text{ and } g'(u) \geq C_{g'} > 0 \quad \forall u \in [0,1].$$

In addition, there exists $c_1, c_2 > 0$ such that $|f(u) - g(u)| \leq c_2 u$ for all $0 \leq u \leq u_*$ and $c_1 (1-u)^{-1} \leq (f(u) - g(u))^{-1} \leq c_2 (1-u)^{-1}$ for all $u_* \leq u < 1$.

(H3) The velocities $\mathbf{V}$ and $\tilde{\mathbf{V}}$ are two measurable functions lying into $(L^\infty(\Omega))^d$.

(H4) The initial condition u_0 satisfies: $0 \leq u_0(\mathbf{x}) \leq 1$ for a.e. $\mathbf{x} \in \Omega$.

A major difficulty of system (1)–(3) is the possible degeneracy of the diffusion term. In assumption (H1), we give the degeneracy assumption for the dissipation function a and we determine the behavior of this degeneracy around 0 and 1. In what follows, we introduce the existence result of weak solutions to system (1)–(3) (verifying a weighted weak formulation) under assumptions (H1)–(H4) and for a particular choice of the initial data. However, for the specific case where the dissipation function a vanishes at only one point (i.e. $a(0) = 0$ or $a(1) = 0$), we give the existence of weak solutions to system (1)–(3) in Remark 1.

In the sequel and for the simplicity, we assume that $\tilde{\mathbf{V}} = \mathbf{V}$ (the same analysis is possible for the case where $\tilde{\mathbf{V}} \neq \mathbf{V}$).

We give now the definition of weak solutions to system (1)–(3) when assumptions (H1)–(H4) are satisfied. Let $\theta, \lambda \geq 0$, then we denote by $j_{\theta,\lambda}$ the continuous function defined by

$$
j_{\theta,\lambda}(u) = \begin{cases} \beta_\theta(u) = u^{r-1+\theta}, & \text{if } 0 \leq u \leq u_* \\ \beta_\theta(u_*)(1-u_*)^{1-\frac{r'}{2}-\lambda}(1-u)^{\frac{r'}{2}-1+\lambda}, & \text{if } u_* \leq u \leq 1, \end{cases} \tag{4}
$$

where, for the fixed two constants r_1 and r_2, we have

$$
r = \begin{cases} r_1 + 2, & \text{if } r_1 \leq 1 \\ r_1, & \text{if } r_1 > 1 \end{cases} \quad \text{and} \quad r' \geq \max(2, r_2). \tag{5}
$$

We denote by $J_{\theta,\lambda}$ the primitive of the function $j_{\theta,\lambda}$, that is

$$
J_{\theta,\lambda} = \int_0^u j_{\theta,\lambda}(y)\, \mathrm{d}y. \tag{6}
$$

Finally, we denote by β, j, and J, the functions defined by

$$
\beta(u) = u^{r-1}, \quad j(u) = \begin{cases} u^{r-1} & \text{if } 0 \leq u \leq u_* \\ c_*(1-u)^{\frac{r'}{2}-1} & \text{if } u_* \leq u \leq 1, \end{cases} \quad J(u) = \int_0^u j(y)\, \mathrm{d}y, \tag{7}
$$

where $c_* = u_*^{r-1}(1-u_*)^{1-\frac{r'}{2}}$. In addition, we consider the continuous functions μ and G, defined by

$$
\begin{cases} \mu(u) = \beta(u), & 0 \leq u \leq u_* \\ \mu'(u) = (f(u) - g(u))^{-1} g'(u) \mu(u), & u_* \leq u < 1. \end{cases} \tag{8}
$$

G is the primitive of μ, that is $G(u) = \int_0^u \mu(y)\, \mathrm{d}y$.

Definition 1.1. For $\theta \geq 7r_1 + 6 - r$, $\lambda \geq 7r_2 + 6 - \frac{r'}{2}$, and under assumptions (H1)–(H4) and assuming that $G(u_0) \in L^1(\Omega)$, we say that u is a degenerate weak solution of system (1)–(3) if

$$
0 \leq u(\mathbf{x}, t) \leq 1 \text{ for a.e. } (\mathbf{x}, t) \in Q_T = \Omega \times (0, T),
$$

$$
J(u) \in L^2(0, T; H_0^1(\Omega)), \quad \sqrt{a(u)\mu'(u)}\nabla u \in (L^2(Q_T))^d,
$$

and such that, the function F defined by

$$
F(u, \chi) = -\int_{Q_T} J_{\theta,\lambda}(u)\, \partial_t \chi\, \mathrm{d}\mathbf{x}\, \mathrm{d}t - \int_\Omega J_{\theta,\lambda}(u_0(\mathbf{x}))\, \chi(\mathbf{x}, 0)\, \mathrm{d}\mathbf{x} + \int_{Q_T} a(u)\, \nabla u \cdot \nabla(j_{\theta,\lambda}(u)\chi)\, \mathrm{d}\mathbf{x}\, \mathrm{d}t
$$

$$+ \int\limits_{Q_T} a\left(u\right) \mathbf{V} \cdot \nabla u\, j_{\theta,\lambda}\left(u\right) \chi\, \mathrm{d}\mathbf{x}\, \mathrm{d}t - \int\limits_{Q_T} \left(f\left(u\right) - g\left(u\right)\right) \mathbf{V} \cdot \nabla\left(j_{\theta,\lambda}\left(u\right) \chi\right) \mathrm{d}\mathbf{x}\, \mathrm{d}t$$

$$+ \int\limits_{Q_T} g'\left(u\right) \mathbf{V} \cdot \nabla u\, j_{\theta,\lambda}\left(u\right) \chi\, \mathrm{d}\mathbf{x}\, \mathrm{d}t - \int\limits_{Q_T} \left(f\left(u\right) - g\left(u\right)\right) \mathbf{V} \cdot \nabla\chi\, j_{\theta,\lambda}\left(u\right) \mathrm{d}\mathbf{x}\, \mathrm{d}t,$$

verifies

$$F\left(u, \chi\right) \le 0, \quad \forall \chi \in \mathscr{C}^1\left([0, T]; H_0^1\left(\Omega\right)\right) \text{ with } \chi\left(\cdot, T\right) = 0 \text{ and } \chi \ge 0, \tag{9}$$

and furthermore,

$$\forall \varepsilon > 0, \exists Q^\varepsilon \subset Q_T \text{ such that } \operatorname{meas}\left(Q^\varepsilon\right) < \varepsilon, \text{ and}$$
$$F\left(u, \chi\right) = 0, \quad \forall \chi \in \mathscr{C}^1\left([0, T]; H_0^1\left(\Omega\right)\right), \operatorname{supp}\chi \subset \left([0, T) \times \Omega\right) \setminus Q^\varepsilon. \tag{10}$$

Theorem 1.1. *Under assumptions* (H1)–(H4), *there exists at least one degenerate weak solution to system* (1)–(3) *in the sense of Definition 1.1.*

Remark 1 *(Classical weak solutions).* Consider the specific case where $a\left(u\right) \approx \left(1 - u\right)^{r_2}$, $0 < r_2 < 2$. Then, a weak solution of system (1)–(3) can be characterized by a classical weak solution verifying

$$0 \le u\left(\mathbf{x}, t\right) \le 1 \text{ a.e. in } Q_T, \quad u \in L^2\left(0, T; H_0^1\left(\Omega\right)\right) \quad \partial_t u \in L^2\left(0, T; H^{-1}\left(\Omega\right)\right),$$

and such that, for all $\varphi \in L^2\left(0, T; H_0^1\left(\Omega\right)\right)$

$$\int\limits_0^T \langle \partial_t u, \varphi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\, \mathrm{d}t + \int\limits_{Q_T} a\left(u\right) \nabla u \cdot \nabla\varphi\, \mathrm{d}\mathbf{x}\, \mathrm{d}t - \int\limits_{Q_T} f\left(u\right) \mathbf{V} \cdot \nabla\varphi\, \mathrm{d}\mathbf{x}\, \mathrm{d}t$$

$$+ \int\limits_{Q_T} g\left(u\right) \mathbf{V} \cdot \nabla\varphi\, \mathrm{d}\mathbf{x}\, \mathrm{d}t + \int\limits_{Q_T} g'\left(u\right) \mathbf{V} \cdot \nabla u\varphi\, \mathrm{d}\mathbf{x}\, \mathrm{d}t + \int\limits_{Q_T} a\left(u\right) \mathbf{V} \cdot \nabla u\varphi\, \mathrm{d}\mathbf{x}\, \mathrm{d}t = 0.$$

Unless stated otherwise, C represents a "generic" nonnegative quantity which need not have the same value through the proofs. Furthermore, C_α represents a nonnegative constant depending only on the subscript α.

In what follows, we give an essential compactness result for degenerate problems, then we introduce and give the existence of at least one solution for the nondegenerate problem associated with the degenerate system (1)–(3). Finally, we prove the existence of solutions for the degenerate problem, and show that they verify a weighted weak formulation.

2. Compactness result

Classical compactness results [19,14,15] for nondegenerate problems cannot be applied in a straightforward way for degenerate problems and they should be adapted to the nature of the degeneracy. Here, we introduce a general preliminary compactness result on the gradient of a degenerate function.

Lemma 2.1. *Consider the differentiable function* $a : [0, 1] \mapsto \mathbb{R}^+$ *satisfying:*

- $a(0) = 0.$

- *there exist $r_1 > 0$, m_1 and $M_1 > 0$ such that:*

$$m_1 r_1 u^{r_1 - 1} \leq a'(u) \leq M_1 r_1 u^{r_1 - 1}, \ \text{for all } 0 \leq u \leq 1.$$

Let us denote by A, B, and b the continuous functions defined by

$$A(u) = \int_0^u a(\tau) \, \mathrm{d}\tau, \quad B(u) = A^2(u), \quad b(u) = B'(u) = 2A(u) a(u). \tag{11}$$

We denote finally by $A_{\eta, \eta'} = A(u_\eta) - A(u_{\eta'})$ and $B_{\eta, \eta'} = B(u_\eta) - B(u_{\eta'})$, for all $\eta, \eta' \in \mathbb{N}$.

For every $\mu > 0$, define the truncation function T_μ by

$$T_\mu(u) = \min(\mu, \max(-\mu, u)), \quad \forall u \in \mathbb{R}. \tag{12}$$

Consider a sequence $(u_\eta)_\eta$ satisfying

(A1) $0 \leq u_\eta \leq 1$ *almost everywhere in Q_T.*

(A2) $(u_\eta)_\eta$ *is strongly convergent in $L^2(Q_T)$.*

(A3) $(a(u_\eta) \nabla u_\eta)_\eta$ *is bounded in $(L^2(Q_T))^d$.*

(A4) $\displaystyle \int_{Q_T} \nabla A_{\eta, \eta'} \cdot \nabla(b(u_\eta) T_\mu(B_{\eta, \eta'})) \mathrm{d}\mathbf{x} \, \mathrm{d}t \longrightarrow 0$, *as $\mu, \eta, \eta' \to 0$.*

Then, the sequence $\left(u_\eta^q \, a(u_\eta) \nabla u_\eta\right)_\eta$ is a Cauchy sequence in measure where $q = 3r_1 + 2$.

Proof. We want to show that the sequence $\left(u_\eta^q \, a(u_\eta) \nabla u_\eta\right)_\eta$ is a Cauchy sequence in measure; this yields, up to extract a subsequence, that $u_\eta^q \, a(u_\eta) \nabla u_\eta \longrightarrow u^q a(u) \nabla u$ for almost everywhere $(\mathbf{x}, t)$ in $\Omega \times (0, T)$.

To do this, it suffices to prove that, for the two sequences $(u_\eta)_\eta$ and $(u_{\eta'})_{\eta'}$ we have

$$\mathrm{meas} \left\{ \left| u_\eta^q \nabla A(u_\eta) - u_{\eta'}^q \nabla A(u_{\eta'}) \right| \geq \delta \right\} \leq \varepsilon, \quad \forall \varepsilon > 0. \tag{13}$$

First, remark that the sequences

$$(\nabla A(u_\eta))_\eta, (\nabla B(u_\eta))_\eta, (\nabla b(u_\eta))_\eta \ \text{are uniformly bounded in } \left(L^2(Q_T)\right)^d. \tag{14}$$

Indeed, we have the following estimates

$$\|\nabla A(u_\eta)\|^2_{(L^2(Q_T))^d} = \|a(u_\eta) \nabla u_\eta\|^2_{(L^2(Q_T))^d} \leq C,$$

$$\|\nabla B(u_\eta)\|^2_{(L^2(Q_T))^d} = \|2A(u_\eta) \nabla A(u_\eta)\|^2_{(L^2(Q_T))^d} \leq (2M_1)^2 \|\nabla A(u_\eta)\|^2_{(L^2(Q_T))^d}.$$

We have, from the definition of b, that $\nabla b(u_\eta) = 2a(u_\eta) \nabla A(u_\eta) + 2a'(u_\eta) A(u_\eta) \nabla u_\eta$. One can get the result using the following statement

$$|a'(u_\eta) A(u_\eta)| \leq M_1^2 \frac{r_1}{r_1 + 1} u_\eta^{r_1 - 1} u_\eta^{r_1 + 1} \leq M_1^2 u_\eta^{2r_1} \leq \frac{M_1^2}{m_1} u_\eta^{r_1} a(u_\eta) \leq \frac{M_1^2}{m_1} a(u_\eta). \tag{15}$$

Now, let s be the continuous function defined by

$$s(u) = \int_0^u b(z) A(z) a(z) \, \mathrm{d}z, \quad \forall u \in \mathbb{R}. \tag{16}$$

Let us prove that the sequence $(\nabla s(u_\eta))_\eta$ is a Cauchy sequence in measure, that is

$$\text{meas}\left\{|\nabla s\left(u_\eta\right) - \nabla s\left(u_{\eta'}\right)| \geq \delta\right\} \xrightarrow[\eta, \eta' \to 0]{} 0.$$

Remark that $\left\{|\nabla s\left(u_\eta\right) - \nabla s\left(u_{\eta'}\right)| \geq \delta\right\} \subset \mathscr{A}_1 \cap \mathscr{A}_2 \cap \mathscr{A}_3 \cap \mathscr{A}_4$, where

$$\mathscr{A}_1 = \left\{|\nabla A\left(u_\eta\right)| \geq k\right\}, \quad \mathscr{A}_2 = \left\{|\nabla A\left(u_{\eta'}\right)| \geq k\right\}, \quad \mathscr{A}_3 = \left\{|B_{\eta, \eta'}| \geq \mu\right\},$$

$$\mathscr{A}_4 = \left\{|\nabla s\left(u_\eta\right) - \nabla s\left(u_{\eta'}\right)| \geq \delta\right\} \cap \left\{|\nabla A\left(u_\eta\right)| \leq k\right\} \cap \left\{|\nabla A\left(u_{\eta'}\right)| \leq k\right\} \cap \left\{|B_{\eta, \eta'}| \leq \mu\right\}.$$

Thanks to statement (14), and to the continuous embedding of $L^2\left(Q_T\right)$ into $L^1\left(Q_T\right)$, we have

$$k \, \text{meas}\left(\mathscr{A}_1\right) \leq \int_{\mathscr{A}_1} |\nabla A\left(u_\eta\right)| \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \leq C.$$

An analogous estimate holds for $\mathscr{A}_2$. Therefore, by choosing k large enough, one gets $\text{meas}\left(\mathscr{A}_1\right) + \text{meas}\left(\mathscr{A}_2\right)$ is arbitrarily small. In the same manner, one gets

$$\text{meas}\left(\mathscr{A}_3\right) \leq \frac{1}{\mu} \left\|B_{\eta, \eta'}\right\|_{L^1(Q_T)},$$

which, for a fixed $\mu > 0$, tends to zero as $\eta, \eta' \to 0$.

It remains to show that $\text{meas}\left(\mathscr{A}_4\right)$ is small enough. Indeed, we have

$$|\nabla s\left(u_\eta\right) - \nabla s\left(u_{\eta'}\right)|^2 = |b\left(u_\eta\right) A\left(u_\eta\right) \nabla A\left(u_\eta\right) - b\left(u_{\eta'}\right) A\left(u_{\eta'}\right) \nabla A\left(u_{\eta'}\right)|^2$$

$$= |b\left(u_\eta\right) A\left(u_\eta\right) \nabla A_{\eta, \eta'} + \left(b\left(u_\eta\right) A\left(u_\eta\right) - b\left(u_{\eta'}\right) A\left(u_{\eta'}\right)\right) \nabla A\left(u_{\eta'}\right)|^2,$$

and therefore, one gets

$$\delta \, \text{meas}\left(\mathscr{A}_4\right) \leq \int_{\mathscr{A}_4} |\nabla s\left(u_\eta\right) - \nabla s\left(u_{\eta'}\right)|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \leq 2 \int_{\mathscr{A}_4} |b\left(u_\eta\right) A\left(u_\eta\right) \nabla A_{\eta, \eta'}|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t$$

$$+ 2 \int_{\mathscr{A}_4} |b\left(u_\eta\right) A\left(u_\eta\right) - b\left(u_{\eta'}\right) A\left(u_{\eta'}\right)|^2 |\nabla A\left(u_{\eta'}\right)|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t$$

$$\leq 4M_1^3 \int_{\mathscr{A}_4} b\left(u_\eta\right) \left(A\left(u_\eta\right) + A\left(u_{\eta'}\right)\right) \nabla A_{\eta, \eta'} \cdot \nabla A_{\eta, \eta'} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t$$

$$+ 2k^2 \int_{Q_T} |b\left(u_\eta\right) A\left(u_\eta\right) - b\left(u_{\eta'}\right) A\left(u_{\eta'}\right)|^2 \, \mathrm{d}\mathbf{x} \, \mathrm{d}t.$$

The parameter k is chosen to be fixed and large enough; then the last term that we denote $W_k\left(\eta, \eta'\right)$ goes to zero as $\eta, \eta' \to 0$. Consequently,

$$\delta \, \text{meas}\left(\mathscr{A}_4\right) \leq W_k\left(\eta, \eta'\right) + 4M_1^3 \int_{Q_T} b\left(u_\eta\right) \left(A\left(u_\eta\right) + A\left(u_{\eta'}\right)\right) \nabla A_{\eta, \eta'} \cdot \nabla A_{\eta, \eta'} \mathbf{1}_{\{|B_{\eta, \eta'}| \leq \mu\}} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t. \qquad (17)$$

We want to show that the second term on the right-hand side of inequality (17) is small enough. For that, we compute

$$\nabla A_{\eta,\eta'} \cdot \nabla \left(b(u_\eta) T_\mu(B_{\eta,\eta'}) \right) = b\left(u_\eta \right) \nabla A_{\eta,\eta'} \cdot \nabla T_\mu\left(B_{\eta,\eta'} \right) + \nabla A_{\eta,\eta'} \cdot \nabla b\left(u_{\eta'} \right) T_\mu\left(B_{\eta,\eta'} \right)$$

$$= b\left(u_\eta \right) \left(A\left(u_\eta \right) + A\left(u_{\eta'} \right) \right) \nabla A_{\eta,\eta'} \cdot \nabla A_{\eta,\eta'} \mathbb{1}_{\{|B_{\eta,\eta'}| \le \mu\}} \tag{18}$$

$$+ \left(b\left(u_\eta \right) A_{\eta,\eta'} \nabla \left(A\left(u_\eta \right) + A\left(u_{\eta'} \right) \right) \mathbb{1}_{\{|B_{\eta,\eta'}| \le \mu\}} + \nabla b\left(u_{\eta'} \right) T_\mu\left(B_{\eta,\eta'} \right) \right) \cdot \nabla A_{\eta,\eta'}.$$

Using the fact that $|T_\mu(\cdot)| \le \mu$, and thanks to estimate (14) and to the Cauchy–Schwarz inequality, we get the following estimate

$$\left\| \nabla b\left(u_{\eta'} \right) T_\mu\left(B_{\eta,\eta'} \right) \cdot \nabla A_{\eta,\eta'} \right\|_{(L^1(Q_T))^d} \le C\mu,$$

and since $|A_{\eta,\eta'}| \le C\mu$, where $|B_{\eta,\eta'}| \le \mu$, one deduces that

$$\left\| b\left(u_\eta \right) \nabla A_{\eta,\eta'} \cdot \nabla \left(A\left(u_\eta \right) + A\left(u_{\eta'} \right) \right) A_{\eta,\eta'} \mathbb{1}_{\{|B_{\eta,\eta'}| \le \mu\}} \right\|_{(L^1(Q_T))^d} \le C\mu,$$

where C is a generic constant independent of η and η'. Consequently, from equation (18), one has

$$\int_{Q_T} b\left(u_\eta \right) \left(A\left(u_\eta \right) + A\left(u_{\eta'} \right) \right) \nabla A_{\eta,\eta'} \cdot \nabla A_{\eta,\eta'} \mathbb{1}_{\{|B_{\eta,\eta'}| \le \mu\}} \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \le \left| \int_{Q_T} \nabla A_{\eta,\eta'} \cdot \nabla \left(b(u_\eta) T_\mu(B_{\eta,\eta'}) \right) \right| + C\mu,$$

and thanks to assumption $(A4)$, then inequality (17) gives

$$\delta \operatorname{meas} \left(\mathscr{A}_4 \right) \le W_k\left(\eta, \eta' \right) + W_\mu\left(\eta, \eta' \right) + C\mu.$$

Using the above results, one can deduce that for all $\varepsilon > 0$, for all $\delta > 0$, there exists $\eta_0 > 0$ such that for all $\eta, \eta' \le \eta_0$, we have

$$\operatorname{meas} \left\{ |\nabla s\left(u_\eta \right) - \nabla s\left(u_{\eta'} \right)| \ge \delta \right\} \le \varepsilon. \tag{19}$$

Now, we can prove statement (13) with the help of inequality (19). Indeed, we have

$$u_\eta^q \nabla A\left(u_\eta \right) - u_{\eta'}^q \nabla A\left(u_{\eta'} \right) = u_\eta^q \nabla A_{\eta,\eta'} + \left(u_\eta^q - u_{\eta'}^q \right) \nabla A\left(u_{\eta'} \right).$$

Since $q = 3r_1 + 2$, then $u_\eta^q = u_\eta^{3r_1+2} \le C_{r_1,m_1} b\left(u_\eta \right) A\left(u_\eta \right)$ where $C_{r_1,m_1} = \dfrac{(r_1 + 1)^2}{2m_1^3}$.

We write

$$\left| u_\eta^q \nabla A_{\eta,\eta'} \right| \le C_{r_1,m_1} b\left(u_\eta \right) A\left(u_\eta \right) |\nabla A_{\eta,\eta'}|$$

$$\le C_{r_1,m_1} |\nabla s\left(u_\eta \right) - \nabla s\left(u_{\eta'} \right)| + C_{r_1,m_1} |\left(b\left(u_\eta \right) A\left(u_\eta \right) - b\left(u_{\eta'} \right) A\left(u_{\eta'} \right) \right) \nabla A\left(u_{\eta'} \right)|.$$

Consequently,

$$\left| u_\eta^q \nabla A\left(u_\eta \right) - u_{\eta'}^q \nabla A\left(u_{\eta'} \right) \right| \le \left| \left(u_{\eta'}^q - u_\eta^q \right) \nabla A\left(u_{\eta'} \right) \right| + C_{r_1,m_1} |\nabla s\left(u_\eta \right) - \nabla s\left(u_{\eta'} \right)|$$

$$+ C_{r_1,m_1} |\left(b\left(u_\eta \right) A\left(u_\eta \right) - b\left(u_{\eta'} \right) A\left(u_{\eta'} \right) \right) \nabla A\left(u_{\eta'} \right)|,$$

which converges to zero as $\eta, \eta' \to 0$. The result is due either to the convergence in $L^1\left(Q_T \right)$ for the first and the last terms on the right-hand side, or either by the help of (19). This ends the proof of Lemma 2.1.

The rest of the paper is devoted to the proof of the main theorem. In the next section, we introduce a nondegenerate problem by adding an artificial diffusion operator.

3. Existence for the nondegenerate case

In this section, we prove the existence of solutions to the nondegenerate problem. To avoid the degeneracy of the dissipation function a, we introduce the modified problem where the dissipation a is replaced by $a_\eta (u) = a (u) + \eta$ in equation (1), with $0 < \eta \ll 1$ being a small parameter strictly positive.

Therefore, we consider the nondegenerate system

$$\partial_t u_\eta - \operatorname{div} \left(a_\eta \left(u_\eta \right) \nabla u_\eta - f(u_\eta) \mathbf{V} \right) - g \left(u_\eta \right) \operatorname{div} \left(\mathbf{V} \right) + a_\eta \left(u_\eta \right) \nabla u_\eta \cdot \mathbf{V} = 0, \quad \text{in } Q_T, \tag{20}$$

$$u_\eta \left(\mathbf{x}, t \right) = 0, \qquad \text{on } \Sigma_T, \tag{21}$$

$$u_\eta \left(\mathbf{x}, 0 \right) = u_0 \left(\mathbf{x} \right), \qquad \text{in } \Omega. \tag{22}$$

We will show (using the Schauder fixed-point theorem) that the nondegenerate problem (20)–(22) has at least one solution.

3.1. Weak nondegenerate solutions

For the existence of a solution to the nondegenerate system, we have the following theorem.

Theorem 3.1 *(Nondegenerate system). For any fixed $\eta > 0$ and under the assumptions* (H1)–(H4), *there exists at least one weak solution u_η to system* (20)–(22) *satisfying*

$$0 \le u_\eta \left(\mathbf{x}, t \right) \le 1 \quad \text{for a.e. } \left(\mathbf{x}, t \right) \in Q_T, \tag{23}$$

$$u_\eta \in L^2 \left(0, T; H_0^1 \left(\Omega \right) \right), \quad \partial_t u_\eta \in L^2 \left(0, T; H^{-1} \left(\Omega \right) \right),$$

and such that for all $\varphi \in L^2 \left(0, T; H_0^1 \left(\Omega \right) \right)$

$$
\int_0^T \left\langle \partial_t u_\eta, \varphi \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, d\tau + \int_{Q_T} a_\eta \left(u_\eta \right) \nabla u_\eta \cdot \nabla \varphi \, d\mathbf{x} \, dt - \int_{Q_T} f \left(u_\eta \right) \mathbf{V} \cdot \nabla \varphi \, d\mathbf{x} \, dt
$$
$$
+ \int_{Q_T} g' \left(u_\eta \right) \mathbf{V} \cdot \nabla u_\eta \varphi \, d\mathbf{x} \, dt + \int_{Q_T} g \left(u_\eta \right) \mathbf{V} \cdot \nabla \varphi \, d\mathbf{x} \, dt + \int_{Q_T} a_\eta \left(u_\eta \right) \mathbf{V} \cdot \nabla u_\eta \varphi \, d\mathbf{x} \, dt = 0. \tag{24}
$$

Proof. The solutions to system (20)–(22) depend on the parameter η. To simplify the notations and for simplicity, we omit the dependence of solutions on the parameter η and we use u instead of u_η in this section. We will apply the Schauder fixed-point theorem to prove the existence of weak solutions to system (20)–(22).

It is necessary to use the continuous extension for the functions depending on u. For instance, we take $f(u) = g(u) = 1$ for all $u \ge 1$ and $f(u) = g(u) = 0$ for all $u \le 0$. Furthermore, we extend the dissipation a outside $[0, 1]$ by taking

$$a(u) = 0, \text{ for } u \le 0, \quad \text{and} \quad a(u) = a(1), \text{ for } u \ge 0.$$

For technical reason, we have that the velocity $\mathbf{V}$ to be more regular. However, we can regularize $\mathbf{V}$ by $\mathbf{V}_\varepsilon$ such that $\operatorname{div} \mathbf{V}_\varepsilon \in L^2(Q_T)$ and $\mathbf{V}_\varepsilon \to \mathbf{V}$ in $L^2(Q_T)$. Here, we omit this step and consider $\mathbf{V} \in L^\infty(Q_T)$ and $\operatorname{div} \mathbf{V} \in L^2(Q_T)$.

3.1.1. Fixed-point method

Let us introduce the closed subset $\mathscr{K}$ of $L^2(Q_T)$ given by

$$\mathscr{K} = \left\{ u \in L^2(Q_T); \ \|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \eta\|u\|^2_{L^2(0,T;H^1_0(\Omega))} \leq A, \ \|\partial_t u\|_{L^2(0,T;H^{-1}(\Omega))} \leq B \right\}.$$

The constants A and B will be fixed later. The set $\mathscr{K}$ is a compact convex of $L^2(0,T;L^2(\Omega))$ (the compactness is due to the Aubin–Simon theorem [19]).

Let $\mathscr{T}$ be a map from $L^2(0,T;L^2(\Omega))$ to $L^2(0,T;L^2(\Omega))$ defined by $\mathscr{T}(\overline{u}) = u$, where u is the unique solution to the following **linear** parabolic equation

$$\partial_t u - \operatorname{div}\left(a_\eta(\overline{u})\nabla u - f(\overline{u})\mathbf{V}\right) - g(\overline{u})\operatorname{div}(\mathbf{V}) + a_\eta(\overline{u})\nabla u \cdot \mathbf{V} = 0, \tag{25}$$

with the associated initial and boundary conditions. The existence of a unique solution to problem (25) is obtained using the Galerkin method [14,9]. Indeed, there exists a unique solution u to problem (25) verifying: $u \in L^2(0,T;H^1_0(\Omega))$, $\partial_t u \in L^2(0,T;H^{-1}(\Omega))$ such that, we have the following weak formulation: $\forall \varphi \in L^2(0,T;H^1_0(\Omega))$,

$$\begin{aligned}
&\int_0^T \langle \partial_t u, \varphi \rangle_{H^{-1}(\Omega),H^1_0(\Omega)} \, dt + \int_{Q_T} a_\eta(\overline{u})\nabla u \cdot \nabla\varphi \, d\mathbf{x}\,dt - \int_{Q_T} f(\overline{u})\mathbf{V}\cdot\nabla\varphi\,d\mathbf{x}\,dt \\
&+ \int_{Q_T} g(\overline{u})\operatorname{div}\mathbf{V}\varphi\,d\mathbf{x}\,dt + \int_{Q_T} a_\eta(\overline{u})\nabla u \cdot \mathbf{V}\varphi\,d\mathbf{x}\,dt = 0.
\end{aligned} \tag{26}$$

Lemma 3.2. *$\mathscr{T}$ is an application from $\mathscr{K}$ to $\mathscr{K}$.*

Proof. Since $u \in L^2(0,T;H^1_0(\Omega))$, one takes the solution u as a test function in the weak formulation (26), and gets, for all $t \in (0,T)$, that

$$E_1 + E_2 = E_3 + E_4 + E_5, \tag{27}$$

where

$$E_1 = \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}, \quad E_2 = \int_0^t\int_\Omega a_\eta(\overline{u})\nabla u \cdot \nabla u\,d\mathbf{x}\,d\tau, \quad E_5 = \int_{Q_T} g(\overline{u})\operatorname{div}\mathbf{V}\varphi\,d\mathbf{x}\,dt,$$

$$E_3 = \int_0^t\int_\Omega (f(\overline{u}) - g(\overline{u}))\nabla u \cdot \mathbf{V}\,d\mathbf{x}\,d\tau, \quad E_4 = -\int_0^t\int_\Omega a_\eta(\overline{u})\nabla u \cdot \mathbf{V}u\,d\mathbf{x}\,d\tau.$$

We rely on the continuous extension of the functions f and g, the Cauchy–Schwarz, and the weighted Young inequality, then one gets

$$|E_3| \leq \int_0^t\int_\Omega |f(\overline{u})|\,|\nabla u \cdot \mathbf{V}|\,d\mathbf{x}\,d\tau \leq \delta\|\nabla u\|^2_{(L^2(Q_t))^d} + \frac{C_{f,g,Q_t}}{4\delta}\|\mathbf{V}\|^2_{(L^\infty(Q_t))^d}, \tag{28}$$

where δ is a constant to be specified later.

In the same manner, we have the following estimate

$$|E_4| \leq \int_0^t \int_\Omega |a_\eta(\overline{u})| |\nabla u \cdot \mathbf{V}u| \, d\mathbf{x} \, d\tau \leq \delta \|\nabla u\|^2_{(L^2(Q_t))^d} + \frac{C_{f,g,\mathbf{V}}}{4\delta} \|u\|^2_{(L^2(Q_t))^d}. \tag{29}$$

Now, we give an estimation for the last term on the right-hand side of equation (27). Indeed, we have

$$|E_5| \leq \int_{Q_T} |g(\overline{u}) \operatorname{div} \mathbf{V}\varphi| \, d\mathbf{x} \, dt \leq C.$$

Choosing the constant $\delta = \frac{\eta}{4}$ and plugging estimate (29) into equation (27) one can conclude that

$$\|u(t)\|^2_{L^2(\Omega)} + \eta \|\nabla u\|^2_{(L^2(Q_t))^d} \leq C_1 + C_2 \int_0^t \|u(\tau)\|^2_{L^2(\Omega)} \, dt, \tag{30}$$

where $C_1 = C + \|u_0\|^2_{L^2(\Omega)} + \frac{2C_{f,g,Q_t}}{\eta} \|\mathbf{V}\|^2_{(L^\infty(Q_t))^d}$ and $C_2 = \frac{C_{f,g,\mathbf{V}}}{\eta}$.

From estimate (30), and thanks to Grönwall's lemma, one can deduce that there exists a constant $C_3 = C_1 \exp(C_2 T) > 0$ such that

$$\|u\|^2_{L^2(Q_t)} \leq C_3, \quad \forall t \in (0, T). \tag{31}$$

Plugging estimate (31) into estimate (30), one has

$$\|u(t)\|^2_{L^2(\Omega)} + \eta \|\nabla u\|^2_{(L^2(Q_t))^d} \leq A, \quad \forall t \in (0, T),$$

where $A = C_1 + C_2 C_3$. Consequently, one deduces that $\|u\|^2_{L^\infty(0,T;L^2(\Omega))} + \eta \|u\|^2_{L^2(0,T;H_0^1(\Omega))} \leq A$.

It remains to show the estimate on $\partial_t u$. To do this, we take $\varphi \in L^2(0, T; H_0^1(\Omega))$ as a test function into the weak formulation (26), then one gets

$$\left| \int_0^T \langle \partial_t u, \varphi \rangle \, dt \right| \leq \int_{Q_T} |f(\overline{u})| |\mathbf{V} \cdot \nabla\varphi| \, d\mathbf{x} \, dt + \int_{Q_T} |a_\eta(\overline{u})| |\nabla u \cdot (\nabla\varphi + \mathbf{V}\varphi)| \, d\mathbf{x} \, dt + \int_{Q_T} |g(\overline{u}) \operatorname{div} \mathbf{V}\varphi| \, d\mathbf{x} \, dt$$

$$\leq C_{f,g} \|\mathbf{V}\|_{(L^2(Q_T))^d} \|\nabla\varphi\|_{(L^2(Q_T))^d} + C_{a,\eta} \|\nabla u\|_{(L^2(Q_T))^d} \|\nabla\varphi\|_{(L^2(Q_T))^d} + C_{g,\mathbf{v}} \|\varphi\|_{L^2(Q_T)}$$

$$+ C_{a,\eta,\mathbf{V}} \|\nabla u\|_{(L^2(Q_T))^d} \|\varphi\|_{L^2(Q_T)}.$$

Note that the Poincaré inequality implies the existence of a constant $C_4 > 0$ (depending only on the domain Ω) such that

$$\|\varphi\|_{L^2(Q_T)} \leq C_4 \|\nabla\varphi\|_{(L^2(Q_T))^d}.$$

Therefore, one can deduce that

$$\left| \int_0^t \langle \partial_t u, \varphi \rangle \, dt \right| \leq B \|\nabla\varphi\|_{(L^2(Q_T))^d}.$$

This ends the proof of the lemma.

Lemma 3.3. $\mathscr{T}$ *is a continuous application.*

Proof. Let $(\overline{u}_n)_n$ be a sequence of $\mathscr{K}$ and $\overline{u} \in \mathscr{K}$ such that $\overline{u}_n \longrightarrow \overline{u}$ converges strongly in $L^2\left(0, T; L^2(\Omega)\right)$. In order to prove the lemma, it suffices to show that

$$\mathscr{T}(\overline{u}_n) = u_n \longrightarrow \mathscr{T}(\overline{u}) = u \text{ converges strongly in } L^2\left(0, T; L^2(\Omega)\right).$$

For all $\varphi \in L^2\left(0, T; H_0^1(\Omega)\right)$, the sequence $(u_n)_n$ satisfies

$$
\int_0^T \langle \partial_t u_n, \varphi \rangle \, \mathrm{d}t + \int_{Q_T} a_\eta(\overline{u}_n) \nabla u_n \cdot \nabla \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t - \int_{Q_T} f(\overline{u}_n) \mathbf{V} \cdot \nabla \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t
$$
$$
+ \int_{Q_T} g(\overline{u}) \operatorname{div} \mathbf{V}\varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t + \int_{Q_T} a_\eta(\overline{u}_n) \mathbf{V} \cdot \nabla u_n \varphi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t = 0. \tag{32}
$$

Let us denote v_n by $v_n = u_n - u$. Then, we subtract equation (26) from equation (32), and take $\varphi = v_n$ as a test function, and a parameter $\delta > 0$ that will be defined later, we get the following equation

$$\sum_{1 \le i \le 7} H_i = 0, \tag{33}$$

where

$$H_1 = \int_0^t \langle \partial_t v_n, v_n \rangle \, \mathrm{d}\tau = \frac{1}{2} \|v_n(t)\|_{L^2(\Omega)}^2, \quad H_2 = \int_{Q_t} a_\eta(\overline{u}_n) \nabla v_n \cdot \nabla v_n \, \mathrm{d}\tau \mathrm{d}\mathbf{x} \ge \eta \|\nabla v_n\|_{(L^2(Q_t))^d}^2,$$

$$H_3 = \left| \int_{Q_t} (a_\eta(\overline{u}_n) - a_\eta(\overline{u})) \nabla u \cdot \nabla v_n \, \mathrm{d}\tau \mathrm{d}\mathbf{x} \right| \le \delta \|\nabla v_n\|_{(L^2(Q_t))^d}^2 + \frac{1}{4\delta} \|(a_\eta(\overline{u}_n) - a_\eta(\overline{u})) \nabla u\|_{(L^2(Q_t))^d}^2,$$

$$H_4 = \left| \int_{Q_t} (f(\overline{u}_n) - f(\overline{u})) \nabla v_n \cdot \mathbf{V} \, \mathrm{d}\tau \mathrm{d}\mathbf{x} \right| \le \delta \|\nabla v_n\|_{(L^2(Q_t))^d}^2 + \frac{1}{4\delta} \|(f(\overline{u}_n) - f(\overline{u})) \mathbf{V}\|_{(L^2(Q_t))^d}^2,$$

$$H_5 = \left| \int_{Q_t} (g(\overline{u}_n) - g(\overline{u})) \operatorname{div} \mathbf{V} v_n \, \mathrm{d}\tau \mathrm{d}\mathbf{x} \right| \le \delta \|\nabla v_n\|_{(L^2(Q_t))^d}^2 + \frac{1}{4\delta} \|(g(\overline{u}_n) - g(\overline{u})) \operatorname{div} \mathbf{V}\|_{(L^2(Q_t))^d}^2,$$

$$H_6 = \left| \int_{Q_t} a_\eta(\overline{u}_n) \nabla v_n \cdot \mathbf{V} v_n \, \mathrm{d}\tau \mathrm{d}\mathbf{x} \right| \le \delta \|\nabla v_n\|_{(L^2(Q_t))^d}^2 + \frac{C_{a,\eta}}{4\delta} \|v_n\|_{L^2(Q_t)}^2,$$

$$H_7 = \left| \int_{Q_t} (a_\eta(\overline{u}_n) - a_\eta(\overline{u})) \nabla u \cdot \mathbf{V} v_n \, \mathrm{d}\tau \mathrm{d}\mathbf{x} \right| \le \delta \|\nabla v_n\|_{(L^2(Q_t))^d}^2 + C_{a,\eta,\mathbf{V}} \|(a_\eta(\overline{u}_n) - a_\eta(\overline{u})) \nabla u\|_{(L^2(Q_t))^d}^2.$$

Plugging these estimates into equation (33) and choosing $\delta = \frac{\eta}{12}$, one can deduce that

$$\|v_n(t)\|_{L^2(\Omega)}^2 \le \overline{C}_n \exp\left(\frac{6 C_{a,\eta} t}{\eta}\right),$$

where $\overline{C}_n$ tends to zero as $n \to \infty$, which implies that

$$u_n \longrightarrow u \text{ strongly in } L^2\left(0, T; L^2\left(\Omega\right)\right).$$

This ends the proof of this lemma.

Using previous results and the Schauder's fixed-point theorem, one can deduce that there exists at least one solution to the nondegenerate problem (20)–(22) in the sense of Theorem 3.1. It remains to show that the solution verifies the maximum principle.

3.2. Maximum principle on the saturation

In this section, we aim to prove that the solution of the nondegenerate problem (20)–(22) is stable in the sense of verifying the maximum principle. Specifically, we have the following lemma.

Lemma 3.4. *Let u be a solution to the nondegenerate system* (20)–(22) *under the assumptions (H1)–(H4). Then, the solution u satisfies*

$$0 \le u\left(\mathbf{x}, t\right) \le 1, \quad \text{for a.e. } \left(\mathbf{x}, t\right) \in Q_T.$$

Proof. Let u^- be the function defined by $u^- = \max\left(-u, 0\right) = \frac{|u| - u}{2} \ge 0$. Stampacchia's Theorem ensures that $u^- \in L^2\left(0, T; H_0^1\left(\Omega\right)\right)$ since $u \in L^2\left(0, T; H_0^1\left(\Omega\right)\right)$. Therefore, one can consider $-u^-$ as a test function into the weak formulation (26) to get

$$\frac{1}{2}\left\|u^-(t)\right\|_{L^2(\Omega)}^2 + \int_{Q_t} a_\eta\left(u\right)\nabla u^- \cdot \nabla u^- \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{Q_t} f\left(u\right)\mathbf{V} \cdot \nabla u^- \,\mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$+ \int_{Q_t} g\left(u\right)\operatorname{div}\left(\mathbf{V}\right)u^- \,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{Q_t} a_\eta\left(u\right)\nabla u^- \cdot \mathbf{V}u^- \,\mathrm{d}\mathbf{x}\,\mathrm{d}t = 0. \tag{34}$$

We use the definition of the function a_η and the degeneracy of the dissipation a to conclude that

$$\int_{Q_t} a_\eta\left(u\right)\nabla u^- \cdot \nabla u^- \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \ge \eta \int_{Q_t} \nabla u^- \cdot \nabla u^- \mathrm{d}\mathbf{x}\,\mathrm{d}t = \eta \left\|\nabla u^-\right\|_{(L^2(Q_t))^d}^2. \tag{35}$$

Furthermore, we rely on the continuous extension by zero of the functions $f\left(u\right)$ and $g\left(u\right)$ for $u \le 0$, to deduce that the third and the fourth terms in equation (34) are equal to zero.

Let us now focus on the last term of equation (34). Indeed, by the Cauchy–Schwarz inequality as well as the weighted Young inequality, one has

$$\int_{Q_t} a_\eta\left(u\right)\nabla u^- \cdot \mathbf{V}u^- \,\mathrm{d}\mathbf{x}\,\mathrm{d}t \le \frac{\eta}{2}\left\|\nabla u^-\right\|_{(L^2(Q_t))^d}^2 + \frac{C_{a,\eta,\mathbf{V}}}{2}\int_0^t \left\|u^-(\tau)\right\|_{L^2(\Omega)}^2 \,\mathrm{d}\tau. \tag{36}$$

Substituting estimates (35)–(36) into equation (34), this yields

$$\left\|u^-(t)\right\|_{L^2(\Omega)}^2 + \eta\left\|\nabla u^-\right\|_{(L^2(Q_t))^d}^2 \le C_{a,\eta,\mathbf{V}}\int_0^t \left\|u^-(\tau)\right\|_{L^2(\Omega)}^2 \,\mathrm{d}\tau,$$

applying, the Grönwall lemma, one can deduce that $u^-\left(\mathbf{x}, t\right) = 0$, for a.e. $\left(\mathbf{x}, t\right) \in Q_T$, i.e. $u\left(\mathbf{x}, t\right) \ge 0$, for almost everywhere $\left(\mathbf{x}, t\right) \in Q_T$.

It remains to show that $u(\mathbf{x}, t) \leq 1$, for a.e. $(\mathbf{x}, t) \in Q_T$. To do this, it suffices to prove that $(u-1)^+ = 0$. Thus, we multiply the saturation equation (20) by the regular function $(u-1)^+ \in L^2(0, T; H_0^1(\Omega))$ and integrate the resulting equation over $\Omega \times (0, t)$; this yields

$$
\frac{1}{2} \left\| (u-1)^+ (t) \right\|_{L^2(\Omega)}^2 + \int_{Q_t} a_\eta (u) \nabla (u-1)^+ \cdot \nabla (u-1)^+ \, d\mathbf{x} \, dt
$$
$$
- \int_{Q_t} f(u) \mathbf{V} \cdot \nabla (u-1)^+ \, d\mathbf{x} \, dt - \int_{Q_t} g(u) \operatorname{div} (\mathbf{V}) (u-1)^+ \, d\mathbf{x} \, dt + \int_{Q_t} a_\eta (u) \nabla u \cdot \mathbf{V} (u-1)^+ \, d\mathbf{x} \, dt = 0.
$$
(37)

Now, we proceed as before and get the estimates for each term of equation (37).

For the third and the fourth terms of equation (37), by using the fact that $f(u) = g(u) = 1$ for all $u \geq 1$, one has

$$
- \int_{Q_t} f(u) \mathbf{V} \cdot \nabla (u-1)^+ \, d\mathbf{x} \, dt - \int_{Q_t} g(u) \operatorname{div} (\mathbf{V}) (u-1)^+ \, d\mathbf{x} \, dt = - \int_{\Sigma_T} (u-1)^+ \mathbf{V} \cdot \mathbf{n} \, d\sigma \, dt = 0.
$$

For the last term of equation (37), we use again the extension by $a(1)$ of the dissipation function a for $u > 1$, the Cauchy–Schwarz inequality and the weighted Young inequality, and get the following estimate

$$
\int_{Q_t} a_\eta (u) \nabla u \cdot \mathbf{V} (u-1)^+ \, d\mathbf{x} \, dt = \int_{Q_t} a_\eta (u) \nabla (u-1) \cdot \mathbf{V} (u-1)^+ \, d\mathbf{x} \, dt
$$
$$
\leq \frac{\eta}{2} \left\| \nabla (u-1)^+ \right\|_{(L^2(Q_t))^2}^2 + \frac{C_{a, \eta, \mathbf{V}}}{2} \int_0^t \left\| (u-1)^+ (\tau) \right\|_{L^2(\Omega)}^2 \, d\tau.
$$

Plugging the previous estimates into equation (37), one has

$$
\left\| (u-1)^+ (t) \right\|_{L^2(\Omega)}^2 + \eta \left\| \nabla (u-1)^+ \right\|_{(L^2(Q_t))^d}^2 \leq C_{a, \eta, \mathbf{V}} \int_0^t \left\| (u-1)^+ (\tau) \right\|_{L^2(\Omega)}^2 \, d\tau.
$$

One can conclude, using the Grönwall lemma, that $u(\mathbf{x}, t) \leq 1$, for a.e. $(\mathbf{x}, t) \in Q_T$. This ends the proof of Lemma 3.4.

The proof of Theorem 3.1 is now completed.

4. Proof of Theorem 1.1

In the previous section, we have shown that the nondegenerate system (20)–(22) admits at least one weak solution. Here, we are going to prove Theorem 1.1, the proof is based on the establishment of estimates on the solutions independent of the parameter η, and next on the passage to the limit as η goes to zero.

From the definition (8) of the continuous function μ, we have

$$
\mu (u_\eta) = \mu (u_*) \exp \left(\int_{u_*}^{u_\eta} (f(\tau) - g(\tau))^{-1} g'(\tau) \, d\tau \right), \quad \text{for all } u_\eta \geq u_*.
$$

As a consequence of assumption (H2), there exist two nonnegative constants c_3 and c_4 depending only on f, g, μ, and u_* such that

$$c_3(1 - u_\eta)^{-1} \leq \mu(u_\eta) \leq c_4(1 - u_\eta)^{-1}, \quad \forall u_* \leq u_\eta < 1. \tag{38}$$

Indeed, we have

$$\mu(u_*) \exp\left(c_1 C_{g'} \int_{u_*}^{u_\eta} \frac{1}{1 - \tau} \mathrm{d}\tau \right) \leq \mu(u_\eta) \leq \mu(u_*) \exp\left(c_2 \|g'\|_\infty \int_{u_*}^{u_\eta} \frac{1}{1 - \tau} \mathrm{d}\tau \right).$$

That is

$$\frac{c_1 C_{g'} \mu(u_*)(1 - u_*)}{1 - u_\eta} \leq \mu(u_\eta) \leq \frac{c_2 \|g'\|_\infty \mu(u_*)(1 - u_*)}{1 - u_\eta}.$$

Denote $c_3 = c_1 C_{g'} \mu(u_*)(1 - u_*)$ and $c_4 = c_2 \|g'\|_\infty \mu(u_*)(1 - u_*)$, then one obtains the confinement (38).

Now, using the confinement (38) of the function μ and denoting by $c_5 = c_1 c_3 C_{g'}$ and $c_6 = c_2 c_4 \|g'\|_\infty$, one can easily obtain that

$$\frac{c_5}{(1 - u_\eta)^2} \leq \mu'(u_\eta) \leq \frac{c_6}{(1 - u_\eta)^2}. \tag{39}$$

Lemma 4.1. *Under the assumptions* (H1)–(H4), *assume that* $G(u_0) = \int_0^{u_0} \mu(y) \, \mathrm{d}y$ *belongs to* $L^1(\Omega)$. *Then the solutions of the saturation equation* (20) *verify*

(i) $0 \leq u_\eta(\mathbf{x}, t) \leq 1$, *for almost everywhere* $(\mathbf{x}, t) \in Q_T$.

(ii) *The sequences* $\left(\sqrt{\mu'(u_\eta) a(u_\eta)} \nabla u_\eta \right)_\eta$ *and* $(a(u_\eta) \nabla u_\eta)_\eta$ *are uniformly bounded in* $\left(L^2(Q_T) \right)^d$.

(iii) *The sequences* $\left(\sqrt{\eta \mu'(u_\eta)} \nabla u_\eta \right)_\eta$ *and* $(\nabla J(u_\eta))_\eta$ *are uniformly bounded in* $\left(L^2(Q_T) \right)^d$.

(iv) *The sequence* $(G(u_\eta))_\eta$ *is uniformly bounded in* $L^\infty\left(0, T; L^1(\Omega) \right)$.

(v) *The sequence* $(\partial_t J(u_\eta))_\eta$ *is uniformly bounded in* $L^1\left(0, T; W^{-1, q'}(\Omega) \right)$.

(vi) *The sequences* $(J(u_\eta))_\eta$ *and* $(u_\eta)_\eta$ *are relatively compact in* $L^2\left(0, T; L^2(\Omega) \right)$.

Proof. The first part, (i), is obtained in section 3.3.

Now, we multiply the saturation equation (20) by $\mu(u_\eta)$ and integrate it over Ω, then one gets

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega G(u_\eta) \, \mathrm{d}\mathbf{x} + \int_\Omega a(u_\eta) \mu'(u_\eta) |\nabla u_\eta|^2 \mathrm{d}\mathbf{x} + \eta \int_\Omega \mu'(u_\eta) |\nabla u_\eta|^2 \mathrm{d}\mathbf{x}$$

$$= \int_\Omega (f(u_\eta) - g(u_\eta)) \mu'(u_\eta) \nabla u_\eta \cdot \mathbf{V} \mathrm{d}\mathbf{x} - \int_\Omega g'(u_\eta) \mu(u_\eta) \nabla u_\eta \cdot \mathbf{V} \mathrm{d}\mathbf{x} \tag{40}$$

$$- \int_\Omega a(u_\eta) \mu(u_\eta) \nabla u_\eta \cdot \mathbf{V} \mathrm{d}\mathbf{x} - \eta \int_\Omega \mu(u_\eta) \nabla u_\eta \cdot \mathbf{V} \mathrm{d}\mathbf{x}.$$

We denote $\Omega_1 = \Omega \cap \{u_\eta < u_*\}$ and $\Omega_2 = \Omega \cap \{u_\eta \geq u_*\}$; then we can split the whole integral appearing in equation (40) into two parts, so we write $\int_\Omega = \int_{\Omega \cap \{u_\eta < u_*\}} + \int_{\Omega \cap \{u_\eta \geq u_*\}}$.

• Into region Ω_1, recall that $\mu = u^{r-1}$ where r is defined in (5), and using assumption (H1), we obtain the following estimates

$$\left| \int_{\Omega_1} (f(u_\eta) - g(u_\eta)) \mu'(u_\eta) \nabla u_\eta \cdot \mathbf{V} \mathrm{dx} - \int_{\Omega_1} g'(u_\eta) \mu(u_\eta) \nabla u_\eta \cdot \mathbf{V} \mathrm{dx} \right|$$

$$\leq \frac{1}{4} \left\| \sqrt{\mu'(u_\eta) a(u_\eta)} \nabla u_\eta \right\|^2_{(L^2(\Omega_1))^d} + 8 \frac{\|g'\|^4_\infty + (c_2(r-1)^2)^2}{(m_1(r-1))^2} \|\mathbf{V}\|^2_{(L^2(\Omega_1))^d},$$

and

$$\left| \int_{\Omega_1} a(u_\eta) \mu(u_\eta) \nabla u_\eta \cdot \mathbf{V} \mathrm{dx} \right| \leq \frac{1}{4} \left\| \sqrt{\mu'(u_\eta) a(u_\eta)} \nabla u_\eta \right\|^2_{(L^2(\Omega_1))^d} + 4 \frac{\|a\|^2_\infty}{(m_1(r-1))^2} \|\mathbf{V}\|^2_{(L^2(\Omega_1))^d},$$

$$\left| \eta \int_{\Omega_1} \mu(u_\eta) \nabla u_\eta \cdot \mathbf{V} \mathrm{dx} \right| \leq \frac{1}{2} \left\| \sqrt{\eta \mu'(u_\eta)} \nabla u_\eta \right\|^2_{(L^2(\Omega_1))^d} + \frac{1}{2(r-1)^2} \|\mathbf{V}\|^2_{(L^2(\Omega_1))^d}.$$

• Into region Ω_2, and from the definition (8) of the function μ, we have

$$\int_{\Omega_2} ((f(u_\eta) - g(u_\eta)) \mu'(u_\eta) - g'(u_\eta) \mu(u_\eta)) \nabla u_\eta \cdot \mathbf{V} \mathrm{dx} = 0.$$

Furthermore, thanks to estimate (39), we have the following estimates

$$\left| \int_{\Omega_2} a(u_\eta) \mu(u_\eta) \nabla u_\eta \cdot \mathbf{V} \mathrm{dx} \right| \leq \frac{1}{2} \left\| \sqrt{\mu'(u_\eta) a(u_\eta)} \nabla u_\eta \right\|^2_{(L^2(\Omega_2))^d} + C \|\mathbf{V}\|^2_{(L^2(\Omega_2))^d},$$

and

$$\left| \eta \int_{\Omega_2} \mu(u_\eta) \nabla u_\eta \cdot \mathbf{V} \mathrm{dx} \right| \leq \frac{1}{2} \left\| \sqrt{\eta \mu'(u_\eta)} \nabla u_\eta \right\|^2_{(L^2(\Omega_2))^d} + C \|\mathbf{V}\|^2_{(L^2(\Omega_2))^d}.$$

Plugging the previous estimates into equation (40), one has

$$2 \frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega G(u_\eta) \mathrm{dx} + \left\| \sqrt{\mu'(u_\eta) a(u_\eta)} \nabla u_\eta \right\|^2_{(L^2(\Omega))^d} + \left\| \sqrt{\eta \mu'(u_\eta)} \nabla u_\eta \right\|^2_{(L^2(\Omega))^d} \leq C. \tag{41}$$

Now, we integrate inequality (41) with respect to the time over $(0, t)$, $t \in (0, T)$; one deduces that the sequences $\left(\sqrt{\mu'(u_\eta) a(u_\eta)} \nabla u_\eta \right)_\eta$ and $\left(\sqrt{\eta \mu'(u_\eta)} \nabla u_\eta \right)_\eta$ are uniformly bounded in $(L^2(Q_T))^d$, and that $(G(u_\eta))_\eta$ is uniformly bounded in $L^\infty(0, T; L^1(\Omega))$.

Let us prove that $(a(u_\eta) \nabla u_\eta)_\eta$ and $(\nabla J(u_\eta))_\eta$ are uniformly bounded in $(L^2(Q_T))^d$. Indeed, since $r \geq r_1$ then for all $0 \leq u_\eta \leq u_* < 1$, we have

$$a(u_\eta) \mu'(u_\eta) \geq m_1(r-1) u_\eta^{r_1} u_\eta^{r-2} \geq m_1(r-1) u_\eta^r u_\eta^{r-2} \geq m_1(r-1) u_\eta^{2r-2} \geq m_1(r-1) j^2(u_\eta),$$

where j is the function defined by (7) and for all $u_\eta \geq u_*$, we have

$$a(u_\eta)\,\mu'(u_\eta) \geq m_1(1-u_\eta)^{r_2}\mu(u_\eta)\,g'(u_\eta)\,(f(u_\eta)-g(u_\eta))^{-1} \geq c_5 m_1(1-u_\eta)^{r_2-2}$$

$$\geq \frac{c_5 m_1}{(1-u_*)^{2-r'}}\left((1-u_*)^{1-\frac{r'}{2}}\right)^2\left((1-u_\eta)^{\frac{r'}{2}-1}\right)^2 \geq C j^2(u_\eta).$$

Therefore, $\|\nabla J(u_\eta)\|^2_{(L^2(Q_T))^d} \leq C \left\|\sqrt{\mu'(u_\eta)\,a(u_\eta)}\nabla u_\eta\right\|^2_{(L^2(Q_T))^d} \leq C.$

For the sequence $(a(u_\eta)\nabla u_\eta)_\eta$, it is easy to see that

$$a(u_\eta) \leq M_1 u_\eta^{r_1} \leq M_1 u_\eta^{r-2} \leq \frac{M_1}{r-1}\mu'(u_\eta), \qquad\qquad \text{if } 0 \leq u_\eta \leq u_*,$$

$$a(u_\eta) \leq M_1(1-u_\eta)^{r_2} \leq M_1(1-u_\eta)^{-2} \leq \frac{M_1}{c_5}\mu'(u_\eta), \qquad \text{if } u_* \leq u_\eta \leq 1.$$

As a consequence, the sequence $(a(u_\eta)\nabla u_\eta)_\eta$ is uniformly bounded in $\left(L^2(Q_T)\right)^d$.

Let us now focus on the fourth part (iv). We want to prove that

$$(\partial_t J(u_\eta))_\eta \text{ is uniformly bounded in } L^2\left(0,T;\left(H^1(\Omega)\right)'\right) + L^1(Q_T).$$

We take a test function $\chi \in L^2\left(0,T;H_0^1(\Omega)\right) \cap L^\infty(Q_T)$ and multiply the saturation equation (20) by $j(u_\eta)\chi$; this yields

$$
\begin{aligned}
\langle \partial_t J(u_\eta), \chi \rangle = {}& -\int_{Q_T} a(u_\eta)\nabla u_\eta \cdot \nabla(j(u_\eta)\chi)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \eta \int_{Q_T} \nabla u_\eta \cdot \nabla(j(u_\eta)\chi)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
& + \int_{Q_T} (f(u_\eta)-g(u_\eta))\,\mathbf{V}\cdot\nabla(j(u_\eta)\chi)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_{Q_T} g'(u_\eta)\nabla u_\eta \cdot \mathbf{V} j(u_\eta)\chi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
& - \int_{Q_T} a(u_\eta)\nabla u_\eta \cdot \mathbf{V} j(u_\eta)\chi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \eta \int_{Q_T} \nabla u_\eta \cdot \mathbf{V} j(u_\eta)\chi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}
\tag{42}
$$

We will give estimates on each integral on the right-hand side of equation (42).

Into region $Q_T \cap \{u_\eta < u_*\}$, we have $j(u_\eta) = \mu(u_\eta)$, thus we give estimates on each integral of the form $\displaystyle\int_{Q_T \cap \{u_\eta < u_*\}}$ on the right-hand side of equation (42) which we denote by I_i, $1 \leq i \leq 6$. To obtain the estimates, we use the Cauchy–Schwarz inequality. For the first term, we have

$$
\begin{aligned}
|I_1| &\leq \int_{Q_T} \left|a(u_\eta)\nabla u_\eta \cdot \left((r-1)u_\eta^{r-2}\nabla u_\eta \chi + u_\eta^{r-1}\nabla\chi\right)\right|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\
&\leq C_{r,a}\left(\int_{Q_T}\left|a(u_\eta)u_\eta^{r-2}\nabla u_\eta \cdot \nabla u_\eta \chi\right|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{Q_T}\left|u_\eta^{r-1}\nabla u_\eta \cdot \nabla\chi\right|\,\mathrm{d}\mathbf{x}\,\mathrm{d}t\right) \\
&\leq C_{r,a}\left\|\sqrt{a(u_\eta)u_\eta^{r-2}}\nabla u_\eta\right\|^2_{(L^2(Q_T))^d}\|\chi\|_{L^\infty(Q_T)} + \left\|u_\eta^{r-1}\nabla u_\eta\right\|_{(L^2(Q_T))^d}\|\nabla\chi\|_{(L^2(Q_T))^d}.
\end{aligned}
$$

In the same manner, we have the estimate on the second term

$$|I_2| \leq \int\limits_{Q_T} \left| \eta \nabla u_\eta \cdot \left((r-1) u_\eta^{r-2} \nabla u_\eta \chi + u_\eta^{r-1} \nabla \chi \right) \right| \mathrm{d}\mathbf{x}\, \mathrm{d}t$$

$$\leq C_{r,a} \left\| \sqrt{\eta u_\eta^{r-2}} \nabla u_\eta \right\|_{(L^2(Q_T))^d} \|\chi\|_{L^\infty(Q_T)} + \left\| u_\eta^{r-1} \nabla u_\eta \right\|_{(L^2(Q_T))^d} \|\nabla \chi\|_{(L^2(Q_T))^d}.$$

The third term is estimated, with the help of assumption (H2) and the Poincaré inequality [3], as follows

$$|I_3| \leq \int\limits_{Q_T} \left| (f(u_\eta) - g(u_\eta)) \mathbf{V} \cdot \left((r-1) u_\eta^{r-2} \nabla u_\eta \chi + u_\eta^{r-1} \nabla \chi \right) \right| \mathrm{d}\mathbf{x}\, \mathrm{d}t$$

$$\leq C_{r,\Omega} \|\mathbf{V}\|_{(L^\infty(Q_T))^d}^2 \left(\left\| u_\eta^{r-1} \nabla u_\eta \right\|_{(L^2(Q_T))^d} + 1 \right) \|\nabla \chi\|_{(L^2(Q_T))^d}.$$

Similarly, we have

$$|I_4| \leq \int\limits_{Q_T} \left| g'(u_\eta) u_\eta^{r-1} \nabla u_\eta \cdot \mathbf{V} \chi \right| \mathrm{d}\mathbf{x}\, \mathrm{d}t \leq C_{g',\Omega} \left(\left\| u_\eta^{r-1} \nabla u_\eta \right\|_{(L^2(Q_T))^d} \|\mathbf{V}\|_{(L^\infty(Q_T))^d}^2 \right) \|\nabla \chi\|_{(L^2(Q_T))^d}.$$

Finally, the last two terms are estimated as follows

$$|I_5 + I_6| \leq \int\limits_{Q_T} \left| a(u_\eta) u_\eta^{r-1} \nabla u_\eta \cdot \mathbf{V} \chi \right| \mathrm{d}\mathbf{x}\, \mathrm{d}t + \int\limits_{Q_T} \left| \eta u_\eta^{r-1} \nabla u_\eta \cdot \mathbf{V} \chi \right| \mathrm{d}\mathbf{x}\, \mathrm{d}t$$

$$\leq C_{a,\Omega} \left(\left\| u_\eta^{r-1} \nabla u_\eta \right\|_{(L^2(Q_T))^d} \|\mathbf{V}\|_{(L^\infty(Q_T))^d}^2 \right) \|\nabla \chi\|_{(L^2(Q_T))^d}.$$

It remains to estimate the terms of the form $\displaystyle\int\limits_{Q_T \cap \{u_\eta \geq u_*\}}$ that we denote by $\{L_i\}_{1 \leq i \leq 6}$ respectively.

For the first term L_1, we have

$$|L_1| \leq \int\limits_{Q_T \cap \{u_\eta \geq u_*\}} \left| a(u_\eta) j'(u_\eta) \nabla u_\eta \cdot \nabla u_\eta \chi \right| + \left| a(u_\eta) j(u_\eta) \nabla u_\eta \cdot \nabla \chi \right| \mathrm{d}\mathbf{x}\, \mathrm{d}t$$

$$\leq \left\| \sqrt{j'(u_\eta) a(u_\eta)} \nabla u_\eta \right\|_{(L^2(Q_T))^d}^2 \|\chi\|_{L^\infty(Q_T)} + \left\| a(u_\eta) \nabla J(u_\eta) \right\|_{(L^2(Q_T))^d} \|\nabla \chi\|_{(L^2(Q_T))^d}.$$

On the other hand, using the definition of μ and j, we have, for all $u_* \leq u_\eta \leq 1$, that

$$|j'(u_\eta)| = \left(\frac{r'}{2} - 1 \right) \beta(u_*)(1 - u_*)^{1 - \frac{r'}{2}} (1 - u_\eta)^{\frac{r'}{2} - 2} \leq C_{u_*, r'} (1 - u_\eta)^{-2} \leq \mu'(u_\eta),$$

thus, thanks to parts (i)–(iii), one deduces that $|L_1| \leq C \left(\|\chi\|_{L^\infty(Q_T)} + \|\nabla \chi\|_{(L^2(Q_T))^d} \right)$.

In the same manner, we obtain the estimates on the remaining terms except the estimate on L_3. Indeed, using assumption (H2) on f and g, one has

$$f(u_\eta) - g(u_\eta) \leq \frac{1}{c_1}(1 - u_\eta), \quad \forall u_* \leq u_\eta \leq 1,$$

and therefore, we obtain the following estimates

$$
\left| \int_{Q_T} (f(u_\eta) - g(u_\eta)) j'(u_\eta) \nabla u_\eta \cdot \mathbf{V} \chi \, d\mathbf{x} \, dt \right| \leq \frac{1}{c_1} \int_{Q_T} |j(u_\eta) \nabla u_\eta \cdot \mathbf{V} \chi| \, d\mathbf{x} \, dt
$$

$$
\leq \frac{1}{c_1} \|\nabla J(u_\eta)\|_{(L^2(Q_T))^d} \|\mathbf{V}\|_{(L^2(Q_T))^d} \|\chi\|_{L^\infty(Q_T)},
$$

and

$$
\left| \int_{Q_T} (f(u_\eta) - g(u_\eta)) j(u_\eta) \mathbf{V} \cdot \nabla \chi \, d\mathbf{x} \, dt \right| \leq C \|\mathbf{V}\|_{(L^2(Q_T))^d} \|\chi\|_{(L^2(Q_T))^d}.
$$

Plugging the previous estimates into equation (42), one gets

$$
|\langle \partial_t J(u_\eta), \chi \rangle| \leq C \left(\|\chi\|_{L^\infty(Q_T)} + \|\chi\|_{L^2(0,T;H^1_0(\Omega))} \right).
$$

One can conclude the proof of part (iii), using the embedding of the Sobolev space $W^{1,q}(\Omega) \subset H^1_0(\Omega) \cap L^\infty(\Omega)$ for $q > d$, and consequently, one has

$$
L^\infty\left(0, T; W^{1,q}(\Omega)\right) \subset L^2\left(0, T; H^1(\Omega)\right) \cap L^\infty\left(0, T; L^2(\Omega)\right), \quad \forall q > d.
$$

To complete the proof of the lemma, we remark that the sequence $(J(u_\eta))_\eta$ is lying into the Sobolev space

$$
\mathscr{W} = \left\{ J(u_\eta); \ J(u_\eta) \in L^2\left(0, T; H^1_0(\Omega)\right) \text{ and } \partial_t J(u_\eta) \in L^1\left(0, T; W^{-1,q'}(\Omega)\right) \right\}.
$$

Thanks to the Aubin–Simon theorem, $\mathscr{W}$ is compactly embedded in $L^2(Q_T)$, and the sequence $(J(u_\eta))_\eta$ is relatively compact in $L^2\left(0, T; L^2(\Omega)\right)$.

Since the differentiable function J is nondecreasing, then J^{-1} exists and it is continuous, then the sequence $(u_\eta)_\eta$ is relatively compact in $L^2\left(0, T; L^2(\Omega)\right)$. The proof of Lemma 4.1 is now accomplished.

Lemma 4.2. *Let $q_1 = 3r_1 + 2$ and $q_2 = 3r_2 + 2$, where r_1 and r_2 are given in assumption (H1). The sequences $\left(1_{\{u_\eta \leq u_*\}} u_\eta^{q_1} a(u_\eta) \nabla u_\eta\right)_\eta$ and $\left(1_{\{u_\eta \geq u_*\}} (1 - u_\eta)^{q_2} a(u_\eta) \nabla u_\eta\right)_\eta$ are two Cauchy sequences in measure.*

Proof. In order to prove Lemma 4.2, we rely on the compactness result given in Lemma 2.1. Indeed, thanks to Lemma 4.1, one deduces that the sequence $(u_\eta)_\eta$ verifies assumptions (A1)–(A3). Then, it suffices to show that

$$
\int_{Q_T} \nabla A_{\eta,\eta'} \cdot \nabla (b(u_\eta) T_\mu(B_{\eta,\eta'})) d\mathbf{x} \, dt \longrightarrow 0, \text{ as } \mu, \eta, \eta' \to 0. \tag{43}
$$

Let us prove statement (43). For that, we consider the primitive Θ_μ of the truncation function T_μ, defined by

$$
\Theta_\mu(u) = \int_0^u T_\mu(\tau) \, d\tau, \quad \forall u \in \mathbb{R}, \ \forall \mu > 0. \tag{44}
$$

We subtract the equations (24) satisfied by $(u_\eta)_\eta$ and $(u_{\eta'})_{\eta'}$, then we multiply by $\sigma_\eta = b(u_\eta) T_\mu(B_{\eta,\eta'})$ and $\sigma_{\eta'} = b(u_{\eta'}) T_\mu(B_{\eta,\eta'})$ respectively, to get

$$\int_\Omega \Theta_\mu \left(B_{\eta,\eta'} \left(t, \mathbf{x} \right) \right) \mathrm{d}\mathbf{x} + \int_{Q_t} \left(\nabla A \left(u_\eta \right) \cdot \nabla \sigma_\eta - \nabla A \left(u_{\eta'} \right) \cdot \nabla \sigma_{\eta'} \right) \mathrm{d}\mathbf{x}\,\mathrm{d}t$$

$$= \int_{Q_t} \left(\left(f \left(u_\eta \right) - g \left(u_\eta \right) \right) \mathbf{V} \cdot \nabla \sigma_\eta - \left(f \left(u_{\eta'} \right) - g \left(u_{\eta'} \right) \right) \mathbf{V} \cdot \nabla \sigma_{\eta'} \right) \mathrm{d}\mathbf{x}\,\mathrm{d}t$$

$$- \int_{Q_t} \left(g' \left(u_\eta \right) \nabla u_\eta \cdot \mathbf{V} \sigma_\eta - g' \left(u_{\eta'} \right) \nabla u_{\eta'} \cdot \mathbf{V} \sigma_{\eta'} \right) \mathrm{d}\mathbf{x}\,\mathrm{d}t$$

$$- \eta \int_{Q_t} \nabla u_\eta \cdot \nabla \sigma_\eta \, \mathrm{d}\mathbf{x}\,\mathrm{d}t + \eta' \int_{Q_t} \nabla u_{\eta'} \cdot \nabla \sigma_{\eta'} \, \mathrm{d}\mathbf{x}\,\mathrm{d}t \qquad (45)$$

$$- \int_{Q_t} \left(\nabla A \left(u_\eta \right) \cdot \mathbf{V} \sigma_\eta - \nabla A \left(u_{\eta'} \right) \cdot \mathbf{V} \sigma_{\eta'} \right) \mathrm{d}\mathbf{x}\,\mathrm{d}t$$

$$- \eta \int_{Q_t} \nabla u_\eta \cdot \mathbf{V} \sigma_\eta \, \mathrm{d}\mathbf{x}\,\mathrm{d}t + \eta' \int_{Q_t} \nabla u_{\eta'} \cdot \mathbf{V} \sigma_{\eta'} \, \mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

We denote by I_i, $i = 1, 7$, the integrals on the right-hand side of equation (45), and let $\left(\delta_\eta \right)_\eta$, $\left(\delta_{\eta'} \right)_{\eta'}$, $\left(\mathbf{V}_\eta \right)_\eta$, and $\left(\mathbf{V}_\eta \right)_{\eta'}$ be the sequences defined by

$$\delta_\eta = \left(f \left(u_\eta \right) - g \left(u_\eta \right) \right), \quad \delta_{\eta'} = \left(f \left(u_{\eta'} \right) - g \left(u_{\eta'} \right) \right), \quad \mathbf{V}_\eta = \delta_\eta \mathbf{V}, \quad \mathbf{V}_{\eta'} = \delta_{\eta'} \mathbf{V}.$$

Using the dominated convergence Lebesgue theorem, we get

$$\left\| \mathbf{V}_\eta - \mathbf{V}_{\eta'} \right\|_{(L^2(Q_T))^d} = \left\| \left(f \left(u_\eta \right) - g \left(u_\eta \right) \right) \mathbf{V} - \left(f \left(u_{\eta'} \right) - g \left(u_{\eta'} \right) \right) \mathbf{V} \right\|_{(L^2(Q_T))^d} \xrightarrow[\eta,\eta'\to 0]{} 0. \qquad (46)$$

Now, we give estimates on each term on the right-hand side of equation (45). For the first term, we have

$$\mathbf{V}_\eta \cdot \nabla \sigma_\eta - \mathbf{V}_{\eta'} \cdot \nabla \sigma_{\eta'} = \left(\mathbf{V}_\eta \cdot \nabla b \left(u_\eta \right) - \mathbf{V}_{\eta'} \cdot \nabla b \left(u_{\eta'} \right) \right) T_\mu \left(B_{\eta,\eta'} \right) + \left(\mathbf{V}_\eta b \left(u_\eta \right) - \mathbf{V}_{\eta'} b \left(u_{\eta'} \right) \right) \nabla T_\mu \left(B_{\eta,\eta'} \right)$$

$$= \left(\mathbf{V}_\eta \cdot \nabla b \left(u_\eta \right) - \mathbf{V}_{\eta'} \cdot \nabla b \left(u_{\eta'} \right) \right) T_\mu \left(B_{\eta,\eta'} \right) + \left(\mathbf{V}_\eta - \mathbf{V}_{\eta'} \right) b \left(u_\eta \right) \nabla T_\mu \left(B_{\eta,\eta'} \right)$$

$$+ \left(b \left(u_\eta \right) - b \left(u_{\eta'} \right) \right) \mathbf{V}_{\eta'} \cdot \nabla T_\mu (B_{\eta,\eta'}).$$

As a consequence,

$$|I_1| \leq \left\| \left(\mathbf{V}_\eta \cdot \nabla b \left(u_\eta \right) - \mathbf{V}_{\eta'} \cdot \nabla b \left(u_{\eta'} \right) \right) T_\mu \left(B_{\eta,\eta'} \right) \right\|_{L^1(Q_T)}$$

$$+ \left\| b \right\|_{L^\infty(Q_T)} \left\| \mathbf{V}_\eta - \mathbf{V}_{\eta'} \right\|_{(L^2(Q_T))^d} \left\| \nabla T_\mu \left(B_{\eta,\eta'} \right) \right\|_{(L^2(Q_T))^d} \qquad (47)$$

$$+ \left\| \left(b \left(u_\eta \right) - b \left(u_{\eta'} \right) \right) \mathbf{V}_{\eta'} \cdot \nabla T_\mu \left(B_{\eta,\eta'} \right) \right\|_{L^1(Q_T)}.$$

The first term on the right-hand side of inequality (47) is estimated as follows

$$\left\| \left(\mathbf{V}_\eta \cdot \nabla b \left(u_\eta \right) - \mathbf{V}_{\eta'} \cdot \nabla b \left(u_{\eta'} \right) \right) T_\mu \left(B_{\eta,\eta'} \right) \right\|_{L^1(Q_T)}$$

$$\leq C \left\| \mathbf{V} \right\|_{(L^\infty(Q_T))^d} \left\| T_\mu \left(B_{\eta,\eta'} \right) \right\|_{(L^2(Q_T))^d} \left(\left\| \nabla b \left(u_\eta \right) \right\|_{(L^2(Q_T))^d} + \left\| \nabla b \left(u_{\eta'} \right) \right\|_{(L^2(Q_T))^d} \right).$$

Taking into account the uniform boundedness in $\left(L^2 \left(Q_T \right) \right)^d$ of the sequence $\left(\nabla b \left(u_\eta \right) \right)_\eta$, and the following overestimate $\left| T_\mu \left(B_{\eta,\eta'} \right) \right| \leq \mu$, one has

$$\left\|\left(\mathbf{V}_\eta \cdot \nabla b\left(u_\eta\right) - \mathbf{V}_{\eta'} \cdot \nabla b\left(u_{\eta'}\right)\right) T_\mu\left(B_{\eta,\eta'}\right)\right\|_{L^1(Q_T)} \xrightarrow[\mu\to0]{} 0, \text{ uniformly on } \eta, \eta'.$$

It is easy to see, using the convergence (46), that the second term on the right-hand side of inequality (47) tends to zero as $\eta, \eta' \to 0$. Using the boundedness of the function b, one has $\left\|\left(b\left(u_\eta\right) - b\left(u_{\eta'}\right)\right) \mathbf{V}_{\eta'}\right\|_{(L^2(Q_T))^d}$ tends to zero as $\eta, \eta' \to 0$. One can conclude that the last term on the right-hand side of inequality (47) tends to zero as $\eta, \eta' \to 0$.

For the second term on the right-hand side of equation (45), we have using the definition of the function b that is

$$g'\left(u_\eta\right) \nabla u_\eta \cdot \mathbf{V} b\left(u_\eta\right) T_\mu\left(B_{\eta,\eta'}\right) - g'\left(u_{\eta'}\right) \nabla u_{\eta'} \cdot \mathbf{V} b\left(u_{\eta'}\right) T_\mu\left(B_{\eta,\eta'}\right)$$
$$= \left(g'\left(u_\eta\right) \mathbf{V} \cdot \nabla B\left(u_\eta\right) - g'\left(u_{\eta'}\right) \mathbf{V} \cdot \nabla B\left(u_{\eta'}\right)\right) T_\mu(B_{\eta,\eta'}),$$

and consequently,

$$\left|I_2\right| \le C_{g',\mathbf{V}} \left\|T_\mu\left(B_{\eta,\eta'}\right)\right\|_{L^2(Q_T)} \left(\left\|\nabla B\left(u_\eta\right)\right\|_{(L^2(Q_T))^d} + \left\|\nabla B\left(u_{\eta'}\right)\right\|_{(L^2(Q_T))^d}\right) \xrightarrow[\mu\to0]{} 0.$$

For the third term I_3 on the right-hand side of equation (45), we write

$$\nabla u_\eta \cdot \nabla\left(b\left(u_\eta\right) T_\mu\left(B_{\eta,\eta'}\right)\right) = 2\nabla A\left(u_\eta\right) \cdot \nabla A\left(u_\eta\right) T_\mu\left(B_{\eta,\eta'}\right) + 2A\left(u_\eta\right) \cdot \nabla A\left(u_\eta\right) \nabla T_\mu\left(B_{\eta,\eta'}\right)$$
$$+ 2A\left(u_\eta\right) a'\left(u_\eta\right) \nabla u_\eta \cdot \nabla u_\eta T_\mu\left(B_{\eta,\eta'}\right).$$

Using the uniform boundedness of the sequences $(\nabla A(u_\eta))_\eta$, $(\nabla B(u_\eta))_\eta$, and $(\nabla b(u_\eta))_\eta$, one can deduce that $|I_3| \le C\eta$, for some constant $C > 0$ independent of η and η'. Therefore, $|I_3| \to 0$ as $\eta, \eta' \to 0$. Similarly, we prove that $|I_4| \le C\eta' \to 0$ as $\eta, \eta' \to 0$. For the fifth term I_5 on the right-hand side of equation (45), we write

$$I_5 = \int_{Q_t} \nabla A\left(u_{\eta'}\right) \cdot \mathbf{V} b\left(u_{\eta'}\right) T_\mu\left(B_{\eta,\eta'}\right) - \nabla A\left(u_\eta\right) \cdot \mathbf{V} b\left(u_\eta\right) T_\mu\left(B_{\eta,\eta'}\right) \mathrm{d}\mathbf{x}\,\mathrm{d}t$$
$$= \int_{Q_t} \left(a\left(u_{\eta'}\right) \mathbf{V} \cdot \nabla B\left(u_{\eta'}\right) - a\left(u_\eta\right) \mathbf{V} \cdot \nabla B\left(u_\eta\right)\right) T_\mu\left(B_{\eta,\eta'}\right) \mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

Obviously, we have

$$\left|I_5\right| \le \left\|\left(a\left(u_\eta\right) \mathbf{V} \cdot \nabla B\left(u_\eta\right) - a\left(u_{\eta'}\right) \mathbf{V} \cdot \nabla B\left(u_{\eta'}\right)\right) T_\mu\left(B_{\eta,\eta'}\right)\right\|_{L^1(Q_T)} \le C\mu.$$

Finally, for the last two terms of equation (45), we have

$$\int_{Q_t} \nabla u_\eta \cdot \mathbf{V} b\left(u_\eta\right) T_\mu\left(B_{\eta,\eta'}\right) \mathrm{d}\mathbf{x}\,\mathrm{d}t = 2\int_{Q_t} \nabla A\left(u_\eta\right) \cdot \mathbf{V} A\left(u_\eta\right) T_\mu\left(B_{\eta,\eta'}\right) \mathrm{d}\mathbf{x}\,\mathrm{d}t.$$

As a consequence,

$$\left|I_6\right| \le \eta\left(\left\|\nabla A\left(u_\eta\right)\right\|_{(L^2(Q_T))^d} \left\|\mathbf{V}\right\|_{(L^2(Q_T))^d} M_1\mu\right) \le C\eta \xrightarrow[\eta,\eta'\to0]{} 0.$$

Similarly, we prove that $|I_7| \le C\eta \to 0$ as $\eta, \eta' \to 0$ for some constant $C > 0$ independent of η and η'.

We denote by $W_\mu\left(\eta, \eta'\right)$ the right-hand side of equation (45) and by $V\left(\mu\right)$ the firm term on the left-hand side of the same equation; from the estimations on the integrals I_i, $W_\mu\left(\eta, \eta'\right)$ goes to zero as $\eta, \eta' \to 0$, for

all $\mu > 0$. We also have $|V(\mu)| \le |\Omega|\,\mu$, which goes to zero as $\mu \to 0$ and uniformly on η and η'. Therefore, we have the following result stemming from equation (45) and the aforementioned definitions

$$\int_{Q_T} (\nabla A(u_\eta) \cdot \nabla (b(u_\eta) T_\mu (B_{\eta,\eta'})) - \nabla A(u_{\eta'}) \cdot \nabla (b(u_{\eta'}) T_\mu (B_{\eta,\eta'}))) \mathrm{d}\mathbf{x}\,\mathrm{d}t = W_\mu(\eta,\eta') + V(\mu). \tag{48}$$

One can get the convergence result (43) using equation (48) and the following equation

$$\int_{Q_t} \nabla A_{\eta,\eta'} \cdot \nabla (b(u_\eta) T_\mu (B_{\eta,\eta'}))\, \mathrm{d}\mathbf{x}\,\mathrm{d}t$$

$$= \int_{Q_T} (\nabla A(u_\eta) \cdot \nabla (b(u_\eta) T_\mu (B_{\eta,\eta'})) - \nabla A(u_{\eta'}) \cdot \nabla (b(u_{\eta'}) T_\mu (B_{\eta,\eta'})))\, \mathrm{d}\mathbf{x}\,\mathrm{d}t$$

$$- \int_{Q_T} \nabla A(u_{\eta'}) \cdot \nabla (b(u_\eta) - b(u_{\eta'})) T_\mu (B_{\eta,\eta'})\, \mathrm{d}\mathbf{x}\,\mathrm{d}t$$

$$- \int_{Q_T} (b(u_\eta) - b(u_{\eta'})) \nabla A(u_{\eta'}) \cdot \nabla A_{\eta,\eta'} 1_{\{|B_{\eta,\eta'}| \le \mu\}}\, \mathrm{d}\mathbf{x}\,\mathrm{d}t$$

that leads to

$$\int_{Q_t} \nabla A_{\eta,\eta'} \cdot \nabla (b(u_\eta) T_\mu (B_{\eta,\eta'}))\, \mathrm{d}\mathbf{x}\,\mathrm{d}t \xrightarrow[\mu,\eta,\eta' \to 0]{} 0.$$

Applying Lemma 2.1, one gets that for all $\eta, \eta' \le \eta_0$, we have meas $\{|\nabla s(u_\eta) - \nabla s(u_{\eta'})| \ge \delta\} \le \varepsilon$, where $\nabla s(u_\eta) = b(u_\eta) A(u_\eta) \nabla A(u_\eta)$. Now we have

$$1_{\{u_\eta \le u_*\}} u_\eta^{q_1} \nabla A(u_\eta) - 1_{\{u_\eta \le u_*\}} u_{\eta'}^{q_1} \nabla A(u_{\eta'})$$

$$= 1_{\{u_\eta \le u_*\}} u_\eta^{q_1} \nabla A_{\eta,\eta'} + \left(1_{\{u_\eta \le u_*\}} u_\eta^{q_1} - 1_{\{u_\eta \le u_*\}} u_{\eta'}^{q_1}\right) \nabla A(u_{\eta'}).$$

The last term of the previous equation goes to zero as η and η' go to zero in $L^1(Q_T)$.

Since $q_1 = 3r_1 + 2$, then $1_{\{u_\eta \le u_*\}} u_\eta^{q_1} = 1_{\{u_\eta \le u_*\}} u_\eta^{3r_1+2} \le C_{r_1,m_1} 1_{\{u_\eta \le u_*\}} b(u_\eta) A(u_\eta)$ where $C_{r_1,m_1} = \dfrac{(r_1+1)^2}{2m_1^3}$.

We write

$$\left|1_{\{u_\eta \le u_*\}} u_\eta^{q_1} \nabla A_{\eta,\eta'}\right| \le C_{r_1,m_1} b(u_\eta) A(u_\eta) |\nabla A_{\eta,\eta'}|$$

$$\le C_{r_1,m_1} |\nabla s(u_\eta) - \nabla s(u_{\eta'})| + C_{r_1,m_1} |(b(u_\eta) A(u_\eta) - b(u_{\eta'}) A(u_{\eta'})) \nabla A(u_{\eta'})|.$$

Consequently,

$$\left|1_{\{u_\eta \le u_*\}} u_\eta^{q_1} \nabla A(u_\eta) - 1_{\{u_{\eta'} \le u_*\}} u_{\eta'}^{q_1} \nabla A(u_{\eta'})\right| \le \left|\left(1_{\{u_\eta \le u_*\}} u_{\eta'}^{q_1} - 1_{\{u_{\eta'} \le u_*\}} u_\eta^{q_1}\right) \nabla A(u_{\eta'})\right| \tag{49}$$

$$+ C_{r_1,m_1} |\nabla s(u_\eta) - \nabla s(u_{\eta'})| + C_{r_1,m_1} |(b(u_\eta) A(u_\eta) - b(u_{\eta'}) A(u_{\eta'})) \nabla A(u_{\eta'})|.$$

One can conclude that the right hand side of inequality (49) goes to zero as η and η' go to zero. Therefore, the sequence $\left(1_{\{u_\eta \le u_*\}} u_\eta^{q_1} a(u_\eta) \nabla u_\eta\right)_\eta$ is a Cauchy sequence in measure. In the same manner, one proves that

$\left(1_{\{u_\eta \geq u_*\}} 1 - u_\eta^{q_2} a\left(u_\eta\right) \nabla u_\eta\right)_\eta$ since $1_{\{u_\eta \geq u_*\}} u_\eta^{q_2} = 1_{\{u_\eta \geq u_*\}} u_\eta^{3r_2+2} \leq C_{r_2,m_1} 1_{\{u_\eta \geq u_*\}} b\left(u_\eta\right) A\left(u_\eta\right)$ where C_{r_2,m_1} is a constant independent of η.

4.1. Convergence and identification as a weak solution

To conclude the proof of Theorem 1.1, we deduce from Lemma 4.1 and Lemma 4.2, that we can extract a subsequence such that we have the following convergences

$$
\begin{aligned}
&0 \leq u\left(\mathbf{x}, t\right) \leq 1 \text{ for almost everywhere } \left(\mathbf{x}, t\right) \in Q_T, \\
&u_\eta \longrightarrow u \text{ strongly in } L^2\left(Q_T\right) \text{ and a.e. in } Q_T, \\
&a(u_\eta)\nabla u_\eta \longrightarrow a(u)\nabla u \text{ weakly } \left(L^2\left(Q_T\right)\right)^d, \\
&J\left(u_\eta\right) \longrightarrow J\left(u\right) \text{ strongly } L^2\left(Q_T\right), \\
&J\left(u_\eta\right) \longrightarrow J\left(u\right) \text{ weakly in } L^2\left(0, T; H_0^1\left(\Omega\right)\right), \\
&\sqrt{a\left(u_\eta\right) \mu'\left(u_\eta\right)}\nabla u_\eta \longrightarrow \sqrt{a\left(u\right) \mu'\left(u\right)}\nabla u \text{ weakly in } \left(L^2\left(Q_T\right)\right)^d, \\
&1_{\{u_\eta \leq u_*\}} u_\eta^{q_1} a\left(u_\eta\right) \nabla u_\eta \longrightarrow 1_{\{u \leq u_*\}} u^{q_1} a\left(u\right) \nabla u \text{ a.e. in } Q_T, \\
&1_{\{u_\eta \geq u_*\}}\left(1 - u_\eta\right)^{q_2} a\left(u_\eta\right) \nabla u_\eta \longrightarrow 1_{\{u \geq u_*\}}\left(1 - u\right)^{q_2} a\left(u\right) \nabla u \text{ a.e. in } Q_T.
\end{aligned}
\tag{50}
$$

We consider the following weak formulation

$$
\begin{aligned}
&-\int_{Q_T} J_{\theta,\lambda}\left(u_\eta\right) \partial_t \chi \, d\mathbf{x} \, dt - \int_\Omega J_{\theta,\lambda}\left(u_0\left(\mathbf{x}\right)\right) \chi\left(\mathbf{x}, 0\right) d\mathbf{x} \\
&+\int_{Q_T} a\left(u_\eta\right) \nabla u_\eta \cdot \nabla j_{\theta,\lambda}\left(u_\eta\right) \chi \, d\mathbf{x} \, dt + \int_{Q_T} a\left(u_\eta\right) \nabla u_\eta \cdot \nabla \chi j_{\theta,\lambda}\left(u_\eta\right) d\mathbf{x} \, dt \\
&+\eta \int_{Q_T} \nabla u_\eta \cdot \nabla j_{\theta,\lambda}\left(u_\eta\right) \chi \, d\mathbf{x} \, dt + \eta \int_{Q_T} \nabla u_\eta \cdot \nabla \chi j_{\theta,\lambda}\left(u_\eta\right) d\mathbf{x} \, dt \\
&-\int_{Q_T}\left(f\left(u_\eta\right) - g\left(u_\eta\right)\right) \mathbf{V} \cdot \nabla j_{\theta,\lambda}\left(u_\eta\right) \chi \, d\mathbf{x} \, dt + \int_{Q_T} g'\left(u_\eta\right) \mathbf{V} \cdot \nabla u_\eta j_{\theta,\lambda}\left(u_\eta\right) \chi \, d\mathbf{x} \, dt \\
&-\int_{Q_T}\left(f\left(u_\eta\right) - g\left(u_\eta\right)\right) \mathbf{V} \cdot \nabla \chi j_{\theta,\lambda}\left(u_\eta\right) d\mathbf{x} \, dt + \int_{Q_T} a\left(u_\eta\right) \mathbf{V} \cdot \nabla u_\eta j_{\theta,\lambda}\left(u_\eta\right) \chi \, d\mathbf{x} \, dt \\
&+\eta \int_{Q_T} \nabla u_\eta \cdot \mathbf{V} j_{\theta,\lambda}\left(u_\eta\right) \chi \, d\mathbf{x} \, dt = 0, \quad \forall \chi \in \mathscr{C}^1\left([0, T]; H_0^1\left(\Omega\right)\right) \text{ with } \chi\left(T, \cdot\right) = 0.
\end{aligned}
\tag{51}
$$

By splitting these integrals into two sub integrals, we denote by L_i, $i = 1, ..., 11$ the integral terms of the form $\displaystyle\int_{Q_T \cap \{u_\eta \leq u_*\}}$ in (51).

From the definition (4) of the function $j_{\theta,\lambda}$, we have

$$
\int_{Q_T \cap \{u_\eta \leq u_*\}} a\left(u_\eta\right) \nabla u_\eta \cdot \nabla \chi j_{\theta,\lambda}\left(u_\eta\right) d\mathbf{x} \, dt = \int_{Q_T \cap \{u_\eta \leq u_*\}} u_\eta^{r-1} \nabla u_\eta \cdot u_\eta^\theta a\left(u_\eta\right) \nabla \chi \, d\mathbf{x} \, dt.
$$

The sequence $\left(u_\eta^{r-1} \nabla u_\eta \right)_\eta$ converges weakly towards $u^{r-1} \nabla u$ in $\left(L^2 \left(Q_T \right) \right)^d$. Further, thanks to Lebesgue's theorem, the sequence $\left(u_\eta^\theta a \left(u_\eta \right) \nabla \chi \right)_\eta$ converges strongly towards $u^\theta a \left(u \right) \nabla \chi$ in $\left(L^2 \left(Q_T \right) \right)^d$; this gives the convergences of terms L_4, and L_{10}. In the same manner, we obtain the convergence of $L_8 + L_9$ towards

$$\int_{Q_T \cap \{u_\eta \leq u_*\}} g' \left(u \right) \mathbf{V} \cdot \nabla u j_{\theta, \lambda} \left(u \right) \chi \, d\mathbf{x} \, dt - \int_{Q_T \cap \{u_\eta \leq u_*\}} \left(f \left(u \right) - g \left(u \right) \right) \mathbf{V} \cdot \nabla \chi j_{\theta, \lambda} \left(u \right) d\mathbf{x} \, dt.$$

Let us focus on the seventh term L_7 of equation (51). Since $\theta > 1$, then we define $\theta_0 = \theta - 1 > 0$. Therefore, using the dominated convergence Lebesgue theorem and the weak convergence (50), one has

$$L_7 = - \left(r - 1 + \theta \right) \int_{Q_T \cap \{u_\eta \leq u_*\}} u_\eta^{r-1} \nabla u_\eta \cdot \mathbf{V} u_\eta^{\theta_0} \left(f \left(u_\eta \right) - g \left(u_\eta \right) \right) \chi \, d\mathbf{x} \, dt$$

$$\xrightarrow[\eta \to 0]{} \int_{Q_T \cap \{u_\eta \leq u_*\}} \left(f \left(u \right) - g \left(u \right) \right) \mathbf{V} \cdot \nabla j_{\theta, \lambda} \left(u \right) \chi \, d\mathbf{x} \, dt.$$

For the fifth term, we have

$$|L_5| = \eta \left| \int_{Q_T \cap \{u_\eta \leq u_*\}} \nabla u_\eta \cdot \nabla j_{\theta, \lambda} \left(u_\eta \right) \chi \, d\mathbf{x} \, dt \right| = C\eta \int_{Q_T} \left| u_\eta^{r-2+\theta} \chi \nabla u_\eta \cdot \nabla u_\eta \right| d\mathbf{x} \, dt$$

$$\leq C\eta \left(\int_{Q_T \cap \{u_\eta \leq \eta^{\frac{1}{2r}}\}} u_\eta^{r-2+\theta} \left| \nabla u_\eta \right|^2 d\mathbf{x} \, dt + \int_{Q_T \cap \{u_\eta > \eta^{\frac{1}{2r}}\}} u_\eta^{r-2+\theta} \left| \nabla u_\eta \right|^2 d\mathbf{x} \, dt \right) \| \chi \|_{L^\infty (Q_T)}$$

$$\leq \left(C\eta^{\frac{\theta}{2r}} \left\| \sqrt{\eta \mu' \left(u_\eta \right)} \nabla u_\eta \right\|_{(L^2 (Q_T))^d} + C\eta^{\frac{1}{2}} \left\| u_\eta^{r-1} \nabla u_\eta \right\|_{(L^2 (Q_T))^d} \right) \| \chi \|_{L^\infty (Q_T)}.$$

As a consequence, $|L_5| \longrightarrow 0$, as η goes to zero.

The convergence to zero for the sixth and the last terms is similar to that of L_7. Indeed, we have

$$|L_6| = \eta \left| \int_{Q_T \cap \{u_\eta \leq u_*\}} \nabla u_\eta \cdot \nabla \chi j_{\theta, \lambda} \left(u_\eta \right) d\mathbf{x} \, dt \right| = \eta \left| \int_{Q_T \cap \{u_\eta \leq u_*\}} u_\eta^{r-1} \nabla u_\eta \cdot u_\eta^\theta \nabla \chi \, d\mathbf{x} \, dt \right|$$

$$\leq C\eta \left\| u_\eta^{r-1} \nabla u_\eta \right\|_{(L^2 (Q_T))^d} \| \nabla \chi \|_{(L^2 (Q_T))^d} \xrightarrow[\eta \to 0]{} 0.$$

Now, let us show the convergence for the remaining terms of the form $\displaystyle\int_{Q_T \cap \{u_\eta \geq u_*\}}$.

We have that the sequence $\left(a \left(u_\eta \right) \nabla u_\eta \right)_\eta$ converges weakly in $\left(L^2 \left(Q_T \right) \right)^d$ towards $a \left(u \right) \nabla u$ and the sequence $\left(\nabla \chi j_{\theta, \lambda} \left(u_\eta \right) \right)_\eta$ converges strongly in $\left(L^2 \left(Q_T \right) \right)^d$ towards $\nabla \chi j_{\theta, \lambda} \left(u \right)$, then

$$\int_{\{u_\eta \geq u_*\}} a \left(u_\eta \right) \nabla u_\eta \cdot \nabla \chi j_{\theta, \lambda} \left(u_\eta \right) d\mathbf{x} \, dt \xrightarrow[\eta \to 0]{} \int_{\{u \geq u_*\}} a \left(u \right) \nabla u \cdot \nabla \chi j_{\theta, \lambda} \left(u \right) d\mathbf{x} \, dt.$$

Furthermore, we have

$$\eta \left| \int\limits_{\{u_\eta \geq u_*\}} \nabla u_\eta \cdot \nabla j_{\theta,\lambda}(u_\eta) \chi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \right| = C\eta \int\limits_{Q_T} \left| (1-u_\eta)^{\frac{r'}{2}-2+\lambda} \chi \nabla u_\eta \cdot \nabla u_\eta \right| \mathrm{d}\mathbf{x} \, \mathrm{d}t$$

$$\leq C\eta \int\limits_{Q_T} \left| (1-u_\eta)^{2r_2} \chi \nabla u_\eta \cdot \nabla u_\eta \right| \mathrm{d}\mathbf{x} \, \mathrm{d}t \leq C\eta \|a(u_\eta)\nabla u_\eta\|_{(L^2(Q_T))^d} \|\chi\|_{L^\infty(Q_T)}.$$

As a consequence

$$\eta \int\limits_{\{u_\eta \geq u_*\}} \nabla u_\eta \cdot \nabla j_{\theta,\lambda}(u_\eta) \chi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \xrightarrow[\eta \to 0]{} 0. \tag{52}$$

In the same manner, we can prove the convergence of the remaining terms on the right-hand except for the third term. Indeed, this term exhibits a product of a sequence which converges weakly in $L^2(Q_T)$ and a sequence that we cannot prove its strong convergence. However, using the convergence almost everywhere of the sequences $1_{\{u_\eta \leq u_*\}} u_\eta^{q_1} a(u_\eta) \nabla u_\eta$ and $1_{\{u_\eta \geq u_*\}}(1-u_\eta)^{q_2} a(u_\eta) \nabla u_\eta$, we can get a result on the convergence of the third term. To do this, we remark that $(a(u_\eta)\nabla u_\eta \cdot \nabla j_{\theta,\lambda}(u_\eta))_\eta$ is a nonnegative sequence and into region Ω_1, we have

$$a(u_\eta)\nabla u_\eta \cdot \nabla j_{\theta,\lambda}(u_\eta) = (r-1+\theta) u_\eta^{r-2+\theta} a(u_\eta) \nabla u_\eta \cdot \nabla u_\eta$$

converges almost everywhere, up to a subsequence, to $a(u)\nabla u \cdot \nabla j_{\theta,\lambda}(u)$, since $r-2+\theta-2q-r_1 \geq 0$, i.e. $\theta \geq 7r_1 + 6 - r$.

In the same manner and into region Ω_2, we have

$$a(u_\eta)\nabla u_\eta \cdot \nabla j_{\theta,\lambda}(u_\eta) = c(u_*)(1-u_\eta)^{\frac{r'}{2}-2+\lambda} a(u_\eta) \nabla u_\eta \cdot \nabla u_\eta$$

which converges almost everywhere, up to a subsequence, to $a(u)\nabla u \cdot \nabla j_{\theta,\lambda}(u)$, since $\frac{r'}{2}-2+\lambda-2q_2-r_2 \geq 0$, i.e. $\lambda \geq 7r_2 + 6 - \frac{r'}{2}$.

Consider a nonnegative test function ($\chi \geq 0$); then the Fatou's lemma ensures that

$$\liminf_{\eta \to 0} \int\limits_{Q_T} a(u_\eta)\nabla u_\eta \cdot \nabla j_{\theta,\lambda}(u_\eta) \chi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \geq \int\limits_{Q_T} a(u)\nabla u \cdot \nabla j_{\theta,\lambda}(u) \chi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t,$$

then the limit solution u verifies inequality (9) into Definition 1.1. Finally, to obtain (10), we apply the Egorov theorem on the sequence $(a(u_\eta)\nabla u_\eta \cdot \nabla j_{\theta,\lambda}(u_\eta))_\eta$ which converges almost everywhere. Indeed, we have

$$\forall \varepsilon > 0, \exists Q^\varepsilon \subset Q_T \text{ such that } \operatorname{meas}(Q^\varepsilon) < \varepsilon, \text{ and}$$

$$a(u_\eta)\nabla u_\eta \cdot \nabla j_{\theta,\lambda}(u_\eta) \xrightarrow[\eta \to 0]{} a(u)\nabla u \cdot \nabla j_{\theta,\lambda}(u) \text{ uniformly in } Q_T \backslash Q^\varepsilon.$$

Now, we take a nonnegative test function χ such that $\operatorname{supp}\chi \subset ([0,T) \times \Omega) \backslash Q^\varepsilon$, then

$$\int\limits_{Q_T \backslash Q^\varepsilon} a(u_\eta)\nabla u_\eta \cdot \nabla j_{\theta,\lambda}(u_\eta) \chi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t \xrightarrow[\eta \to 0]{} \int\limits_{Q_T \backslash Q^\varepsilon} a(u)\nabla u \cdot \nabla j_{\theta,\lambda}(u) \chi \, \mathrm{d}\mathbf{x} \, \mathrm{d}t.$$

This ends the proof of Theorem 1.1. $\quad \square$

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